A Discussion of Financial Economics in Actuarial Models
A Preparation for the Actuarial Exam MFE/3F

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To Pallavi and Amin

Preface

This is the third of a series of books intended to help individuals to pass actuarial exams. The present manuscript covers the financial economics segment of Exam M referred to by MFE/3F.
The flow of topics in the book follows very closely that of McDonald's *Derivatives Markets*. The book covers designated sections from this book as suggested by the 2009 SOA Syllabus.
The recommended approach for using this book is to read each section, work on the embedded examples, and then try the problems. Answer keys are provided so that you check your numerical answers against the correct ones.
Problems taken from previous SOA/CAS exams will be indicated by the symbol ‡.
This manuscript can be used for personal use or class use, but not for commercial purposes. If you find any errors, I would appreciate hearing from you: mfinan@atu.edu

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Contents

Parity and Other Price Options Properties

Parity is one of the most important relations in option pricing. In this chapter we discuss different versions of parity for different underlying assets. The main parity relation for European options can be rearranged to create synthetic securities. Also, options where the underlying asset and the strike asset can be anything are discussed. Bounds of option prices for European and American options are also discussed.
Since many of the discussions in this book are based on the no-arbitrage principle, we will remind the reader of this concept.

The concept of no-arbitrage:
No arbitrage principle assumes there are no transaction costs such as tax and commissions.
Arbitrage is possible when one of three conditions is met:
1. The same asset does not trade at the same price on all markets ("the law of one price").
2. Two assets with identical cash flows do not trade at the same price.
3. An asset with a known price in the future does not today trade at its future price discounted at the risk-free interest rate.

1 A Review of Options

Derivative securities are financial instruments that derive their value from the value of other assets. Examples of derivatives are forwards, options, and swaps. In this section we discuss briefly the basic vocabulary of options.
A **forward contract** is the obligation to buy or sell something at a pre-specified time (called the **expiration date**, the **delivery date**, or the **maturity date**) and at a pre-specified price, known as the **forward price** or **delivery price.** A forward contract requires no initial premium. In contrast, an **option** is a contract that gives the owner the right, but not the obligation, to buy or sell a specified asset at a specified price, on or by a specified date. The underlying assets include stocks, major currencies, and bonds. The majority of options are traded on an exchange (such as Chicago Board of Exchange) or in the over-the-counter market.
There are two types of options: A **call option** gives the right to the owner to buy the asset. A **put** option gives the right to the owner to sell the asset. Option trading involves two parties: a buyer and a seller. The **buyer** or **owner** of a call (put) option obtains the right to buy (sell) an asset at a specified price by paying a **premium** to the **writer** or **seller** of the option, who assumes the collateral obligation to sell (buy) the asset, should the owner of the option choose to exercise it. The buyer of an option is said to take a **long** position in the option whereas the seller is said to take a **short** position in the option. A **short-sale** of an asset (or **shorting** the asset) entails borrowing the asset and then selling it, receiving the cash. Some time later, we buy back the asset, paying cash for it, and return it to the lender.
Additional terms are needed in understanding option contracts. The **strike price** or **exercise price** (denoted by K) is the fixed price specified in the option contract for which the holder can buy or sell the underlying asset. The **expiration date, exercise date,** or **maturity** (denoted by T with $T = 0$ for "today") is the last date on which the contract is still valid. After this date the contract no longer exists. By **exercising the option** we mean enforcing the contract,i.e., buy or sell the underlying asset using the option. Two types of exercising options (also known as **style**) are: An **American option** may be exercised at any time up to the expiration date. A **European option** on the other hand, may be exercised only on the expiration date. Unless otherwise stated, options are considered to be Europeans.

Example 1.1

A call option on ABC Corp stock currently trades for \$6. The expiration date is December 17, 2005. The strike price of the option is \$95.
(a) If this is an American option, on what dates can the option be exercised?
(b) If this is a European option, on what dates can the option be exercised?

Solution.
(a) Any date before and including the expiration date, December 17, 2005.
(b) Only on Dec 17, 2005 ■

The **payoff** or **intrinsic value** from a call option at the expiration date is a function of the strike price K and the spot (or market) price S_T of the underlying asset on the delivery date. It is given by $\max\{0, S_T - K\}$. From this definition we conclude that if a call option is held until expiration (which must be so for a European option, but not an American option) then the option will be exercised if, and only if, $S_T > K$, in which case the owner of the option will realize a net payoff $S_T - K > 0$ and the writer of the option will realize a net payoff(loss) $K - S_T < 0$.
Likewise, the payoff from a put option at the expiration date is a function of the strike price K and the spot price S_T of the underlying asset on the delivery date. It is given by $\max\{0, K - S_T\}$. From this definition we conclude that if a put option is held until expiration (which must be so for a European option, but not an American option) then the option will be exercised if, and only if, $S_T < K$, in which case the owner of the option will realize a net payoff $K - S_T > 0$ and the writer of the option will realize a net payoff(loss) $S_T - K < 0$.

Example 1.2
Suppose you buy a 6-month call option with a strike price of \$50. What is the payoff in 6 months for prices \$45, \$50, \$55, and \$60?

Solution.
The payoff to a purchased call option at expiration is:

$$\text{Payoff to call option} = \max\{0, \text{spot price at expiration} - \text{strike price}\}$$

The strike is given: It is \$50. Therefore, we can construct the following table:

Price of assets in 6 months	Strike price	Payoff to the call option
45	50	0
50	50	0
55	50	5
60	50	10 ■

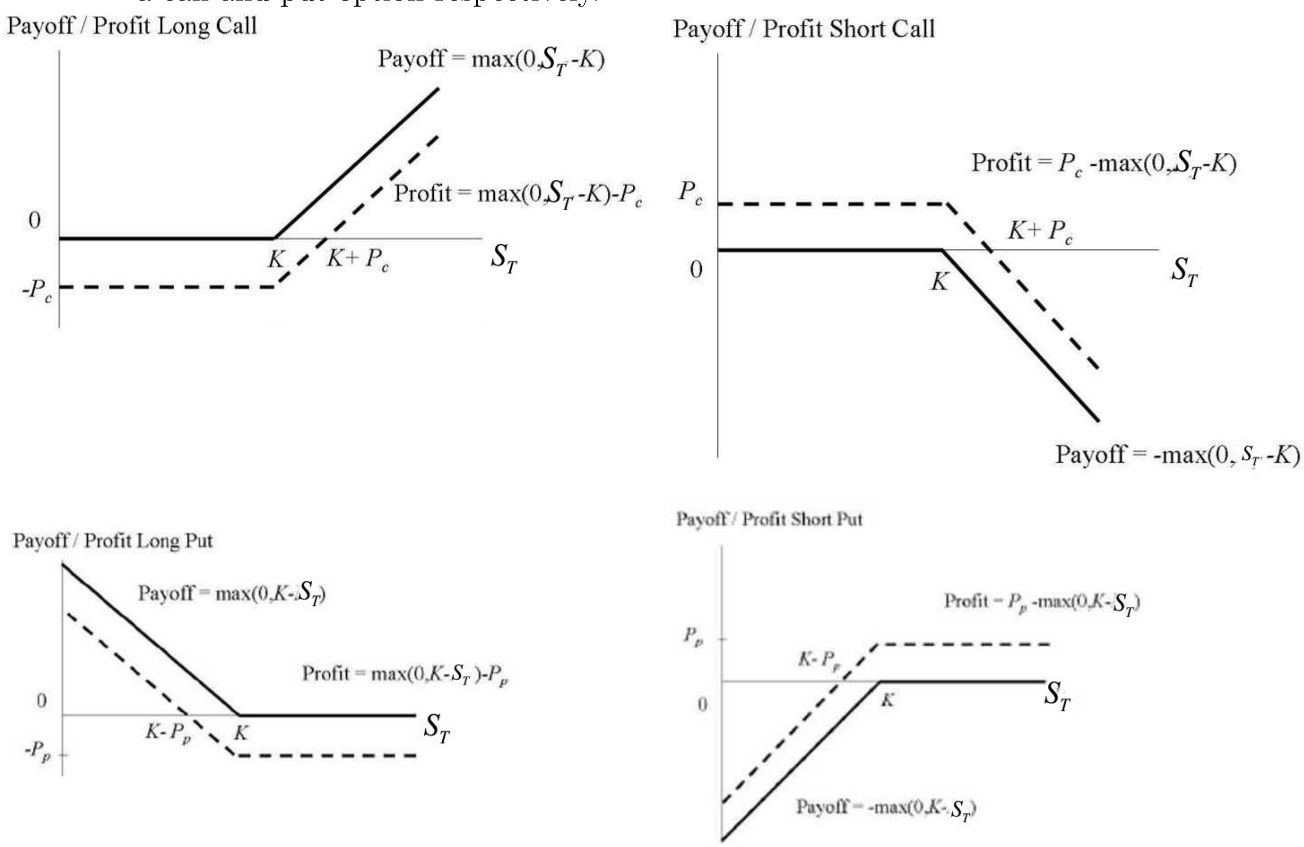
Payoff does not take into consideration the premium which is paid to acquire an option. Thus, the payoff of an option is not the money earned (or lost). For a call option we define the profit earned by the owner of the option by

$$\text{Buyer's call profit} = \text{Buyer's call payoff} - \text{future value of premium.}$$

Likewise, we define the profit of a put option by

$$\text{Buyer's put profit} = \text{Buyer's put payoff} - \text{future value of premium.}$$

Payoff and profit diagrams of a call option and a put option are shown in the figure below. P_c and P_p will denote the future value of the premium for a call and put option respectively.



Example 1.3
You hold a European call option on 100 shares of Coca-Cola stock. The exercise price of the call is \$50. The option will expire in moments. Assume

there are no transactions costs or taxes associated with this contract.
(a) What is your profit on this contract if the stock is selling for \$51?
(b) If Coca-Cola stock is selling for \$49, what will you do?

Solution.
(a) The profit is $1 \times 100 = \$100$.
(b) Do not exercise. The option is thus expired without exercise ■

Example 1.4
One can use options to insure assets we own (or purchase) or assets we short sale. An investor who owns an asset (i.e. being long an asset) and wants to be protected from the fall of the asset's value can insure his asset by buying a put option with a desired strike price. This combination of owning an asset and owning a put option on that asset is called a **floor**. The put option guarantees a minimum sale price of the asset equals the strike price of the put.
Show that buying a stock at a price S and buying a put option on the stock with strike price K and time to expiration T has payoff equals to the payoff of buying a zero-coupon bond with par-value K and buying a call on the stock with strike price K and expiration time T.

Solution.
We have the following payoff tables.

		Payoff at Time T	
Transaction	Payoff at Time 0	$S_T < K$	$S_T \geq K$
Buy a stock	$-S$	S_T	S_T
Buy a put	$-P$	$K - S_T$	0
Total	$-S - P$	K	S_T

		Payoff at Time T	
Transaction	Payoff at Time 0	$S_T \leq K$	$S_T > K$
Buy a Bond	$-PV_{0,T}(K)$	K	K
Buy a Call	$-C$	0	$S_T - K$
Total	$-PV_{0,T}(K) - C$	K	S_T

Both positions guarantee a payoff of $\max\{K, S_T\}$. By the no-arbitrage principle they must have same payoff at time $t = 0$. Thus,

$$P + S = C + PV_{0,T}(K) \blacksquare$$

An option is said to be **in-the-money** if its immediate exercise would produce positive cash flow. Thus, a put option is in the money if the strike price exceeds the spot price or (market price) of the underlying asset and a call option is in the money if the spot price of the underlying asset exceeds the strike price. An option that is not in the money is said to be **out-of-the-money.** An option is said to be **at-the-money** if its immediate exercise produces zero cash flow. Letting S_T denote the market price at time T we have the following chart.

	Call	Put
$S_T > K$	In-the-money	out-of-the-money
$S_T = K$	At-the-money	At-of-the-money
$S_T < K$	Out-of-the-money	In-the-money

An option with an exercise price significantly below (for a call option) or above (for a put option) the market price of the underlying asset is said to be **deep in-the-money**. An option with an exercise price significantly above (for a call option) or below (for a put option) the market price of the underlying asset is said to be **deep out-of-the money.**

Example 1.5
If the underlying stock price is \$25, indicate whether each of the options below is in-the-money, at-the-money, or out-of-the-money.

Strike	Call	Put
\$20		
\$25		
\$30		

Solution.

Strike	Call	Put
\$20	In-the-money	out-of-the-money
\$25	At-the-money	At-the-money
\$30	Out-of-the-money	In-the-money ■

Practice Problems

Problem 1.1
When you short an asset, you borrow the asset and sell, hoping to replace them at a lower price and profit from the decline. Thus, a short seller will experience loss if the price rises. He can insure his position by purchasing a call option to protect against a higher price of repurchasing the asset. This combination of short sale and call option purchase is called a **cap**.
Show that a short-sale of a stock and buying a call has a payoff equals to that of buying a put and selling a zero-coupon bond with par-value the strike price of the options. Both options have the same strike price and time to expiration.

Problem 1.2
Writing an option backed or covered by the underlying asset (such as owning the asset in the case of a call or shorting the asset in the case of a put) is referred to as **covered writing** or **option overwriting**. The most common motivation for covered writing is to generate additional income by means of premium.
A **covered call** is a call option which is sold by an investor who owns the underlying assets. An investor's risk is limited when selling a covered call since the investor already owns the underlying asset to cover the option if the covered call is exercised. By selling a covered call an investor is attempting to capitalize on a neutral or declining price in the underlying stock. When a covered call expires without being exercised (as would be the case in a declining or neutral market), the investor keeps the premium generated by selling the covered call. The opposite of a covered call is a **naked call**, where a call is written without owned assets to cover the call if it is exercised.
Show that the payoff of buying a stock and selling a call on the stock is equal to the payoff of buying a zero-coupon bond with par-value the strike price of the options and selling a put. Both options have a strike price K and time to expiration T.

Problem 1.3
A **covered put** is a put option which is sold by an investor and which is covered (backed) by a short sale of the underlying assets. A covered put may also be covered by deposited cash or cash equivalent equal to the exercise price of the covered put. The opposite of a covered put would be a **naked**

put.
Show that the payoff of shorting a stock and selling a put on the stock is equal to the payoff of selling a zero-coupon bond with par-value the strike price of the options and selling a call. Both options have a strike price K and time to expiration T.

Problem 1.4
A position that consists of buying a call with strike price K_1 and expiration T and selling a call with strike price $K_2 > K_1$ and same expiration date is called a **bull call spread.** In contrast, buying a put with strike price K_1 and expiration T and selling a put with strike price $K_2 > K_1$ and same expiration date is called a **bull put spread.** An investor who enters a bull spread is speculating that the stock price will increase.
(a) Find formulas for the payoff and profit functions of a bull call spread. Draw the diagrams.
(b) Find formulas for the payoff and profit functions of a bull put spread. Draw the diagrams.

Problem 1.5
A **bear spread is precisely** the opposite of a bull spread. An investor who enters a bull spread is hoping that the stock price will increase. By contrast, an investor who enters a bear spread is hoping that the stock price will decline. Let $0 < K_1 < K_2$. A bear spread can be created by either selling a K_1-strike call and buying a K_2-strike call, both with the same expiration date (bear call spread), or by selling a K_1-strike put and buying a K_2-strike put, both with the same expiration date (bear put spread).
Find formulas for the payoff and the profit of a bear spread created by selling a K_1-strike call and buying a K_2-strike call, both with the same expiration date. Draw the diagram.

Problem 1.6
A **(call) ratio spread** is achieved by buying a certain number of calls with a high strike and selling a different number of calls at a lower strike. By replacing the calls with puts one gets a **(put) ratio spread.** All options under considerations have the same expiration date and same underlying asset. If m calls were bought and n calls were sold we say that the ratio is $\frac{m}{n}$.
An investor buys one \$70-strike call and sells two \$85-strike call of a stock.

All the calls have expiration date one year from now. The risk free annual effective rate of interest is 5%. The premiums of the \$70-strike and \$85-strike calls are \$10.76 and \$3.68 respectively. Draw the profit diagram.

Problem 1.7
A **collar** is achieved with the purchase of a put option with strike price K_1 and expiration date T and the selling of a call option with strike price $K_2 > K_1$ and expiration date T. Both options use the same underlying asset. A collar can be used to speculate on the decrease of price of an asset. The difference $K_2 - K_1$ is called the **collar width**. A **written collar** is the reverse of collar (sale of a put and purchase of a call).
Find the profit function of a collar as a function of the spot price S_T and draw its graph.

Problem 1.8
Collars can be used to insure assets we own. This position is called a **collared stock.** A collared stock involves buying the index, buy an at-the-money K_1-strike put option (which insures the index) and selling an out-of-the-money K_2-strike call option (to reduce the cost of the insurance), where $K_1 < K_2$.
Find the profit function of a collared stock as a function of S_T.

Problem 1.9
A **long straddle** or simply a **straddle** is an option strategy that is achieved by buying a $K-$strike call and a $K-$strike put with the same expiration time T and same underlying asset. Find the payoff and profit functions of a straddle. Draw the diagrams.

Problem 1.10
A **strangle** is a position that consists of buying a K_1-strike put and a K_2-strike call with the same expiration date and same underlying asset and such that $K_1 < K_2$. Find the profit formula for a strangle and draw its diagram.

Problem 1.11
Given $0 < K_1 < K_2 < K_3$. A **butterfly spread** is created by using the following combination:
(1) Create a written straddle by selling a K_2-strike call and a K_2-strike put.

(2) Create a long strangle by buying a K_1-strike call and a K_3-strike put. Find the initial cost and the profit function of the position. Draw the profit diagram.

Problem 1.12
Given $0 < K_1 < K_2 < K_3$. When a symmetric butterfly is created using these strike prices the number K_2 is the midpoint of the interval with endpoints K_1 and K_3. What if K_2 is not midway between K_1 and K_3? In this case, one can create a butterfly-like spread with the peak tilted either to the left or to the right as follows: Define a number λ by the formula

$$\lambda = \frac{K_3 - K_2}{K_3 - K_1}.$$

Then λ satisfies the equation

$$K_2 = \lambda K_1 + (1 - \lambda)K_3.$$

Thus, for every written K_2-strike call, a butterfly-like spread can be constructed by buying λ K_1-strike calls and $(1-\lambda)$ K_3-strike calls. The resulting spread is an example of an **asymmetric butterfly spread.**
Construct an asymmetric butterfly spread using the 35-strike call (with premium \$6.13), 43-strike call (with premium \$1.525) and 45-strike call (with premium \$0.97). All options expire 3 months from now. The risk free annual effective rate of interest is 8.33%. Draw the profit diagram.

Problem 1.13
Suppose the stock price is \$40 and the effective annual interest rate is 8%.
(a) Draw a single graph payoff diagram for the options below.
(b) Draw a single graph profit diagram for the options below.
(i) A one year 35-strike call with a premium of \$9.12.
(ii) A one year 40-strike call with a premium of \$6.22.
(iii) A one year 45-strike call with a premium of \$4.08.

Problem 1.14
A trader buys a European call option and sells a European put option. The options have the same underlying asset, strike price, and maturity date. Show that this position is equivalent to a long forward.

Problem 1.15
Suppose the stock price is \$40 and the effective annual interest rate is 8%.
(a) Draw the payoff diagrams on the same window for the options below.
(b) Draw the profit diagrams on the same window for the options below.
(i) A one year 35-strike put with a premium of \$1.53.
(ii) A one year 40-strike put with a premium of \$3.26.
(iii) A one year 45-strike put with a premium of \$5.75.

Problem 1.16
Complete the following table.

Derivative Position	Maximum Loss	Maximum Gain	Position with Respect to Underlying Asset	Strategy
Long Forward				
Short Forward				
Long Call				
Short Call				
Long Put				
Short Put				

Problem 1.17 ‡
An insurance company sells single premium deferred annuity contracts with return linked to a stock index, the time$-t$ value of one unit of which is denoted by $S(t)$. The contracts offer a minimum guarantee return rate of $g\%$. At time 0, a single premium of amount π is paid by the policyholder, and $\pi \times y\%$ is deducted by the insurance company. Thus, at the contract maturity date, T, the insurance company will pay the policyholder

$$\pi \times (1 - y\%) \times \text{Max}[S(T)/S(0), (1 + g\%)^T].$$

You are given the following information:
(i) The contract will mature in one year.
(ii) The minimum guarantee rate of return, g%, is 3%.
(iii) Dividends are incorporated in the stock index. That is, the stock index is constructed with all stock dividends reinvested.
(iv) $S(0) = 100$.
(v) The price of a one-year European put option, with strike price of \$103, on the stock index is \$15.21.

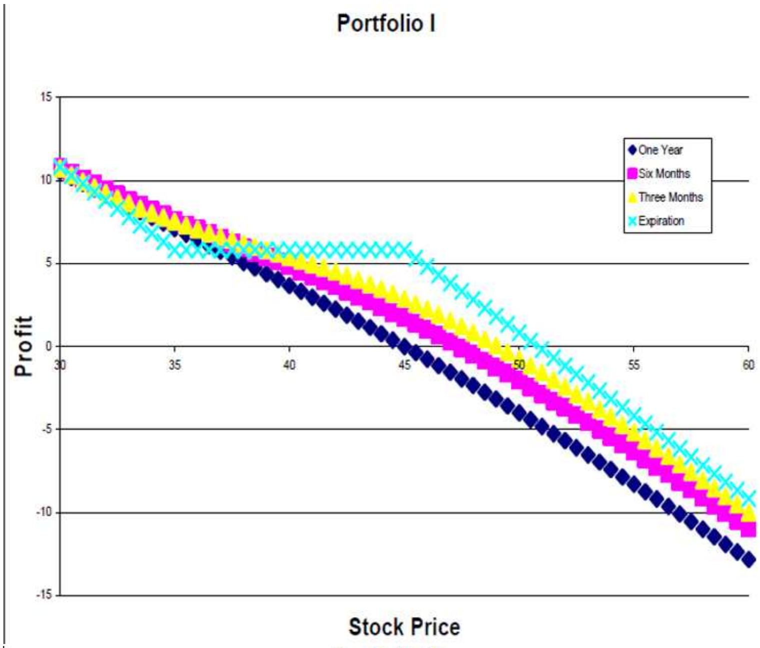
Determine y%, so that the insurance company does not make or lose money on this contract (i.e., the company breaks even.)

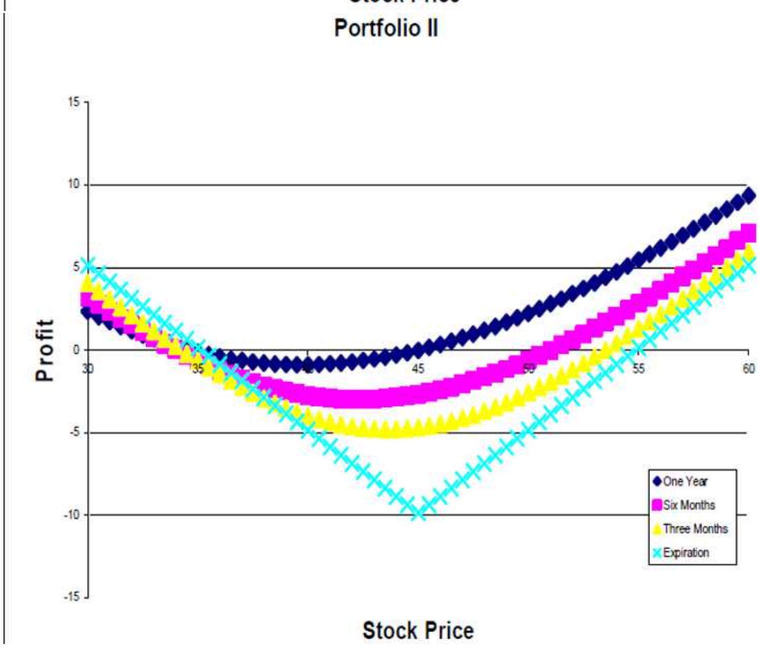
Problem 1.18 ‡
You are given the following information about a securities market:
(i) There are two nondividend-paying stocks, X and Y.
(ii) The current prices for X and Y are both \$100.
(iii) The continuously compounded risk-free interest rate is 10%.
(iv) There are three possible outcomes for the prices of X and Y one year from now:

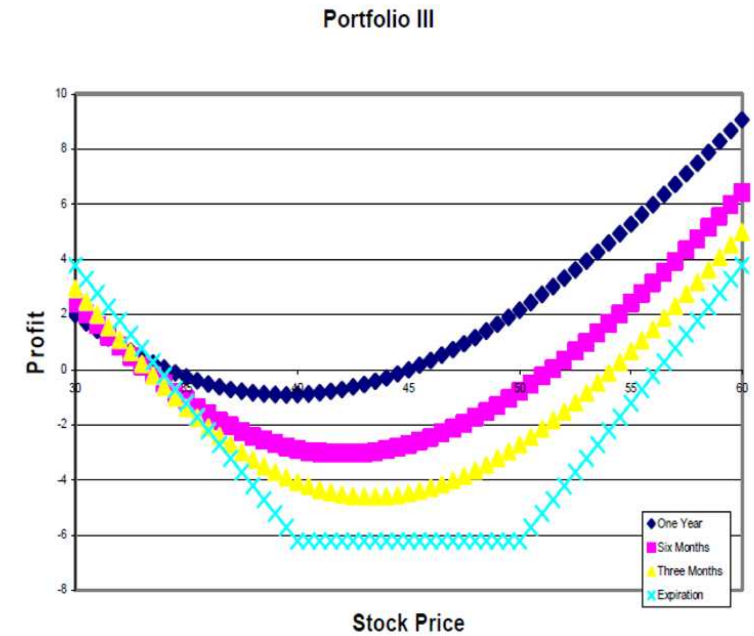
Outcome	X	Y
1	\$200	\$0
2	\$50	\$0
3	\$0	\$300

Consider the following two portfolios:
• Portfolio A consists of selling a European call on X and buying a European put on Y. Both options expire in one year and have a strike price of \$95. The cost of this portfolio at time 0 is then $C_X - P_Y$.
• Portfolio B consists of buying M shares of X, N shares of Y and lending \$P. The time 0 cost of this portfolio is $-100M - 100N - P$.
Determine $P_Y - C_X$ if the time-1 payoff of Portfolio B is equal to the time-1 payoff of Portfolio A.

Problem 1.19 ‡
The following four charts are profit diagrams for four option strategies: Bull Spread, Collar, Straddle, and Strangle. Each strategy is constructed with the purchase or sale of two 1-year European options. Match the charts with the option strategies.


Portfolio I
One Year
Six Months
Three Months
Expiration
Profit
Stock Price
15
10
5
0
-5
-10
-15
30
35
40
45
50
55
60


Portfolio II
One Year
Six Months
Three Months
Expiration
Profit
Stock Price
15
10
5
0
-5
-10
-15
30
35
45
55
60


Portfolio III
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One Year
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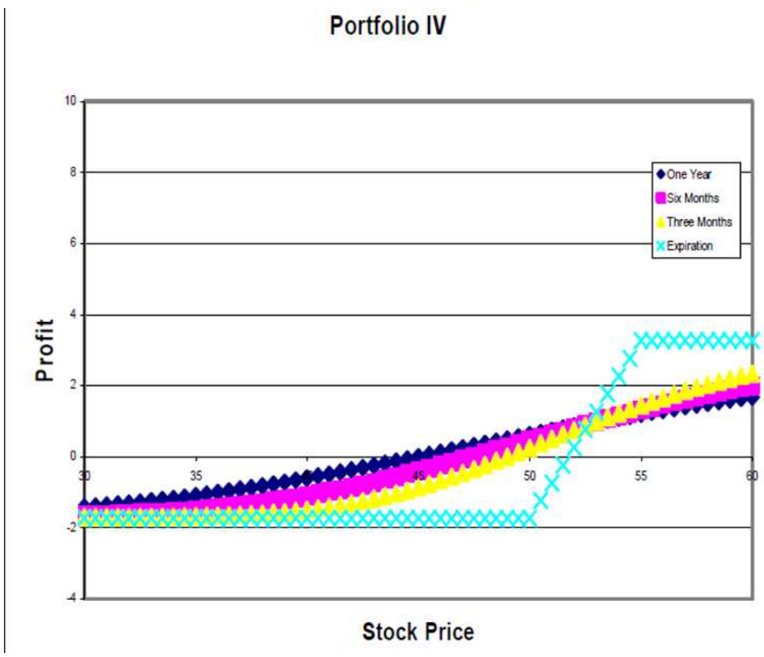
Expiration


Portfolio IV
Profit
Stock Price
One Year
Six Months
Three Months
Expiration

2 Put-Call Parity for European Options

Put-call parity is an important principle in options pricing.[1] It states that the premium of a call option implies a certain fair price for the corresponding put option having the same strike price and expiration date, and vice versa. To begin understanding how the put-call parity is established, we consider the following position: An investor buys a call option with strike price K and expiration date T for the price of $C(K,T)$ and sells a put option with the same strike price and expiration date for the price of $P(K,T)$. If the spot price at expiration is greater than K, the put will not be exercised (and thus expires worthless) but the investor will exercise the call. So the investor will buy the asset for the price of K. If the spot price at expiration is smaller than K, the call will not be exercised and the investor will be assigned on the short put, if the owner of the put wishes to sell (which is likely the case) then the investor is obliged to buy the asset for K. Either way, the investor is obliged to buy the asset for K and the long call-short put combination induces a long forward contract that is synthetic since it was fabricated from options.

Now, the portfolio consists of buying a call and selling a put results in buying the stock. The payoff of this portfolio at time 0 is $P(K,T) - C(K,T)$. The portfolio of buying the stock through a long forward contract and borrowing $PV_{0,T}(K)$ has a payoff at time 0 of $PV_{0,T}(K) - PV_{0,T}(F_{0,T})$, where $F_{0,T}$ is the forward price. Using the no-arbitrage pricing theory[2], the net cost of owning the asset must be the same whether through options or forward contracts, that is,

$$P(K,T) - C(K,T) = PV_{0,T}(K - F_{0,T}). \tag{2.1}$$

Equation (2.1) is referred to as the **put-call parity.** We point out here that $PV_{0,T}(F_{0,T})$ is the prepaid forward price for the underlying asset and $PV_{0,T}(K)$ is the prepaid forward price of the strike.

It follows from (2.1), that selling a put and buying a call with the same strike price K and maturity date create a synthetic long forward contract with forward price $F_{0.T}$ and net premium $P(K,T) - C(K,T)$. To mimic an actual forward contract the put and call premiums must be equal.

[1] **Option pricing model** is a mathemtical formula that uses the factors that determine an option's price as inputs to produce the theoretical fair value of an option.

[2] The **no-arbitrage pricing** theory asserts that acquiring an asset at time T should cost the same, no matter how you achieve it.

Since American style options allow early exercise, put-call parity will not hold for American options unless they are held to expiration. Note also that if the forward price is higher than the strike price of the options, call is more expensive than put, and vice versa.

Remark 2.1
We will use the sign "+" for selling and borrowing positions and the sign "−" for buying and lending positions.

Remark 2.2
In the literature, (2.1) is usually given in the form

$$C(K,T) - P(K,T) = PV_{0,T}(F_{0,T} - K).$$

Thus, we will use this form of parity in the rest of the book.

Remark 2.3
Note that for a nondividend paying stock we have $PV_{0,T}(F_{0,T}) = S_0$, the stock price at time 0.

Example 2.1
The current price of a nondividend-paying stock is \$80. A European call option that expires in six months with strike price of \$90 sells for \$12. Assume a continuously compounded risk-free interest rate of 8%, find the value to the nearest penny of a European put option that expires in six months and with strike price of \$90.

Solution.
Using the put-call parity (2.1) with $C = 12$, $T = 6$ months$= 0.5$ years, $K = 90$, $S_0 = 80$, and $r = 0.08$ we find

$$P = C - S_0 + e^{-rT}K = 12 - 80 + 90e^{-0.08(0.5)} = \$18.47 \blacksquare$$

Example 2.2
Suppose that the current price of a nondividend-paying stock is S_0. A European put option that expires in six months with strike price of \$90 is \$6.47 more expensive than the corresponding European call option. Assume a continuously compounded interest rate of 15%, find S_0 to the nearest dollar.

Solution.
The put-call parity

$$C(K,T) - P(K,T) = S_0 - Ke^{-rT}$$

gives

$$-6.47 = S_0 - 90e^{-0.15\times 0.5}.$$

Solving this equation we find $S_0 = \$77$ ■

Example 2.3
Consider a European call option and a European put option on a nondividend-paying stock. You are given:
(i) The current price of the stock is \$60.
(ii) The call option currently sells for \$0.15 more than the put option.
(iii) Both the call option and put option will expire in T years.
(iv) Both the call option and put option have a strike price of \$70.
(v) The continuously compounded risk-free interest rate is 3.9%.
Calculate the time of expiration T.

Solution.
Using the put-call parity

$$C(K,T) - P(K,T) = S_0 - Ke^{-rT}$$

we have

$$0.15 = 60 - 70e^{-0.039T}.$$

Solving this equation for T we find $T \approx 4$ years ■

Practice Problems

Problem 2.1
If a synthetic forward contract has no initial premium then
(A) The premium you pay for the call is larger than the premium you receive from the put.
(B) The premium you pay for the call is smaller than the premium you receive from the put.
(C) The premium you pay for the call is equal to the premium you receive from the put.
(D) None of the above.

Problem 2.2
In words, the Put-Call parity equation says that
(A) The cost of buying the asset using options must equal the cost of buying the asset using a forward.
(B) The cost of buying the asset using options must be greater than the cost of buying the asset using a forward.
(C) The cost of buying the asset using options must be smaller than the cost of buying the asset using a forward.
(D) None of the above.

Problem 2.3
State two features that differentiate a synthetic forward contract from a no-arbitrage (actual) forward contract.

Problem 2.4
Recall that a **covered call** is a call option which is sold by an investor who owns the underlying asset. Show that buying an asset plus selling a call option on the asset with strike price K (i.e. selling a covered call) is equivalent to selling a put option with strike price K and buying a zero-coupon bond with par value K.

Problem 2.5
Recall that a **covered put** is a put option which is sold by an investor and which is backed by a short sale of the underlying asset(s). Show that short selling an asset plus selling a put option with strike price K (i.e. selling a covered put) is equivalent to selling a call option with strike price K and taking out a loan with maturity value of K.

Problem 2.6
The current price of a nondividend-paying stock is \$40. A European call option on the stock with strike price \$40 and expiration date in three months sells for \$2.78 whereas a 40-strike European put with the same expiration date sells for \$1.99. Find the annual continuously compounded interest rate r. Round your answer to two decimal places.

Problem 2.7
Suppose that the current price of a nondividend-paying stock is S_0. A European call option that expires in six months with strike price of \$90 sells for \$12. A European put option with strike price of \$90 and expiration date in six months sells for \$18.47. Assume a continuously compounded interest rate of 15%, find S_0 to the nearest dollar.

Problem 2.8
Consider a European call option and a European put option on a nondividend-paying stock. You are given:
(i) The current price of the stock is \$60.
(ii) The call option currently sells for \$0.15 more than the put option.
(iii) Both the call option and put option will expire in 4 years.
(iv) Both the call option and put option have a strike price of \$70.
Calculate the continuously compounded risk-free interest rate.

Problem 2.9
The put-call parity relationship

$$C(K,T) - P(K,T) = PV(F_{0,T} - K)$$

can be rearranged to show the equivalence of the prices (and payoffs and profits) of a variety of different combinations of positions.
(a) Show that buying an index plus a put option with strike price K is equivalent to buying a call option with strike price K and a zero-coupon bond with par value of K.
(b) Show that shorting an index plus buying a call option with strike price K is equivalent to buying a put option with strike price K and taking out a loan with maturity value of K.

Problem 2.10
A call option on XYZ stock with an exercise price of \$75 and an expiration

date one year from now is worth \$5.00 today. A put option on XYZ stock with an exercise price of \$75 and an expiration date one year from now is worth \$2.75 today. The annual effective risk-free rate of return is 8% and XYZ stock pays no dividends. Find the current price of the stock.

Problem 2.11 ‡
You are given the following information:
• The current price to buy one share of XYZ stock is 500.
• The stock does not pay dividends.
• The risk-free interest rate, compounded continuously, is 6%.
• A European call option on one share of XYZ stock with a strike price of K that expires in one year costs \$66.59.
• A European put option on one share of XYZ stock with a strike price of K that expires in one year costs \$18.64.
Using put-call parity, determine the strike price K.

Problem 2.12
The current price of a stock is \$1000 and the stock pays no dividends in the coming year. The premium for a one-year European call is \$93.809 and the premium for the corresponding put is \$74.201. The annual effective risk-free interest rate is 4%. Determine the forward price of the synthetic forward. Round your answer to the nearest dollar.

Problem 2.13
The actual forward price of a stock is \$1020 and the stock pays no dividends in the coming year. The premium for a one-year European call is \$93.809 and the premium for the corresponding put is \$74.201. The forward price of the synthetic forward contract is \$1000. Determine the annual risk-free effective rate r.

Problem 2.14 ‡
You are given the following information:
• One share of the PS index currently sells for 1,000.
• The PS index does not pay dividends.
• The effective annual risk-free interest rate is 5%.
You want to lock in the ability to buy this index in one year for a price of 1,025. You can do this by buying or selling European put and call options with a strike price of 1,025. Which of the following will achieve your objective

and also gives the cost today of establishing this position.
(A) Buy the put and sell the call, receive 23.81.
(B) Buy the put and sell the call, spend 23.81.
(C) Buy the put and sell the call, no cost.
(D) Buy the call and sell the put, receive 23.81.
(E) Buy the call and sell the put, spend 23.81.

3 Put-Call Parity of Stock Options

Recall that a prepaid forward contract on a stock is a forward contract with payment made at time 0. The forward price is the future value of the prepaid forward price and this is true regardless of whether there are discrete dividends, continuous dividends, or no dividends.
If the expiration time is T then for a stock with no dividends the future value of the prepaid forward price is given by the formula

$$F_{0,T} = FV_{0,T}(S_0),$$

where S_0 is the stock's price at time $t = 0$.
In the case of a stock with discrete dividends, the prepaid forward price is

$$F_{0,T}^P = S_0 - PV_{0,T}(Div)$$

so that

$$F_{0,T} = FV_{0,T}(F_{0,T}^P) = FV_{0,T}(S_0 - PV_{0,T}(Div)).$$

In particular, if the dividends $D_1, D_2, \cdots, D_n$ made at time $t_1, t_2, \cdots, t_n$ prior to the maturity date T, then

$$\begin{aligned}F_{0,T} =& FV_{0,T}(S_0 - \sum_{i=1}^{n} PV_{0,t_i}(D_i))\\ =& FV_{0,T}(S_0) - FV_{0,T}(\sum_{i=1}^{n} PV_{0,t_i}D_i)\end{aligned}$$

In this case, Equation (2.1) takes the form

$$C(K,T) - P(K,T) = S_0 - \mathrm{PV_{0,T}(Div)} - \mathrm{PV_{0,T}(K)}$$

or in the case of a continuously compounded risk-free interest rate r

$$C(K,T) - P(K,T) = S_0 - \sum_{i=1}^{n} D_i e^{-rt_i} - e^{-rT}K.$$

Example 3.1
Suppose ABC stock costs \$75 today and is expected to pay semi-annual dividend of \$1 with the first coming in 4 months from today and the last just prior to the delivery of the stock. Suppose that the annual continuously compounded risk-free rate is 8%. Find the cost of a 1-year prepaid forward contract.

Solution.
The cost is

$$e^{-rT}F_{0,T} = S_0 - \mathrm{PV_{0,T}(Div)} = 75 - \mathrm{e}^{-0.08\times\frac{4}{12}} - \mathrm{e}^{-0.08\times\frac{10}{12}} = \$73.09 \blacksquare$$

Example 3.2
A one year European call option with strike price of K on a share of XYZ stock costs \$11.71 while a one year European put with the same strike price costs \$5.31. The share of stock pays dividend valued at \$3 six months from now and another dividend valued at \$5 one year from now. The current share price is \$99 and the continuously-compounded risk-free rate of interest is 5.6%. Determine K.

Solution.
The put-call parity of stock options with discrete dividends is

$$C - P = S_0 - \sum_{i=1}^{n} D_i e^{-rt_i} - e^{-rT}K.$$

We are given $C = 11.71,\ P = 5.31,\ T = 1,\ S_0 = 99, D_1 = 3,\ D_2 = 5, t_1 = \frac{6}{12}, t_2 = 1$ and $r = 0.056$. Solving for K we find $K = \$89.85 \blacksquare$

For a stock with continuous dividends we have

$$F_{0,T} = FV_{0,T}(S_0e^{-\delta T})$$

where δ is the continuously compounded dividend yield (which is defined to be the annualized dividend payment divided by the stock price). In this case, Equation (2.1) becomes

$$C(K,T) - P(K,T) = S_0e^{-\delta T} - PV_{0,T}(K).$$

Example 3.3
A European call option and a European put option on a share of XYZ stock have a strike price of \$100 and expiration date in nine months. They sell for \$11.71 and \$5.31 respectively. The price of XYZ stock is currently \$99, and the stock pays a continuous dividend yield of 2%. Find the continuously compounded risk-free interest rate r.

Solution.
The put-call parity of stock options with continuous dividends is

$$C - P = S_0 e^{-\delta T} - e^{-rT} K.$$

We are given $C = 11.71,\ P = 5.31,\ K = 100,\ T = \frac{9}{12},\ S_0 = 99$, and $\delta = 0.02$. Solving for r we find $r = 12.39\%$ ■

Parity provides a way for the creation of a synthetic stock. Using the put-call parity, we can write

$$-S_0 = -C(K,T) + P(K,T) - \mathrm{PV_{0,T}(Div)} - \mathrm{PV_{0,T}(K)}.$$

This equation says that buying a call, selling a put, and lending the present value of the strike and dividends to be paid over the life of the option is equivalent to the outright purchase[1] of the stock.

Example 3.4
A European call option and a European put option on a share of XYZ stock have a strike price of \$100 and expiration date in nine months. They sell for \$11.71 and \$5.31 respectively. The price of XYZ stock is currently \$99 and the continuously compounded risk-free rate of interest is 12.39%. A share of XYZ stock pays \$1.62 dividends in nine months. Determine the amount of cash that must be lent at the given risk-free rate of return in order to replicate the stock.

Solution.
By the previous paragraph, in replicating the stock, we must lend $\mathrm{PV_{0,T}(Div)} + \mathrm{PV_{0,T}(K)}$ to be paid over the life of the option. We are given $C = 11.71,\ P = 5.31,\ K = 100,\ T = \frac{9}{12},\ S_0 = 99$, and $r = 0.1239$. Thus,

$$\mathrm{PV_{0,T}(Div)} + \mathrm{PV_{0,T}(K)} = -11.701 + 5.31 + 99 = \$92.609 \blacksquare$$

The put-call parity provides a simple test of option pricing models. Any pricing model that produces option prices which violate the put-call parity is considered flawed and leads to arbitrage.

[1] outright purchase occurs when an investor simultaneously pays S_0 in cash and owns the stock.

Example 3.5
A call option and a put option on the same nondividend-paying stock both expire in three months, both have a strike price of 20 and both sell for the price of 3. If the continuously compounded risk-free interest rate is 10% and the stock price is currently 25, identify an arbitrage.

Solution.
We are given that $C = P = 3$, $r = 0.10$, $T = \frac{3}{12}$, $K = 20$, and $S_0 = 25$. Thus, $C - P = 0$ and $S_0 - Ke^{-rT} = 25 - 20e^{-0.10\times 0.25} > 0$ so that the put-call parity fails and therefore an arbitrage opportunity exists. An arbitrage can be exploited as follows: Borrow 3 at the continuously compounded annual interest rate of 10%. Buy the call option. Short-sell a stock and invest that 25 at the risk-free rate. Three months later purchase the stock at strike price 20 and pay the bank $3e^{0.10\times 0.25} = 3.08$. You should have $25e^{0.10\times 0.25} - 20 - 3.08 =$ \$2.55 in your pocket ■

Remark 3.1
Consider the put-call parity for a nondividend-paying stock. Suppose that the option is at-the-money at expiration, that is, $S_0 = K$. If we buy a call and sell a put then we will differ the payment for owning the stock until expiration. In this case, $P - C = Ke^{-rT} - K < 0$ is the present value of the interest on K that we pay for deferring the payment of K until expiration. If we sell a call and buy a put then we are synthetically short-selling the stock. In this case, $C - P = K - Ke^{-rT} > 0$ is the compensation we receive for deferring receipt of the stock price.

Practice Problems

Problem 3.1

According to the put-call parity, the payoffs associated with ownership of a call option can be replicated by
(A) shorting the underlying stock, borrowing the present value of the exercise price, and writing a put on the same underlying stock and with the same exercise price
(B) buying the underlying stock, borrowing the present value of the exercise price, and buying a put on the same underlying stock and with the same exercise price
(C) buying the underlying stock, borrowing the present value of the exercise price, and writing a put on the same underlying stock and with the same exercise price
(D) none of the above.

Problem 3.2

A three-year European call option with a strike price of \$50 on a share of ABC stock costs \$10.80. The price of ABC stock is currently \$42. The continuously compounded risk-free rate of interest is 10%. Find the price of a three-year European put option with a strike price of \$50 on a share of ABC stock.

Problem 3.3

Suppose ABC stock costs \$X today. It is expected that 4 quarterly dividends of \$1.25 each will be paid on the stock with the first coming 3 months from now. The 4th dividend will be paid one day before expiration of the forward contract. Suppose the annual continuously compounded risk-free rate is 10%. Find X if the cost of the forward contract is \$95.30. Round your answer to the nearest dollar.

Problem 3.4

Suppose ABC stock costs \$75 today and is expected to pay semi-annual dividend of \$X with the first coming in 4 months from today and the last just prior to the delivery of the stock. Suppose that the annual continuously compounded risk-free rate is 8%. Find X if the cost of a 1-year prepaid forward contract is \$73.09. Round your answer to the nearest dollar.

Problem 3.5
Suppose XYZ stock costs \$50 today and is expected to pay quarterly dividend of \$1 with the first coming in 3 months from today and the last just prior to the delivery of the stock. Suppose that the annual continuously compounded risk-free rate is 6%. What is the price of a prepaid forward contract that expires 1 year from today, immediately after the fourth-quarter dividend?

Problem 3.6
An investor is interested in buying XYZ stock. The current price of stock is \$45 per share. This stock pays dividends at an annual continuous rate of 5%. Calculate the price of a prepaid forward contract which expires in 18 months.

Problem 3.7
A nine-month European call option with a strike price of \$100 on a share of XYZ stock costs \$11.71. A nine-month European put option with a strike price of \$100 on a share of XYZ stock costs \$5.31. The price of XYZ stock is currently \$99, and the stock pays a continuous dividend yield of δ. The continuously compounded risk-free rate of interest is 12.39%. Find δ.

Problem 3.8
Suppose XYZ stock costs \$50 today and is expected to pay 8% continuous dividend. What is the price of a prepaid forward contract that expires 1 year from today?

Problem 3.9
Suppose that annual dividend of 30 on the stocks of an index valued at \$1500. What is the continuously compounded dividend yield?

Problem 3.10
You buy one share of Ford stock and hold it for 2 years. The dividends are paid at the annualized daily continuously compounded rate of 3.98% and you reinvest in the stocks all the dividends[3] when they are paid. How many shares do you have at the end of two years?

Problem 3.11
A 6-month European call option on a share of XYZ stock with strike price

[3] In this case, one share today will grow to $e^{\delta T}$ at time T. See [2] Section 70.

of \$35.00 sells for \$2.27. A 6-month European put option on a share of XYZ stock with the same strike price sells for \$P. The current price of a share is \$32.00. Assuming a 4% continuously compounded risk-free rate and a 6% continuous dividend yield, find P.

Problem 3.12
A nine-month European call option with a strike price of \$100 on a share of XYZ stock costs \$11.71. A nine-month European put option with a strike price of \$100 on a share of XYZ stock costs \$5.31. The price of XYZ stock is currently \$99. The continuously-compounded risk-free rate of interest is 12.39%. What is the present value of dividends payable over the next nine months?

Problem 3.13
A European call option and a European put option on a share of XYZ stock have a strike price of \$100 and expiration date in T months. They sell for \$11.71 and \$5.31 respectively. The price of XYZ stock is currently \$99. A share of Stock XYZ pays \$2 dividends in six months. Determine the amount of cash that must be lent in order to replicate the purchased stock.

Problem 3.14 ‡
On April 30, 2007, a common stock is priced at \$52.00. You are given the following:
(i) Dividends of equal amounts will be paid on June 30, 2007 and September 30, 2007.
(ii) A European call option on the stock with strike price of \$50.00 expiring in six months sells for \$4.50.
(iii) A European put option on the stock with strike price of \$50.00 expiring in six months sells for \$2.45.
(iv) The continuously compounded risk-free interest rate is 6%.
Calculate the amount of each dividend.

Problem 3.15 ‡
Given the following information about a European call option about a stock Z.
- The call price is 5.50
- The strike price is 47.
- The call expires in two years.

• The current stock price is 45.
• The continuously compounded risk-free is 5%.
• Stock Z pays a dividend of 1.50 in one year.
Calculate the price of a European put option on stock Z with strike price 47 that expires in two years.

Problem 3.16 ‡
An investor has been quoted a price on European options on the same nondividend-paying stock. The stock is currently valued at 80 and the continuously compounded risk- free interest rate is 3%. The details of the options are:

	Option 1	Option 2
Type	Put	Call
Strike	82	82
Maturity	180 days	180 days

Based on his analysis, the investor has decided that the prices of the two options do not present any arbitrage opportunities. He decides to buy 100 calls and sell 100 puts. Calculate the net cost of his transaction.

Problem 3.17 ‡
For a dividend paying stock and European options on this stock, you are given the following information:
• The current stock price is \$49.70.
• The strike price of the option is \$50.00.
• The time to expiration is 6 months.
• The continuously compunded risk-free interest rate is 3%.
• The continuous dividend yield is 2%.
• The call price is \$2.00.
• The put price is \$2.35.
Using put-call parity, calculate the present value arbitrage profit per share that could be generated, given these conditions.

Problem 3.18 ‡
The price of a European call that expires in six months and has a strike price of \$30 is \$2. The underlying stock price is \$29, and a dividend of \$0.50 is expected in two months and again in five months. The continuously compounded risk-free interest rate is 10%. What is the price of a European put option that expires in six months and has a strike price of \$30?

4 Conversions and Reverse Conversions

In a **forward conversion** or simply a **conversion** a trader buys a stock, buys a put option and sells a call option with the same strike price and expiration date. The stock price is usually close to the options' strike price. If the stock price at expiration is above the strike price, the short call is exercised against the trader, which automatically offsets his long position in the stock; the long put option expires unexercised.
If the stock price at expiration is below the strike price, the long put option is exercised by the trader, which automatically offsets his long position in the stock; the short call option expires unexercised.
Parity shows us

$$-S_0 - P(K,T) + C(K,T) = -(\mathrm{PV_{0,T}(K)} + \mathrm{PV_{0,T}(Div)}).$$

We have thus created a position that costs $\mathrm{PV_{0,T}(K)} + \mathrm{PV_{0,T}(Div)}$ and that pays $K + \mathrm{FV_{0,T}(Div)}$ at expiration. This is a **synthetic long $T-$bill.**[1] You can think of a conversion as a synthetic short stock (Long put/Short call) hedged with a long stock position.

Example 4.1
Suppose that the price of a stock is \$52. The stock pays dividends at the continuous yield rate of 7%. Options have 9 months to expiration. A 50-strike European call option sells for \$6.56 and a 50-strike put option sells for \$3.61. Calculate the cost of buying a conversion $T-$bill that matures for \$1,000 in nine months.

Solution.
To create an asset that matures for the strike price of \$50, we must buy $e^{-\delta T}$ shares of the stock, sell a call option, and buy a put option. Using the put-call parity, the cost today is

$$\mathrm{PV_{0,T}(K) = S_0e^{-\delta T} + P(K,T) - C(K,T).}$$

We are given that $C = 6.56$, $P = 3.61$, $T = 0.75$, $K = 50$, $S_0 = 52$, and $\delta = 0.07$. Substituting we find $PV(K) = \$46.39$. Thus, the cost of a long synthetic $T-$bill that matures for \$1000 in nine months is $20 \times 46.39 =$ \$927.80 ■

[1]T-bills are purchased for a price that is less than their par (face) value; when they mature, the issuer pays the holder the full par value. Effectively, the interest earned is the difference between the purchase price of the security and what you get at maturity.

Example 4.2
Suppose the S&R index is 800, the continuously compounded risk-free rate is 5%, and the dividend yield is 0%. A 1-year European call with a strike of \$815 costs \$75 and a 1-year European put with a strike of \$815 costs \$45. Consider the strategy of buying the stock, selling the call, and buying the put.
(a) What is the rate of return on this position held until the expiration of the options?
(b) What is the arbitrage implied by your answer to (a)?
(c) What difference between the call and put will eliminate arbitrage?

Solution.
(a) By buying the stock, selling the 815-strike call, and buying the 815 strike put, your cost will be

$$-800 + 75 - 45 = -\$770.$$

After one year, you will have for sure \$815 because either the sold call commitment or the bought put cancel out the stock price (create a payoff table). Thus, the continuously compounded rate of return r satisfies the equation $770e^r = 815$. Solving this equation we find $r = 0.0568$.
(b) The conversion position pays more interest than the risk-free interest rate. Therefore, we should borrow money at 5%, and buy a large amount of the aggregate position of (a), yielding a sure return of 0.68%. To elaborate, you borrow \$770 from a bank to buy one position. After one year, you owe the bank $770e^{0.05} = 809.48$. Thus, from one single position you make a profit of $815 - 809.40 = \$5.52$.
(c) To eliminate arbitrage, the put-call parity must hold. In this case $C - P = S_0 - Ke^{-rT} = 800 - 815e^{-0.05\times 1} = \$24.748$ ■

A **reverse conversion** or simply a **reversal** is the opposite to conversion. In this strategy, a trader short selling a stock[2], sells a put option and buys a call option with the same strike price and expiration date. The stock price is usually close to the options' strike price. If the stock price at expiration is above the strike price, the long call option is exercised by the trader, which

[2]Remember that shorting an asset or short selling an asset is selling an asset that have been borrowed from a third party (usually a broker) with the intention of buying identical assets back at a later date to return to the lender.

automatically offsets his short position in the stock; the short put option expires unexercised.
If the stock price at expiration is below the strike price, the short put option is exercised against the trader, which automatically offsets his short position in the stock; the long call option expires unexercised.
Parity shows us

$$-C(K,T) + P(K,T) + S_0 = \mathrm{PV_{0,T}(K)} + \mathrm{PV_{0,T}(Div)}.$$

Thus, a reversal creates a synthetic short $T-$bill position that sells for $\mathrm{PV_{0,T}(K)} + \mathrm{PV_{0,T}(Div)}$. Also, you can think of a reversal as selling a bond with par-value $K + \mathrm{FV_{0,T}(Div)}$ for the price of $\mathrm{PV_{0,T}(K)} + \mathrm{PV_{0,T}(Div)}$.

Example 4.3
A reversal was created by selling a nondividend-paying stock for \$42, buying a three-year European call option with a strike price of \$50 on a share of stock for \$10.80, and selling a three-year European put option with a strike price of \$50 for \$P. The continuously compounded risk-free rate of interest is 10%. Find P.

Solution.
We have

$$P(K,T) = -S_0 + C(K,T) + \mathrm{PV_{0,T}(K)} = -42 + 10.80 + 50\mathrm{e}^{-0.1\times 3} = \$5.84 \blacksquare$$

From the discussion of this and the previous sections, we have noticed that the put-call parity can be rearranged to create synthetic securities. Summarizing, we have
• Synthetic long stock: buy call, sell put, lend present value of the strike and dividends:

$$-S_0 = -C(K,T) + P(K,T) - (\mathrm{PV_{0,T}(Div)} + \mathrm{PV_{0,T}(K)}).$$

• Synthetic long T-bill: buy stock, sell call, buy put (conversion):

$$-(\mathrm{PV_{0,T}(Div)} + \mathrm{PV_{0,T}(K)}) = -\mathrm{S_0} - \mathrm{P(K,T)} + \mathrm{C(K,T)}.$$

• Synthetic short T-bill: short-sell stock, buy call, sell put (reversal):

$$\mathrm{PV_{0,T}(Div)} + \mathrm{PV_{0,T}(K)} = \mathrm{S_0} + \mathrm{P(K,T)} - \mathrm{C(K,T)}.$$

• Synthetic short call: sell stock and put, lend present value of strike and dividends:

$$C(K,T) = S_0 + P(K,T) - (\mathrm{PV_{0,T}(Div)} + \mathrm{PV_{0,T}(K)}).$$

• Synthetic short put: buy stock, sell call, borrow present value of strike and dividends:

$$P(K,T) = C(K,T) - S_0 + \mathrm{PV_{0,T}(Div)} + \mathrm{PV_{0,T}(K)}.$$

Practice Problems

Problem 4.1
Which of the following applies to a conversion?
(A) Selling a call, selling a put, buying the stock
(B) Selling a call, buying a put, short-selling the stock
(C) Buying a call, buying a put, short-selling the stock
(D) Selling a call, buying a put, buying the stock
(E) Buying a call, selling a put, short-selling the stock
(F) Buying a call, selling a put, buying the stock

Problem 4.2
Which of the following applies to a reverse conversion?
(A) Selling a call, selling a put, buying the stock
(B) Selling a call, buying a put, short-selling the stock
(C) Buying a call, buying a put, short-selling the stock
(D) Selling a call, buying a put, buying the stock
(E) Buying a call, selling a put, short-selling the stock
(F) Buying a call, selling a put, buying the stock

Problem 4.3
A conversion creates
(A) A synthetic stock
(B) A synthetic short stock hedged with a long stock position
(C) A synthetic long stock hedged with a short stock position
(D) A synthetic long $T-$bill

Problem 4.4
Suppose that the price of a nondividend-paying stock is \$41. A European call option with strike price of \$40 sells for \$2.78. A European put option with strike price \$40 sells for \$1.09. Both options expire in 3 months. Calculate the annual continuously compounded risk-free rate on a synthetic T-bill created using these options.

Problem 4.5
The current price of a nondividend-paying stock is \$99. A European call option with strike price \$100 sells for \$13.18 and a European put option with strike price \$100 sells for \$5.31. Both options expire in nine months. The continuously compounded risk-free interest rate is 12.39%. What is the cost of a synthetic $T-$bill created using these options?

Problem 4.6
A synthetic $T-$bill was created by buying a share of XYZ stock, selling a call option with strike price \$20 and underlying asset the share of XYZ stock for \$5.43, and buying a put option with strike price of \$20 for \$2.35. Both options expire in one year. The current stock price is \$23. What is the continuously compounded risk-free interest involved in this transaction? Assume that the stock pays no dividends.

Problem 4.7
Using the put-call parity relationship, what should be done to replicate a short sale of a nondividend-paying stock?

Problem 4.8 *(Synthetic Options)*
Using the put-call parity relationship how would you replicate a long call option with underlying asset a share of stock that pays dividends?

Problem 4.9 *(Synthetic Options)*
Using the put-call parity relationship how would you replicate a long put option with underlying asset a share of stock that pays dividends?

Problem 4.10
A reversal was created by selling a nondividend-paying stock for \$99, buying a 5-month European call option with a strike price of \$96 on a share of stock for \$6.57, and selling a 5-month European put option with a strike price of \$96 for \$P. The annual effective rate of interest is 3%. Find P.

Problem 4.11 ‡
The price of a nondividend-paying stock is \$85. The price of a European call with a strike price of \$80 is \$6.70 and the price of a European put with a strike price of \$80 is \$1.60. Both options expire in three months.
Calculate the annual continuously compounded risk-free rate on a synthetic $T-$Bill created using these options.

5 Parity for Currency Options

In the previous section we studied the parity relationship that involves stock options. In this section we consider the parity relationship applied to options with currencies as underlying assets.
Currencies are in general exchanged at an exchange rate. For example, the market today lists the exchange rate between dollar and euro as €1 = $ 1.25 and $1 = €0.80. Exchange rates between currencies fluctuate over time. Therefore, it is useful for businesses involved in international markets to be involved in currency related securities in order to be secured from exchange rate fluctuation. One such security is the currency forward contract.
Currency contracts are widely used to hedge against changes in exchange rates. A **Forward currency contract** is an agreement to buy or sell a specified amount of currency at a set price on a future date. The contract does not require any payments in advance. In contrary, a **currency prepaid forward** is a contract that allows you to pay today in order to acquire the currency at a later time. What is the prepaid forward price? Suppose at a time T in the future you want to acquire €1. Let $r_€$ be the euro-denominated interest rate (to be defined below) and let x_0 be the exchange rate today in $/€, that is €1 = $$x_0$. If you want €1 at time T you must have $e^{-r_€ T}$ in euros today. To obtain that many euros today, you must exchange $x_0e^{-r_€ T}$ dollars into euros. Thus, the prepaid forward price, in dollars, for a euro is

$$F_{0,T}^P = x_0e^{-r_€ T}.$$

Example 5.1
Suppose that $r_€ = 4\%$ and $x_0 = \$1.25/€$. How much should be invested in dollars today to have €1 in one year?

Solution.
We must invest today

$$1.25e^{-0.04} = \$1.2 \blacksquare$$

Now, the prepaid forward price is the dollar cost of obtaining €1 in the future. Thus, the forward price in dollars for one euro in the future is

$$F_{0,T} = x_0e^{(r-r_€)T}$$

where r is the dollar-denominated interest rate.

Example 5.2
Suppose that $r_€ = 4\%,\ r = 6\%,\ T = 1$ and $x_0 = \$1.25/€$. Find the forward exchange rate after one year.

Solution.
The forward exchange rate is $1.25e^{0.06-0.04} = \$1.275$ per one euro after one year ■

Another currency related security that hedges against changes in exchange rates is the currency option. A **currency option** is an option which gives the owner the right to sell or buy a specified amount of foreign currency at a specified price and on a specified date. Currency options can be either dollar-denominated or foreign-currency denominated. A **dollar-denominated** option on a foreign currency would give one the option to sell or buy the foreign currency at some time in the future for a specified number of dollars. For example, a dollar-denominated call on yen would give one the option to obtain yen at some time in the future for a specified number of dollars. Thus, a 3-year, \$0.008 strike call on yen would give its owner the option in 3 years to buy one yen for \$0.008. The owner would exercise this call if 3 years from now the exchange rate is greater than \$0.008 per yen.
A dollar-denominated put on yen would give one the option to sell yen at some time in the future for a specified number of dollars. Thus, a 3-year, \$0.008 strike put on yen would give its owner the option in 3 years to sell one yen for \$0.008. The owner would exercise this put if 3 years from now the exchange rate is smaller than \$0.008 per yen.
The payoff on a call on currency has the same mathematical expression as for a call on stocks, with S_T being replaced by x_T where x_T is the (spot) exchange rate at the expiration time. Thus, the payoff of a currency call option is $\max\{0, x_T - K\}$ and that for a currency put option is $\max\{0, K - x_T\}$.
Consider again the options of buying euros by paying dollars. Since

$$F_{0,T} = x_0 e^{(r-r_€)T}$$

the put-call parity

$$P(K,T) - C(K,T) = e^{-rT}K - e^{-rT}F_{0,T}$$

reduces to

$$P(K,T) - C(K,T) = e^{-rT}K - x_0 e^{-r_€ T}.$$

Thus, buying a euro call and selling a euro put has the same payoff to lending euros and borrowing dollars. We will use the following form of the previous equation:

$$C(K,T) - P(K,T) = x_0 e^{-r_{€}T} - e^{-rT}K.$$

Example 5.3
Suppose the (spot) exchange rate is \$1.25/€, the euro-denominated continuously compounded interest rate is 4%, the dollar-denominated continuously compounded interest rate is 6%, and the price of 2-year \$1.20-strike European put on the euro is \$0.10. What is the price of a 2-year \$1.20-strike European call on the Euro?

Solution.
Using the put-call parity for currency options we find

$$C = 0.10 + 1.25e^{-0.04\times 2} - 1.20e^{-0.06\times 2} = \$0.19 \blacksquare$$

A **foreign currency**-denominated option on the dollar is defined in a similar way as above by interchanging dollar with the foreign currency. Thus, a put-call parity of options to buy dollars by paying euros takes the form

$$C(K,T) - P(K,T) = x_0 e^{-rT} - e^{-r_{€}T}K$$

where x_0 is the exchange rate in €/\$.

Example 5.4
Suppose the (spot) exchange rate is €0.8/\$, the euro-denominated continuously compounded interest rate is 4%, the dollar-denominated continuously compounded interest rate is 6%, and the price of 2-year €0.833-strike European put on the dollar is €0.08. What is the price of a 2-year €0.833-strike European call on the dollar?

Solution.
Using the put-call parity for currency options we find

$$C = 0.08 + 0.8e^{-0.06\times 2} - 0.833e^{-0.04\times 2} = € \ 0.0206 \blacksquare$$

Remark 5.1
Note that in a dollar-denominated option, the strike price and the premium are in dollars. In contrast, in a foreign currency-denominated option, the strike price and the premium are in the foreign currency.

Practice Problems

Problem 5.1
A U.S. based company wants to buy some goods from a firm in Switzerland and the cost of the goods is 62,500 SF. The firm must pay for the goods in 120 days. The exchange rate is $x_0 = \$0.7032/SF$. Given that $r_{SF} = 4.5\%$ and $r_{\$} = 3.25\%$, find the forward exchange price.

Problem 5.2
Suppose the exchange rate is \$1.25/€, the euro-denominated continuously compounded interest rate is 4%, the dollar-denominated continuously compounded interest rate is 6%, and the price of 2-year \$1.20-strike European call on the euro is \$0.19. What is the price of a 2-year \$1.20-strike European put on the Euro?

Problem 5.3
Suppose the exchange rate is £0.4/SF, the Swiss Franc-denominated continuously compounded interest rate is 5%, the British pound-denominated continuously compounded interest rate is 3%, and the price of 3-year £0.5-strike European call on the SF is £0.05. What is the price of a 3-year £0.5-strike European put on the SF?

Problem 5.4
Suppose the current \$/€exchange rate is 0.95 \$/€, the euro-denominated continuously compounded interest rate is 4%, the dollar-denominated continuously compounded interest rate is r. The price of 1-year \$0.93-strike European call on the euro is \$0.0571. The price of a 1-year \$0.93-strike put on the euro is \$0.0202. Find r.

Problem 5.5
A six-month dollar-denominated call option on euros with a strike price of \$1.30 is valued at \$0.06. A six-month dollar-denominated put option on euros with the same strike price is valued at \$0.18. The dollar-denominated continuously compounded interest rate is 5%.
(a) What is the 6-month dollar-euro forward price?
(b) If the euro-denominated continuously compounded interest rate is 3.5%, what is the spot exchange rate?

Problem 5.6
A nine-month dollar-denominated call option on euros with a strike price of \$1.30 is valued at \$0.06. A nine-month dollar-denominated put option on euros with the same strike price is valued at \$0.18. The current exchange rate is \$1.2/€ and the dollar-denominated continuously compounded interest rate is 7%. What is the continuously compounded interest rate on euros?

Problem 5.7
Currently one can buy one Swiss Franc for \$0.80. The continuously-compounded interest rate for Swiss Francs is 4%. The continuously-compounded interest rate for dollars is 6%. The price of a SF 1.15-strike 1-year call option is SF 0.127. The spot exchange rate is SF 1.25/\$. Find the cost in dollar of a 1-year SF 1.15-strike put option.

Problem 5.8
The price of a \$0.02-strike 1 year call option on an Indian Rupee is \$0.00565. The price of a \$0.02 strike 1 year put option on an Indian Rupee is \$0.00342. Dollar and rupee interest rates are 4.0% and 7.0%, respectively. How many dollars does it currently take to purchase one rupee?

Problem 5.9
Suppose the (spot) exchange rate is \$0.009/¥, the yen-denominated continuously compounded interest rate is 1%, the dollar-denominated continuously compounded interest rate is 5%, and the price of 1-year \$0.009-strike European call on the yen is \$0.0006. What is the dollar-denominated European yen put price such that there is no arbitrage opportunity?

Problem 5.10
Suppose the (spot) exchange rate is \$0.009/¥, the yen-denominated continuously compounded interest rate is 1%, the dollar-denominated continuously compounded interest rate is 5%, and the price of 1-year \$0.009-strike European call on the yen is \$0.0006. The price of a 1-year dollar-denominated European yen put with a strike of \$0.009 has a premium of \$0.0004. Demonstrate the existence of an arbitrage opportunity.

6 Parity of European Options on Bonds

In this section we construct the put-call parity for options on bonds. We start the section by recalling some vocabulary about bonds. A **bond** is an interest bearing security which promises to pay a stated amount (or amounts) of money at some future date (or dates). The company or government branch which is issuing the bond outlines how much money it would like to borrow and specifies a length of time, along with an interest rate it is willing to pay. Investors who then lend the requested money to the issuer become the issuer's creditors through the bonds that they hold.
The **term** of the bond is the length of time from the date of issue until the date of final payment. The date of the final payment is called the **maturity date.** Any date prior to, or including, the maturity date on which a bond may be redeemed is termed a **redemption date.**
The **par value** or **face value** of a bond is the amount that the issuer agrees to repay the bondholder by the maturity date.
Bonds with coupons are periodic payments of interest made by the issuer of the bond prior to redemption. **Zero coupon bonds** are bonds that pay no periodic interest payments. It just pays a lump sum at the redemption date.
Like loans, the price of a bond is defined to be the present value of all future payments. The basic formula for price is given by

$$B_0 = Fra_{\overline{n}|i} + C(1+i)^{-n}$$

where F is the par value of the bond, r is the rate per coupon payment period, C is the amount of money paid at a redemption date to the holder of the bond.

Example 6.1
A 15-year 1000 par value bond bearing a 12% coupon rate payable annually is bought to yield 10% convertible continuously. Find the price of the bond.

Solution.
The price of the bond is

$$\begin{aligned}B_0 =&120a_{\overline{15}|} + 1000e^{-0.10\times 15}\\ =&120\left(\frac{1-e^{-0.10\times 15}}{e^{0.10}-1}\right) + 1000e^{-0.10\times 15}\\ =&\$1109.54 \blacksquare\end{aligned}$$

Example 6.2
A 10-year 100 par value bond bearing a 10% coupon rate payable semi-annually, and redeemable at 105, is bought to yield 8% convertible semi-annually. Find the price of the bond.

Solution.
The semi-annual coupon payment is $Fr = 100 \times 0.05 = \$5$. The price of the bond is

$$B_0 = 5a_{\overline{20}|0.04} + 105(1.04)^{-20} = \$115.87 \blacksquare$$

A **bond option** is an option that allows the owner of the option to buy or sell a particular bond at a certain date for a certain price. Because of coupon payments (which act like stock dividends), the prepaid forward price differs from the bond's price. Thus, if we let B_0 be the bond's price then the put-call parity for bond options is given by

$$C(K,T) = P(K,T) + [B_0 - \mathrm{PV_{0,T}(Coupons)}] - \mathrm{PV_{0,T}(K)}.$$

Notice that for a non-coupon bond, the parity relationship is the same as that for a nondividend-paying stock.

Example 6.3
A 15-year 1000 par value bond bearing a 12% coupon rate payable annually is bought to yield 10% convertible continuously. A \$1000-strike European call on the bond sells for \$150 and expires in 15 months. Find the value of a \$1000-strike European put on the bond that expires in 15 months.

Solution.
The price of the bond was found in Example 6.1. Using the put-call parity for options on bonds, we have

$$C(K,T) = P(K,T) + [B_0 - \mathrm{PV_{0,T}(Coupons)}] - \mathrm{PV_{0,T}(K)}.$$

or

$$150 = P(1000, 1.25) + [1109.54 - 120e^{-0.1}] - 1000e^{-0.1\times 1.25}.$$

Solving for P we find $P(1000, 1.25) = \$31.54 \blacksquare$

Example 6.4
A 15-year 1000 par value bond pays annual coupon of X dollars. The annual continuously compounded interest rate is 10%. The bond currently sells for \$1109.54. A \$1000-strike European call on the bond sells for \$150 and expires in 15 months. A \$1000-strike European put on the bond sells for \$31.54 and expires in 15 months. Determine the value of X.

Solution.
Using the put-call parity for options on bonds we have

$$C(K,T) = P(K,T) - \mathrm{PV_{0,T}(Coupons)} + \mathrm{B_0} - \mathrm{Ke^{-rT}}$$

or

$$150 = 31.54 - Xe^{-0.1} + 1109.54 - 1000e^{-0.1\times 1.25}.$$

Solving this equation we find $X = \$120$ ∎

Example 6.5
A 23-year bond pays annual coupon of \$2 and is currently priced at \$100. The annual continuously compounded interest rate is 4%. A \$K-strike European call on the bond sells for \$3 and expires in 23 years. A \$K-strike European put on the bond sells for \$5 and expires in 23 years.
(a) Find the present value of the coupon payments.
(b) Determine the value of K.

Solution.
(a) The present value of the coupon payments is the present value of an annuity that pays \$2 a year for 23 years at the continuous interest rate of 4%. Thus,

$$\mathrm{PV_{0,T}(Coupons)} = 2\left(\frac{1-\mathrm{e}^{-0.04\times 23}}{\mathrm{e}^{0.04}-1}\right) = \$29.476.$$

(b) Using the put-call parity of options on bonds we have

$$Ke^{-rT} = P(K,T) - C(K,T) + B_0 - \mathrm{PV_{0,T}(Coupons)}$$

or

$$Ke^{-0.04\times 23} = 5 - 3 + 100 - 29.476.$$

Solving for K we find $K = \$181.984$ ∎

Practice Problems

Problem 6.1
Find the price of a \$1000 par value 10-year bond maturing at par which has coupons at 8% convertible semi-annually and is bought to yield 6% convertible quarterly.

Problem 6.2
Find the price of a \$1000 par value 10-year bond maturing at par which has coupons at 8% convertible quarterly and is bought to yield 6% convertible semi-annually.

Problem 6.3
Find the price of a 1000 par value 10-year bond with coupons of 8.4% convertible semi-annually, which will be redeemed at 1050. The bond yield rate is 10% convertible semi-annually for the first five years and 9% convertible semi-annually for the second five years.

Problem 6.4
A zero-coupon bond currently sells for \$67. The annual effective interest rate is 4%. A \$80-strike European call on the bond costs \$11.56. Find the price of a \$80-strike put option on the bond. Both options expire in 9 years.

Problem 6.5
The annual coupon payment of a 10-year bond is \$10. A \$200-strike European call expiring in 10 years costs \$20. A \$200-strike European put expiring in 10 years costs \$3. Assume an annual effective interest rate of 3%.
(a) Calculate the present value of the coupons.
(b) Find the price of the bond.

Problem 6.6
A 76-year bond pays annual coupon of X dollars. The annual continuously compounded interest rate is 5%. The bond currently sells for \$87. A \$200-strike European call on the bond sells for \$5 and expires in 76 years. A \$200-strike European put on the bond sells for \$2 and expires in 76 years. Determine the value of X.

Problem 6.7
A 1-year bond is currently priced at \$90. The bond pays one dividend of

\$5 at the end of one year. A \$100-strike European call on the bond sells for \$6 and expires in one year. A \$100-strike European put on the bond sells for \$5 and expires in one year. Let i be the annual effective rate of interest. Determine i.

Problem 6.8
A 15-year 1000 par value bond bearing a 12% coupon rate payable semi-annually is bought to yield 10% continuously compounded interest. One month after the bond is issued, a \$950-strike European put option on the bond has a premium of \$25 and expiration date of 1 year. Calculate the value of a \$950-strike European call option on the bond expiring in one year, one month after the bond is issued.

Problem 6.9
A 10-year bond with par value \$1000 pays semi-annual coupon of \$35 and is currently priced at \$1000. The annual interest rate is 7% convertible semi-annually. A \$K-strike European call on the bond that expires in nine months has a premium \$58.43 greater than a \$K-strike European put on the bond that expires in nine months. Determine the value of K.

Problem 6.10
Amorphous Industries issues a bond with price 100 and annual coupons of 2, paid for 23 years. The annual effective interest rate is 0.04. A put option with a certain strike price and expiring in 23 years has price 5, whereas a call option with the same strike price and time to expiration has price 3. Find the strike price of both options.

7 Put-Call Parity Generalization

The options considered thus far have a strike asset consisting of cash. That is, the option's underlying asset is exchanged for cash. Options can be designed to exchange any asset for any other asset not just asset for cash. In this section we derive a put-call parity of such options. This parity version includes all previous versions as special cases. Options where the underlying assets and the strike assets can be anything are called **exchange options**. Let the underlying asset be called asset A, and the strike asset, asset B. Let S_t be the price at time t of asset A and Q_t that of asset B. Let $F_{t,T}^P(S)$ denote the time t price of the prepaid forward on asset A, paying S_T at time T. Let $F_{t,T}^P(Q)$ denote the time t price of the prepaid forward on asset B, paying Q_T at time T.

Example 7.1
If the underlying asset is a share of stock, find a formula for $F_{t,T}^P(S)$.

Solution.
We consider the following cases:
• If the stock pays no dividends then $F_{t,T}^P(S) = S_t$.
• If the stock pays discrete dividends then $F_{t,T}^P(S) = S_t - PV_{t,T}(Div)$ where $PV_{t,T}$ stands for the present value with $T-t$ periods to expiration.
• If the stock pays continuous dividends then $F_{t,T}^P(S) = e^{-\delta(T-t)}S_t$ ■

Now, let $C(S_t, Q_t, T-t)$ be the time t price of an option with $T-t$ periods to expiration, which gives us the privilege to give asset B and get asset A at the expiration time T. Let $P(S_t, Q_t, T-t)$ be time t price of the corresponding put option which gives us the privilege to give asset A and get asset B. The payoff of the call option at the expiration date T is

$$C(S_T, Q_T, 0) = \max\{0, S_T - Q_T\}$$

and that of the put option is

$$P(S_T, Q_T, 0) = \max\{0, Q_T - S_T\}.$$

Now, to understand how the generalized put-call parity is established, we consider the following portfolio:
1) buying a call at the time t price of $C(S_t, Q_t, T-t)$;

2) selling a put at the time t price of $P(S_t, Q_t, T-t)$;
3) selling a prepaid forward on A at the price $F^P_{t,T}(S)$;
4) buying a prepaid forward on B at the price $F^P_{t,T}(Q)$.
The payoff table of this portfolio is given next:

		Expiration	
Transaction	Cost at Time t	$S_T \le Q_T$	$S_T > Q_T$
Buy Call	$-C(S_t, Q_t, T-t)$	0	$S_T - Q_T$
Sell Put	$P(S_t, Q_t, T-t)$	$S_T - Q_T$	0
Sell Prepaid Forward on A	$F^P_{t,T}(S)$	$-S_T$	$-S_T$
Buy Prepaid Forward on B	$-F^P_{t,T}(Q)$	Q_T	Q_T
Net cash flow	$-C(S_t, Q_t, T-t) + F^P_{t,T}(S)$ $+ P(S_t, Q_t, T-t) - F^P_{t,T}(Q)$	0	0

These transactions, as a whole, create a portfolio that provides a sure payoff of zero at time T. In order to avoid arbitrage, the net cost of these transactions at time t must be zero, implying

$$C(S_t, Q_t, T-t) - P(S_t, Q_t, T-t) = F^P_{t,T}(S) - F^P_{t,T}(Q). \qquad (7.1)$$

This equation is referred to as the **generalized put-call parity of options**. It follows from this equation that relative call and put premiums are determined by prices of prepaid forwards on the underlying asset and the strike asset.

Remark 7.1
All European options satisfy equation (7.1) whatever the underlying asset.

Example 7.2
A nondividend-paying stock A is currently selling for \$34 a share whereas a nondividend-paying stock B is selling for \$56 a share. Suppose A is the underlying asset and B is the strike asset.
(a) Find $F^P_{t,T}(S)$ and $F^P_{t,T}(Q)$.
(b) Show that the put option is more expensive than the call for any time to expiration T of the options.

Solution.
(a) Since the stocks pay no dividends, we have $F^P_{t,T}(S) = \$34$ and $F^P_{t,T}(Q) =$

\$56.
(b) Using the put-call parity we have

$$C(A_t, B_t, T-t) - P(A_t, B_t, T-t) = F_{t,T}^P(S) - F_{t,T}^P(Q) = 34 - 56 = -\$22.$$

Thus, the put is \$22 more expensive than the call ■

Example 7.3
A share of Stock A pays quarterly dividend of \$1.2 and is currently selling for \$55 per share. A share of Stock B pays dividends at the continuously compounded yield of 8% and is currently selling for \$72 per share. The continuously compounded risk-free interest rate is 6% per year. A European exchange call option with stock B as the underlying asset and stock A as the strike asset sells for \$27.64. The option expires in one year. Find the premium of the corresponding put option.

Solution.
The quarterly effective rate of interest is $i = e^{0.06\times 0.25} - 1 = 1.5113\%$. We have

$$F_{t,T}^P(S) = 72e^{-0.08} = 66.46437694$$

and

$$F_{t,T}^P(Q) = 55 - 1.2a_{\overline{4}|i} = \$50.38.$$

Now, using the generalized put-call parity we find

$$27.64 - P(S_t, Q_t, T-t) = 66.46437694 - 50.38$$

or

$$P(S_t, Q_t, T-t) = 11.55562306 \blacksquare$$

Example 7.4
A share of stock A pays dividends at the continuously compounded yield of 5% and is currently selling for \$110. A share of a nondividend-paying stock B is currently selling for \$30. A 10-month European call option with underlying asset four shares of Stock B and strike asset one share of stock A sells for \$35.95. Find the price of a 10-month put option with underlying asset twelve shares of stock B and strike asset three shares of stock A.

Solution.
The current value of the four shares of stock B is $4 \times 30 = \$120$. Using the put-call parity for exchange options to find the price of a 10-month put option with underlying asset four shares of stock B and strike asset one share of stock A to find

$$C(120, 110, \frac{10}{12}) - P(120, 110, \frac{10}{12}) = F^P_{0,\frac{10}{12}}(4B) - F^P_{0,\frac{10}{12}}(A)$$

or

$$35.95 - P(120, 110, \frac{10}{12}) = 120 - 110e^{-0.05\times\frac{10}{12}}.$$

Solving this equation we find

$$P(120, 110, \frac{10}{12}) = 21.4608.$$

Hence, the price of a 10-month put option with underlying asset twelve shares of stock B and strike asset three shares of stock A is

$$3 \times 21.4608 = \$64.3824 \blacksquare$$

Practice Problems

Problem 7.1
A share of stock A pays dividends at the continuous compounded dividend yield of 3%. Currently a share of stock A sells for \$65. A nondividend-paying stock B sells currently for \$85 a share. A 9-month call option with stock A as underlying asset and stock B as strike asset costs \$40. Find the price of the corresponding 9-month put option.

Problem 7.2
A nondividend-paying stock A is currently selling for \$34 a share whereas a nondividend-paying stock B is selling for \$56 a share. Suppose A is the underlying asset and B is the strike asset. Suppose that a put option on A costs \$25. Find the price of the corresponding call option.

Problem 7.3
A share of stock A pays dividends at the continuous compounded dividend yield of δ. Currently a share of stock A sells for \$67. A nondividend-paying stock B sells currently for \$95 a share. A 13-month call option with stock A as underlying asset and stock B as strike asset costs \$45. The corresponding 13-month put option costs \$74.44. Determine δ.

Problem 7.4
A share of stock A pays dividends at the continuously compounded yield of 10%. A share of stock B pays dividends at the continuously compounded yield of 25% . A share of stock A currently sells for \$2000 and that of stock B sells for \$D. A 12-year call option with underlying asset A and strike asset B costs \$543. A 12-year put option with underlying asset A and strike asset B costs \$324. Determine the value of D.

Problem 7.5
A share of stock A pays dividends at the continuously compounded yield of 4.88%. Currently a share of stock A costs \$356. A share of stock B pays no dividends and currently costs \$567. A $T-$year call option with underlying asset A and strike asset B costs \$34. A $T-$year put option with underlying asset A and strike asset B costs \$290. Determine the time of expiration T.

Problem 7.6
A share of stock A pays no dividends and is currently selling for \$100. Stock

B pays dividends at the continuously compounded yield of 4% and is selling for \$100. A 7-month call option with stock A as underlying asset and stock B as strike asset costs \$10.22. Find the price of the corresponding 7-month put option.

Problem 7.7
A share of stock A pays \$3 dividends every two months at the continuously compounded interest rate of 6% and is currently selling for \$50. Stock B pays dividends at the continuously compounded yield of 3% and is selling for \$51. A 6-month call option with stock A as underlying asset and stock B as strike asset costs \$2.70. Find the price of the corresponding 6-month put option.

Problem 7.8
A share of stock A pays no dividends and is currently selling for \$40. A share of Stock B pays no dividends and is selling for \$45. A 3-month European exchange call option with stock A as underlying asset and stock B as strike asset costs \$6. Find the price of the corresponding 3-month exchange put option.

Problem 7.9
A share of stock A pays dividends at the continuously compounded yield of 4%. A share of stock B pays dividends at the continuously compounded yield of 4% . A share of stock A currently sells for \$40 and that of stock B sells for \$D. A 5-month put option with underlying asset A and strike asset B is \$7.76 more expensive than the call option. Determine the value of D.

Problem 7.10
A share of stock B pays dividends at the continuously compounded yield of 4% and is currently selling for \$100. A share of a nondividend-paying stock A is currently selling for \$100. A 7-month American call option with underlying asset one share of stock A and strike asset one share of stock B sells for \$10.22. Find the price of the corresponding 7-month European put option. Hint: For a stock with no dividends the price of an American call option is equal to the price of a European call option.

8 Labeling Options: Currency Options

Any option can be labeled as a put (the privilege to sell) or a call (the privilege to buy). The labeling process is a matter of perspective depending upon what we label as the underlying asset and what we label as the strike asset. Thus, a call option can be viewed as a put option and vice versa. To elaborate, you purchase an option that gives you the privilege of receiving one share of stock A by surrendering one share of stock B in one year. This option can be viewed as either a call or a put. If you view stock A as the underlying asset and stock B as the strike asset then the option is a call option. This option gives you the privilege to buy, at $T = 1$, one share of stock A by paying one share of stock B. If you view stock B as the underlying asset and stock A as the strike asset then the option is a put option. This option gives you the privilege to sell, at $T = 1$, one share of stock B for the price of one share of stock A.

Example 8.1
You purchase an option that gives you the privilege to sell one share of Intel stock for \$35. Explain how this option can be viewed either as a call option or as a put option.

Solution.
If you view the share of Intel stock as the underlying asset, the option is a put. The option gives you the privilege to sell the stock for the price of \$35. If you view \$35 as the underlying asset then the option is a call option. This option gives you the right to buy \$35 by selling one share of the stock ■

Options on Currencies
The idea that calls can be relabeled as put is used frequently by currency traders. Let x_t be the value in dollars of one foreign currency (f) at time t. That is, 1 f= $\$x_t$. A $K-$strike dollar-denominated call option on a foreign currency that expires at time T gives the owner the right to receive one (f) in exchange of K dollars. This option has a payoff at expiration, in dollars, given by

$$\max\{0, x_T - K\}.$$

In the foreign currency, this payoff at the expiration time T would be

$$\frac{1}{x_T}\max\{0, x_T - K\} = \max\{0, 1 - \frac{K}{x_T}\} = K\max\{0, \frac{1}{K} - \frac{1}{x_T}\}.$$

Now, the payoff at expiration of a $\frac{1}{K}-$strike foreign-denominated put on dollars is in that foreign currency

$$\max\{0, \frac{1}{K} - \frac{1}{x_T}\}.$$

It follows that on the expiration time T, the payoff in the foreign currency of one $K-$strike dollar-denominated call on foreign currency is equal to the payoff in foreign currency of K $\frac{1}{K}-$strike foreign-denominated puts on dollars. Thus, the two positions must cost the same at time $t = 0$, or else there is an arbitrage opportunity. Hence,

$K\times$(premium in foreign currency of $\frac{1}{K}-$strike foreign-denominated put on dollars) = (premium in foreign currency of $K-$strike dollar-denominated call on foreign currency) = (premium in dollars of $K-$strike dollar-denominated call on foreign currency)$/x_0$.

In symbols,

$$\frac{C_\$(x_0, K, T)}{x_0} = KP_f\left(\frac{1}{x_0}, \frac{1}{K}, T\right)$$

or

$$C_\$(x_0, K, T) = x_0KP_f\left(\frac{1}{x_0}, \frac{1}{K}, T\right).$$

Likewise,

$$P_\$(x_0, K, T) = x_0KC_f\left(\frac{1}{x_0}, \frac{1}{K}, T\right).$$

Example 8.2

A 1-year dollar-denominated call option on euros with a strike price of \$0.92 has a payoff, in dollars, of $\max\{0, x_1 - 0.92\}$, where x_1 is the exchange rate in dollars per euro one year from now. Determine the payoff, in euros, of a 1-year euro-denominated put option on dollars with strike price $\frac{1}{0.92} =$ €1.0870.

Solution.

The payoff in euros is given by

$$\max\{0, \frac{1}{0.92} - \frac{1}{x_1}\} \blacksquare$$

Example 8.3
The premium of a 1-year 100-strike yen-denominated put on the euro is ¥ 8.763. The current exchange rate is ¥95/€. What is the strike of the corresponding euro-denominated yen call, and what is its premium?

Solution.
The strike price of the euro-denominated yen call is $\frac{1}{100} = €0.01$. The premium, in euros, of the euro-denominated yen call satisfies the equation

$$P_{\text{yen}}(95, 100, 1) = 95 \times 100 C_{€}\left(\frac{1}{95}, \frac{1}{100}, 1\right).$$

Thus,

$$C_{€}\left(\frac{1}{95}, \frac{1}{100}, 1\right) = \frac{1}{95} \times 0.01 \times 8.763 = €0.00092242 \blacksquare$$

Example 8.4
Suppose the (spot) exchange rate is \$0.009/¥, the yen-denominated continuously compounded interest rate is 1%,the dollar-denominated continuously compounded interest rate is 5%, and the price of 1-year \$0.009-strike dollar-denominated European call on the yen is \$0.0006. What is the price of a yen-denominated dollar call?

Solution.
The dollar-denominated call option is related to the yen-denominated put option by the equation

$$C_{\$}(x_0, K, T) = x_0 K P_{¥}\left(\frac{1}{x_0}, \frac{1}{K}, T\right).$$

Thus,

$$P_{¥}\left(\frac{1}{0.009}, \frac{1}{0.009}, 1\right) = \frac{0.0006}{0.009} \times \frac{1}{0.009} = ¥7.4074.$$

Using the put-call parity of currency options we have

$$P_{¥}\left(\frac{1}{0.009}, \frac{1}{0.009}, 1\right) - C_{¥}\left(\frac{1}{0.009}, \frac{1}{0.009}, 1\right) = e^{-r_{¥}T} K - x_0 e^{-r_{\$}T}$$

where $x_0 = ¥\frac{1}{0.009}/\$$. Thus,

$$C_{¥}\left(\frac{1}{0.009}, \frac{1}{0.009}, 1\right) = 7.4074 - \frac{1}{0.009} e^{-0.01} + \frac{1}{0.009} e^{-0.05} = ¥3.093907 \blacksquare$$

Practice Problems

Problem 8.1
You purchase an option that gives you the privilege to buy one share of Intel stock for \$35. Explain how this option can be viewed either as a call option or a put option.

Problem 8.2
A 1-year dollar-denominated call option on euros with a strike price of \$0.92 costs \$0.0337. The current exchange rate is $x_0 = \$0.90/€$. What is the premium, in euros, of this call option?

Problem 8.3
A 1-year dollar-denominated call option on euros with a strike price of \$0.92 costs \$0.0337. The current exchange rate is $x_0 = \$0.90/€$. What is the strike of the corresponding euro-denominated dollar put, and what is its premium?

Problem 8.4
The premium on a 1-year dollar-denominated call option on euro with a strike price \$1.50 is \$0.04. The current exchange rate is \$1.52/€. Calculate the premium in euros of the corresponding 1-year euro-denominated put option on dollars.

Problem 8.5
Suppose the (spot) exchange rate is £0.6/\$, the pound-denominated continuously compounded interest rate is 8%,the dollar-denominated continuously compounded interest rate is 7%, and the price of 1-year £0.58-strike pound-denominated European put on the dollar is £0.0133. What is the price of a dollar-denominated pounds put?

Problem 8.6
Suppose the (spot) exchange rate is \$1.20/€. The price of a 6-month \$1.25-strike dollar-denominated European call on the euro is \$0.083. What is the premium in euro of the euro-denominated dollar put?

Problem 8.7
Suppose the (spot) exchange rate is €0.85/\$. The price of a 6-month €0.80-strike euro-denominated European call on the dollar is €.0898. What is the premium in dollars of the dollar-denominated euro put?

Problem 8.8
Suppose the (spot) exchange rate is franc f 1.65/€. The price of a 1-year f 1.60-strike franc-denominated European put on the euro is f 0.0918. What is the premium in euros of the euro-denominated franc call?

Problem 8.9
Let \$ denote the Australian dollars. Suppose the (spot) exchange rate is £0.42/\$, the pound-denominated continuously compounded interest rate is 8%,the dollar-denominated continuously compounded interest rate is 7%, and the price of 1-year £0.40-strike pound-denominated European put on the dollar is £0.0133. What is the price of a dollar-denominated pounds put?

Problem 8.10
Suppose the (spot) exchange rate is €0.70/\$, the euro-denominated continuously compounded interest rate is 8%,the dollar-denominated continuously compounded interest rate is 7%, and the price of 6-month €0.625-strike euro-denominated European call on the dollar is €0.08. What is the price, in euros, of a dollar-denominated euro call?

Problem 8.11
Suppose the (spot) exchange rate is £0.38/\$. The price of a 6-month £1.40-strike pound-denominated European put on the dollar is £0.03. What is the premium in dollars of the dollar-denominated pounds call?

Problem 8.12 ‡
You are given:
(i) The current exchange rate is \$0.011/¥.
(ii) A four-year dollar-denominated European put option on yen with a strike price of \$0.008 sells for \$0.0005.
(iii) The continuously compounded risk-free interest rate on dollars is 3%.
(iv) The continuously compounded risk-free interest rate on yen is 1.5%.
Calculate the price of a four-year yen-denominated European put option on dollars with a strike price of ¥125.

9 No-Arbitrage Bounds on Option Prices

In this section lower and upper bounds of option prices are determined. Recall that American options are options that can be exercised at any time up to (and including) the maturity date. On the contrary, European options can be exercised only at the maturity. Since an American option can duplicate a European option by exercising the American option at the maturity date, it follows that an American option is always worth at least as much as a European option on the same asset with the same strike price and maturity date. In symbols,

$$C_{\text{Amer}}(K,T) \geq C_{\text{Eur}}(K,T)$$

and

$$P_{\text{Amer}}(K,T) \geq P_{\text{Eur}}(K,T).$$

Example 9.1
Show that for a nondividend-paying stock one has $C_{\text{Amer}}(K,T) = C_{\text{Eur}}(K,T)$.

Solution.
Suppose that $C_{\text{Amer}}(K,T) > C_{\text{Eur}}(K,T)$. We will show that this creates an arbitrage opportunity. Consider the position of selling the American call and buying the European call. The net cash flow $C_{\text{Amer}}(K,T) - C_{\text{Eur}}(K,T)$ would be invested at the risk-free rate r.
If the owner of the American call chooses to exercise the option at some time $t \leq T$, sell short a share of the security for amount K and add the proceeds to the amount invested at the risk-free rate. At time T close out the short position in the security by exercising the European option. The amount due is

$$(C_{\text{Amer}}(K,T) - C_{\text{Eur}}(K,T))e^{rT} + K(e^{r(T-t)} - 1) > 0.$$

If the American option is not exercised, the European option can be allowed to expire and the amount due is

$$(C_{\text{Amer}}(K,T) - C_{\text{Eur}}(K,T))e^{rT} > 0.$$

In either case, an arbitrage opportunity occurs ■

We next establish some bounds on the option prices. We first consider call options:

• Since the best one can do with a call stock option is to own the stock so the call price cannot exceed the current stock price. Thus,

$$S_0 \geq C_{\text{Amer}}(K,T) \geq C_{\text{Eur}}(K,T).$$

Example 9.2
Use a no-arbitrage argument to establish that $C_{\text{Amer}}(K,T) \leq S_0$.

Solution.
Suppose that $C_{\text{Amer}}(K,T) > S_0$. Consider the position of buying a stock and selling an American call on the stock. The payoff table for this position is given next.

		Exercise or Expiration	
Transaction	Time 0	$S_t \leq K$	$S_t > K$
Buy a stock	$-S_0$	S_t	S_t
Sell a call	C_{Amer}	0	$K - S_t$
Total	$C_{\text{Amer}} - S_0 > 0$	S_t	K

Every entry in the row "Total" is nonnegative with $C_{\text{Amer}} - S_0 > 0$. Thus, an arbitrage occurs ■

• The price of a call or a put option has to be non-negative because with these options you are offered the possibility for a future gain with no liability. In symbols, we have

$$0 \leq C_{\text{Eur}}(K,T) \leq C_{\text{Amer}}(K,T).$$

• The price of a European call option must satisfy the put-call parity. Thus, for a nondividend-paying stock we have

$$C_{\text{Eur}}(K,T) = P_{\text{Eur}}(K,T) + S_0 - PV_{0,T}(K) \geq S_0 - PV_{0,T}(K).$$

Combining, all of the above results we can write

$$S_0 \geq C_{\text{Amer}}(K,T) = C_{\text{Eur}}(K,T) \geq \max\{0, S_0 - PV_{0,T}(K)\}.$$

Likewise, for a discrete-dividend paying stock we have

$$S_0 \geq C_{\text{Amer}}(K,T) \geq C_{\text{Eur}}(K,T) \geq \max\{0, S_0 - PV_{0,T}(Div) - PV_{0,T}(K)\}$$

and for a continuous-dividend paying stock we have

$$S_0 \geq C_{\text{Amer}}(K,T) \geq C_{\text{Eur}}(K,T) \geq \max\{0, S_0e^{-\delta T} - PV_{0,T}(K)\}.$$

Next, we consider bounds on put options:
• Put options are nonnegative: $0 \leq P_{\text{Eur}}(K,T) \leq P_{\text{Amer}}(K,T)$.
• The best one can do with a European put option is to get the strike price K at the maturity date T. So a European put cannot be worth more than the present value of the strike price. That is,

$$P_{\text{Eur}}(K,T) \leq PV_{0,T}(K).$$

• The best one can do with an American put option is to exercise it immediately after time zero and receive the strike price K. So an American put cannot be worth more than the strike price. In symbols,

$$P_{\text{Amer}}(K,T) \leq K.$$

Combining the above results we find

$$0 \leq P_{\text{Eur}}(K,T) \leq P_{\text{Amer}}(K,T) \leq K.$$

• The price of a European put option must obey the put-call parity. For a nondividend-paying stock we have

$$P_{\text{Eur}}(K,T) = C_{\text{Eur}}(K,T) + PV_{0,T}(K) - S_0 \geq PV_{0,T}(K) - S_0.$$

Hence,

$$K \geq P_{\text{Amer}}(K,T) \geq P_{\text{Eur}}(K,T) \geq \max\{0, PV_{0,T}(K) - S_0\}.$$

For a discrete-dividend paying stock we have

$$K \geq P_{\text{Amer}}(K,T) \geq P_{\text{Eur}}(K,T) \geq \max\{0, PV_{0,T}(K) + PV_{0,T}(Div) - S_0\}.$$

For a continuous-dividend paying stock we have

$$K \geq P_{\text{Amer}}(K,T) \geq P_{\text{Eur}}(K,T) \geq \max\{0, PV_{0,T}(K) - S_0e^{-\delta T}\}.$$

Example 9.3
The price of a $K-$strike European put option on a share of stock A that expires at time T is exactly K. What is the price of an American put option on the same stock with the same strike and expiration date?

Solution.
We have $K = P_{\text{Eur}}(K,T) \le P_{\text{Amer}}(K,T) \le K$. Thus, $P_{\text{Amer}}(K,T) = K$ ■

Example 9.4
What is a lower bound for the price of a 1-month European put option on a nondividend-paying stock when the stock price is \$12, the strike price is \$15, and the risk-free interest rate is 6% per annum?

Solution.
A lower bound is given by

$$\max\{0, PV_{0,T}(K) - S_0\} = \max\{0, 15e^{-0.06\times\frac{1}{12}} - 12\} = \$2.93 \blacksquare$$

Example 9.5
A four-month European call option on a dividend-paying stock has a strike price of \$60. The stock price is \$64 and a dividend of \$0.80 is expected in one month. The continuously compounded risk-free interest rate is 12%. Find a lower bound for the price of the call.

Solution.
A lower bound is given by

$$\max\{0, S_0 - PV_{0,T}(Div) - PV_{0,T}(K)\} = 64 - 0.80e^{-0.12\times\frac{1}{12}} - 60e^{-0.12\times\frac{4}{12}} = \$5.56 \blacksquare$$

Example 9.6
Show that for a nondividend-paying stock we have

$$C_{\text{Amer}}(K,T) \ge C_{\text{Eur}}(K,T) \ge S_0 - K.$$

Solution.
Using the put-call parity we have

$$C_{\text{Eur}}(K,T) = S_0 - K + P_{\text{Eur}}(K,T) + K(1 - e^{-rT}) \ge S_0 - K$$

where we used the fact that $P_{\text{Eur}}(K,T) \ge 0$ and $K(1 - e^{-rT}) \ge 0$ ■

Practice Problems

Problem 9.1
The maximum value of a call stock option is equal to:
(A) the strike price minus the initial cost of the option
(B) the exercise price plus the price of the underlying stock
(C) the strike price
(D) the price of the underlying stock.

Problem 9.2
The lower bound of a call option:
(A) can be a negative value regardless of the stock or exercise price
(B) can be a negative value but only when the exercise price exceeds the stock price
(C) can be a negative value but only when the stock price exceeds the exercise price
(D) must be greater than zero
(E) can be equal to zero.

Problem 9.3
What is a lower bound for the price of a 4-month call option on a nondividend-paying stock when the stock price is \$28, the strike price is \$25, and the continuously compounded risk-free interest rate is 8%?

Problem 9.4
A six-month European put option on a nondividend-paying stock has a strike price of \$40. The stock price is \$37. The continuously compounded risk-free interest rate is 5%. Find a lower bound for the price of the put.

Problem 9.5
A share of stock currently sells for \$96. The stock pays two dividends of \$8 per share, one six months from now and the other one year from now. The annual continuously compounded rate of interest is 5.6%. What is a lower bound for a 1-year put option with strike price of \$86?

Problem 9.6
A share of stock is currently selling for \$99. The stock pays a continuous dividend yield of 2%. The continuously compounded risk-free interest rate is 12.39%. Find a lower bound for a nine-month European call on the stock with strike price of \$100.

Problem 9.7 ‡
Which of the following effects are correct on the price of a stock option?
I. The premiums would not decrease if the options were American rather than European.
II. For European put, the premiums increase when the stock price increases.
III. For American call, the premiums increase when the strike price increases.

Problem 9.8
Show that the assumption

$$C_{\text{Amer}}(K,T) < C_{\text{Eur}}(K,T)$$

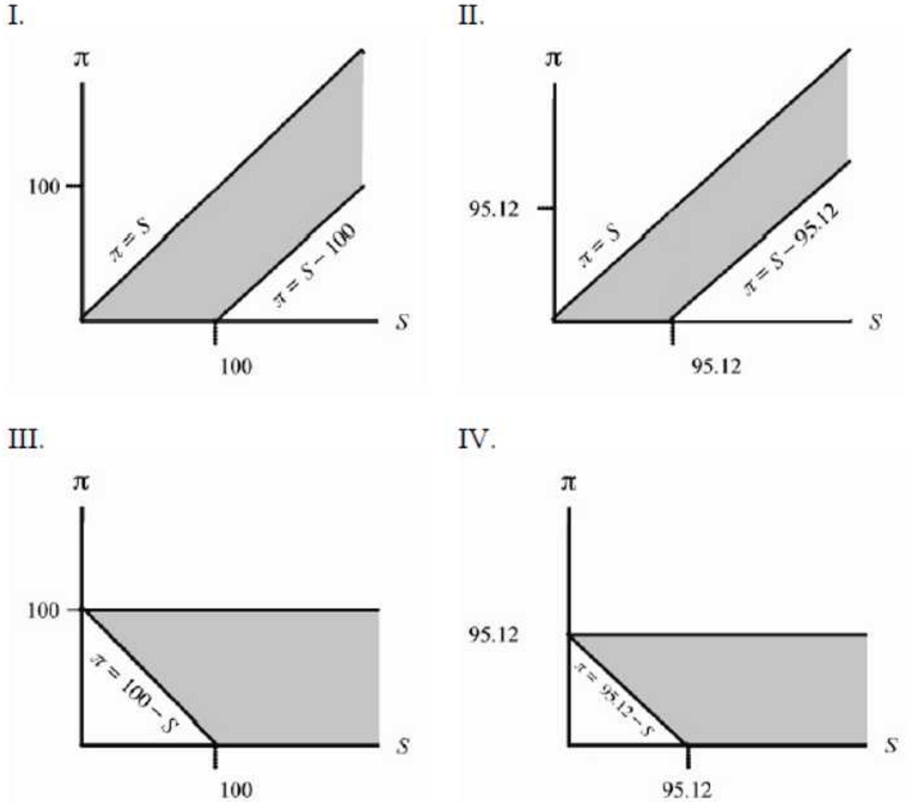
creates an arbitrage opportunity.

Problem 9.9
Use a no-arbitrage argument to establish that $P_{\text{Eur}} \leq PV_{0,T}(K)$.

Problem 9.10
Use a no-arbitrage argument to establish that $C_{\text{Eur}} \geq S_0 - K$.

Problem 9.11
Use a no-arbitrage argument to establish that $C_{\text{Eur}}(K,T) + PV_{0,T}(K) \geq S_0$.

Problem 9.12 ‡
Consider European and American options on a nondividend-paying stock. You are given:
(i) All options have the same strike price of 100.
(ii) All options expire in six months.
(iii) The continuously compounded risk-free interest rate is 10%.
You are interested in the graph for the price of an option as a function of the current stock price. In each of the following four charts I-IV, the horizontal axis, S, represents the current stock price, and the vertical axis, π, represents the price of an option.
Match the option (i.e. American, European calls and puts) with the shaded region in which its graph lies. If there are two or more possibilities, choose the chart with the smallest shaded region.


I.
π
100
π = S
π = S − 100
S
100
II.
π
95.12
π = S
π = S − 95.12
S
95.12
III.
π
100
π = 100 − S
S
100
IV.
π
95.12
π = 95.12 − S
S
95.12

10 General Rules of Early Exercise on American Options

American options are contracts that may be exercised early, prior to expiration date. In this section, we derive general sets of rules about when early exercise is not optimal, or under what conditions it may be optimal. Further discussions about early exercising that require option pricing models will be discussed in future sections. In this section the word option stands for American options unless indicated otherwise.
First, we consider calls on a nondividend-paying stock. Suppose you buy a call option at time $t = 0$. Let t denote today's date so that $0 \leq t < T$. Let S_t denote today's stock price and the stock price at expiration be denoted by S_T. Let r denote the annual continuously compounded risk-free interest rate. We state our first result in the form of a proposition.

Proposition 10.1
It is NEVER optimal to early exercise an American call option on a nondividend-paying stock.

Proof.
An American call option can be exercised early if the exercise price is higher than the option value. Using the generalized put-call parity for European options and the fact that $P_{\text{Eur}}(S_t, K, T-t) \geq 0$ we can write

$$\begin{aligned}
C_{\text{Eur}}(S_t, K, T-t) =& P_{\text{Eur}}(S_t, K, T-t) + S_t - Ke^{-r(T-t)} \\
\geq& S_t - Ke^{-r(T-t)} \\
\geq& S_t - K
\end{aligned}$$

where we use the fact that for $0 \leq t \leq T$ we have $0 < e^{-r(T-t)} \leq 1$. Now, since $C_{\text{Amer}}(S_t, K, T-t) \geq C_{\text{Eur}}(S_t, K, T-t)$, we have

$$C_{\text{Amer}}(S_t, K, T-t) \geq S_t - K. \tag{10.1}$$

This shows that the exercise value $S_t - K,\ 0 \leq t \leq T$ can never be higher than the option value $C_{\text{Amer}}(S_t, K, T-t)$. Thus, if you sell the call option you receive C_{Amer} while if you exercise the option you will receive $S_t - K$ and therefore you would either break even or lose money since $C_{\text{Amer}}(S_t, K, T-t) \geq S_t - K$. Hence, it is never optimal to early exercise an American call option on a nondividend-paying stock ■

Remark 10.1
The above proposition does not say that you must hold the option until expiration. It says that if no longer you wish to hold the call, you should sell it rather than exercise it. Also, it follows from the previous proposition that an American call on a nondividend-paying stock is like a European call so that we can write $C_{\text{Amer}} = C_{\text{Eur}}$.

Remark 10.2
One of the effects of early exercising is the time value of money: When you exercise the call option at time t you pay the strike price K and you own the stock. However, you lose the interest you could have earned during the time interval $[t, T]$ had you put K in a savings account, since owning the stock has no gain during $[t, T]$ (stock pays no dividends).

The case $C_{\text{Amer}}(S_t, K, T-t) < S_t - K$ leads to an arbitrage opportunity as shown next.

Example 10.1
Suppose an American call option is selling for $C_{\text{Amer}}(S_t, K, T-t) < S_t - K$. Demonstrate an arbitrage opportunity.

Solution.
First note that $C_{\text{Amer}}(S_t, K, T-t) < S_t - K$ implies $C_{\text{Amer}}(S_t, K, T-t) < S_t - Ke^{-r(T-t)}$. Consider the position of buying the call option for the price of $C_{\text{Amer}}(S_t, K, T-t)$, short selling the stock, and lending $Ke^{-r(T-t)}$. The payoff table of this position is given next.

		Exercise or Expiration	
Transaction	Time t	$S_T \leq K$	$S_T > K$
Short sell a stock	S_t	$-S_T$	$-S_T$
Buy a call	$-C_{\text{Amer}}$	0	$S_T - K$
Lend $Ke^{-r(T-t)}$	$-Ke^{-r(T-t)}$	K	K
Total	$S_t - C_{\text{Amer}} - Ke^{-r(T-t)} > 0$	$K - S_T$	0

Every entry in the row "Total" is nonnegative with $S_t - C_{\text{Amer}} - Ke^{-r(T-t)} > 0$. Thus, an arbitrage occurs ■

Now, what if you can not sell the call option, should you ever exercise early

then? By short-selling the stock we receive S_t. If at expiration, $S_T > K$, you exercise the call paying K and receiving the stock. You return the stock to the broker. In this case, the present value of your profit at time t is $S_t - Ke^{-r(T-t)} > S_t - K$. If $S_T \leq K$, you let your call option expire worthless, purchase a stock in the market, and return it to the broker. The present value of your profit is $S_t - S_Te^{-r(T-t)} > S_t - K$.
In conclusion, you never want to exercise a call when the underlying asset does not pay dividends.

Example 10.2
Suppose you own an American call option on a nondividend-paying stock with strike price of \$100 and due in three months from now. The continuously compounded risk-free interest rate is 1% and the stock is currently selling for \$105.
(a) What is the payoff from exercising the option now?
(b) What is the least payoff from selling the option now?
Suppose that you can not sell your call.
(c) What happens if you exercise now?
(d) What advantage do you have by delaying the exercise until maturity?
(e) What if you know the stock price is going to fall? Shouldn't you exercise now and take your profits (by selling the stock), rather than wait and have the option expire worthless?

Solution.
(a) The payoff from exercising the option now is $105 - 100 = \$5$.
(b) We know that $C_{\text{Amer}} \geq \max\{0, 105 - 100e^{-0.01\times 0.25}\}$ so that the least payoff from selling the option now is

$$105 - 100e^{-0.01\times 0.25} = \$5.25.$$

(c) If you exercise now, you pay \$100 and owns the stock which will be worth S_T in three months.
(d) You can deposit the \$100 into a savings account which will accumulate to $100e^{0.01\times 0.25} = \$100.25$. At the maturity, if $S_T > 100$ you exercise the call and pay \$100 and own the stock. In this case, you have an extra \$0.25 as opposed to exercising now. If $S_T < 100$, you don't exercise the call and you have in your pocket \$100.25 as opposed to exercising it now (a loss of $S_T - 100$.)

(e) Say you exercise now. You pay \$100 for the stock, sell it for \$105 and deposit the \$5 in a savings account that will accumulate to $5e^{0.01\times0.25} = \$5.0125$. If you delay exercising until maturity, you can short the stock now, deposit \$105 into a savings account that accumulates to $105e^{0.01\times0.25} = \$105.263$. At maturity you cover the short by exercising the call option and in this case your profit is $\$5.263 > \$5.0125$ ■

We next consider calls on a dividend paying stock. We first establish a condition under which early exercise is never optimal.

Proposition 10.2
Early exercise for calls on dividend paying stock can not be optimal if

$$K - PV_{t,T}(K) > PV_{t,T}(Div).$$

Proof.
Using put-call parity relationship for stocks with dividends given by

$$C_{\text{Eur}}(S_t, K, T-t) = P_{\text{Eur}}(S_t, K, T-t) + S_t - PV_{t,T}(Div) - PV_{t,T}(K)$$

which can be written as

$$\begin{aligned} C_{\text{Eur}}(S_t, K, T-t) =& S_t - K + P_{\text{Eur}}(S_t, K, T-t) - PV_{t,T}(Div) + K - PV_{t,T}(K) \\ \geq& S_t - K - [PV_{t,T}(Div) - K + PV_{t,T}(K)]. \end{aligned}$$

Thus,

$$C_{\text{Amer}} \geq S_t - K - [PV_{t,T}(Div) - K + PV_{t,T}(K)].$$

To avoid early exercise we expect that selling the call (getting C_{Amer}) to be more profitable than exercising the call (getting $S_t - K$). Thus, in order to achieve that we require

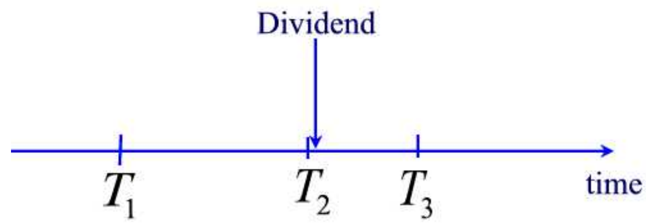
$$K - PV_{t,T}(K) > PV_{t,T}(Div). \tag{10.2}$$

In this case, we obtain

$$C_{\text{Amer}}(S_t, K, T-t) > S_t - K \blacksquare \tag{10.3}$$

As in the discussion for call options with no dividends, selling the call is better than early-exercising it. The condition (10.2) says that early exercise should not occur if the interest on the strike price exceeds the value of the

dividends through early exercise. If (10.2) is violated, this does not tell us that we will exercise, only that we cannot rule it out.
When does early exercising optimal then? If it's optimal to exercise an American call early, then the best time to exercise the call is immediately before the dividend payment. To further elaborate, consider the picture below.



We notice that by exercising at T_1 instead of T_2 we lose the interest that can be earned on K during $[T_1, T_2]$, we lose the remaining call option on $[T_1, T_2]$, and gain nothing since there are no dividends between T_1 and T_2. Thus, exercising at T_1 cannot be optimal. On the other hand, if we exercise at T_3, we lose the dividend, the remaining call option, lose the interest that can be earned on K during $[T_3, T]$,and gain just a tiny interest of the dividend on $[T_2, T_3]$. Again, exercising at time T_3 is not optimal.
We conclude that for a dividend paying stock, if it's ever worthwhile to exercise an American call early, you should exercise the call immediately before the dividend payment, no sooner or later.

Remark 10.3
It follows from our discussion that early exercise on American calls with dividends has its advantages and disadvantages. Namely,
$(+)$ You gain the dividends between t and T and the interest on the dividends.
$(-)$ You lose the time value of money on the strike. That is, we lose the interest on K from time t to T.
$(-)$ You lose the remaining call option on the time interval $[t, T]$.
$(-)$ We pay K for a stock that might be worth less than K at T.

The second half of this section concerns early exercise of American put options. Consider an American put option on a nondividend-paying stock. Contrary to American call options for nondividend paying stocks, an American put option on a nondividend paying stock may be exercised early.
To avoid early exercise, selling the put (getting P_{Amer}) should be more profitable than exercising (getting $K - S_t$), that is, $P_{\text{Amer}} > K - S_t$. Now, the

put-call parity for European options says

$$P_{\text{Eur}} = C_{\text{Eur}} + (K - S_t) - (K - PV_{t,T}(K)).$$

It follows from this equation that if

$$C_{\text{Eur}} > K - PV_{t,T}(K) \tag{10.4}$$

then $P_{\text{Amer}} > K - S_t$ and therefore selling is better than exercising. This means that there is no early exercise if the European call price is high (high asset price compared to strike price), the strike price is low, or if the discounting until expiration is low (low interest rate or small time to expiration).
It should be noted that if condition (10.4) is not satisfied, we will not necessarily exercise, but we cannot rule it out.

Example 10.3
Consider an American put option on a stock. When the stock is bankrupt then $S_t = 0$ and it is known that it will stay $S_t = 0$.
(a) What is the payoff from early exercising?
(b) What is the present value at time t if put is exercised at maturity?
(c) Is early exercising optimal?

Solution.
Note that $C_{\text{Eur}} = 0 < K - PV_{0,T}(K)$.
(a) The payoff will be K.
(b) If the put is exercised at maturity, the present value of K at time t is $PV_{t,T}(K) < K$.
(c) From (a) and (b) we see that early exercise is optimal ■

Finally, using an argument similar to the one considered for call options with dividends, it is easy to establish (left as an exercise) that it is not optimal to early exercise an American put option with dividends satisfying

$$K - PV_{t,T}(K) < PV_{t,T}(Div) \tag{10.5}$$

If condition (10.5) is not satisfied, we will not necessarily exercise, but we cannot rule it out.

Example 10.4
For call options on a stock that pays no dividends, early exercise is never optimal. However, this is not true in general for put options. Why not?

Solution.
Delaying exercise of a call gains interest on the strike, but delaying exercise of a put loses interest on the strike ∎

Practice Problems

Problem 10.1
For which of the following options it is never optimal to early exercise?
(A) American put on a dividend paying stock
(B) American put on a nondividend-paying stock
(C) American call on a dividend paying stock
(D) American call on a nondividend-paying stock.

Problem 10.2
XYZ stock pays annual dividends of \$5 per share of stock, starting one year from now. For which of these strike prices of American call options on XYZ stocks might early exercise be optimal? All of the call options expire in 5 years. The annual effective risk-free interest rate is 3%.
(A) \$756
(B) \$554
(C) \$396
(D) \$256
(E) \$43
(F) \$10

Problem 10.3
American put options on XYZ stocks currently cost \$56 per option. We know that it is never optimal to exercise these options early. Which of these are possible values of the strike price K on these options and the stock price S_t of XYZ stock?
(A) $S = 123, K = 124$
(B) $S = 430, K = 234$
(C) $S = 234, K = 430$
(D) $S = 1234, K = 1275$
(E) $S = 500, K = 600$
(F) $S = 850, K = 800$

Problem 10.4
A 1-year European call option on a nondividend-paying stock with a strike price of \$38 sells for \$5.5 and is due six months from now. A 1-year European put option with the same underlying stock and same strike sells for \$0.56. The continuously compounded risk-free interest rate is 5%. The stock is currently selling for \$42. Would exercising a 1-year American put option with strike \$38 be optimal if exercised now?

Problem 10.5
A 1-year European call option on a dividend-paying stock with a strike price of \$38 sells for \$5.5 now and is due six months from now. A 1-year European put option with the same underlying stock and same strike sells for \$0.56. The continuously compounded risk-free interest rate is 5%. A dividend of \$0.95 is due at $t = 10$ months. Would it be optimal to exercise a 1-year American call with the same stock and strike price now?

Problem 10.6
A 1-year European call option on a dividend-paying stock with a strike price of \$38 sells for \$5.5 now and is due six months. A 1-year European put option with the same underlying stock and same strike sells for \$0.56. The continuously compounded risk-free interest rate is 5%. A dividend of \$0.95 is due at $t = 6.1$ months. Would it be optimal to exercise a 1-year American call with the same stock and strike price now?

Problem 10.7
A share of stock pays monthly dividends of \$4, starting one month from now. 1-year American call options with the underlying stock are issued for a strike price of \$23. The annual effective interest rate is 3%. Is it optimal to early exercise such American call options now?

Problem 10.8
A nondividend-paying stock is currently trading for \$96 per share. American put options are written with the underlying stock for a strike price of \$100. What is the maximum price of the put options at which early exercise might be optimal?

Problem 10.9
The following table list several American call options with a dividend paying stock. The stock is currently selling fo \$58. The first dividend payment is \$3 and due now whereas the second payment due in nine months. Which of the options might early exercise be optimal? Assume a continuously compounded risk-free interest rate of 5%.

Option	Strike	Expiration(in years)
A	40	1.5
B	50	1.5
C	50	1.0
D	52	1.0
E	59	0.75

Problem 10.10
Suppose that you have an American call option that permits you to receive one share of stock A by giving up one share of stock B. Neither stock pays a dividend. In what circumstances might you early exercise this call?

Problem 10.11
Suppose that you have an American put option that permits you to give up one share of stock A by receiving one share of stock B. Neither stock pays a dividend. In what circumstances might you early exercise this put?

Problem 10.12 ‡
For a stock, you are given:
(i) The current stock price is \$50.00.
(ii) $\delta = 0.08$
(iii) The continuously compounded risk-free interest rate is $r = 0.04$.
(iv) The prices for one-year European calls (C) under various strike prices (K) are shown below:

K	C
40	9.12
50	4.91
60	0.71
70	0.00

You own four special put options each with one of the strike prices listed in (iv). Each of these put options can only be exercised immediately or one year from now.
Determine the lowest strike price for which it is optimal to exercise these special put option(s) immediately.

11 Effect of Maturity Time Growth on Option Prices

Recall that the time of expiration of an option is the time after which the option is worthless. In this section, we discuss how time to expiration affects the prices of American and European options.

Consider two American calls with the same underlying asset and strike price K but with expiration time $T_1 < T_2$. The call with expiration time T_2 can be converted into a call with expiration time T_1 by voluntarily exercising it at time T_1. Thus, the price of the call with expiration time T_2 is at least as valuable as the call with expiration time T_1. In symbols, we have

$$C_{\text{Amer}}(K, T_1) \leq C_{\text{Amer}}(K, T_2).$$

A similar result holds for American put options

$$P_{\text{Amer}}(K, T_1) \leq P_{\text{Amer}}(K, T_2).$$

The above results are valid for all American options regardless whether the underlying asset pays dividends or not.

Next, we consider European call options. If the underlying asset pays no dividends, then the European call has the same price as an American call with the same underlying asset, same strike, and same expiration date (See Example 9.1). Thus, for $T_1 < T_2$, we have

$$C_{\text{Eur}}(K, T_1) \leq C_{\text{Eur}}(K, T_2).$$

The above inequality for European call options may not be valid for European call options with dividend-paying underlying asset. We illustrate this in the next example.

Example 11.1

Consider a stock that will pay a liquidating dividend two weeks from today. This means that the stock is worthless after the dividend payment. Show that a 1-week European call on the stock is more valuable than a 3-week European call.

Solution.

For $T > 2$ the stock is worthless, so that $C_{\text{Eur}}(K, T) = 0$. If $T \leq 2$, the call might be worth something depending on how high the strike price K is. That is, $C_{\text{Eur}}(K,T) > 0$. Let $T_1 = 1$ and $T_2 = 3$. Then $T_1 < T_2$ but $C_{\text{Eur}}(K,T_1) > 0 = C_{\text{Eur}}(K,T_2)$ ■

European Options when the strike price grows overtime
Next, let's consider European options on non-dividend paying stock with strike price growing over time. Let K be the original ($t = 0$) strike price. Let $C(t)$ denote the time 0 price for a European call maturing at time t and with strike price $K_t = Ke^{rt}$. Suppose $t < T$. Suppose that $C(t) \geq C(T)$. Then we buy the call with T years to expiration and sell the call with t years to expiration. The payoff of the longer-lived call at time T is $\max\{0, S_T - K_T\}$. The payoff of the shorter-lived call at time T is $-\max\{0, S_t - K_t\}$ accumulated from t to T.
• Suppose $S_t < K_t$. Then the payoff of the shorter-lived call is 0 and accumulates to 0 at time T. If $S_T < K_T$ then the payoff of the longer-lived call is 0 at time T. Thus, the total payoff of the position is 0 at time T.
• Suppose $S_t < K_t$. Then the payoff of the shorter-lived call is 0 and accumulates to 0 at time T. If $S_T \geq K_T$ then the payoff of the longer-lived call is $S_T - K_T$ at time T. Thus, the total payoff of the position is $S_T - K_T$ at time T.
• Suppose $S_t \geq K_t$. Borrow the stock and deliver it and collect the strike price K_t. Invest this amount at the risk-free rate, so you will have $K_te^{r(T-t)}$ at time T. If $S_T \leq K_T$, you buy back the stock give it to the owner so that the total payoff of this position is $K_T - S_T$ at time T.
• Suppose $S_t \geq K_t$. Borrow the stock and deliver it and collect the strike price K_t. Invest this amount at the risk-free rate, so you will have $K_te^{r(T-t)}$ at time T. If $S_T > K_T$, exercise the option and receive $S_T - K_T$. Buy the stock. For this position, the total payoff is 0 at time T.
It follows that in order to avoid arbitrage, we must have

$$\text{If } T > t \text{ then } C_{\text{Eur}}(K_T, T) > C_{\text{Eur}}(K_t, t).$$

Likewise,

$$\text{If } T > t \text{ then } P_{\text{Eur}}(K_T, T) > P_{\text{Eur}}(K_t, t).$$

Example 11.2
The premium of a 6-month European call with strike price \$100 is \$22.50. The premium of a 9-month European call with strike price \$102.53 is \$20.00.

The continuously compounded risk-free interest rate is 10%.
(a) Demonstrate an arbitarge opportunity.
(b) Given $S_{0.5} = \$98$ and $S_{0.75} = \$101$. Find the value of the accumulated arbitrage strategy after 9 months?
(c) Given $S_{0.5} = \$98$ and $S_{0.75} = \$103$. Find the value of the accumulated arbitrage strategy after 9 months?

Solution.
(a) First, notice that $102.53 = 100e^{0.1\times 0.25}$. We have $C(0.5) = 22.50 > C(0.75)$. We buy one call option with strike price \$102.53 and time to expiration nine months from now. We sell one call option with strike price of \$100 and time to expiration six months from now. The payoff table is given next.

		Payoff at Time 9 months			
		$S_{0.75} \leq 102.53$		$S_{0.75} > 102.53$	
		Payoff at Time 6 months			
Transaction	Time 0	$S_{0.5} \leq 100$	$S_{0.5} > 100$	$S_{0.5} \leq 100$	$S_{0.5} > 100$
Sell $C(0.5)$	$C(0.5)$	0	$102.53 - S_{0.75}$	0	$102.53 - S_{0.75}$
Buy $C(0.75)$	$-C0.75)$	0	0	$S_{0.75} - 102.53$	$S_{0.75} - 102.53$
Total Payoff	$C(0.5) - C(0.75)$	0	$102.53 - S_{0.75}$	$S_{0.75} - 102.53$	0

(b) The cash flow of the two options nets to zero after 9 months. Thus, the accumulated value of the arbitrage strategy at the end of nine months is

$$(22.50 - 20.00)e^{0.10\times 0.75} = \$2.6947$$

(c) The cash flow of the two options nets to $S_{0.75} - 102.53 = 103 - 102.53 = 0.47$. The accumulated value of the arbitrage strategy at the end of nine months is

$$2.6947 + 0.47 = \$3.1647 \blacksquare$$

Practice Problems

Problem 11.1
True or false:
(A) American call and put options (with same strike price) become more valuable as the time of expiration increases.
(B) European call options (with same strike price) become more valuable as the time of expiration increases.
(C) A long-lived $K-$strike European call on a dividend paying stock is at least as valuable as a short-lived $K-$strike European call.
(D) A long-lived $K-$strike European call on a nondividend-paying stock is at least as valuable as a short-lived $K-$strike European call.

Problem 11.2
Which of these statements about call and put options on a share of stock is always true?
(A) An American call with a strike price of \$43 expiring in 2 years is worth at least as much as an American call with a strike price of \$52 expiring in 1 year.
(B) An American call with a strike price of \$43 expiring in 2 years is worth at least as much as an American call with a strike price of \$43 expiring in 1 year.
(C) An American call with a strike price of \$43 expiring in 2 years is worth at least as much as a European call with a strike price of \$43 expiring in 2 years.
(D) An American put option with a strike price of \$56 expiring in 3 years is worth at least as much as an American put option with a strike price of \$56 expiring in 4 years.
(E) A European put option with a strike price of \$56 expiring in 3 years is worth at least as much as a European put option with a strike price of \$56 expiring in 2 years.

Problem 11.3
Gloom and Doom, Inc. will be bankrupt in 2 years, at which time it will pay a liquidating dividend of \$30 per share. You own a European call option on Gloom and Doom, Inc. stock expiring in 2 years with a strike price of \$20. The annual continuously compounded interest rate is 2%. How much is the option currently worth? (Assume that, for $T = 2$, the option can be exercised just before the dividend is paid.)

Problem 11.4
European put options are written on the stock of a bankrupt company. That is, the stock price is \$0 per share. What is the price of a put option that has strike price \$92 and expires 2 years from now? The annual continuously compounded interest rate is 21%.

Problem 11.5
Imprudent Industries plans to pay a liquidating dividend of \$20 in one year. Currently, European calls and puts on Imprudent Industries are traded with the following strike prices (K) and times to expiration (T). Which of the following European options will have the highest value? (Assume that, for $T = 1$, the options can be exercised just before the dividend is paid.) The annual effective interest rate is 3%.
(A) Call; $K = 3; T = 1.1$
(B) Call; $K = 19; T = 1$
(C) Call; $K = 19; T = 1.1$
(D) Put; $K = 10; T = 2$
(E) Put; $K = 10; T = 0.4$.

Problem 11.6
European call options on Oblivious Co. are issued with a certain strike price that grows with time according to the formula $K_t = Ke^{rt}$. Which of these times to maturity will result in the highest call premium? (A higher number indicates later time to maturity.) Assume Oblivious Co. pays no dividend.
(A) 2 months
(B) 4 months
(C) 1 year
(D) 2 years
(E) 2 weeks
(F) 366 days

Problem 11.7
Let $P(t)$ denote the time 0 price for a European put maturing at time t and with strike price $K_t = Ke^{rt}$. Suppose $t < T$. Suppose that $P(t) \geq P(T)$. Show that this leads to an arbitrage opportunity. The underlying stock does not pay any dividends.

Problem 11.8
The premium of a 6-month European put with strike price \$198 is \$4.05.

The premium of a 9-month European put with strike price \$206.60 is \$3.35. The continuously compounded risk-free interest rate is 17%.
(a) Demonstrate an arbitrage opportunity.
(b) Given that $S_{0.5} = \$196$ and $S_{0.75} = \$205$. What is the value of the accumulated arbitrage strategy after 9 months?

Problem 11.9
Two American call options on the same stock both having a striking price of \$100. The first option has a time to expiration of 6 months and trades for \$8. The second option has a time to expiration of 3 months and trades for \$10. Demonstrate an arbitrage.

Problem 11.10
The premium of a 6-month European call on a non-dividend paying stock with strike price \$198 is \$23.50. The premium of a 9-month European call on the same non-dividend paying stock with strike price \$207.63 is \$22.00. The continuously compounded risk-free interest rate is 19%.
(a) Demonstrate an arbitrage opportunity.
(b) Given that $S_{0.5} = \$278$ and $S_{0.75} = \$205$. What is the value of the accumulated arbitrage strategy after 9 months?

12 Options with Different Strike Prices but Same Time to Expiration

In this section we explore properties of options with different strike prices but the same time to expiration.
The first result shows that the call premium decreases as the strike price increases.

Proposition 12.1
Suppose $K_1 < K_2$ with corresponding call option (American or European) prices $C(K_1)$ and $C(K_2)$ then

$$C(K_1) \geq C(K_2). \tag{12.1}$$

Moreover,

$$C(K_1) - C(K_2) \leq K_2 - K_1. \tag{12.2}$$

Proof.
We will show (12.1) using the usual strategy of a no-arbitrage argument. Let's assume that the inequality above does not hold, that is, $C(K_1) < C(K_2)$. We want to set up a strategy that pays us money today. We can do this by selling the high-strike call option and buying the low-strike call option (this is a bull spread). We then need to check if this strategy ever has a negative payoff in the future. Consider the following payoff table:

		Expiration or Exercise		
Transaction	Time 0	$S_t < K_1$	$K_1 \leq S_t \leq K_2$	$S_t > K_2$
Sell $C(K_2)$	$C(K_2)$	0	0	$K_2 - S_t$
Buy $C(K_1)$	$-C(K_1)$	0	$S_t - K_1$	$S_t - K_1$
Total	$C(K_2) - C(K_1)$	0	$S_t - K_1$	$K_2 - K_1$

Every entry in the row labeled "Total" is nonnegative. Thus, by selling the high-strike call and buying the low-strike call we are guaranteed not to lose money. This is an arbitrage. Hence, to prevent arbitrage, (12.1) must be satisfied. If the options are Americans then we have to take in consideration the possibility of early exercise of the written call. If that happens at time $t < T$, we can simply exercise the purchased option, earning the payoffs in the table. If it is not optimal to exercise the purchased option, we can sell it, and the payoff table becomes

		Expiration or Exercise		
Transaction	Time 0	$S_t < K_1$	$K_1 \leq S_t \leq K_2$	$S_t > K_2$
Sell $C(K_2)$	$C(K_2)$	0	0	$K_2 - S_t$
Buy $C(K_1)$	$-C(K_1)$	0	$S_t - K_1$	$C_t(K_1)$
Total	$C(K_2) - C(K_1)$	0	$S_t - K_1$	$C(K_1) + K_2 - S_t$

Since $C_t(K_1) \geq S_t - K_1$ we find that $C_t(K_1) + K_2 - S_t \geq K_2 - K_1$. That is, we get even higher payoffs than exercising the purchased option ∎

Example 12.1
Establish the relationship (12.2). That is, the call premium changes by less than the change in the strike price.

Solution.
We will use the strategy of a no-arbitrage argument. Assume $C(K_1) - C(K_2) - (K_2 - K_1) > 0$. We want to set up a strategy that pays us money today. We can do this by selling the low-strike call option, buying the high-strike call option (this is a call bear spread), and lending the amount $K_2 - K_1$. We then need to check if this strategy ever has a negative payoff in the future. Consider the following payoff table:

		Expiration or Exercise		
Transaction	Time 0	$S_t < K_1$	$K_1 \leq S_t \leq K_2$	$S_t > K_2$
Sell $C(K_1)$	$C(K_1)$	0	$K_1 - S_t$	$K_1 - S_t$
Buy $C(K_2)$	$-C(K_2)$	0	0	$S_t - K_2$
Lend $K_2 - K_1$	$K_1 - K_2$	$e^{rt}(K_2 - K_1)$	$e^{rt}(K_2 - K_1)$	$e^{rt}(K_2 - K_1)$
Total	$C(K_1) - C(K_2)$ $-(K_2 - K_1)$	$e^{rt}(K_2 - K_1)$	$e^{rt}(K_2 - K_1)$ $-(S_t - K_1)$	$e^{rt}(K_2 - K_1)$ $-(K_2 - K_1)$

Every entry in the row labeled "Total" is nonnegative. Thus, by selling the low-strike call, buying the high-strike call and lending $K_2 - K_1$ we are guaranteed not to lose money at time T. This is an arbitrage. Hence, to prevent arbitrage, (12.2) must be satisfied. In the case of American options, if the written call is exercised, we can duplicate the payoffs in the table by throwing our option away (if $K_1 \leq S_t \leq K_2$) or exercising it (if $S_t > K_2$). Since it never makes sense to discard an unexpired option, and since exercise may not be optimal, we can do at least as well as the payoff in the table if the options are American ∎

Remark 12.1
If the options are European, we can put a tighter restriction on the difference in call premiums, namely $C_{\text{Eur}}(K_1) - C_{\text{Eur}}(K_2) \leq PV_{0,T}(K_2 - K_1)$. We would show this by lending $PV_{0,T}(K_2 - K_1)$ instead of $K_2 - K_1$. This strategy does not work if the options are American, since we don't know how long it will be before the options are exercised, and, hence, we don't know what time to use in computing the present value.

We can derive similar relationships for (American or European) puts as we did for calls. Namely, we have

Proposition 12.2
Suppose $K_1 < K_2$ with corresponding put option prices $P(K_1)$ and $P(K_2)$ then

$$P(K_2) \geq P(K_1). \tag{12.3}$$

Moreover,

$$P(K_2) - P(K_1) \leq K_2 - K_1 \tag{12.4}$$

and

$$P_{\text{Eur}}(K_2) - P_{\text{Eur}}(K_1) \leq PV_{0,T}(K_2 - K_1).$$

Example 12.2
The premium of a 50-strike call option is 9 and that of a 55-strike call option is 10. Both options have the same time to expiration.
(a) What no-arbitrage property is violated?
(b) What spread position would you use to effect arbitrage?
(c) Demonstrate that the spread position is an arbitrage.

Solution.
(a) We are given that $C(50) = 9$ and $C(55) = 10$. This violates (12.1).
(b) Since $C(55) - C(50) = 1 > 0$, to profit from an arbitrage we sell the 55-strike call option and buy the 50-strike call option. This is a call bull spread position.
(c) We have the following payoff table.

		Expiration or Exercise		
Transaction	Time 0	$S_t < 50$	$50 \leq S_t \leq 55$	$S_t > 55$
Sell $C(55)$	10	0	0	$55 - S_t$
Buy $C(50)$	-9	0	$S_t - 50$	$S_t - 50$
Total	$+1$	0	$S_t - 50$	5

Note that we initially receive money, and that at expiration the profit is non-negative. We have found arbitrage opportunities ■

Example 12.3
The premium of a 50-strike put option is 7 and that of a 55-strike option is 14. Both options have the same time of expiration.
(a) What no-arbitrage property is violated?
(b) What spread position would you use to effect arbitrage?
(c) Demonstrate that the spread position is an arbitrage.

Solution.
(a) We are given that $P(50) = 7$ and $P(55) = 14$. This violates (12.4).
(b) Since $P(55) - P(50) = 7 > 0$, to profit from an arbitrage we sell the 55-strike put option, buy the 50-strike call option. This is a put bull spread position. This positive cash flow can be lent resulting in the following table
(c) We have the following payoff table.

		Expiration or Exercise		
Transaction	Time 0	$S_t < 50$	$50 \leq S_t \leq 55$	$S_t > 55$
Sell $P(55)$	14	$S_t - 55$	$S_t - 55$	0
Buy $P(50)$	-7	$50 - S_t$	0	0
Total	7	-5	$S_t - 55 \geq -5$	0

Note that we initially receive more money than our biggest possible exposure in the future ■

Example 12.4
A 90-strike European call with maturity date of 2 years sells for \$10 and a 95-strike European call with the same underlying asset and same expiration date sells for \$5.25. The continuously compounded free-risk interest rate is 10%. Demonstrate an arbitrage opportunity.

Solution.
We are given $C(90) = 10$ and $C(95) = 5.25$. We sell the 90-strike call, buy the 95-strike call (this is a call bear spread), and loan \$4.75. The payoff table is shown next.

		Expiration or Exercise		
Transaction	Time 0	$S_T < 90$	$90 \leq S_t \leq 95$	$S_t > 95$
Sell $C(90)$	10	0	$90 - S_T$	$90 - S_T$
Buy $C(95)$	-5.25	0	0	$S_T - 95$
Lend 4.75	-4.75	5.80	5.80	5.80
Total	0	5.80	$95.80 - S_T > 0$	0.8

In all possible future states, we have a strictly positive payoff. We demonstrated an arbitrage ■

Practice Problems

Problem 12.1
Given two options with strike prices \$77 and \$89 respectively and with the same time to expiration. Which of these statements are true about the following American call and put options?
(A) $P(77) \leq P(89)$
(B) $P(89) - P(77) \geq 12$
(C) $C(77) \geq C(89)$
(D) $P(77) \geq P(89)$
(E) $C(89) - C(77) \geq 12$
(F) $C(77) - C(89) \geq 12$

Problem 12.2
54-strike Call options on a share of stock have a premium of \$19. Which of the following are possible values for call options on the stock with a strike price of \$32 and the same time to expiration?
(A) 16
(B) 20
(C) 34
(D) 25
(E) 45
(F) It is impossible to have call options for this stock with a strike price of \$32.

Problem 12.3
Suppose $K_1 < K_2$ with corresponding put option prices $P(K_1)$ and $P(K_2)$. Show that

$$P(K_2) \geq P(K_1)$$

Problem 12.4
Establish the relationship $P(K_2) - P(K_1) \leq K_2 - K_1$.

Problem 12.5
Establish the relationship $P_{\text{Eur}}(K_2) - P_{\text{Eur}}(K_1) \leq PV_{0,T}(K_2 - K_1)$.

Problem 12.6
The premium of a 50-strike put option is 7 and that of a 55-strike put option is 6. Both options have the same time to expiration.

(a) What no-arbitrage property is violated?
(b) What spread position would you use to effect arbitrage?
(c) Demonstrate that the spread position is an arbitrage.

Problem 12.7
The premium of a 50-strike call option is 16 and that of a 55-strike call option is 10. Both options have the same time to expiration.
(a) What no-arbitrage property is violated?
(b) What spread position would you use to effect arbitrage?
(c) Demonstrate that the spread position is an arbitrage.

Problem 12.8 ‡
Given the following chart about call options on a particular dividend paying stock, which options has the highest value?

Option	Option Style	Maturity	Strike Price	Stock Price
A	European	1 year	50	42
B	American	1 year	50	42
C	European	2 years	50	42
D	American	2 years	50	42
E	American	2 years	55	42

Problem 12.9
A 70-strike European put with maturity date of 1 year sells for \$8.75 and a 75-strike European put with the same underlying asset and same expiration date sells for \$13.50. The continuously compounded free-risk interest rate is 6.9%.
(a) Demonstrate an arbitrage opportunity.
(b) What are the accumulated arbitrage profits if the final stock price is \$68?
(c) What are the accumulated arbitrage profits if the final stock price is \$73?

Problem 12.10 ‡
You are given:
(i) $C(K,T)$ denotes the current price of a $K-$strike $T-$year European call option on a nondividend-paying stock.
(ii) $P(K,T)$ denotes the current price of a $K-$strike $T-$year European put option on the same stock.
(iii) S denotes the current price of the stock.

(iv) The continuously compounded risk-free interest rate is r.
Which of the following is (are) correct?
(I) $0 \leq C(50, T) - C(55, T) \leq 5e^{-rT}$
(II) $50e^{-rT} \leq P(45, T) - C(50, T) + S \leq 55e^{-rT}$
(III) $45e^{-rT} \leq P(45, T) - C(50, T) + S \leq 50e^{-rT}$.

13 Convexity Properties of the Option Price Functions

In this section, we consider the call option price and the put option price as functions of the strike price while keeping the time to expiration fixed. We will show that these functions are convex. The results of this section are true for both American and European options.
We recall the reader that a function f with domain D is said to be **convex** if for all x, y in D and $0 < \lambda < 1$ such that $\lambda x + (1-\lambda)y \in D$ we must have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

Example 13.1
Let f be a convex function with domain D. Let x, y be in D such that $\frac{x+y}{2}$ is in D. Show that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}.$$

Solution.
This follows from the definition of convexity with $\lambda = \frac{1}{2}$ ■
We next show that the call premium is a convex function of the strike price K.

Proposition 13.1
Consider two call options with strike prices $K_1 < K_2$ and the same time to expiration. Let $0 < \lambda < 1$. Then

$$C(\lambda K_1 + (1-\lambda)K_2) \leq \lambda C(K_1) + (1-\lambda)C(K_2).$$

Proof.
We will show the required inequality by using the usual strategy of a no-arbitrage argument. Let's assume that the inequality above does not hold, that is, $C(\lambda K_1 + (1-\lambda)K_2) > \lambda C(K_1) + (1-\lambda)C(K_2)$ or $C(K_3) > \lambda C(K_1) + (1-\lambda)C(K_2)$ where $K_3 = \lambda K_1 + (1-\lambda)K_2$. Note that $K_3 = \lambda(K_1 - K_2) + K_2 < K_2$ and $K_1 = (1-\lambda)K_1 + \lambda K_1 < (1-\lambda)K_2 + \lambda K_1 = K_3$ so that $K_1 < K_3 < K_2$. We want to set up a strategy that pays us money today. We can do this by selling one call option with strike price K_3, buying λ call options with strike price K_1, and buying $(1-\lambda)$ call options with strike K_2. The payoff table of this position is given next.

		Expiration or exercise			
Transaction	Time 0	$S_t < K_1$	$K_1 \le S_t \le K_3$	$K_3 \le S_t \le K_2$	$S_t > K_2$
Sell 1 K_3- strike call	$C(K_3)$	0	0	$K_3 - S_t$	$K_3 - S_t$
Buy λ K_1-strike calls	$-\lambda C(K_1)$	0	$\lambda(S_t - K_1)$	$\lambda(S_t - K_1)$	$\lambda(S_t - K_1)$
Buy $(1-\lambda)$ K_2-strike calls	$-(1-\lambda)C(K_2)$	0	0	0	$(1-\lambda)(S_t - K_2)$
Total	$C(K_3) - \lambda C(K_1)$ $-(1-\lambda)C(K_2)$	0	$\lambda(S_t - K_1)$	$(1-\lambda)(K_2 - S_t)$	0

Note that

$$\begin{aligned} K_3 - S_t + \lambda(S_t - K_1) =& K_3 - \lambda K_1 - (1-\lambda)S_t \\ =&(1-\lambda)K_2 - (1-\lambda)S_t \\ =&(1-\lambda)(K_2 - S_t) \end{aligned}$$

and

$$K_3 - S_t + \lambda(S_t - K_1) + (1-\lambda)(S_t - K_2) = \lambda K_1 + (1-\lambda)K_2 - S_t + \lambda(S_t - K_1) + (1-\lambda)(S_t - K_2) = 0.$$

The entries in the row "Total" are all nonnegative. In order to avoid arbitrage, the initial cost must be non-positive. That is

$$C(\lambda K_1 + (1-\lambda)K_2) \le \lambda C(K_1) + (1-\lambda)C(K_2) \blacksquare$$

Example 13.2
Consider three call options with prices $C(K_1), C(K_2)$, and $C(K_3)$ where $K_1 < K_2 < K_3$. Show that

$$\frac{C(K_1) - C(K_2)}{K_2 - K_1} \ge \frac{C(K_2) - C(K_3)}{K_3 - K_2}. \tag{13.1}$$

Solution.
Let $\lambda = \frac{K_3 - K_2}{K_3 - K_1}$. We have $K_1 < K_2$ so that $K_3 - K_2 < K_3 - K_1$ and hence $0 < \lambda < 1$. Also, we note that

$$K_2 = \frac{K_3 - K_2}{K_3 - K_1}K_1 + \frac{K_2 - K_1}{K_3 - K_1}K_3 = \lambda K_1 + (1-\lambda)K_3.$$

Using convexity we can write

$$C(K_2) \leq \frac{K_3 - K_2}{K_3 - K_1}C(K_1) + \frac{K_2 - K_1}{K_3 - K_1}C(K_3)$$

which is equivalent to

$$(K_3 - K_1)C(K_2) \leq (K_3 - K_2)C(K_1) + (K_2 - K_1)C(K_3)$$

or

$$(K_3-K_1)C(K_2)-(K_3-K_2)C(K_2)-(K_2-K_1)C(K_3) \leq (K_3-K_2)C(K_1)-(K_3-K_2)C(K_2).$$

Hence,

$$(K_2 - K_1)[C(K_2) - C(K_3)] \leq (K_3 - K_2)[C(K_1) - C(K_2)]$$

and the result follows by dividing through by the product $(K_2 - K_1)(K_3 - K_2)$ ∎

Example 13.3
Consider three call options with the same time to expiration and with strikes: \$50, \$55, and \$60. The premiums are \$18, \$14, and \$9.50 respectively.
(a) Show that the convexity property (13.1) is violated.
(b) What spread position would you use to effect arbitrage?
(c) Demonstrate that the spread position is an arbitrage.

Solution.
(a) We are given $K_1 = 50, K_2 = 55$, and $K_3 = 60$. Since

$$\frac{C(K_1) - C(K_2)}{K_2 - K_1} = 0.8 < \frac{C(K_3) - C(K_2)}{K_3 - K_2} = 0.9$$

the convexity property (13.1) is violated.
(b) We find

$$\lambda = \frac{K_3 - K_2}{K_3 - K_1} = \frac{60 - 55}{60 - 50} = 0.5.$$

Thus, a call butterfly spread[1] is constructed by selling one 55-strike call, buying 0.5 units of 50-strike calls and 0.5 units of 60-strike calls. To buy and sell round lots, we multiply all the option trades by 2.
(c) The payoff table is given next.

[1] A **call butterfly spread** is an option strategy that involves selling several call options and at the same time buying several call options with the same underlying asset and different strike prices. This strategy involves three different strike prices.

		Expiration or exercise			
Transaction	Time 0	$S_t < 50$	$50 \leq S_t \leq 55$	$55 \leq S_t \leq 60$	$S_t > 60$
Sell 2 55−strike calls	28	0	0	$110 - 2S_t$	$110 - 2S_t$
Buy one 50−strike call	−18	0	$S_t - 50$	$S_t - 50$	$S_t - 50$
Buy one 60−strike call	−9.50	0	0	0	$S_t - 60$
Total	0.50	0	$S_t - 50$	$60 - S_t$	0

Note that we initially receive money and have non-negative future payoffs. Therefore, we have found an arbitrage possibility, independent of the prevailing interest rate ■

Similar convexity results hold for put options. Namely, we have

Proposition 13.2
(a) For $0 < \lambda < 1$ and $K_1 < K_2$ we have

$$P(\lambda K_1 + (1-\lambda)K_2) \leq \lambda P(K_1) + (1-\lambda)P(K_2).$$

(b) For $K_1 < K_2 < K_3$ we have

$$\frac{P(K_2) - P(K_1)}{K_2 - K_1} \leq \frac{P(K_3) - P(K_2)}{K_3 - K_2}. \tag{13.2}$$

(c) For $\lambda = \frac{K_3 - K_2}{K_3 - K_1}$ we have

$$P(K_2) \leq \lambda P(K_1) + (1-\lambda)P(K_3).$$

Example 13.4
Consider three put options with the same time to expiration and with strikes: \$50, \$55, and \$60. The premiums are \$7, \$10.75, and \$14.45 respectively.
(a) Show that the convexity property (13.2) is violated.
(b) What spread position would you use to effect arbitrage?
(c) Demonstrate that the spread position is an arbitrage.

Solution.
(a) We are given $K_1 = 50, K_2 = 55$, and $K_3 = 60$. Since

$$\frac{P(K_2) - P(K_1)}{K_2 - K_1} = 0.75 > \frac{P(K_3) - P(K_2)}{K_3 - K_2} = 0.74$$

the convexity property for puts is violated.
(b) We find

$$\lambda = \frac{K_3 - K_2}{K_3 - K_1} = \frac{60 - 55}{60 - 50} = 0.5.$$

Thus, a put butterfly spread is constructed by selling one 55-strike put, buying 0.5 units of 50-strike puts and 0.5 units of 60-strike puts. To buy and sell round lots, we multiply all the option trades by 2.
(c) The payoff table is given next.

		Expiration or exercise			
Transaction	Time 0	$S_t < 50$	$50 \leq S_t \leq 55$	$55 \leq S_t \leq 60$	$S_t > 60$
Sell 2 55−strike puts	21.50	$2S_t - 110$	$2S_t - 110$	0	0
Buy one 50−strike put	−7	$50 - S_t$	0	0	0
Buy one 60−strike put	−14.45	$60 - S_t$	$60 - S_t$	$60 - S_t$	0
Total	0.05	0	$S_t - 50$	$60 - S_t$	0

Note that we initially receive money and have non-negative future payoffs. Therefore, we have found an arbitrage possibility, independent of the prevailing interest rate ■

Example 13.5 ‡
Near market closing time on a given day, you lose access to stock prices, but some European call and put prices for a stock are available as follows:

Strike Price	Call Price	Put Price
\$40	\$11	\$3
\$50	\$6	\$8
\$55	\$3	\$11

All six options have the same expiration date.
After reviewing the information above, John tells Mary and Peter that no arbitrage opportunities can arise from these prices.
Mary disagrees with John. She argues that one could use the following portfolio to obtain arbitrage profit: Long one call option with strike price 40; short three call options with strike price 50; lend \$1; and long some calls

with strike price 55.
Peter also disagrees with John. He claims that the following portfolio, which is different from Mary's, can produce arbitrage profit: Long 2 calls and short 2 puts with strike price 55; long 1 call and short 1 put with strike price 40; lend \$2; and short some calls and long the same number of puts with strike price 50.
Who is correct?

Solution.
One of the requirements for an arbitrage position is that it costs nothing on net to enter into. The second requirement is that it will make the owner a profit, irrespective of future price movements.
First consider Mary's proposal. The call option prices do not satisfy the convexity condition

$$\frac{C(K_1)-C(K_2)}{K_2-K_1} \geq \frac{C(K_2)-C(K_3)}{K_2-K_3}$$

so there is an arbitrage opportunity: Purchasing one 40-strike call costs \$11, while selling 3 50-strike call options gives her $3\times 6=\$18$. She also buys X \$55-strike calls at a price of \$3 each and lends out \$1. So her net cost is $-11+18-1-3X$ or $6-3X$. In order for $6-3X$ to be 0, X must equal 2. In this case, we have the following payoff table for Mary's position

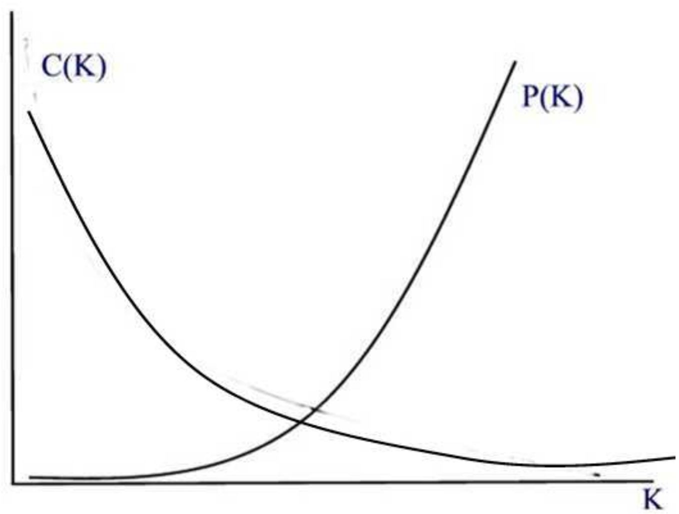
Transaction	Time 0	$S_T<40$	$40\leq S_T\leq 50$	$50\leq S_T\leq 55$	$S_T>55$
Buy 1 40-Strike call	-11	0	S_T-40	S_T-40	S_T-40
Sell 3 50-strike calls	18	0	0	$-3(S_T-50)$	$-3(S_T-50)$
Lend \$	-1	e^{rT}	e^{rT}	e^{rT}	e^{rT}
Buy 2 55-strike calls	-6	0	0	0	$2(S_T-55)$
Total	0	$e^{rT}>0$	$e^{rT}+S_T-40>0$	$e^{rT}+2(55-S_T)>0$	$e^{rT}>0$

Thus, Mary's portofolio yields an arbitrage profit.
Now consider Peter's proposal. Purchasing one 40-strike call and selling one 40-strike put option cost $-11+3=-\$8$. Purchasing 2 55-strike puts and selling 2 55-strike calls cost $2(-3+11)=\$16$. Lending \$2 costs $-\$2$ at time $=0$. Selling X 50-strike calls and purchasing X 50-strike puts cost $X(6-8)=-2X$. Thus, her net cost at time $t=0$ is $-8+16-2-2X=6-2X$. In order for $6-2X$ to be zero we must have $X=3$. In this case, we have the following payoff table for Peter's position

Transaction	Time 0	$S_T < 40$	$40 \leq S_T \leq 50$	$50 \leq S_T \leq 55$	$S_T > 55$
Buy 2 55-strike calls Sell 2 55-strike puts	16	$2(S_T - 55)$	$2(S_T - 55)$	$2(S_T - 55)$	$2(S_T - 55)$
Buy 1 40-strike call Sell 1 40-strike put	-8	$S_T - 40$	$S_T - 40$	$S_T - 40$	$S_T - 40$
Lend \$	-2	$2e^{rT}$	$2e^{rT}$	$2e^{rT}$	$2e^{rT}$
Buy 3 50-strike puts Sell 3 50-strike calls	-6	$3(50 - S_T)$	$3(50 - S_T)$	$3(50 - S_T)$	$3(50 - S_T)$
Total	0	$2e^{rT} > 0$	$2e^{rT} > 0$	$2e^{rT} > 0$	$2e^{rT} > 0$

Thus, Peter's portofolio yields an arbitrage profit. So Mary and Peter are right while John is wrong ■

Remark 13.1
We conclude from this and the previous section that a call(put) price function is decreasing (increasing) and concave up. A graph is given below.



Practice Problems

Problem 13.1
Three call options on a stock with the same time to expiration trade for three strike prices: \$46, \$32, and \$90. Determine λ so that $C(46) \leq \lambda C(32) + (1 - \lambda)C(90)$.

Problem 13.2
Call options on a stock trade for three strike prices: \$43, \$102, and \$231. The price of the \$43-strike option is \$56. The price of the \$231-strike option is \$23. What is the maximum possible price of the \$102-strike option? All options have the same time to expiration.

Problem 13.3
Call options on a stock trade for three strike prices, \$32, \$34, and \$23. The \$32-strike call currently costs \$10, while the \$34-strike call costs \$7. What is the least cost for the \$23-strike call option?

Problem 13.4
Put options on a stock trade for three strike prices: \$102, \$105, and $K_3 > 105$. Suppose that $\lambda = 0.5$. The \$102-strike put is worth \$20, the \$105-strike put is worth \$22, and the K_3-strike put is worth \$24. Find the value of K_3.

Problem 13.5
Consider two put options with strike prices $K_1 < K_2$ and the same time to expiration. Let $0 < \lambda < 1$. Show that

$$P(\lambda K_1 + (1 - \lambda)K_2) \leq \lambda P(K_1) + (1 - \lambda)P(K_2)$$

for some $0 < \lambda < 1$.

Problem 13.6
Consider three put options with prices $P(K_1), P(K_2)$, and $P(K_3)$ where $K_1 < K_2 < K_3$. Show that

$$P(K_2) \leq \lambda P(K_1) + (1 - \lambda)P(K_3).$$

Problem 13.7
Consider three put options with prices $P(K_1), P(K_2)$, and $P(K_3)$ where $K_1 < K_2 < K_3$. Show that

$$\frac{P(K_2) - P(K_1)}{K_2 - K_1} \leq \frac{P(K_3) - P(K_2)}{K_3 - K_2}.$$

Problem 13.8
Consider three call options with the same time to expiration and with strikes:\$80, \$100, and \$105. The premiums are \$22, \$9, and \$5 respectively.
(a) Show that the convexity property is violated.
(b) What spread position would you use to effect arbitrage?
(c) Demonstrate that the spread position is an arbitrage.

Problem 13.9
Consider three put options with the same time to expiration and with strikes: \$80, \$100, and \$105. The premiums are \$4, \$21, and \$24.80 respectively.
(a) Show that the convexity property is violated.
(b) What spread position would you use to effect arbitrage?
(c) Demonstrate that the spread position is an arbitrage.

Problem 13.10
Consider three Europeans put options that expire in 6 months and with strikes: \$50, \$55, and \$60. The premiums are \$7, \$10.75, and \$14.45 respectively. The continuously compounded risk free interest rate is 10%. Complete the following table

Stock at Expiration	Accumulated Strategy Profits
48	
53	
59	
63	

Option Pricing in Binomial Models

The binomial option pricing model is a model founded by Cox, Ross, and Rubenstein in 1979. This model computes the no-arbitrage price of an option when the price of the underlying asset has exactly two-state prices at the end of a period. Binomial pricing model displays the option pricing in a simple setting that involves only simple algebra. In this chapter, we introduce the binomial model and use it in computing European and American option prices for a variety of underlying assets.

14 Single-Period Binomial Model Pricing of European Call Options: Risk Neutral Approach

In this section we develop the one period binomial model to exactly price a long call option. For simplicity, we will assume that the underlying asset is a share of a stock that pays continuous dividends. We will make the assumption that for a given period the stock price either goes up (up state) or down (down state) in value. No other outcomes are possible for this stock's price. The restriction to only two possible values justifies the use of the word "binomial".

We will introduce the following notation: Let h be the length of one period. Let S be the stock price at the beginning of the period. We define the **up ratio** by $u = 1 + g = \frac{S_h}{S}$ where g is the rate of capital gain when the stock goes up and S_h is the stock price at the end of the period. We define the **down ratio** by $d = 1 + l = \frac{S_h}{S}$ where l is the rate of capital loss when the stock goes down. For example, if $S = 100$ and the stock price at the end of the period is 175 then $g = 75\%$. If at the end of the period the stock drops to 75 then $l = -25\%$. When the stock goes up we shall write uS and when it goes down we shall write dS. These two states can be described by a tree known as a **binomial tree** as shown in Figure 14.1.

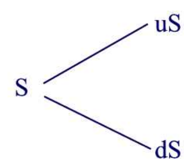


Figure 14.1

Now, let K be the strike price of a call option on the stock that pays continuous dividends and that matures at the end of the period. At the end of the period, we let $C_u = \max\{0, uS - K\}$ denote the value of the call option in the up state and $C_d = \max\{0, dS - K\}$ the value of the call option in the down state. Let C denote the price of the option at the beginning of the period. Let r be the continuously compounded risk-free annual rate. Thus, the periodic rate is rh. Let the dividend yield be δ. We will assume that the dividends are reinvested in the stock so that one share at the beginning of the period grows to $e^{\delta h}$ shares at the end of the period.[1]

[1]See [2] Section 70.

The major question is to determine the current price C of the option. For that, we are going to use two different approaches: The risk-neutral approach and the replicating portfolio approach. Either approach will yield the same answer. In this section, we will discuss the former one leaving the latter one to the next section.
The basic argument in the risk neutral approach is that investors are risk-neutral, i.e., assets expected return is the risk-free rate.[1] In this world of risk-neutral investors, one computes risk-neutral probabilities associated with the stock. These probabilities are used to compute the expected option payoff, and this payoff is discounted back to the present at the risk-free rate in order to get today's option price.
Let p_u denote the **risk-neutral probability**[2] of an increase in the stock price. Then the risk neutral probability of a decrease in the stock price is $p_d = 1 - p_u$. Let X be the discrete random variable representing the call option price. Thus, the range of X is the set $\{C_u, C_d\}$ with C_u occurring with a probability p_u and C_d occurring with a probability $p_d = 1 - p_u$. Thus, the expected future price of the option is

$$p_uC_u + (1 - p_u)C_d.$$

In the risk neutral approach, we assume that the expected stock price is just the forward price of the stock from time t to time $t + h$. In this case, we can think of p_u as the probability for which the expected stock price is the forward price. Hence,

$$p_u uS + (1 - p_u)dS = Se^{(r-\delta)h}.$$

Solving this equation for p_u we find

$$p_u = \frac{e^{(r-\delta)h} - d}{u-d} \text{ and } p_d = \frac{u - e^{(r-\delta)h}}{u-d}.$$

Today's option price is

$$C = e^{-rh}\left(p_uC_u + (1 - p_u)C_d\right). \tag{14.1}$$

Example 14.1
Using the condition $d < e^{(r-\delta)h} < u$, show that $p_u > 0, p_d > 0$, and $p_u + p_d = 1$.

[1] Risk-neutral will be discussed in more details in Section 23.
[2] p_u and p_d look like probabilites but they are not in general. See Section 23.

Solution.
We have $e^{(r-\delta)h} - d > 0$, and $u - e^{(r-\delta)h} > 0$, and $u - d > 0$. Hence, $p_u > 0$ and $p_d > 0$. Now, adding p_u and p_d we find

$$p_u + p_d = \frac{e^{(r-\delta)h} - d}{u - d} + \frac{u - e^{(r-\delta)h}}{u - d} = \frac{u - d}{u - d} = 1.$$

Thus, p_u and p_d can be interpreted as probabilities ■

Example 14.2
Consider a European call option on the stock of XYZ with strike $95 and six months to expiration. XYZ stock does not pay dividends and is currently worth $100. The annual continuously compounded risk-free interest rate is 8%. In six months the price is expected to be either $130 or $80. Find C using the risk-neutral approach as discussed in this section.

Solution.
We have

$$p_u = \frac{e^{(0.08-0)\times 0.5} - 0.8}{1.3 - 0.8} = 0.4816215$$

This is the risk-neutral probability of the stock price increasing to $130 at the end of six months. The probability of it going down to $80 is $p_d = 1 - p_u = 0.5183785$. Now given that if the stock price goes up to $130, a call option with an exercise price of $95 will have a payoff of $35 and $0 if the stock price goes to $80, a risk-neutral individual would assess a 0.4816215 probability of receiving $35 and a 0.5183785 probability of receiving $0 from owning the call option. As such, the risk neutral value would be:

$$C = e^{-0.08\times 0.5}[0.4816215 \times 35 + 0.5183785 \times 0] = \$16.1958 \blacksquare$$

Practice Problems

Problem 14.1
Consider a European call option on the stock of XYZ with strike \$65 and one month to expiration. XYZ stock does not pay dividends and is currently worth \$75. The annual continuously compounded risk-free interest rate is 6%. In one month the price is expected to be either \$95 or \$63. Find C using the risk-neutral approach as discussed in this section.

Problem 14.2
Stock XYZ price is expected to be either \$75 or \$40 in one year. The stock is currently valued at \$51. The stock pays continuous dividends at the yield rate of 9%. The continuously compounded risk-free interest rate is 12%. Find the risk-neutral probability of an increase in the stock price.

Problem 14.3
GS Inc pays continuous dividends on its stock at an annual continuously-compounded yield of 9%. The stock is currently selling for \$10. In one year, its stock price could either be \$15 or \$7. The risk-neutral probability of the increase in the stock price is 41.3%. Using the one-period binomial option pricing model, what is the annual continuously-compounded risk-free interest rate?

Problem 14.4
The stock of GS, which current value of \$51, will sell for either \$75 or \$40 one year from now. The annual continuously compounded interest rate is 7%. The risk-neutral probability of an increase in the stock price (to \$75) is 0.41994. Using the one-period binomial option pricing model, find the current price of a call option on GS stock with a strike price of \$50.

Problem 14.5
The probability that the stock of GS Inc will be \$555 one year from now is 0.6. The probability that the stock will be \$521 one year from now is 0.4. Using the one-period binomial option pricing model, what is the forward price of a one-year forward contract on the stock?

Problem 14.6
Consider a share of nondividend-paying stock in a one-year binomial framework with annual price changes, with the current price of the stock being

55, and the price of the stock one year from now being either 40 or 70. The annual continuously compounded risk-free interest rate is 12%. Calculate risk-neutral probability that the price of the stock will go up.

Problem 14.7
A nondividend-paying stock, currently priced at \$120 per share, can either go up \$25 or down \$25 in any year. Consider a one-year European call option with an exercise price of \$130. The continuously-compounded risk-free interest rate is 10%. Use a one-period binomial model and a risk-neutral probability approach to determine the current price of the call option.

Problem 14.8 ‡
For a one-year straddle on a nondividend-paying stock, you are given:
(i) The straddle can only be exercised at the end of one year.
(ii) The payoff of the straddle is the absolute value of the difference between the strike price and the stock price at expiration date.
(iii) The stock currently sells for \$60.00.
(iv) The continuously compounded risk-free interest rate is 8%.
(v) In one year, the stock will either sell for \$70.00 or \$45.00.
(vi) The option has a strike price of \$50.00.
Calculate the current price of the straddle.

Problem 14.9 ‡
You are given the following regarding stock of Widget World Wide (WWW):
(i) The stock is currently selling for \$50.
(ii) One year from now the stock will sell for either \$40 or \$55.
(iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 10%.
(iv) The continuously compounded risk-free interest rate is 5%.
While reading the Financial Post, Michael notices that a one-year at-the-money European call written on stock WWW is selling for \$1.90. Michael wonders whether this call is fairly priced. He uses the binomial option pricing model to determine if an arbitrage opportunity exists.
What transactions should Michael enter into to exploit the arbitrage opportunity (if one exists)?

Problem 14.10
A binomial tree can be constructed using the equations

$$u = e^{(r-\delta)h+\sigma\sqrt{h}}$$

and

$$d = e^{(r-\delta)h - \sigma\sqrt{h}},$$

where σ denotes the volatility of the stock to be discussed in Section 16. Show that

$$p_u = \frac{1}{e^{\sigma\sqrt{h}} + 1}.$$

Problem 14.11 ‡

On January 1 2007, the Florida Property Company purchases a one-year property insurance policy with a deductible of \$50,000. In the event of a hurricane, the insurance company will pay the Florida Property Company for losses in excess of the deductible. Payment occurs on December 31 2007. For the last three months of 2007, there is a 20% chance that a single hurricane occurs and an 80% chance that no hurricane occurs. If a hurricane occurs, then the Florida Property company will experience \$1000000 in losses. The continuously compounded risk free rate is 5%.

On October 1 2007, what is the risk neutral expected value of the insurance policy to the Florida Property Company?

15 The Replicating Portfolio Method

The **replicating portfolio approach** consists of creating a portfolio that replicates the actual call option. Let Portfolio A consists of buying a call option on the stock and Portfolio B consists of buying Δ shares of the stock and borrowing $\$B > 0$. The payoff tables of these positions are shown below.

Transaction	Time 0	Up State	Down State
Buy a Call	$-C$	C_u	C_d
Total	$-C$	C_u	C_d

Transaction	Time 0	Up State	Down State
Buy Δ Shares	$-\Delta S$	$\Delta e^{\delta h} uS$	$\Delta e^{\delta h} dS$
Borrow \$B	B	$-Be^{rh}$	$-Be^{rh}$
Total	$-\Delta S + B$	$\Delta e^{\delta h} uS - Be^{rh}$	$\Delta e^{\delta h} dS - Be^{rh}$

If Portfolio B is to replicate portfolio A we must have

$$C = \Delta S - B$$

$$C_u = \Delta e^{\delta h} uS - Be^{rh}$$
$$C_d = \Delta e^{\delta h} dS - Be^{rh}.$$

Solving the last two equations we find

$$\Delta = e^{-\delta h} \frac{C_u - C_d}{S(u-d)} \tag{15.1}$$

and

$$B = e^{-rh} \frac{uC_d - dC_u}{d - u}. \tag{15.2}$$

Thus,

$$C = \Delta S - B = e^{-rh} \left(C_u \frac{e^{(r-\delta)h} - d}{u - d} + C_d \frac{u - e^{(r-\delta)h}}{u - d} \right). \tag{15.3}$$

Remark 15.1
Note that ΔS is not a change in S. It is monetary value of the shares of stock in the portfolio. Note that

$$\Delta = e^{-\delta h} \frac{C_u - C_d}{S_u - S_d}$$

where $S_u = uS$ and $S_d = dS$. As h approaches zero, $S_u - S_d$ approaches zero so that we can write $\Delta = \frac{\partial C}{\partial S}$. Hence, Δ measures the sensitivity of the option to a change in the stock price. Thus, if the stock price increases by \$1, then the option price, $\Delta S - B$, changes by Δ.

Example 15.1
Consider a European call option on the stock of XYZ with strike \$95 and six months to expiration. XYZ stock does not pay dividends and is currently worth \$100. The annual continuously compounded risk-free interest rate is 8%. In six months the price is expected to be either \$130 or \$80. Using the single-period binomial option pricing model, find Δ, B and C.

Solution.
We are given: $S = 100,\ K = 95,\ r = 0.08,\ \delta = 0,\ h = 0.5,\ u = \frac{130}{100} = 1.3$, and $d = \frac{80}{100} = 0.8$. Thus, $C_u = 35$ and $C_d = 0$. Hence,

$$\Delta = e^{-\delta h}\frac{C_u - C_d}{S(u-d)} = 0.7$$

$$B = e^{-rh}\frac{uC_d - dC_u}{d-u} = \$53.8042$$

and the price of the call option

$$C = e^{-rh}\left(C_u\frac{e^{(r-\delta)h} - d}{u-d} + C_d\frac{u - e^{(r-\delta)h}}{u-d}\right) = \$16.1958$$

which agrees with the answer obtained by using the risk-neutral approach ■

The no-arbitrage principle was a key factor in establishing formulas (15.1) - (15.3). The following result exhibits a condition for which the principle can be valid.

Proposition 15.1 *(no-arbitrage profits condition)*
The no-arbitrage condition is

$$d < e^{(r-\delta)h} < u. \tag{15.4}$$

Proof.
Suppose first that $e^{(r-\delta)h} \geq u$ (i.e. $Se^{(r-\delta)h} \geq Su$). Consider the following position: Short-sell the stock and collect S. Now, invest the proceeds for a length of time h while paying dividends to the original owner

of the stock, should they occur. At the end of the period you will get $Se^{(r-\delta)h}$. At the end of the period, if the stock ends up at uS, buy back the stock and return it to its original owner. Now, pocket the difference $Se^{(r-\delta)h} - Su \geq 0$. If the stock goes down to dS instead, the payoff is even larger: $Se^{(r-\delta)h} - Sd > Se^{(r-\delta)h} - Su \geq 0$. Thus, the condition $e^{(r-\delta)h} \geq u$ demonstrates an arbitrage.
Next, suppose that $d \geq e^{(r-\delta)h}$ (i.e., $Sde^{\delta h} \geq Se^{rh}$.) Consider the following position: Borrow \$S at the risk-free rate and use it to buy one share of stock. Now, hold the stock for a period of h. If the stock goes down, collect $Sde^{\delta h}$ (sale of stock + dividends received.) Next, repay the loan plus interest, i.e., Se^{rh}. In this case, you pocket $Sde^{\delta h} - Se^{rh} \geq 0$. If the stock goes up to uS instead the payoff is even larger $Sue^{\delta h} - Se^{rh} > Sde^{\delta h} - Se^{rh} \geq 0$. Again, the condition $d \geq e^{(r-\delta)h}$ demonstrates an arbitrage opportunity ■

Arbitrage opportunities arise if the options are mispriced, that is, if the actual option price is different from the theoretical option price:
• If an option is overpriced, that is, the actual price is greater than the theoretical price, then we can sell the option. However, the risk is that the option will be in the money at expiration, and we will be required to deliver the stock. To hedge this risk, we can buy a synthetic option at the same time we sell the actual option.
• If the option is underpriced, that is, the actual price is smaller than the theoretical price, then we buy the option. To hedge the risk associated with the possibility of the risk price falling at expiration, we sell a synthetic option at the same time we buy the actual option.
We illustrate these results in the next example.

Example 15.2
Consider the option of Example 15.1.
(a) Suppose you observe a call price of \$17 (i.e. option is overpriced). What is the arbitrage?
(b) Suppose you observe a call price of \$15.50 (i.e. option is underpriced). What is the arbitrage?

Solution.
(a) The observed price is larger than the theoretical price. We sell the actual call option for \$17 and synthetically create a call option by buying 0.7 of one share and borrowing \$53.8042. This synthetic option hedges the written call as shown in the payoff table.

	Stock Price in Six Months	
Transaction	\$80	\$130
Written Call	0	−\$35
0.7 Purchased shares	\$56	\$91
Repay Loan	−\$56	−\$56
Total	\$0	\$0

Now, the initial cash flow is

$$17.00 - 0.7 \times 100 + 53.8042 = \$0.8042.$$

Thus, we earn \$0.8042 which is the amount the option is mispriced.
(b) The observed price is smaller than the theoretical price. We buy the option and synthetically create a short position in an option. In order to do so, we sell 0.7 units of the share and lend \$53.8042. The initial cash flow is

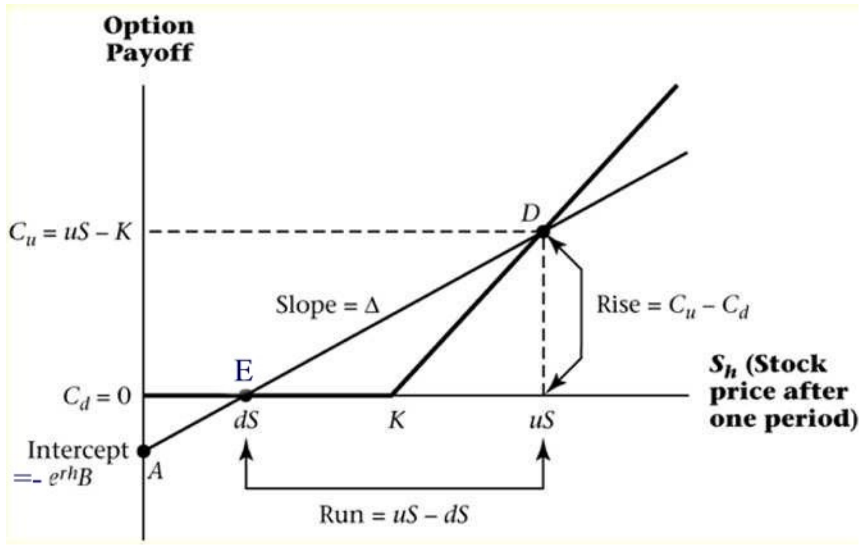
$$-15.50 + 0.7 \times 100 - 53.8042 = \$0.6958$$

Thus, we earn \$0.6958 which is the amount the option is mispriced ■

Graphical Interpretation of the Binomial Formula
Consider graphing the price function after one period for a nondividend-paying stock. Let C_h be the option price after one period with corresponding stock price S_h. Then

$$C_h = \Delta S_h - e^{rh}B.$$

The graph is the straight line going through the points A, E, D as shown below with slope Δ and vertical intercept $-e^{rh}B$.



Thus, any line replicating a call must have a positive slope and a negative vertical intercept.

Example 15.3
Consider a nondividend paying stock and a 1-year call on the stock. The stock is currently trading for \$60. Suppose that $C_h = -32.38$ when $S_h = 0$ and $C_h = 26.62$ when $S_h = 88.62$. Find the current price of the call option if the continuously compounded risk-free rate is 9%.

Solution.
The current price of the call is given by

$$C = \Delta S - B = 60\Delta - B$$

where

$$\Delta = \frac{26.62 - (-32.38)}{88.62 - 0} = 0.6658$$

and

$$-32.38 = 0.6658(0) - e^{0.09}B.$$

Solving for B we find $B = 29.5931$. Thus, the final answer is

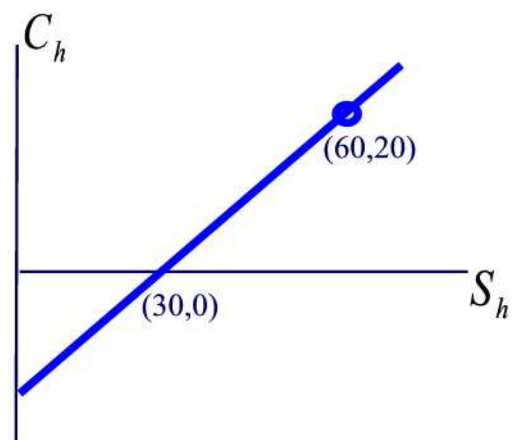
$$C = 0.6658(60) - 29.5931 = \$10.35 \blacksquare$$

Practice Problems

Problem 15.1
XYZ currently has a stock price of \$555 per share. A replicating portfolio for a particular call option on XYZ stock involves borrowing \$B and buying $\frac{3}{4}$ of one share. The price of the call option is \$360.25. Calculate B using the one-period binomial option pricing model.

Problem 15.2
Show that $\Delta \leq 1$.

Problem 15.3
The graph of the value of a replicating portfolio of a nondividend-paying stock is given below.



Determine Δ and B given that $h = 1$ and the continuously compounded risk-free interest rate is 8%.

Problem 15.4
XYZ currently has a stock price of \$41 per share. A replicating portfolio for a particular call option on XYZ stock involves borrowing \$18.462 and buying $\frac{2}{3}$ of one share. Calculate the price of the call option using the one-period binomial option pricing model.

Problem 15.5
XYZ currently has a stock price of \$555 per share. A replicating portfolio for a particular call option on XYZ stock involves borrowing \$56 and buying Δ of one share. The price of the call option is \$360.25. Calculate Δ using the one-period binomial option pricing model.

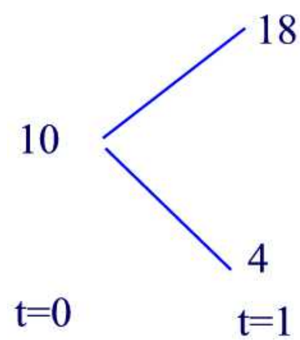
Problem 15.6
Consider a European call option on the stock of XYZ with strike \$110 and 1 year to expiration. XYZ stock does not pay dividends and is currently worth \$100. The annual continuously compounded risk-free interest rate is 5%. In one year the price is expected to be either \$120 or \$90. Using the single-period binomial option pricing model, find Δ and B.

Problem 15.7
A call option on a stock currently trades for \$45. The stock itself is worth \$900 per share. Using the one-period binomial option pricing model, a replicating portfolio for the call option is equal to buying $\frac{1}{5}$ of one share of stock and borrowing \$B. Calculate B.

Problem 15.8
A share of stock XYZ pays continuous dividends at the annual yield rate of δ. The stock currently trades for \$65. A European call option on the stock has a strike price of \$64 and expiration time of one year. Suppose that in one year, the stock will be worth either \$45 or \$85. Assume that the portfolio replicating the call consists of $\frac{9}{20}$ of one share. Using the one-period binomial option pricing model, what is the annual continuously-compounded dividend yield?

Problem 15.9
Consider a European call option on the stock of XYZ, with a strike price of \$25 and two months to expiration. The stock pays continuous dividends at the annual yield rate of 5%. The annual continuously compounded risk free interst rate is 11%. The stock currently trades for \$23 per share. Suppose that in two months, the stock will trade for either \$18 per share or \$29 per share. Use the one-period binomial option pricing to find the today's price of the call.

Problem 15.10 ‡
A nondividend-paying stock S is modeled by the tree shown below.



A European call option on S expires at $t = 1$ with strike price $K = 12$. Calculate the number of shares of stock in the replicating portfolio for this option.

Problem 15.11 ‡
You are given the following information:
• A particular non-dividend paying stock is currently worth 100
• In one year, the stock will be worth either 120 or 90
• The annual continuously-compounded risk-free rate is 5%
Calculate the delta for a call option that expires in one year and with strike price of 105.

Problem 15.12
Which of the following binomial models with the given parameters represent an arbitrage?
(A) $u = 1.176, d = 0.872, h = 1, r = 6.3\%, \delta = 5\%$
(B) $u = 1.230, d = 0.805, h = 1, r = 8\%, \delta = 8.5\%$
(C) $u = 1.008, d = 0.996, h = 1, r = 7\%, \delta = 6.8\%$
(D) $u = 1.278, d = 0.783, h = 1, r = 5\%, \delta = 5\%$
(E) $u = 1.100, d = 0.982, h = 1, r = 4\%, \delta = 6\%$.

16 Binomial Trees and Volatility

The goal of a binomial tree is to characterize future uncertainty about the stock price movement. In the absence of uncertainty (i.e. stock's return is certain at the end of the period), a stock must appreciate at the risk-free rate less the dividend yield. Thus, from time t to time $t+h$ we must have

$$S_{t+h} = F_{t,t+h} = S_t e^{(r-\delta)h}.$$

In other words, under certainty, the price next period is just the forward price.

What happens in the presence of uncertainty (i.e. stock's return at the end of the period is uncertain)? First, a measure of uncertainty about a stock's return is its **volatility** which is defined as the annualized standard deviation of the return of the stock when the return is expressed using continuous compounding. Thus, few facts about continuously compounded returns are in place.

Let S_t and S_{t+h} be the stock prices at times t and $t+h$. The **continuously compounded rate of return** in the interval $[t, t+h]$ is defined by

$$r_{t,t+h} = \ln \frac{S_{t+h}}{S_t}.$$

Example 16.1

Suppose that the stock price on three consecutive days are \$100, \$103, \$97. Find the daily continuously compounded returns on the stock.

Solution.

The daily continuously compounded returns on the stock are

$$\ln \tfrac{103}{100} = 0.02956 \text{ and } \ln \tfrac{97}{103} = -0.06002 \blacksquare$$

Now, if we are given S_t and $r_{t,t+h}$ we can find S_{t+h} using the formula

$$S_{t+h} = S_t e^{r_{t,t+h}}.$$

Example 16.2

Suppose that the stock price today is \$100 and that over 1 year the continuously compounded return is -500%. Find the stock price at the end of the year.

Solution.
The answer is

$$S_1 = 100e^{-5} = \$0.6738 \blacksquare$$

Now, suppose $r_{t+(i-1)h,t+ih}, 1 \leq i \leq n$, is the continuously compounded rate of return over the time interval $[t+(i-1)h, t+ih]$. Then the continuously compounded return over the interval $[t, t+nh]$ is

$$r_{t,t+nh} = \sum_{i=1}^{n} r_{t+(i-1)h,t+ih}. \tag{16.1}$$

Example 16.3
Suppose that the stock price on three consecutive days are \$100, \$103, \$97. Find the continuously compounded returns from day 1 to day 3.

Solution.
The answer is

$$0.02956 - 0.06002 == -0.03046 \blacksquare$$

The rate of returns on the sample space of stock prices are random variables. Suppose a year is splitted into n periods each of length h. Let $r_{(i-1)h,ih}$ be the rate of return on the time interval $[(i-1)h, ih]$. Define the random variable r_{Annual} to be the annual continuously compounded rate of return. Thus, we can write

$$r_{\text{Annual}} = \sum_{i=1}^{n} r_{(i-1)h,ih}.$$

The variance of the annual return is therefore

$$\begin{aligned}\sigma^2 =&\text{Var}(r_{\text{Annual}}) = \text{Var}\left(\sum_{i=1}^{n} r_{(i-1)h,ih}\right)\\ =&\sum_{i=1}^{n}\text{Var}(r_{(i-1)h,ih}) = \sum_{i=1}^{n}\sigma_{h,i}^2\end{aligned}$$

where we assume that the return in one period does not affect the expected returns in subsequent periods. That is, periodic rates are independent. If we assume that each period has the same variance of return σ_h then we can write

$$\sigma^2 = n\sigma_h^2 = \frac{\sigma_h^2}{h}.$$

Thus, the standard deviation of the period of length h is

$$\sigma_h = \sigma\sqrt{h}.$$

Now, one way to incorporate uncertainty in the future stock price is by using the model

$$\begin{aligned} uS_t =& F_{t,t+h}e^{\sigma\sqrt{h}} \\ dS_t =& F_{t,t+h}e^{-\sigma\sqrt{h}} \end{aligned}$$

Note that in the absence of uncertainty, $\sigma = 0$ and therefore $uS_t = dS_t = F_{t,t+h}$. Now using the fact that $F_{t,t+h} = S_t e^{(r-\delta)h}$ we obtain

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} \tag{16.2}$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} \tag{16.3}$$

We will refer to a tree constructed using equations (16.2)-(16.3) as a **forward tree.**

Remark 16.1
From the relation

$$S_{t+h} = S_t e^{(r-\delta)h\pm\sigma\sqrt{h}}$$

we find that the continuously compound return to be equal to

$$(r-\delta)h \pm \sigma\sqrt{h}.$$

That is, the continuously compound return consists of two parts, one of which is certain $(r-\delta)h$, and the other of which is uncertain and generates the up and down stock moves $\pm\sigma\sqrt{h}$.

Example 16.4
The current price of a stock is \$41. The annual continuously compounded interest rate is 0.08, and the stock pays no dividends. The annualized standard deviation of the continuously compounded stock return is 0.3. Find the price of a European call option on the stock with strike price \$40 and that matures in one year.

Solution.
Using equations (16.1)-(16.2) we find

$$\begin{aligned} uS =& 41e^{(0.08-0)\times 1+0.3\times\sqrt{1}} = \$59.954 \\ dS =& 41e^{(0.08-0)\times 1-0.3\times\sqrt{1}} = \$32.903 \end{aligned}$$

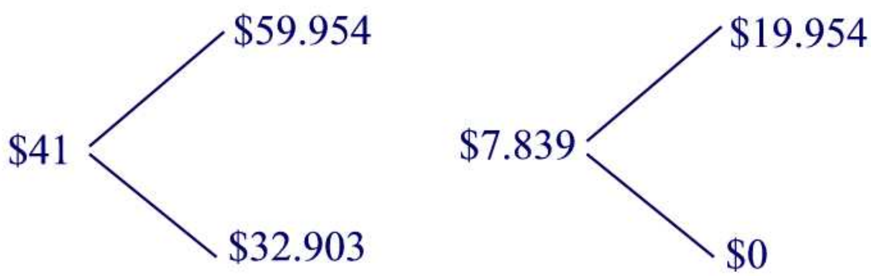
It follows that

$$\begin{aligned} u =& \frac{59.954}{41} = 1.4623 \\ d =& \frac{32.903}{41} = 0.8025 \\ C_u =& 59.954 - 40 = \$19.954 \\ C_d =& 0 \\ \Delta =& \frac{19.954 - 0}{41 \times (1.4623 - 0.8025)} = 0.7376 \\ B =& e^{-0.08}\frac{1.4623 \times 0 - 0.8025 \times 19.954}{0.8025 - 1.4623} = \$22.405. \end{aligned}$$

Hence, the option price is given by

$$C = \Delta S - B = 0.7376 \times 41 - 22.405 = \$7.839.$$

Forward trees for the stock price and the call price are shown below.



Remark 16.2
Volatility measures how sure we are that the stock return will be close to the expected return. Stocks with a larger volatility will have a greater chance of return far from the expected return.

Remark 16.3
A word of caution of the use of volatility when the underlying asset pays dividends. For a paying dividend stock, volatility is for the prepaid forward price

$S - PV_{0,T}(Div)$ and not for the stock price. Thus, for nondividend-paying stock the stock price volatility is just the prepaid forward price volatility. Mcdonald gives the following relationship between stock price volatility and prepaid forward price volatility

$$\sigma_F = \sigma_{\text{stock}} \times \frac{S}{F^P}.$$

Note that when the stock pays no dividends then $F^P = S$ and so $\sigma_F = \sigma_{\text{stock}}$

Practice Problems

Problem 16.1
Given the daily continuously compound returns on three consecutive days: 0.02956, −0.06002, and $r_{2,3}$. The three-day continuously compounded return is -0.0202. Determine $r_{2,3}$.

Problem 16.2
Suppose that the stock price today is \$S and at the end of the year it is expected to be \$0.678. The annual continuously compounded rate of return is -500%. Find today's stock price.

Problem 16.3
Establish equality (16.1).

Problem 16.4
Given that the volatility of a prepaid forward price on the stock is 90%, the annual continuously compounded interest rate is 7%. The stock pays dividends at an annual continuously compounded yield of 5%. Find the factors by which the price of the stock might increase or decrease in 4 years?

Problem 16.5
(a) Find an expression of p_u in a forward tree.
(b) Show that p_u decreases as h increases. Moreover, p_u approaches 0.5 as $h \to 0$.

Problem 16.6
The forward price on a 9-year forward contract on GS stock is \$567. The annualized standard deviation of the continuously compounded stock return is currently 0.02. Find the price of the stock after 9 years if we know that it is going to decrease.

Problem 16.7
GS stock may increase or decline in 1 year under the assumptions of the one-period binomial option pricing model. The stock pays no dividends and the annualized standard deviation of the continuously compounded stock return is 81%. A 1-year forward contract on the stock currently sells for \$100. GS stock currently sells for \$90. What is the annual continuously compounded risk-free interest rate?

Problem 16.8
A stock currently sell for \$41. Under the assumptions of the one-period binomial option pricing model the stock is expected to go up to \$59.954 in one year. The annual continuously compounded return is 8% and the stock pays no dividends. Determine the annualized standard deviation of the continuously compounded stock return.

Problem 16.9 ‡
Consider the following information about a European call option on stock ABC:
• The strike price is \$100
• The current stock price is \$110
• The time to expiration is one year
• The annual continuously-compounded risk-free rate is 5%
• The continuous dividend yield is 3.5%
• Volatility is 30%
• The length of period is 4 months.
Find the risk-neutral probability that the stock price will increase over one time period.

Problem 16.10 ‡
A three month European call on a stock is modeled by a single period binomial tree using the following parameters
• The annual continuously-compounded risk-free rate is 4%
• Stock pays no dividends
• Annual volatility is 15%
• Current stock price is 10
• Strike price is 10.5
Calculate the value of the option.

17 Multi-Period Binomial Option Pricing Model

The single-period binomial model extends easily to a multiperiod model. In this section, we examine the special cases of the two- and three-period models.
The binomial trees of the stock prices as well as the call prices of the two-period model are shown in Figure 17.1.

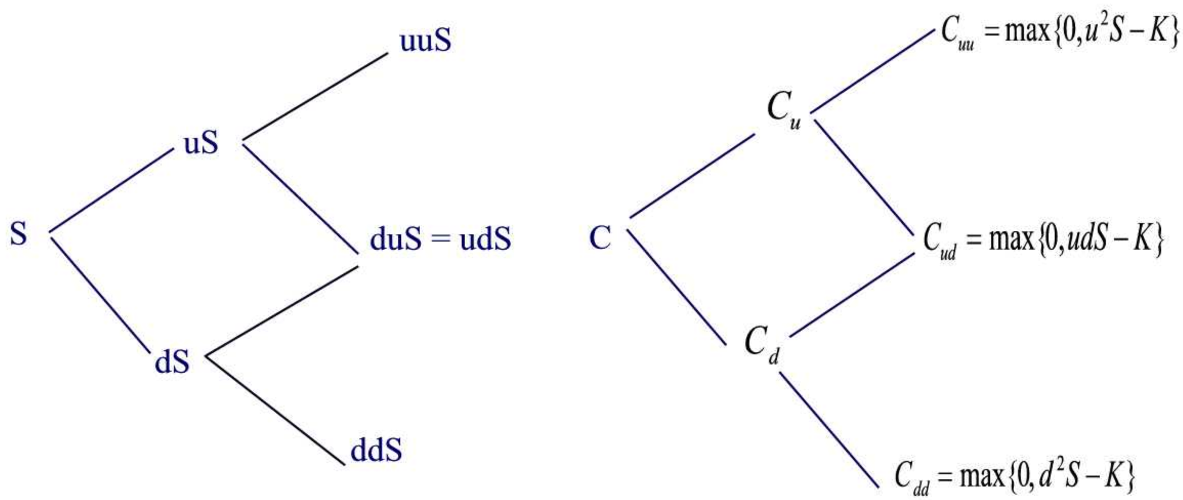


Figure 17.1

Note that an up move in the stock price for one period followed by a down move in the stock price in the next period generates the same stock price to a down move in the first period followed by an up move in the next. A binomial tree with this property is called a **recombining tree**.
Note that we work backward when it comes to pricing the option since formula (15.3) requires knowing the option prices resulting from up and down moves in the subsequent periods. At the outset, the only period where we know the option price is at expiration.
Knowing the price at expiration, we can determine the price in period 1. Having determined that price, we can work back to period 0. We illustrate this process in the following example.

Example 17.1
A stock is currently worth \$56. Every year, it can increase by 30% or decrease by 10%. The stock pays no dividends, and the annual continuously-

compounded risk-free interest rate is 4%. Find the price today of one two-year European call option on the stock with a strike price of \$70.

Solution.
We are given $u = 1.3, d = 0.9, r = 0.04, h = 1$, and $K = 70$. In one year the stock is worth either $uS = 1.3 \times 56 = \$72.8$ or $dS = 0.9 \times 56 = \$50.4$. In two years, the stock is worth either $u^2S = 72.8 \times 1.3 = \$94.64$ or $udS = 1.3 \times 50.4 = \$65.52$ or $d^2S = 0.9 \times 50.4 = 45.36$.
Year 2, Stock Price = \$94.64 Since we are at expiration, the option value is $C_{uu} = 94.64 - 70 = \$24.64$.
Year 2, Stock Price = \$65.52 Again we are at expiration and the option is out of the money so that $C_{ud} = 0$.
Year 2, Stock Price = \$45.36 So at expiration we have $C_{dd} = 0$.
Year 1, Stock Price = \$72.8 At this node we use (15.3) to compute the option value:

$$C_u = e^{-0.04}\left(24.64 \times \frac{e^{0.04} - 0.9}{1.3 - 0.9} + 0 \times \frac{1.3 - e^{0.04}}{1.3 - 0.9}\right) = \$8.3338.$$

Year 1, Stock Price = \$50.4 At this node we use (15.3) to compute the option value:

$$C_d = e^{-0.04}\left(0 \times \frac{e^{0.04} - 0.9}{1.3 - 0.9} + 0 \times \frac{1.3 - e^{0.04}}{1.3 - 0.9}\right) = \$0.$$

Year 0, Stock Price = \$56 At this node we use (15.3) to compute the option value:

$$C = e^{-0.04}\left(8.3338 \times \frac{e^{0.04} - 0.9}{1.3 - 0.9} + 0 \times \frac{1.3 - e^{0.04}}{1.3 - 0.9}\right) = \$2.8187 \blacksquare$$

The content of this section can be extended to any number of periods.

Example 17.2
Find the current price of a 60-strike 1.5-year (18-month) European call option on one share of an underlying dividend-paying stock. Let $S = 60, r = 0.03, \sigma = 0.25, \delta = 0.03$, and $h = 0.50$ (the binomial interval is 6 months – thus, you need a three-step tree).

Solution.
We first find u and d. We have

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.03-0.03)\times 0.5+0.25\sqrt{0.5}} = 1.1934$$

and

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.03-0.03)\times 0.5-0.25\sqrt{0.5}} = 0.8380$$

and

$$p_u = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{1-0.8380}{1.1934-0.8380} = 0.4558.$$

We have
Period 3, Stock Price $= u^3S = 101.9787$. Since we are at expiration, the option value is $C_{uuu} = 101.9787 - 60 = \$41.9787$.
Period 3, Stock Price $= u^2dS = 71.6090$ and $C_{uud} = 11.6090$.
Period 3, Stock Price $= ud^2S = 50.2835$ and $C_{udd} = C_{ddu} = 0$.
Period 3, Stock Price $= d^3S = 35.3088$ and $C_{ddd} = 0$.
Period 2, Stock Price $= u^2S = 85.4522$ and $C_{uu} = e^{-0.03\times 0.5}[0.4558 \times 41.9787 + (1-0.4558) \times 11.6090] = 25.0726$.
Period 2, Stock Price $= udS = 60.0042$ and $C_{ud} = e^{-0.03\times 0.5}[0.4558 \times 11.6090 + (1-0.4588) \times 0] = 5.2126$.
Period 2, Stock Price $= d^2S = 42.1346$ and $C_{dd} = e^{-0.03\times 0.5}[0.4558 \times 0 + (1-0.4558) \times 0] = 0$.
Period 1, Stock Price $= uS = 71.604$ and $C_u = e^{-0.03\times 0.5}[0.4558 \times 25.0726 + (1-0.4558) \times 5.2126] = 14.0524$.
Period 1, Stock Price $= dS = 50.28$ and $C_d = e^{-0.03\times 0.5}[0.4558 \times 5.2126 + (1-0.4558) \times 0] = 2.3405$.
Period 0, Stock Price = \$60 and the current option value is:

$$C = e^{-0.03\times 0.5}[0.4558 \times 14.0524 + (1-0.4558) \times 2.3405] = 7.5645 \blacksquare$$

Example 17.3
Consider a two-period binomial model. Show that the current price of a call option is given by the formula

$$C = e^{-2rh}[p_u^2C_{uu} + 2p_u(1-p_u)C_{ud} + (1-p_u)^2C_{dd}].$$

Solution.
We have

$$
\begin{aligned}
C =& e^{-rh}[p_uC_u + (1-p_u)C_d] \\
=& e^{-rh}\left(p_ue^{-rh}[p_uC_{uu} + (1-p_u)C_{ud}] + (1-p_u)e^{-rh}[p_uC_{du} + (1-p_u)C_{dd}]\right) \\
=& e^{-2rh}(p_u^2C_{uu} + p_u(1-p_u)C_{ud} + p_u(1-p_u)C_{du} + (1-p_u)^2C_{dd}) \\
=& e^{-2rh}[p_u^2C_{uu} + 2p_u(1-p_u)C_{ud} + (1-p_u)^2C_{dd}] \blacksquare
\end{aligned}
$$

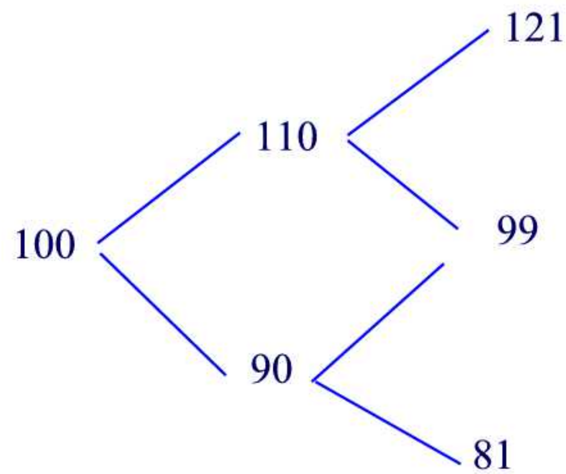
Practice Problems

Problem 17.1
A stock is currently worth \$41. Every year, it can increase by 46.2% or decrease by 19.7%. The stock pays no dividends, and the annual continuously-compounded risk-free interest rate is 8%. Find the price today of one two-year European call option on the stock with a strike price of \$40.

Problem 17.2
The annualized standard deviation of the continuously compounded stock return on GS Inc is 23%. The annual continuously compounded rate of interest is 12%, and the annual continuously compounded dividend yield on GS Inc. is 7%. The current price of GS stock is \$35 per share. Using a two-period binomial model, find the price of GS Inc., stock if it moves up twice over the course of 7 years.

Problem 17.3 ‡
Consider the following information about a European call option on stock ABC:
- The strike price is \$95
- The current stock price is \$100
- The time to expiration is two years
- The annual continuously-compounded risk-free rate is 5%
- The stock pays non dividends
- The price is calculated using two-step binomial model where each step is one year in length.

The stock price tree is



Calculate the price of a European call on the stock of ABC.

Problem 17.4
Find the current price of a 120-strike six-month European call option on one share of an underlying nondividend-paying stock. Let $S = 120, r = 0.08, \sigma = 0.30$, and $h = 0.25$ (the binomial interval is 3 months).

Problem 17.5
GS Inc., pays dividends on its stock at an annual continuously compounded yield of 6%. The annual continuously compounded risk-free rate is 9%. GS stock is currently worth \$100. Every two years, it can change by a factor of 0.7 or 1.5. Using a two-period binomial option pricing model, find the price today of one four-year European call option on GS, Inc., stock with a strike price of \$80.

Problem 17.6
Find the current price of a 95-strike 3-year European call option on one share of an underlying stock that pays continuous dividends. Let $S = 100, r = 0.05, \sigma = 0.3, \delta = 0.03$ and $h = 1$.

Problem 17.7
Given the following information about a European call option: $S = \$40, r = 3\%, \delta = 5\%, u = 1.20, d = 0.90, K = \$33$, and $T = 3$ months. Using a three-period binomial tree, find the price of the call option.

Problem 17.8
A European call option on a stock has a strike price \$247 and expires in eight months. The annual continuously compounded risk-free rate is 7% and the compounded continuously dividend yield is 2%. The current price of the stock is \$130. The price volatility is 35%. Use a 8-period binomial model to find the price of the call.

Problem 17.9
A European put option on a stock has a strike price \$247 and expires in eight months. The annual continuously compounded risk-free rate is 7% and the compounded continuously dividend yield is 2%. The current price of the stock is \$130. The price volatility is 35%. Use a 8-period binomial model to find the price of the put. Hint: Put-call parity and the previous problem.

18 Binomial Option Pricing for European Puts

Binomial option pricing with puts can be done using the exact same formulas and conceptual tools developed for European call options except of one difference that occurs at expiration: Instead of computing the price as $\max\{0, S-K\}$, we use $\max\{0, K-S\}$. The objective of this section is to establish the conceptual approach to binomial option pricing with puts.

The **replicating portfolio approach** consists of creating a portfolio that replicates the actual short put option. Let Portfolio A consists of selling a put option on the stock and Portfolio B consists of selling Δ shares of the stock and lending $\$B < 0$. The payoff tables of these positions are shown below.

Transaction	Time 0	Up State	Down State
Sell a Put	P	$-P_u$	$-P_d$
Total	P	$-P_u$	$-P_d$

Transaction	Time 0	Up State	Down State
Sell Δ Shares	ΔS	$-\Delta e^{\delta h} uS$	$-\Delta e^{\delta h} dS$
Lend \$B	$-B$	Be^{rh}	Be^{rh}
Total	$\Delta S - B$	$-\Delta e^{\delta h} uS + Be^{rh}$	$-\Delta e^{\delta h} dS + Be^{rh}$

If Portfolio B is to replicate portfolio A we must have

$$P = \Delta S - B$$

$$P_u = \Delta e^{\delta h} uS - Be^{rh}$$

$$P_d = \Delta e^{\delta h} dS - Be^{rh}.$$

Solving the last two equations we find

$$\Delta = e^{-\delta h} \frac{P_u - P_d}{S(u-d)} \tag{18.1}$$

and

$$B = e^{-rh} \frac{uP_d - dP_u}{d-u}. \tag{18.2}$$

Thus,

$$P = \Delta S - B = e^{-rh} \left(P_u \frac{e^{(r-\delta)h} - d}{u-d} + P_d \frac{u - e^{(r-\delta)h}}{u-d} \right). \tag{18.3}$$

Using risk-neutral probability we can write

$$P = e^{-rh}[p_u P_u + (1 - p_u)P_d].$$

Example 18.1

Consider a European put option on the stock of XYZ with strike \$50 and one year to expiration. XYZ stock does not pay dividends and is currently worth \$51. The annual continuously compounded risk-free interest rate is 7%. In one year the price is expected to be either \$75 or \$40.
(a) Using the single-period binomial option pricing model, find Δ, B and P.
(b) Find P using the risk-neutral approach.

Solution.

(a) We have: $u = \frac{75}{51}$ and $d = \frac{40}{51}$. Also, $P_d = 10$ and $P_u = 0$. Hence,

$$\Delta = e^{-\delta h}\frac{P_u - P_d}{S(u-d)} = \frac{0-10}{51\left(\frac{75}{51} - \frac{40}{51}\right)} = -0.2857.$$

Also,

$$B = e^{-rh}\frac{uP_d - dP_u}{d-u} = e^{-0.07\times 1}\frac{\frac{75}{51}\times 10 - 0}{\frac{40}{51} - \frac{75}{51}} = -19.9799.$$

The current price of the put is

$$P = \Delta S - B = -0.2857 \times 51 + 19.9799 = \$5.41.$$

(b) We have

$$p_u = \frac{e^{(r-\delta)h} - d}{u-d} = 0.41994$$

and

$$P = e^{-rh}[p_u P_u + (1-p_u)P_d] = \$5.41 \blacksquare$$

Example 18.2

Consider a European put option on the stock of XYZ with strike \$95 and six months to expiration. XYZ stock does not pay dividends and is currently worth \$100. The annual continuously compounded risk-free interest rate is 8%. In six months the price is expected to be either \$130 or \$80.
(a) Using the single-period binomial option pricing model, find Δ, B and P.
(b) Suppose you observe a put price of \$8 (i.e. option is overpriced). What is the arbitrage?
(c) Suppose you observe a put price of \$6 (i.e. option is underpriced). What is the arbitrage?

Solution.
(a) We are given: $S = 100,\ K = 95,\ r = 0.08,\ \delta = 0,\ h = 0.5,\ u = \frac{130}{100} = 1.3,$ and $d = \frac{80}{100} = 0.8$. Thus, $P_u = 0$ and $P_d = 95 - 80 = 15$. Hence,

$$\Delta = e^{-\delta h}\frac{C_u - C_d}{S(u-d)} = -0.3$$

$$B = e^{-rh}\frac{uP_d - dP_u}{d-u} = -\$37.470788$$

and the price of the put option

$$P = \Delta S - B = \$7.4707.$$

(b) The observed price is larger than the theoretical price. We sell the actual put option for \$8 and synthetically create a long put option by selling 0.3 units of one share and lending \$37.471. This synthetic option hedges the written put as shown in the payoff table.

	Stock Price in Six Months	
Transaction	\$80	\$130
Written Put	−\$15	0
0.3 Written shares	−\$24	−\$39
Lending	\$39	\$39
Total	\$0	\$0

Now, the initial cash flow is

$$8.00 + 0.3 \times 100 - 37.471 = \$0.529.$$

Thus, we earn \$0.529 which is the amount the option is mispriced.
(c) The observed price is smaller than the theoretical price. We buy the option and synthetically create a short put option. In order to do so, we buy 0.3 units of the share and borrow \$37.471. The initial cash flow is

$$-6.00 - 0.3 \times 100 + 37.471 = \$1.471$$

Thus, we earn \$1.471 which is the amount the option is mispriced ■

Example 18.3
The current price of a stock is \$65. The annual continuously compounded interest rate is 0.08, and the stock pays dividends at the continuously compounded rate of 0.05. The annualized standard deviation of the continuously compounded stock return is 0.27. Find the price of a European put option on the stock with strike price \$63 and that matures in one year.

Solution.
Using equations (16.1)-(16.2) we find

$$uS = 65e^{(0.08-0.05)\times 1+0.27\times\sqrt{1}} = \$87.7408$$
$$dS = 65e^{(0.08-0.05)\times 1-0.27\times\sqrt{1}} = \$51.1308$$

It follows that

$$u = \frac{87.7408}{65} = 1.3499$$
$$d = \frac{51.1308}{65} = 0.7866$$
$$P_u = \$0$$
$$P_d = 63 - 51.1308 = 11.8692$$
$$\Delta = e^{-\delta h}\frac{P_u - P_d}{S(u-d)} = -0.3084$$
$$B = e^{-rh}\frac{uP_d - dP_u}{d-u} = e^{-0.08}\frac{1.3499 \times 11.8692 - 0.7866 \times 0}{0.7866 - 1.3499} = -\$26.2567.$$

Hence, the option price is given by

$$P = \Delta S - B = -0.3084 \times 65 + 26.2567 = \$6.2107 \blacksquare$$

Practice Problems

Problem 18.1
Consider a European put option on the stock of XYZ with strike \$130 and one year to expiration. XYZ stock does not pay dividends and is currently worth \$100. The annual continuously compounded risk-free interest rate is 5%. In one year the price is expected to be either \$120 or \$90. Using, the one-period binomial option pricing model, find the price today of one such put option.

Problem 18.2
Consider a European put option on the stock of XYZ, with a strike price of \$30 and two months to expiration. The stock pays continuous dividends at the annual continuously compounded yield rate of 5%. The annual continuously compounded risk free interst rate is 11%. The stock currently trades for \$23 per share. Suppose that in two months, the stock will trade for either \$18 per share or \$29 per share. Use the one-period binomial option pricing to find the today's price of the put.

Problem 18.3
One year from today, GS stock will sell for either \$130 or \$124. The annual continuously compounded interest rate is 11%. The risk-neutral probability of an increase in the stock price (to \$130) is 0.77. Using the one-period binomial option pricing model, find the current price of a one-year European put option on the stock with a strike price of \$160.

Problem 18.4
Consider a two-period binomial model. Show that the current price of a put option is given by the formula

$$P = e^{-2rh}[p_u^2 P_{uu} + 2p_u(1 - p_u)P_{ud} + (1 - p_u)^2 P_{dd}].$$

Problem 18.5
GS stock is currently worth \$56. Every year, it can increase by 30% or decrease by 10%. The stock pays no dividends, and the annual continuously-compounded risk-free interest rate is 4%. Using a two-period binomial option pricing model, find the price today of one two-year European put option on the stock with a strike price of \$120

Problem 18.6
GS stock pays dividends at an annual continuously compounded yield of 6%. The annual continuosuly compounded interest rate is 9%. The stock is currently worth \$100. Every two years, it can increase by 50% or decrease by 30%. Using a two-period binomial option pricing model, find the price today of one four-year European put option on the stock with a strike price of \$130.

Problem 18.7
Given the following information about a stock: $S = \$100, \sigma = 0.3, r = 0.05, \delta = 0.03, K = 95$. Using a three-period binomial tree, find the current price of a European put option with strike \$95 and expiring in three years.

Problem 18.8 ‡
For a two-year European put option, you are given the following information:
- The stock price is \$35
- The strike price is \$32
- The continuously compounded risk-free interest rate is 5%
- The stock price volatility is 35%
- The stock pays no dividends.

Find the price of the put option using a two-period binomial pricing.

Problem 18.9
A European put option on a nondividend-paying stock has a strike price \$88 and expires in seven months. The annual continuously compounded risk-free rate is 8%. The current price of the stock is \$130. The price volatility is 30%. Use a 7-period binomial model to find the price of the put. Hint: Compare d^7S and ud^6S.

Problem 18.10
A European call option on a nondividend-paying stock has a strike price \$88 and expires in seven months. The annual continuously compounded risk-free rate is 8%. The current price of the stock is \$130. The price volatility is 30%. Use a 7-period binomial model to find the price of the call.

19 Binomial Option Pricing for American Options

Binomial trees are widely used within finance to price American type options. The binomial method constructs a tree lattice which represents the movements of the stock and prices the option relative to the stock price by working backward through the tree. At each node, we compare the value of the option if it is held to expiration to the gain that could be realized upon immediate exercise. The higher of these is the American option price.
Thus, for an American call the value of the option at a node is given by

$$C(S_t, K, t) = \max\{S_t - K, e^{-rh}[p_u C(uS_t, K, t+h) + (1-p_u)C(dS_t, K, t+h)]\}$$

and for an American put it is given by

$$P(S_t, K, t) = \max\{K - S_t, e^{-rh}[p_u P(uS_t, K, t+h) + (1-p_u)P(dS_t, K, t+h)]\}$$

where
- t is the time equivalent to some node in the tree
- S_t is the stock price at time t
- h the length of a period
- r is the continuously compounded risk-free interest rate
- K is the strike price
- p_u is the risk-neutral probability on an increase in the stock
- $P(S_t, K, t)$ is the price of an American put with strike price K and underlying stock price S_t
- $C(S_t, K, t)$ is the price of an American call with strike price K and underlying stock price S_t.

Example 19.1
Given the following information about a stock: $S = \$100, \sigma = 0.3, r = 0.05, \delta = 0.03, K = 95$. Using a three-period binomial tree, find the current price of an American call option with strike \$95 and expiring in three years.

Solution.
We first find u and d. We have

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.05-0.03)+0.3} = 1.3771$$

and

$$d = e^{(r-\delta)h - \sigma\sqrt{h}} = e^{(0.05-0.03)-0.3} = 0.7558.$$

Thus,

$$p_u = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{0.02} - 0.7558}{1.3771 - 0.7558} = 0.4256$$

Now, we have
Year 3, Stock Price $= u^3S = 261.1539$ Since we are at expiration, the option value is $C_{uuu} = 261.1539 - 95 = \$166.1539$.
Year 3, Stock Price $= u^2dS = 143.3302$ and $C_{uud} = 143.3302 - 95 = 48.3302$.
Year 3, Stock Price $= ud^2S = 78.6646$ and $C_{udd} = C_{ddu} = 0$.
Year 3, Stock Price $= d^3S = 43.1738$ and $C_{ddd} = 0$.
Year 2, Stock Price $= u^2S = 189.6404$

$$\begin{aligned}
C_{uu} =& \max\{189.6404 - 95, e^{-0.05}[0.4256 \times 166.1539 + (1 - 0.4256) \times 48.3302]\} \\
=& 189.6404 - 95 \\
=& 94.6404.
\end{aligned}$$

Year 2, Stock Price $= udS = 104.0812$

$$\begin{aligned}
C_{ud} =& \max\{104.0812 - 95, e^{-0.05}[0.4256 \times 48.3302 + (1 - 0.4256) \times 0]\} \\
=& e^{-0.05}[0.4256 \times 48.3302 + (1 - 0.4256) \times 0] \\
=& 19.5662.
\end{aligned}$$

Year 2, Stock Price $= d^2S = 57.1234$

$$\begin{aligned}
C_{dd} =& \max\{57.1234 - 95, e^{-0.05}[0.4256 \times 0 + (1 - 0.4256) \times 0]\} \\
=& e^{-0.05}[0.4256 \times 0 + (1 - 0.4256) \times 0] = 0
\end{aligned}$$

Year 1, Stock Price $= uS = 137.71$

$$\begin{aligned}
C_u =& \max\{137.71 - 95, e^{-0.05}[0.4256 \times 94.6404 + (1 - 0.4256) \times 19.5662]\} \\
=& e^{-0.05}[0.4256 \times 94.6404 + (1 - 0.4256) \times 19.5662] \\
=& 49.0052
\end{aligned}$$

Year 1, Stock Price $= dS = 75.58$

$$\begin{aligned}
C_d =& \max\{75.58 - 95, e^{-0.05}[0.4256 \times 19.5662 + (1 - 0.4256) \times 0]\} \\
=& e^{-0.05}[0.4256 \times 19.5662 + (1 - 0.4256) \times 0] \\
=& 7.9212
\end{aligned}$$

Year 0, Stock Price = \$100 and the current option value is:

$$\begin{aligned}C =&\max\{100-95, e^{-0.05}[0.4256\times 49.0052+(1-0.4256)\times 7.9212]\}\\ =&e^{-0.05}[0.4256\times 49.0052+(1-0.4256)\times 7.9212]\\ =&24.17 \blacksquare\end{aligned}$$

Example 19.2
Given the following information about a stock: $S = \$100, \sigma = 0.3, r = 0.05, \delta = 0.03, K = 95$. Using a three-period binomial tree, find the current price of an American put option with strike \$95 and expiring in three years.

Solution.
The values of u, d, and p_u are the same as in the previous example. We have
Year 3, Stock Price $= u^3S = 261.1539$ Since we are at expiration, the option value is $P_{uuu} = 0$.
Year 3, Stock Price $= u^2dS = 143.3302$ and $P_{uud} = 0$.
Year 3, Stock Price $= ud^2S = 78.6646$ and $P_{udd} = P_{ddu} = 95 - 78.6646 = 16.3354$.
Year 3, Stock Price $= d^3S = 43.1738$ and $P_{ddd} = 95 - 43.1738 = 51.8262$.
Year 2, Stock Price $= u^2S = 189.6404$ and

$$\begin{aligned}P_{uu} =&\max\{95-189.6404, e^{-0.05}[0.4256\times 0+(1-0.4256)\times 0]\}\\ =&0.\end{aligned}$$

Year 2, Stock Price $= udS = 104.0812$ and

$$\begin{aligned}P_{ud} =&\max\{95-104.08212, e^{-0.05}[0.4256\times 0+(1-0.4256)\times 16.3554]\}\\ =&e^{-0.05}[0.4256\times 0+(1-0.4256)\times 16.3354] = 8.9254\end{aligned}$$

Year 2, Stock Price $= d^2S = 57.1234$ and

$$\begin{aligned}P_{dd} =&\max\{95-57.123, e^{-0.05}[0.4256\times 16.3354+(1-0.4256)\times 51.8262]\}\\ =&95-57.123 = 37.877\end{aligned}$$

Year 1, Stock Price $= uS = 137.71$ and

$$\begin{aligned}P_{u} =&\max\{95-137.71, e^{-0.05}[0.4256\times 0+(1-0.4256)\times 9.3831]\}\\ =&e^{-0.05}[0.4256\times 0+(1-0.4256)\times 8.9254] = 4.8767\end{aligned}$$

Year 1, Stock Price = $dS = 75.58$ and

$$
\begin{aligned}
P_d =& \max\{95 - 75.58, e^{-0.05}[0.4256 \times 8.9254 + (1 - 0.4256) \times 37.877]\} \\
=& e^{-0.05}[0.4256 \times 8.9254 + (1 - 0.4256) \times 37.877] = 24.3089
\end{aligned}
$$

Year 0, Stock Price = \$100 and the current option value is:

$$
\begin{aligned}
P =& \max\{95 - 100, e^{-0.05}[0.4256 \times 4.8767 + (1 - 0.4256) \times 24.3089]\} \\
=& e^{-0.05}[0.4256 \times 4.8767 + (1 - 0.4256) \times 24.3089] = 15.2563 \blacksquare
\end{aligned}
$$

Practice Problems

Problem 19.1
A stock is currently worth \$100. In one year the stock will go up to \$120 or down to \$90. The stock pays no dividends, and the annual continuously-compounded risk-free interest rate is 5%. Find the price today of a one year America put option on the stock with a strike price of \$130.

Problem 19.2
One year from now, GS stock is expected to sell for either \$130 or \$124. The annual continuously compounded interest rate is 11%. The risk-neutral probability of the stock price being \$130 in one year is 0.77. What is the current stock price for which a one-year American put option on the stock with a strike price of \$160 will have the same value whether calculated by means of the binomial option pricing model or by taking the difference between the stock price and the strike price?

Problem 19.3
GS Inc., pays dividends on its stock at an annual continuously compounded yield of 6%. The annual effective interest rate is 9%. GS Inc., stock is currently worth \$100. Every two years, it can go up by 50% or down by 30%. Using a two-period binomial option pricing model, find the price two years from today of one four-year American call option on the stock with a strike price of \$80 in the event that the stock price increases two years from today.

Problem 19.4
Repeat the previous problem in the event that the stock price decreases two years from today.

Problem 19.5
The current price of a stock is \$110. The annual continuously compounded interest rate is 0.10, and the stock pays continuous dividends at the continuously compounded yield 0.08. The annualized standard deviation of the continuously compounded stock return is 0.32. Using a three-period binomial pricing model, find the price of an American call option on the stock with strike price \$100 and that matures in nine months.

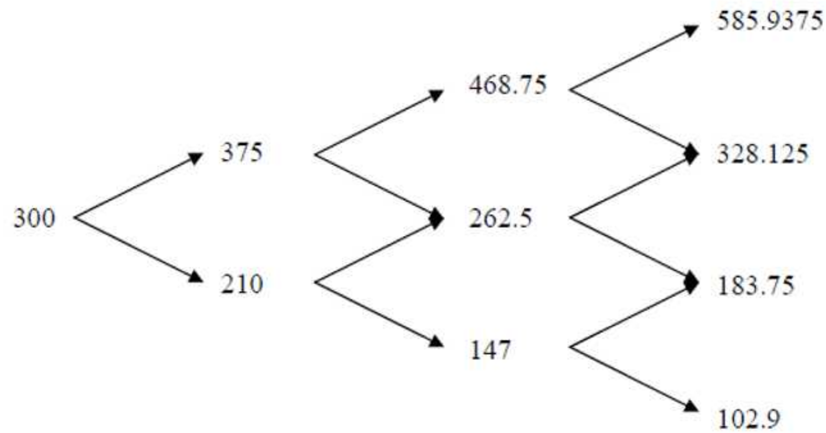
Problem 19.6
Repeat the previous problem with an American put.

Problem 19.7
Given the following: $S = \$72, r = 8\%, \delta = 3\%, \sigma = 23\%, h = 1, K = 74$. Use two-period binomial pricing to find the current price of an American put.

Problem 19.8 ‡
You are given the following information about American options:
• The current price for a non-dividend paying stock is 72
• The strike price is 80
• The continuously-compounded risk-free rate is 5%
• Time to expiration is one year
• Every six months, the stock price either increases by 25% or decreases by 15%.
Using a two-period binomial tree, calculate the price of an American put option.

Problem 19.9 ‡
For a two-period binomial model, you are given:
(i) Each period is one year.
(ii) The current price for a nondividend-paying stock is \$20.
(iii) $u = 1.2840$, where u is one plus the rate of capital gain on the stock per period if the stock price goes up.
(iv) $d = 0.8607$, where d is one plus the rate of capital loss on the stock per period if the stock price goes down.
(v) The continuously compounded risk-free interest rate is 5%.
Calculate the price of an American call option on the stock with a strike price of \$22.

Problem 19.10 ‡
Consider the following three-period binomial tree model for a stock that pays dividends continuously at a rate proportional to its price. The length of each period is 1 year, the continuously compounded risk-free interest rate is 10%, and the continuous dividend yield on the stock is 6.5%.
Calculate the price of a 3-year at-the-money American put option on the stock.



Problem 19.11 ‡

For a two-period binomial model for stock prices, you are given:
(i) Each period is 6 months.
(ii) The current price for a nondividend-paying stock is \$70.00.
(iii) $u = 1.181$, where u is one plus the rate of capital gain on the stock per period if the price goes up.
(iv) $d = 0.890$, where d is one plus the rate of capital loss on the stock per period if the price goes down.
(v) The continuously compounded risk-free interest rate is 5%.
Calculate the current price of a one-year American put option on the stock with a strike price of \$80.00.

Problem 19.12 ‡

Given the following information for constructing a binomial tree for modeling the price movements of a non-dividend paying stock. (This tree is sometimes called a forward tree.)
(i) The length of each period is one year.
(ii) The current stock price is 100.
(iii) Volatility is 30%.
(iv) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 5%.
(v) The continuously compounded risk-free interest rate is 5%.
Calculate the price of a two-year 100-strike American call option on the stock.

Problem 19.13 ‡

Given the following information for constructing a binomial tree for modeling

the price movements of a stock:
(i) The period is 3 months.
(ii) The initial stock price is \$100.
(iii) The stock's volatility is 30%.
(iv) The continuously compounded risk-free interest rate is 4%.
At the beginning of the period, an investor owns an American put option on the stock. The option expires at the end of the period.
Determine the smallest integer-valued strike price for which an investor will exercise the put option at the beginning of the period.

20 Binomial Option Pricing on Currency Options

In the case of currency options with underlying asset a foreign currency and strike asset dollars, the replicating portfolio consists of buying Δ units of the foreign currency and borrowing cash in the amount of B dollars. Letting r_f denote the foreign currency denominated interest rate, one unit of the foreign currency now increases to $e^{r_f h}$ units at the end of one period. Thus, after one period, the payoff of the portfolio is $\Delta \times ux \times e^{r_f h} - e^{rh}B$ in the up state and $\Delta \times dx \times e^{r_f h} - e^{rh}B$ in the down state where x is the exchange rate at the beginning of the period and r is the dollar-denominated interest rate. Now, if this portfolio is to replicate the call option then by the no-arbitrage principle we must have

$$\Delta \times ux \times e^{r_f h} - e^{rh}B = C_u \text{ and } \Delta \times dx \times e^{r_f h} - e^{rh}B = C_d.$$

This is a system of two equations in the unknowns Δ and B. Solving the system we find

$$\Delta = e^{-r_f h}\frac{C_u - C_d}{x(u-d)} \tag{20.1}$$

$$B = e^{-rh}\frac{uC_d - dC_u}{d-u} \tag{20.2}$$

Now, the call price at the beginning of the period is defined to be the total amount invested in the equivalent portfolio. That is,

$$C = \Delta x - B = e^{-rh}\left(C_u\frac{e^{(r-r_f)h} - d}{u-d} + C_d\frac{u - e^{(r-r_f)h}}{u-d}\right). \tag{20.3}$$

These equations look similar to the ones for stock with the dividend yield being replaced by the foreign denominated interest rate.
Now, from equation (20.3) we can define the risk-neutral probability of an up move by the formula

$$p_u = \frac{e^{(r-r_f)h} - d}{u-d}.$$

By Section 16, we have the following equations

$$\begin{aligned} ux =& F_{t,t+h}e^{\sigma\sqrt{h}} \\ dx =& F_{t,t+h}e^{-\sigma\sqrt{h}} \end{aligned}$$

where $F_{t,t+h} = xe^{(r-r_f)h}$. Hence u and d are found using the equations

$$u = e^{(r-r_f)h+\sigma\sqrt{h}}$$
$$d = e^{(r-r_f)h-\sigma\sqrt{h}}$$

Example 20.1
You are given the following information: $x =$\$1.20/€, $r = 5\%, r_{€} = 9\%, \sigma = 15\%$. Using a three-period binomial tree, calculate the price of a nine months European call on the euro, denominated in dollars, with a strike price of \$1.10.

Solution.
We first find u and v given by

$$u = e^{(r-r_f)h+\sigma\sqrt{h}} = e^{(0.05-0.09)\times 0.25+0.15\sqrt{0.25}} = 1.06716$$

and

$$d = e^{(r-r_f)h-\sigma\sqrt{h}} = e^{(0.05-0.09)\times 0.25-0.15\sqrt{0.25}} = 0.91851$$

The risk-neutral probability of an up move is

$$p_u = \frac{e^{(r-r_f)h} - d}{u-d} = \frac{e^{-0.04\times 0.25} - 0.918151}{1.06716 - 0.91851} = 0.48126.$$

We have
9-month, Exchange rate $= u^3x =$ \$1.4584/€ Since we are at expiration, the option value is $C_{uuu} = 1.4584 - 1.1 = \$0.3584$.
9-month, Exchange rate $= u^2dx =$ \$1.2552/€ and $C_{uud} = 1.2552 - 1.10 = \$0.1552$.
9-month, Exchange rate $= ud^2x =$ \$1.0804/€ and $C_{udd} = C_{ddu} = 0$.
9-month, Exchange rate $= d^3x =$ \$0.9299/€ and $C_{ddd} = 0$.
6-month, Exchange rate $= u^2x =$ \$1.3666/€ and

$$C_{uu} = e^{-0.05\times 0.25}[0.48126 \times 0.3584 + (1 - 0.48126) \times 0.1552] = \$0.2499.$$

6-month, Exchange rate $= udx =$ \$1.1762/€ and

$$C_{ud} = e^{-0.05\times 0.25}[0.48126 \times 0.1552 + (1 - 0.48126) \times 0] = \$0.0738.$$

6-month, Exchange rate $= d^2x =$ \$1.0124/€ and

$$C_{dd} = e^{-0.05\times 0.25}[0.48126 \times 0 + (1 - 0.48126) \times 0] = 0.$$

3-month, Exchange rate $= ux = \$1.2806/€$ and

$$C_u = e^{-0.05\times 0.25}[0.48126 \times 0.2499 + (1 - 0.48216) \times 0.0738] = \$0.1565.$$

3-month, Exchange rate $= dx = \$1.1022/€$ and

$$C_d = e^{-0.05\times 0.25}[0.48216 \times 0.0738 + (1 - 0.48216) \times 0] = \$0.0351.$$

The current option value is:

$$C = e^{-0.05\times 0.25}[0.48216 \times 0.1565 + (1 - 0.48216) \times 0.0351] = \$0.0924 \blacksquare$$

Example 20.2
You are given the following information: $x = \$1.20/€$, $r = 5\%, r_€ = 9\%, \sigma = 15\%$. Using a three-period binomial tree, calculate the price of a nine months American call on the euro, denominated in dollars, with a strike price of \$1.10.

Solution.
From the previous example, we have $u = 1.06716, d = 0.91851$, and $p_u = 0.48126$. Now, we have
9-month, Exchange rate $= u^3x = \$1.4584/€$ Since we are at expiration, the option value is $C_{uuu} = 1.4584 - 1.1 = \$0.3584$.
9-month, Exchange rate $= u^2dx = \$1.2552/€$ and $C_{uud} = 1.2552 - 1.10 = \$0.1552$.
9-month, Exchange rate $= ud^2x = \$1.0804/€$ and $C_{udd} = C_{ddu} = 0$.
9-month, Exchange rate $= d^3x = \$0.9299/€$ and $C_{ddd} = 0$.
6-month, Exchange rate $= u^2x = \$1.3666/€$ and

$$\begin{aligned}C_{uu} =& \max\{1.3666 - 1.10, e^{-0.05\times 25}[0.48216 \times 0.3584 + (1 - 0.48126) \times 0.1552]\}\\ =&1.3666 - 1.10\\ =&0.2530.\end{aligned}$$

6-month, Exchange rate $= udx = \$1.1762/€$ and

$$\begin{aligned}C_{ud} =& \max\{1.1762 - 1.10, e^{-0.05\times 0.25}[0.48126 \times 0.1552 + (1 - 0.48126) \times 0]\}\\ =&1.1762 - 1.10\\ =&0.0752.\end{aligned}$$

6-month, Exchange rate $= d^2x = \$1.0124/€$ and

$$\begin{aligned}C_{dd} =& \max\{1.0124 - 1.1, e^{-0.05\times0.25}[0.48126 \times 0 + (1 - 0.4826) \times 0]\}\\ =& e^{-0.05\times0.25}[0.48126 \times 0 + (1 - 0.48216) \times 0] = 0\end{aligned}$$

3-month, Exchange rate $= ux = \$1.2806/€$ and

$$\begin{aligned}C_u =& \max\{1.2806 - 1.1, e^{-0.05\times0.25}[0.48126 \times 0.2530 + (1 - 0.48126) \times 0.0752]\}\\ =& 1.2806 - 1.1\\ =& 0.1806\end{aligned}$$

3-month, Exchange rate $= dx = \$1.1022/€$ and

$$\begin{aligned}C_d =& \max\{1.1022 - 1.10, e^{-0.05\times0.25}[0.48126 \times 0.0752 + (1 - 0.48126) \times 0]\}\\ =& e^{-0.05\times0.25}[0.48126 \times 0.0752 + (1 - 0.48126) \times 0]\\ =& 0.0357\end{aligned}$$

The current option value is:

$$\begin{aligned}C =& \max\{1.20 - 1.10, e^{-0.05\times0.25}[0.48126 \times 0.1806 + (1 - 0.48126) \times 0.0357]\}\\ =& e^{-0.05\times0.25}[0.48126 \times 0.1806 + (1 - 0.48126) \times 0.0357]\\ =& 0.1041 \blacksquare\end{aligned}$$

Practice Problems

Problem 20.1
One euro currently trades for \$1.56. The dollar-denominated annual continuously-compounded risk-free interest rate is 2%, and the euro-denominated annual continuously-compounded risk-free interest rate is 9%. Calculate the price of a 10-year forward contract on euros, denominated in dollars.

Problem 20.2
One euro currently trades for \$1.56. The dollar-denominated annual continuously-compounded risk-free interest rate is 0.02, and the euro-denominated annual continuously-compounded risk-free interest rate is 0.09. The annualized standard deviation of the continuously compounded return on the euro is 0.54. Using a one-period binomial model, calculate what the euro price in dollars will be in two years if the euro's price increases.

Problem 20.3
One dollar is currently trading for FC 45. The dollar-denominated annual continuously compounded risk-free interest rate is 0.13, while the FC-denominated annual continuously compounded risk-free interest rate is 0.05. The annualized standard deviation of the continuously compounded return on dollars is 0.81. Using the one-period binomial option pricing model, what is the risk-neutral probability that the price of a dollar will increase in two months?

Problem 20.4
You are given the following information: $x =\$0.92/€$, $r = 4\%, r_€ = 3\%, u = 1.2, d = 0.9$, and $h = 0.25$. Using a three-period binomial tree, calculate the price of a nine months European put on the euro, denominated in dollars, with a strike price of \$0.85.

Problem 20.5
You are given the following information: $x =\$0.92/€$, $r = 4\%, r_€ = 3\%, u = 1.2, d = 0.9$, and $h = 0.25$. Using a three-period binomial tree, calculate the price of a nine months American put on the euro, denominated in dollars, with a strike price of \$1.00.

Problem 20.6
One yen currently trades for \$0.0083. The dollar-denominated annual continuously-compounded risk-free interest rate is 0.05, and the yen-denominated annual

continuously-compounded risk-free interest rate is 0.01. The annualized standard deviation of the continuously compounded return on the dollar is 0.10. Using a one-period binomial model, calculate what the yen price in dollars will be in two years if the yen's price increases.

Problem 20.7
Given the following information: $r = 0.05, r_f = 0.01, \sigma = 0.10, h = \frac{1}{3}$. Find the risk-neutral probability of an up move.

Problem 20.8
One dollar is currently trading FC 45. The dollar-denominated annual continuously-compounded risk-free interest rate is 13%, while the FC-denominated annual continuously-compounded risk-free interest rate is 5%. The annualized standard deviation of the continuously compounded return on dollars is 0.81. For a certain two-month European call option on one dollar, the replicating portfolio involves buying $\frac{3}{4}$ dollars and borrowing FC 23. Using the one-period binomial option pricing model, what would the price of this call option (in FC) be in two months in the case there is a decrease in the dollar value?

Problem 20.9
A dollar is currently selling for ¥118. The dollar-denominated annual continuously-compounded risk-free interest rate is 0.06, and the yen-denominated annual continuously-compounded risk-free interest rate is 0.01. The annualized standard deviation of the continuously compounded return on the dollar is 0.11. Using a three-period binomial tree, find the current price of a yen-denominated 1-year American call on the dollar with strike price of ¥118.

Problem 20.10 ‡
Consider a 9-month dollar-denominated American put option on British pounds. You are given that:
(i) The current exchange rate is 1.43 US dollars per pound.
(ii) The strike price of the put is 1.56 US dollars per pound.
(iii) The volatility of the exchange rate is $\sigma = 0.3$.
(iv) The US dollar continuously compounded risk-free interest rate is 8%.
(v) The British pound continuously compounded risk-free interest rate is 9%.
Using a three-period binomial model, calculate the price of the put.

21 Binomial Pricing of Futures Options

In this section we consider applying the binomial model to price options with underlying assets futures contracts.

Recall that a futures contract, or simply a futures, is a pledge to purchase at a future date a given amount of an asset at a price agreed on today. A futures contract requires no initial payment. At the maturity date, cash is exchanged for the asset. For example, a 3-month futures contract for 1000 tons of soybean at a forward price of \$165/ton is a committment from the owner of the contract to buy 1000 tons of soybean in three months for a price of \$165 a ton.

The replicating portfolio consists of buying Δ units of the futures contracts and lending cash in the amount of B dollars. At the end of a given period, a futures' contract payoff is just the change in the futures price[1]. Thus, after one period, the payoff of the portfolio is $\Delta \times (uF - F) + e^{rh}B$ in the up state and $\Delta \times (dF - F) + e^{rh}B$ in the down state where F is the delivery price of the contract at the beginning of the period and r is the continuously compounded risk-free interest rate. Now, if this portfolio is to replicate a call option on the futures then by the no-arbitrage principle we must have

$$\Delta \times (uF - F) + e^{rh}B = C_u \text{ and } \Delta \times (dF - F) + e^{rh}B = C_d.$$

This is a system of two equations in the unknowns Δ and B. Solving the system we find

$$\Delta = \frac{C_u - C_d}{F(u-d)} \tag{21.1}$$

$$B = e^{-rh}\left(C_u\frac{1-d}{u-d} + C_d\frac{u-1}{u-d}\right). \tag{21.2}$$

Now, the call price at the beginning of the period is time 0 value of the replicating portfolio. That is,

$$C = B = e^{-rh}\left(C_u\frac{1-d}{u-d} + C_d\frac{u-1}{u-d}\right) \tag{21.3}$$

since a futures requires no initial premium. Now, from equation (21.2) we can define the risk-neutral probability of an up move by the formula

$$p_u = \frac{1-d}{u-d}.$$

[1] See [2], Section 59.

By Section 16, the up and down move of the forward price are modeled by the following equations

$$\begin{aligned} uF =& F_{t,t+h}e^{\sigma\sqrt{h}} \\ dF =& F_{t,t+h}e^{-\sigma\sqrt{h}} \end{aligned}$$

where $F_{t,t+h} = F$ is the forward price. Hence u and d are found by using the equations

$$\begin{aligned} u =& e^{\sigma\sqrt{h}} \\ d =& e^{-\sigma\sqrt{h}} \end{aligned}$$

Example 21.1
An option has a gold futures contract as the underlying asset. The current 1-year gold futures price is \$300/oz, the strike price is \$290, the continuously compounded risk-free interest rate is 6%, volatility is 10%, and the time to expiration is 1 year. Using a one-period binomial model, find Δ, B, and the price of the call.

Solution.
We first find u and d. We have

$$u = e^{\sigma\sqrt{h}} = e^{0.10} = 1.1052$$

and

$$d = e^{-\sigma\sqrt{h}} = e^{-0.10} = 0.9048.$$

We also have

$$C_u = 1.1052 \times 300 - 290 = \$41.56$$

and

$$C_d = 0.$$

Thus,

$$\Delta = \frac{C_u - C_d}{F(u-d)} = \frac{41.56 - 0}{300(1.1052 - 0.9048)} = 0.6913$$

and

$$C = B = e^{-rh}\left(C_u\frac{1-d}{u-d} + C_d\frac{u-1}{u-d}\right) = e^{-0.06}\left[41.56 \times \frac{1-0.9048}{1.1052-0.9048}\right] = \$18.5933$$

Alternatively, we could have found the risk-neutral probability of an up move

$$p_u = \frac{1-d}{u-d} = \frac{1-0.9048}{1.1052-0.9048} = 0.4751$$

and then find the price of the call to be

$$C = B = e^{-rh}[p_uC_u + (1-p_u)C_d] = e^{-0.06} \times 0.4751 \times 41.56 = \$18.5933 \blacksquare$$

Example 21.2
An option has a futures contract as the underlying asset. The current 1-year futures price is \$1000, the strike price is \$1000, the continuously compounded risk-free interest rate is 7%, volatility is 30%, and the time to expiration is one year. Using a three-period binomial model, find the price of an American call.

Solution.
We have

$$u = e^{\sigma\sqrt{h}} = e^{0.30\sqrt{\frac{1}{3}}} = 1.18911$$

and

$$d = e^{-\sigma\sqrt{h}} = e^{-0.30\sqrt{\frac{1}{3}}} = 0.84097.$$

The risk-neutral probability of an up move is

$$p_u = \frac{1-d}{u-d} = \frac{1-0.84097}{1.18911-0.84097} = 0.45681.$$

Now, we have
12-month, Futures Price $= u^3F = 1681.3806$ Since we are at expiration, the option value is $C_{uuu} = 1681.3806 - 1000 = \$681.3806$.
12-month, Futures Price $= u^2dF = 1189.1099$ and $C_{uud} = 1189.1099 - 1000 = 189.1099$.
12-month, Futures Price $= ud^2F = 840.9651$ and $C_{udd} = C_{ddu} = 0$.
12-month, Futures Price $= d^3F = 594.7493$ and $C_{ddd} = 0$.
8-month, Futures Price $= u^2F = 1413.9825$

$$\begin{aligned}C_{uu} =& \max\{1413.9825 - 1000, e^{-0.07\times\frac{1}{3}}[0.45681 \times 681.3806 + (1-0.45681) \times 189.1099]\}\\ =&1413.9825 - 1000\\ =&413.9825.\end{aligned}$$

8-month, Futures Price $= udF = 1000$

$$\begin{aligned}C_{ud} =& \max\{10000 - 1000, e^{-0.07\times\frac{1}{3}}[0.45681 \times 189.1099 + (1 - 0.45681) \times 0]\}\\ =&e^{-0.07\times\frac{1}{3}}[0.45681 \times 189.1099 + (1 - 0.45681) \times 0]\\ =&84.3943.\end{aligned}$$

8-month, Futures Price $= d^2F = 707.2224$

$$\begin{aligned}C_{dd} =& \max\{707.2224 - 1000, e^{-0.07\times\frac{1}{3}}[0.45681 \times 0 + (1 - 0.4256) \times 0]\}\\ =&e^{-0.07\times\frac{1}{3}}[0.45681 \times 0 + (1 - 0.4256) \times 0 = 0\end{aligned}$$

4-month, Futures Price $= uF = 1189.1099$

$$\begin{aligned}C_u =& \max\{1189.1099 - 1000, e^{-0.07\times\frac{1}{3}}[0.45681 \times 413.9825 + (1 - 0.45681) \times 84.3943]\}\\ =&e^{-0.07\times\frac{1}{3}}[0.45681 \times 413.9825 + (1 - 0.45681) \times 84.3943]\\ =&229.5336.\end{aligned}$$

4-month, Futures Price $= dF = 840.9651$

$$\begin{aligned}C_d =& \max\{849.9651 - 1000, e^{-0.07\times\frac{1}{3}}[0.45681 \times 84.3943 + (1 - 0.45681) \times 0]\}\\ =&e^{-0.07\times\frac{1}{3}}[0.45681 \times 84.3943 + (1 - 0.45681) \times 0]\\ =&37.6628.\end{aligned}$$

0-month, Futures Price = \$1000 and the current option value is:

$$\begin{aligned}C =& \max\{1000 - 1000, e^{-0.07\times\frac{1}{3}}[0.45681 \times 229.5336 + (1 - 0.45681) \times 37.6628]\}\\ =&e^{-0.07\times\frac{1}{3}}[0.45681 \times 229.5336 + (1 - 0.45681) \times 37.6628]\\ =&122.4206 \blacksquare\end{aligned}$$

Practice Problems

Problem 21.1
An option has a gold futures contract as the underlying asset. The current 1-year gold futures price is \$600/oz, the strike price is \$620, the continuously compounded risk-free interest rate is 7%, volatility is 12%, and the time to expiration is 1 year. Using a one-period binomial model, find the price of the call.

Problem 21.2
An option has a futures contract as the underlying asset. The current 1-year futures price is \$1000, the strike price is \$1000, the continuously compounded risk-free interest rate is 7%, volatility is 30%, and the time to expiration is one year. Using a three-period binomial model, find the time-0 number of futures contract Δ in the replicating portfolio of an American call on the futures.

Problem 21.3
Consider a European put option on a futures contract. Suppose that $d = \frac{3}{4}u$ when using a two-period binomial model and the risk-neutral probability of an increase in the futures price is $\frac{1}{3}$. Determine u and d.

Problem 21.4
Consider a European put option on a futures contract with expiration time of 1 year and strike price of \$80. The time-0 futures price is \$80. Suppose that $u = 1.2, d = 0.9, r = 0.05$. Using a two-period binomial model, find the current price of the put.

Problem 21.5
Consider an American put option on a futures contract with expiration time of 1 year and strike price of \$80. The time-0 futures price is \$80. Suppose that $u = 1.2, d = 0.9, r = 0.05$. Using a two-period binomial model, find the current price of the put.

Problem 21.6
Consider an option on a futures contract. The time-0 futures price is \$90. The annualized standard deviation of the continuously compounded return on the futures contract is 34%. Using a three-period binomial option pricing model, find the futures contract price after 6 years if the contract always increases in price every time period.

Problem 21.7
Find the risk-neutral probability of a down move in Problem 21.6.

Problem 21.8
Consider an option on a futures contract. The time-0 futures price is \$90. The annualized standard deviation of the continuously compounded return on the futures contract is 0.34. The annual continuously compounded risk-free interest rate is 0.05. Using a one-period binomial option pricing model, find Δ for a replicating portfolio equivalent to one two-year European call option on futures contract with a strike price of \$30.

Problem 21.9
Consider an option on a futures fontract. The time-0 futures price is \$49. The annual continuously compounded risk-free interest rate is 15%. The price today of one particular three-month European call option on the contract is \$10. The value of Δ in a replicating portfolio equivalent to one such option is 0.4. If in three months, futures prices will be worth 0.85 of its present amount, what will the price of the call option be? Use a one-period binomial option pricing model.

Problem 21.10 ‡
You are to price options on a futures contract. The movements of the futures price are modeled by a binomial tree. You are given:
(i) Each period is 6 months.
(ii) $\frac{u}{d} = \frac{4}{3}$, where u is one plus the rate of gain on the futures price if it goes up, and d is one plus the rate of loss if it goes down.
(iii) The risk-neutral probability of an up move is $\frac{1}{3}$.
(iv) The initial futures price is 80.
(v) The continuously compounded risk-free interest rate is 5%.
Let C_I be the price of a 1-year 85-strike European call option on the futures contract, and C_{II} be the price of an otherwise identical American call option. Determine $C_{II} - C_I$.

22 Further Discussion of Early Exercising

As you have seen by now, when American options are valued using the binomial tree, one compares the value of exercising immediately with the value of continuing holding the option and this is done at each binomial node. In this section, we examine early exercise in more detail. More specifically, we want to know when is it rational to early exercise.

The early exercise decision weighs three economic considerations:

- The dividends on the underlying asset.
- The interest on the strike price.
- The insurance value of keeping the option alive.

Example 22.1

Discuss the three considerations of early exercise for an American call option holder.

Solution.

By exercising, the option holder

- Receives the underlying asset and captures all related future dividends.
- Pays the strike price and therefore loses the interest from the time of exercising to the time of expiration.
- Loses the insurance/flexibility implicit in the call. The option holder is not protected anymore when the underlying asset has a value less than the strike price at expiration ■

It follows that for a call, dividends encourage early exercising while interest and insurance weigh against early exercise.

Example 22.2

Consider an American call option on a stock with strike price of \$100. The stock pays continuous dividends at the continuous yield rate of 5%. The annual continuously compounded risk-free interest rate is 5%.

(a) Suppose that one year is left to expiration and that the stock price is currently \$200. Compare the amount of dividends earned from acquiring the stock by early exercise to the amount of interest saved by not exercising.

(b) According to your answer to (a), what is the only economic consideration for an option holder not to exercise early?

Solution.

(a) The amount of dividends earned from early exercise is $200e^{0.05} - 200 =$

\$10.25. The amount of interest saved from not exercising is $100e^{0.05} - 100 =$ \$5.13. Thus, the dividends lost by not exercising exceed interest saved by deferring exercise.
(b) The only reason for not exercising in this case is to keep the implicit insurance provided by the call against a drop in the stock price below the strike price ■

According to the previous example, one of the reasons an option holder may defer early exercising is the insurance feature of the option. However, with zero volatility this insurance has zero value. In this case, what will be the optimal decision? Obviously, it is optimal to defer exercise as long as the interest savings on the strike exceeds the dividends lost. In symbol, we want

$$e^{r(T-t)}K - K > e^{\delta(T-t)}S_t - S_t.$$

Example 22.3
Show that the condition $rK > \delta S_t$ implies $e^{r(T-t)}K - K > e^{\delta(T-t)}S_t - S_t$.

Solution.
Suppose that $rK > \delta S_t$. Using the Taylor series expansion of the function e^t around zero we can write $e^{r(T-t)}K - K \approx Kr(T-t)$ and $e^{\delta(T-t)}S_t - S_t \approx \delta S_t(T-t)$. Thus, $rK > \delta S_t$ implies $e^{r(T-t)}K - K > e^{\delta(T-t)}S_t - S_t$ ■

It follows that for an American call option where the volatility is zero, it is optimal to defer exercise as long as the following condition holds:

$$rK > \delta S_t.$$

It is optimal to exercise whenever[1]

$$S_t > \frac{rK}{\delta}.$$

With American put options, the reverse holds. It is optimal to exercise early when

$$S_t < \frac{rK}{\delta}$$

and it is optimal to defer exercise when

$$rK < \delta S_t.$$

[1]This condition may be wrong in some instances. However, this condition is always correct for infinitely-lived American options.

Example 22.4
Consider an American call option with zero volatility. Suppose that $r = 2\delta$. When it is optimal to exercise?

Solution.
It is optimal to exercise whenever $S_t > \frac{rK}{\delta} = 2K$. That is, when the stock price is at least twice the strike price ■

When volatility is positive, the implicit insurance has value, and the value varies with time to expiration. Figure 22.1 shows, for a fixed time, the lowest stock price above which early exercise is optimal for a 5-year American call with strike \$100, $r = \delta = 5\%$, for three different volatilities. Recall that if it is optimal to exercise a call at a given stock price then it is optimal to exercise at all higher stock prices. The figure shows the effect of volatility. The exercise bounds for lower volatility are lower than the exercise bounds for higher volatility. This stems from the fact that the insurance value lost by early-exercise is greater when volatility is greater. Moreover, for a fixed volatility, the passage of time decreases the exercise bounds. This stems from the fact that the value of insurance diminishes as the options approach expiration.

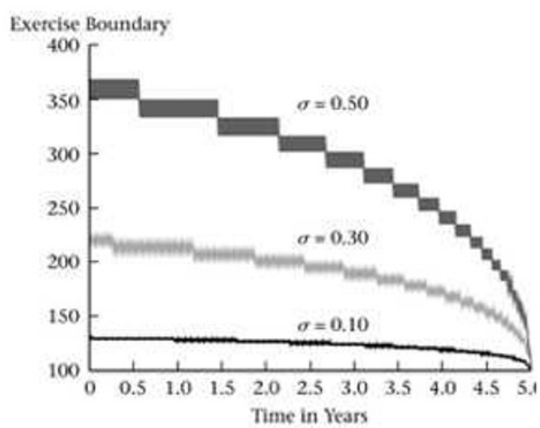


Figure 22.1

In the case of an American put, the higher the volatility the lower the exercise bound. Moreover, the passage of time increases the exercise bounds. See Figure 22.2.

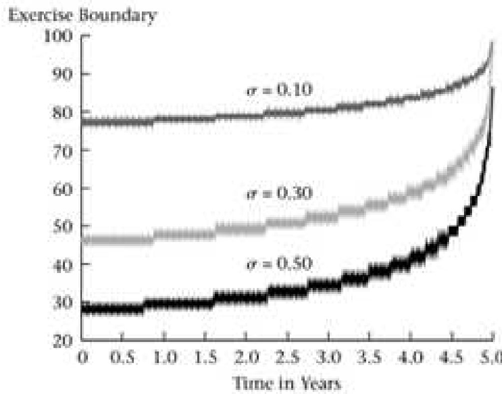


Figure 22.2

Example 22.5
Suppose that for a volatility of 10%, an early exercise for an American call option is optimal when the lowest price of the underlying asset is \$130. For a volatility of 30%, would the lowest price of the underlying asset be larger or smaller than \$130 in order for early exercise to be optimal?

Solution.
It has to be larger than \$130 ■

Example 22.6
You are given the following information about an American call on a stock:
- The current stock price is \$70.
- The strike price is \$68
- The continuously compounded risk-free interest rate is 7%.
- The continuously compounded dividend yield is 5%.
- Volatility is zero.

Find the time until which early exercise is optimal.

Solution.
Early exercise is optimal if and only of

$$S > \frac{rK}{\delta} = \frac{0.07 \times 68}{0.05} = \$95.20.$$

That is, when the stock price reaches \$95.20. Now,

$$u = e^{(r-\delta)h + \sigma\sqrt{h}} = e^{0.02h} = d.$$

Thus, early exercise is optimal when $uS = 95.20$ or $70e^{0.02T} = 95.20$. Solving this equation we find $T = 15.37$ ■

Practice Problems

Problem 22.1
Discuss the three considerations of early exercise for an American put option holder.

Problem 22.2
Suppose that for a volatility of 10%, an early exercise for an American put option is optimal when the highest price of the underlying asset is \$80. For a volatility of 30%, would the highest price of the underlying asset be larger or smaller than \$80 in order for early exercise to be optimal?

Problem 22.3
Consider an American call option with strike K such that $r = \delta$. If at time t, $S_t > K$, would it be optimal to exercise?

Problem 22.4
Given the following information about an American call option on a stock:
• The current price of the stock is \$456.
• The continuously compounded risk-free interest rate is 5%.
• The continuously compounded dividend yield rate is 4%.
• Volatility is zero.
What is the least strike value so that to defer early exercise?

Problem 22.5
Given the following information about an American call option on a stock:
• The strike price is \$30.
• The continuously compounded risk-free interest rate is 14%.
• The continuously compounded dividend yield rate is 11%.
• Volatility is zero.
What is the lowest stock price for which early exercise is optimal?

Problem 22.6
You own an American put option on the non-volatile GS Inc. stock. The strike price of the option is 23, and the stock's annual continuously compounded dividend yield is 0.4. The annual continuously compounded interest rate is 0.1. For which of these stock prices would it be optimal to exercise the option?
(A) $S = 20$

(B) $S = 15.65$
(C) $S = 13.54$
(D) $S = 12.34$
(E) $S = 6.78$
(F) $S = 3.45$

Problem 22.7
GS Co. stock prices currently have a volatility of $\sigma = 0.3$. For American call options with a strike price of \$120 and time to expiration 1 year, you know that the lowest stock price where exercise is optimal is \$160. If the stock price volatility changes, for which of these volatilities will exercise still be optimal at a stock price of \$160?
(A) $\sigma = 0.1$
(B) $\sigma = 0.2$
(C) $\sigma = 0.4$
(D) $\sigma = 0.5$
(E) $\sigma = 0.6$

Problem 22.8
GS Co. stock prices currently have a volatility of $\sigma = 0.3$. For American put options with a strike price of \$90 and time to expiration 1 year, you know that the highest exercise bound for optimal exercise price is \$40. If the stock price volatility changes, for which of these volatilities will exercise still be optimal at a stock price of \$40?
(A) $\sigma = 0.1$
(B) $\sigma = 0.2$
(C) $\sigma = 0.4$
(D) $\sigma = 0.5$
(E) $\sigma = 0.6$

Problem 22.9
You are given the following information about an American call on a stock:
• The current stock price is \$50.
• The strike price is \$48.
• The continuously compounded risk-free interest rate is 9%.
• The continuously compounded dividend yield is 6%.
• Volatility is zero.
Find the time until which early exercise is optimal.

23 Risk-Neutral Probability Versus Real Probability

The risk-neutral pricing approach was first introduced in Section 15. In this section, we examine this approach in more detail and we compare the risk-neutral approach with the pricing approach using true probability.
We first discuss the meaning of risk-neutral. Suppose you are offered the following two scenarios. In the first scenario you are guaranteed to receive \$100. In the second scenario, a coin is flipped and you receive \$200 if the coin shows head or \$0 otherwise. The expected payoff for both scenarios is \$100. Investors may have different risk attitudes. A **risk-averse** investor prefers the sure thing which is the first scenario. A **risk-neutral** investor is indifferent between the bet and a certain \$100 payment since both have the same expected payoff. That is, a risk-neutral investor will be equally happy with either scenarios.
In most of the future pricing calculations that will occur in the rest of the book, investors are assumed to be risk-averse unless otherwise indicated. However, let's consider a hypothetical world of risk-neutral investors only. In such a world, investors would only be concerned with expected returns, and not about the level of risk. Hence, investors will not charge or require a premium for risky securities. Therefore, risky securities would have the same expected rate of return as riskless securities. In other words, investors hold assets with an expected return equal to the risk-free rate.
In this imaginary risk-neutral world, we let p_u denote the probability of the stock going up such that the stock is expected to earn the risk-free rate. In the binomial model, p_u for one period satisfies the equation

$$p_u uSe^{\delta h} + (1-p_u)dSe^{\delta h} = e^{rh}S.$$

Solving for p_u we find

$$p_u = \frac{e^{(r-\delta)h} - d}{u-d}.$$

This is why we referred to p_u introduced in Section 15 as a risk-neutral probability. It is the probability that the stock price would increase in a risk-neutral world. Also, in the hypothetical world, the option price valuation for a call is given by

$$C = e^{-rh}[p_u C_u + (1-p_u)C_d].$$

Example 23.1
You are given the following information about a stock:
• The stock pays continuous dividend at the continuously compounded yield of 6%.
• The continuously compounded risk-free interest rate is 9%.
• Every h years the stock price increases by 90% or decreases by 80%.
• The risk-neutral probability of the stock price's increase in h years is 0.72.
Find h.

Solution.
We know that
$$p_u = \frac{e^{(r-\delta)h} - d}{u - d}$$
or
$$0.72 = \frac{e^{(0.09-0.06)h} - 0.2}{1.9 - 0.2}.$$
Solving this equation we find $h = 11.7823$ years ■

What is the option pricing in the risk-averse world? In this world, we let p denote the real probability of the stock going up. Let α be the continuously compounded expected return on the stock. Then p satisfies the equation
$$puSe^{\delta h} + (1-p)dSe^{\delta h} = e^{\alpha h}S.$$
Solving for p we find
$$p = \frac{e^{(\alpha-\delta)h} - d}{u - d}.$$
The real probability for the stock to go down is then
$$1 - p = \frac{u - e^{(\alpha-\delta)h}}{u - d}.$$
Imposing the condition $d < e^{(\alpha-\delta)h} < u$ we obtain $0 < p < 1$. Now, using p we can find the actual expected payoff at the end of the period:
$$pC_u + (1-p)C_d.$$

Example 23.2
Consider a nondividend-paying stock. Every two years, the stock price either increases by 5% or decreases by 6%. Find an upper constraint on the annual continuously compounded expected return on the stock.

Solution.
We use the condition $d < e^{\alpha h} < u$ where we write $u = e^{ah}$. Thus, $1.05 = e^{2a} \to a = \frac{\ln 1.05}{2} = 0.024395$ ■

Example 23.3
Consider a nondividend-paying stock. Every two years, the stock price either increases by 7% or decreases by 8%. The annual continuously compounded expected return on the stock is 3%. Find the real probability that the stock will increase in price in two years.

Solution.
We have

$$p = \frac{e^{\alpha h} - d}{u - d} = \frac{e^{0.03 \times 2} - 0.92}{1.07 - 0.92} = 0.9456 \blacksquare$$

For risk-neutral probabilities we discounted the expected payoff at the risk-free rate in order to obtain the current option price. At what rate do we discount the actual expected payoff? Definitely not at the rate α since the option is a type of leveraged investment in the stock so that it is riskier than the stock.
Let γ denote the appropriate per-period discount rate.[1] We use the result of Brealey and Meyer which states that the return on any portfolio is the weighted average of the returns on the assets in the portfolio. We apply this result to the portfolio consisting of Δ shares of nondividend-paying stock and borrowing $\$B$ that mimic the payoff of the call option to obtain

$$e^{\gamma h} = \frac{S\Delta}{S\Delta - B} e^{\alpha h} - \frac{B}{S\Delta - B} e^{rh}.$$

Since an option is equivalent to holding a portfolio consisting of buying Δ shares and borrowing $\$B$, the denominator of the previous relation is just the option price. Thus, discounted cash flow is not used in practice to price options: It is necessary to compute the option price in order to compute the correct discount rate.
We can now compute the option price as the discounted expected payoff at the rate γ to obtain

$$C = e^{-\gamma h}[pC_u + (1 - p)C_d].$$

[1]This means that in a multiperiod-binomial model, the per-period discount rate is different at each node.

Example 23.4
Show that the option price obtained with real probabilities is the same as the one with risk-neutral probabilities.

Solution.
We have

$$e^{-\gamma h}[pC_u+(1-p)C_d] = \frac{S\Delta - B}{S\Delta e^{\alpha h} - Be^{rh}}\left[\frac{e^{rh}-d}{u-d}C_u + \frac{u-e^{rh}}{u-d}C_d + \frac{e^{\alpha h}-e^{rh}}{u-d}(C_u-C_d)\right].$$

But

$$\frac{e^{rh}-d}{u-d}C_u + \frac{u-e^{rh}}{u-d}C_d = e^{rh}(S\Delta - B)$$

and

$$\frac{e^{\alpha h}-e^{rh}}{u-d}(C_u-C_d) = (e^{\alpha h}-e^{rh})\Delta S.$$

Thus

$$\frac{e^{rh}-d}{u-d}C_u + \frac{u-e^{rh}}{u-d}C_d + \frac{e^{\alpha h}-e^{rh}}{u-d}(C_u-C_d) = S\Delta e^{\alpha h} - Be^{rh}.$$

Hence,

$$e^{-\gamma h}[pC_u + (1-p)C_d] = S\Delta - B$$

which is the same result as the one obtained using risk-neutral probabilities ■

Example 23.5
Given the following information about a 1-year European call option on a stock:
- The strike price is \$40.
- The current price of the stock is \$41.
- The expected rate of return is 15%.
- The stock pays no dividends.
- The continuously compounded risk-free rate is 8%.
- Volatility is 30%.

Use a one-period binomial model to compute the price of the call
(a) Using true probabilities on the stock.
(b) Using risk-neutral probabilities.

Solution.
We first compute u and d. We have

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.08-0)\times 1+0.30\sqrt{1}} = 1.4623$$

and

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.08-0)\times 1-0.30\sqrt{1}} = 0.8025.$$

The number of shares in the replicating portfolio is

$$\Delta = \frac{C_u - C_d}{S(u-d)} = \frac{[1.4623 \times 41 - 40] - 0}{41(1.4623 - 0.8025)} = 0.738.$$

The amount of money borrowed is

$$B = e^{-rh}\frac{uC_d - dC_u}{d-u} = e^{-0.08} \times \frac{-0.8025 \times 19.954}{0.8025 - 1.4623} = \$22.404.$$

(a) The true probability of the stock going up in value is

$$p = \frac{e^{\alpha h} - d}{u-d} = \frac{e^{0.15} - 0.8025}{1.4623 - 0.8025} = 0.5446.$$

Now,

$$\begin{aligned}
e^{\gamma h} =& \frac{S\Delta}{S\Delta - B}e^{\alpha h} - \frac{B}{S\Delta - B}e^{rh}\\
=& \frac{41 \times 0.738}{0.738 \times 41 - 22.404}e^{0.15} - \frac{22.404}{0.738 \times 41 - 22.404}e^{0.08}\\
=& 1.386
\end{aligned}$$

Thus,

$$\gamma = \ln 1.386 = 0.3264.$$

The price of the option is

$$C = e^{-\gamma h}[pC_u + (1-p)C_d] = e^{-0.3264}[0.5446 \times 19.954 + (1-0.5446)\times 0] = \$7.839.$$

(b) The risk-neutral probability of a move up state is

$$p_u = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{0.08} - 0.8025}{1.4625 - 0.8025} = 0.4256.$$

Thus, the price of the call is

$$C = e^{-rh}[p_u C_u + (1-p_u)C_d] = e^{-0.08}[0.4256 \times 19.954 + (1-0.4256)\times 0] = \$7.839 \blacksquare$$

Remark 23.1
(1) Notice that in order to find γ we had to find Δ and B. But then the option price is just $C = S\Delta - B$ and there is no need for any further computations. It can be helpful to know how to find the actual expected return, but for valuation it is pointless.
(2) In general, expected rate of returns α are hard to estimate, thus by Example 23.4, risk-neutral approach is easier to use in price valuation.

Example 23.6
Consider the following information about a 2-year American call on a stock:
- Stock pays dividends at the continuously compounded yield rate of 10%.
- The continuously compounded risk-free rate is 11%.
- The continuously compounded rate of return is 24%.
- Volatility is 40%.
- The current price of the stock is \$50.
- The strike price is \$40.

Using a two-period binomial model, find the continuously compounded discount rate at each node.

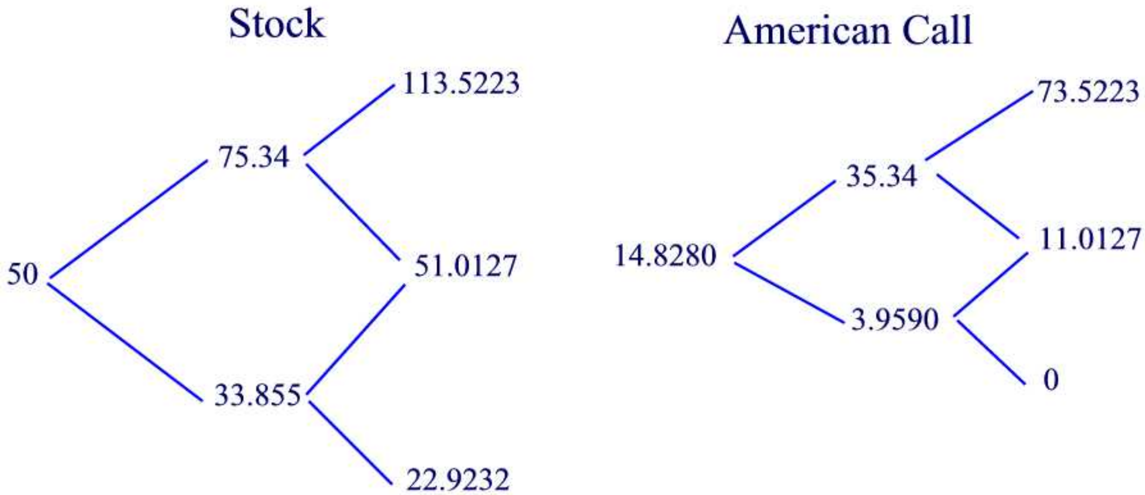
Solution.
We first compute u and d. We have

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{0.11-0.10+0.4\sqrt{1}} = 1.5068$$

and

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.11-0.10-0.4\sqrt{1}} = 0.6771.$$

The binomial trees are shown below.



The risk-neutral probability is

$$p_u = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.11-0.10)\times 1} - 0.6771}{1.5068 - 0.6771} = 0.4013.$$

The true probability for an upward movement is

$$p = \frac{e^{(\alpha-\delta)h} - d}{u-d} = \frac{e^{0.24-0.10} - 0.6771}{1.5068 - 0.6771} = 0.5703.$$

Node with Stock price = \$113.5223 $\gamma =$ N/A.
Node with Stock price = \$51.0127 $\gamma =$ N/A.
Node with Stock price = \$22.9232 $\gamma =$ N/A.
Node with Stock price = \$75.34 We have

$$e^{-\gamma}[pC_{uu} + (1-p)C_{ud}] = 35.34$$

or

$$e^{-\gamma}[0.5703 \times 73.5223 + (1 - 0.5703) \times 11.0127] = 35.34.$$

Solving this equation we find $\gamma = 0.2779$.
Node with Stock price = \$33.855 We have

$$e^{-\gamma}[pC_{ud} + (1-p)C_{dd}] = 3.9590$$

or

$$e^{-\gamma}[0.5703 \times 11.0127 + (1 - 0.5703) \times 0] = 3.9590.$$

Solving this equation we find $\gamma = 0.4615$.
Node with Stock price = \$50 We have

$$e^{-\gamma}[pC_u + (1-p)C_d] = 14.8280$$

or

$$e^{-\gamma}[0.5703 \times 35.34 + (1 - 0.5703) \times 3.9590] = 14.8280.$$

Solving this equation we find $\gamma = 0.3879$ ■

Remark 23.2
A discussion of why the risk-neutral pricing works is covered in Section 85.

Practice Problems

Problem 23.1
Consider the following two scenarios:
Scenario 1: You have \$1000 which you deposit into a savings account that pays annual continuously compounded risk-free interest rate of r.
Scenario 2: You purchase a stock for \$1000. The stock pays dividends with an annual continuously compounded yield of 12%. You hold the stock for 13 years, at the end of which it will be either 3.4 or 0.1 of its present price. The risk-neutral probability of a stock price increase is 0.82.
Find the balance in the savings account, 13 years from now.

Problem 23.2
Given the following information about a non-dividend paying stock:
• After one period, the stock price will either increase by 30% or decrease by 20%.
• The annual compounded continuously expected rate of return is 15%.
• The true probability of increase in the stock price is 0.5446.
Determine the length of the period in years.

Problem 23.3
Given the following information of a 1-year European call on a stock:
• Stock does not pay dividends.
• The continuously compounded risk-free rate is 8%.
• Volatility is 30%.
• The current price of the stock is \$50.
• The strike price is \$48.
• The true probability of an upward movement is 0.46.
Using a one-period binomial model, find the continuously compounded discount rate γ.

Problem 23.4
Consider the following information about a 1-year European put on a stock:
• Stock does not pay dividends.
• The continuously compounded risk-free rate is 8%.
• Volatility is 30%.
• The current price of the stock is \$50.
• The strike price is \$48.

• The true probability of an upward movement is 0.46.
Using a one-period binomial model, find the continuously compounded discount rate γ.

Problem 23.5
Consider the following information about a 1-year European put on a stock:
• Stock pays dividends at the continuously compounded yield rate of 4%.
• The continuously compounded risk-free rate is 8%.
• Volatility is 24%.
• The current price of the stock is \$62.
• The strike price is \$64.
• The continuously compounded expected rate of return is 12%.
Using a one-period binomial model, find the continuously compounded expected rate of discount γ.

Problem 23.6
Consider the following information about a 1-year European call on a stock:
• Stock pays dividends at the continuously compounded yield rate of 2%.
• The continuously compounded risk-free rate is 7%.
• Volatility is 27%.
• The current price of the stock is \$38.
• The strike price is \$40.
• The continuously compounded discount rate of return is 34.836%.
Using a one-period binomial model, find
(a) the true probability of a downward movement.
(b) the expected return on the stock α.

Problem 23.7
Consider the following information about a 1-year American call on a stock:
• Stock pays dividends at the continuously compounded yield rate of 5%.
• The continuously compounded risk-free rate is 6%.
• The continuously compounded rate of return is 10%.
• Volatility is 30%.
• The current price of the stock is \$50.
• The strike price is \$47.
Using a two-period binomial model, find the continuously compounded discount rate after one upward movement.

Problem 23.8 ‡
For a one-period binomial model for the price of a stock, you are given:
(i) The period is one year.
(ii) The stock pays no dividends.
(iii) $u = 1.433$, where u is one plus the rate of capital gain on the stock if the price goes up.
(iv) $d = 0.756$, where d is one plus the rate of capital loss on the stock if the price goes down.
(v) The continuously compounded annual expected return on the stock is 10%.
Calculate the true probability of the stock price going up.

Problem 23.9
For a one-period binomial model, the up and down moves are modeled by the equations $u = e^{\sigma\sqrt{h}}$ and $d = e^{-\sigma\sqrt{h}}$. Given that the period is 6 months, the continuously compounded expected return on the stock is 15%, the continuous compounded yield is 5%, and the stock price volatility is 30%. Find the true probability of the stock going up in price.

24 Random Walk and the Binomial Model

Stock market price movements cannot be predicted since they move randomly. It is conjectured that stock prices follow a random walk model. In this section we introduce the random walk model and we show that the binomial model is a variant of a random walk and that it approximates a lognormal distribution.

Symmetric Random Walk
The one-dimensional random walk is constructed as follows: Starting from location 0, you randomly want to walk along a line either forward or backward, each pace being the same length. To decide whether you want to move forward or backward, you flip a coin. That is, the walk is simulated with a coin flip. If it's a head, you take one step forward(add 1 to your current position). If it's a tail, you take one step back(subtract 1 from your current position). The coin is unbiased, so the chances of heads or tails are equal. The problem is to find the probability of landing at a given spot after a given number of steps, and, in particular, to find how far away you are on average from where you started.

Example 24.1
Suppose that the flip of the coin after 10 steps shows the following outcomes: $HTHTTTHHHT$. What is your current position?

Solution.
We have the following chart

Time	1	2	3	4	5	6	7	8	9	10
Coin	H	T	H	T	T	T	H	H	H	T
Step	1	-1	1	-1	-1	-1	1	1	1	-1
Position	1	0	1	0	-1	-2	-1	0	1	0

Thus, you are at the initial starting point ■

Consider the case of n steps. For the ith step we define the random variable Y_i by

$$Y_i = \begin{cases} +1 & \text{if coin displays head} \\ -1 & \text{if coin displays tail} \end{cases}$$

We define the cumulative total to be the sum

$$Z_n = \sum_{i=1}^{n} Y_i.$$

So Z_n gives your position from the starting point after n steps of the walk. The random walk model states that the larger the number of steps, the likelier it is that we will be farther away from where we started (either from the left or the right of the starting point).

Example 24.2
Suppose that $Z_n = 5$. What are the possible values for Z_{n+1}?

Solution.
If $Z_n = 5$, then $Z_{n+1} = 4$ or 6 with probability of $\frac{1}{2}$ each ■

Example 24.3
You flip a coin five times and you get the following outcomes: Head, Head, Tail, Head, Tail. What is the Value of Z_5?

Solution.
We have $Y_1 = 1, Y_2 = 1, Y_3 = -1, Y_4 = 1$, and $Y_5 = -1$. Thus, $Z_5 = 1 + 1 - 1 + 1 - 1 = 1$ ■

How does the random walk model relate to asset price movements? In "efficient markets", an asset price should reflect all available information. Thus, in response to new information, the asset price should move up or down with equal probability, as with the coin flip. The asset price after a period must be the initial price plus the cumulative up and down movements during the period which are the results of new information.

Modeling Stock Prices as a Random Walk
The random model described above is a bad model for stock price movements. We name three problems with this model:
(I) Negative stock prices are possible: If by chance we get enough cumulative down movements, the stock price becomes negative, something that does not happen. A stock price is either positive or zero (case of a bankrupt firm).
(II) The magnitude of the move should depend upon how quickly the coin

flips occur and the level of the stock price.
(III) The expected return on the stock is always zero. The stock on average should have a positive return.

The binomial model is a variant of the the random model that solves all of the above problems. The binomial model assumes that continuously compounded returns[1] can be modeled by a random walk.
Recall that the binomial model for stock pricing is given by

$$S_{t,t+h} = S_t e^{(r-\delta)h \pm \sigma\sqrt{h}}.$$

Taking logs, we obtain

$$r_{t,t+h} = \ln\left(\frac{S_{t,t+h}}{S_t}\right) = (r-\delta)h \pm \sigma\sqrt{h} \tag{24.1}$$

That is, the continuously compounded returns consist of two parts, one of which is certain $(r-\delta)h$, and the other of which is uncertain and generates the up and down stock moves $\pm\sigma\sqrt{h}$. Thus, the binomial model is a particular way to model the continuously compounded return.
Equation (24.1) solves the three problems mentioned above:
(I) The stock price cannot be negative since the price at the beginning of the period is multiplied by the exponential function $e^{(r-\delta)h\pm\sigma\sqrt{h}} > 0$.
(II) As stock price moves occur more frequently, h gets smaller, therefore the up move $\sigma\sqrt{h}$ and the down move $-\sigma\sqrt{h}$ get smaller. Thus, the magnitude depends upon the frequency of moves. Also for a given h, the percentage price change is the same. Hence, the moves are proportional to the stock price.
(III) There is a $(r-\delta)h$ term so we can choose the probability of an up move, so we can guarantee that the expected return on the stock is positive.

Lognormality and the Binomial Model

What is a lognormal distribution[2]? In probability theory, a random variable X is said to have the **lognormal distribution,** with parameters μ and σ, if $\ln X$ has the normal distribution with mean μ and standard deviation σ. Equivalently,

$$X = e^Y$$

[1]See Section 16 for a discusssion of the continuously compounded returns.
[2]See Section 47 for a further discussion of lognormal distributions.

where Y is normally distributed with mean μ and standard deviation σ.
It is known that the continuously compounded returns at the nodes of a binomial tree have a normal distribution[3]. It follows that the stock prices in the binomial model tree approximate a lognormal distribution. Figure 24.1 compares the probability distribution for a 25-period binomial tree (assuming that the initial stock price is \$100) with the corresponding lognormal distribution.

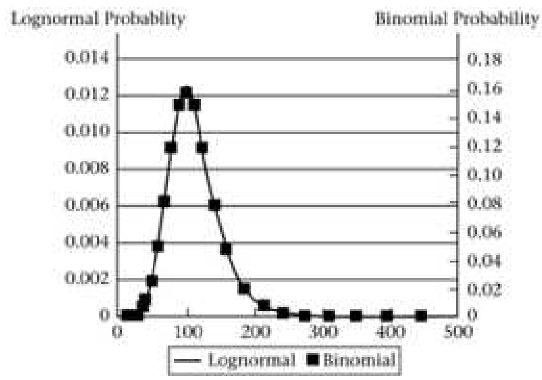
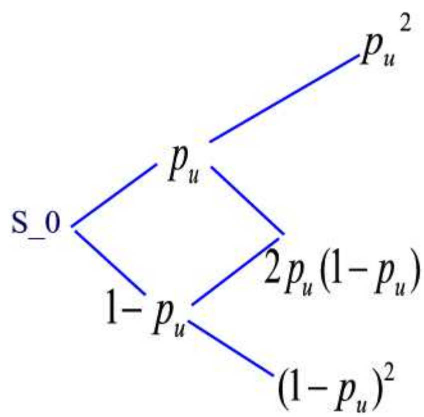


Figure 24.1

The binomial model implicitly assigns probabilities to the various nodes. Let p_u be the risk-neutral probability of an upward movement in the stock's price. For a binomial tree with two periods we find the following tree of probabilities



Example 24.4

[3]See Section 48

Find the probability assigned to the node *uud* in a three-period binomial tree.

Solution.
There are three paths to reach the node: uud, udu, and duu. At each node the probability is $p_u^2(1-p_u)$. Thus, the answer is $3p_u^2(1-p_u)$ ■

Remark 24.1
We have seen that the binomial model requires that the volatility be constant, large stock price movements do not suddenly occur, and the periodic stock returns are independent of each other. These assumptions are not considered realistic.

Practice Problems

Problem 24.1
Find the expected value and the standard deviation of the random variable Y_i.

Problem 24.2
Find $\text{Var}(Z_n)$.

Problem 24.3
Show that $Z_{n+1} = Z_n + Y_{n+1}$.

Problem 24.4
Let S_n represent the price of the stock on day n with S_0 representing the initial stock price. Find a relation between S_n and Y_n.

Problem 24.5
Suppose that the annualized standard deviation of returns on a stock is 0.67. What is the standard deviation of returns on the same stock after 12 years?

Problem 24.6
The standard deviation of returns on a stock over 10 years is 0.02. The standard deviation of returns on the same stock over Z years is 0.15. Find Z.

Problem 24.7
The standard deviation of returns on a stock over 10 years is 0.02; the annual continuously-compounded interest rate is 0.03, and the stock pays dividends at an annual continuously-compounded rate of 0.01. The stock price is currently \$120/share. If the stock price increases in 10 years, what will it be?

Problem 24.8
A coin was flipped 13 times, and you know that $Z_{12} = 6, Y_{12} = -1, Y_{11} = -1, Y_{10} = 1$, and $Y_9 = 1$. Find Z_8.

Problem 24.9
Construct a three-period binomial tree depicting stock price paths, along with risk-neutral probabilities of reaching the various terminal prices.

Problem 24.10

For an $n-$period binomial tree, the probability for reaching the i^{th} node (where $0 \leq i \leq n$) from the top in the terminal prices column (i.e. the last column in the tree) is given by the formula

$$p_u^{n-i}(1-p_u)^i \frac{n!}{(n-i)!i!}.$$

Use a 15-period binomial tree to model the price movements of a certain stock. For each time period, the risk-neutral probability of an upward movement in the stock price is 0.54. Find the probability that stock price will be at the 8th node of the binomial tree at the end of 15 periods.

25 Alternative Binomial Trees

Up to this point, a binomial tree is constructed based on the formulas

$$u = e^{(r-\delta)h+\sigma\sqrt{h}}$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}}$$

$$p_u = \frac{e^{(r-\delta)h} - d}{u-d}.$$

In this section, we discuss two more additional ways for constructing binomial trees that approximate a lognormal distribution.

The Cox-Ross-Rubinstein Binomial Tree
The Cox-Ross-Rubinstein binomial tree is constructed based on the formulas

$$u = e^{\sigma\sqrt{h}}$$

$$d = e^{-\sigma\sqrt{h}}$$

$$p_u = \frac{e^{(r-\delta)h} - d}{u-d}.$$

The CRR approach is often used in practice. However, the approach breaks down for any choice of h and σ such that $e^{rh} > e^{\sigma\sqrt{h}}$. In practice, h is usually small so that such a problem does not occur.

Example 25.1
The current price of a stock is \$1230. The price volatility 30%. Use a CRR three-period binomial tree to find the price of the stock at the node *uud*. A period consists of two months.

Solution.
We have

$$u = e^{\sigma\sqrt{h}} = e^{0.30\sqrt{\frac{1}{6}}} = 1.1303 \text{ and } d = 1.1303^{-1} = 0.8847.$$

Thus, the price of the stock at node *uud* is $u^2dS = 1.1303^2 \times 0.8847 \times 1230 =$ \$1390.24 ■

Example 25.2
During 54 periods in a binomial model, the stock price of GS LLC has gone up 33 times and gone down 21 times at the end of which the price of the stock is \$32/share. The price volatility is 0.2, and one time period in the binomial model is 6 months. Using a Cox-Rubinstein binomial tree, calculate the original price of the stock.

Solution.
We are asked to find S such that $u^{33}d^{21}S = 32$. Since $ud = 1$, we just need to solve the equation $u^{12}S = 32$ or $S = \frac{32}{u^{12}} = \frac{32}{e^{12\times 0.2\times\sqrt{0.5}}} = \$5.86$ ■

Example 25.3
You are given the following information about a 1-year European call on a stock:
- The current price of the stock is \$38.
- The strike price is \$40.
- The stock price volatility is 30%
- The continuously compounded risk-free rate is 7%.
- The stock pays no dividends.

Use a one-period CRR binomial tree to find the current price of the call.

Solution.
We have
$$u = e^{\sigma\sqrt{h}} = e^{0.30\sqrt{1}} = 1.3498$$
and
$$d = e^{-\sigma\sqrt{h}} = e^{-0.30\sqrt{1}} = 0.7408.$$
The risk-neutral probability is
$$p_u = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{0.07} - 0.7408}{1.3498 - 0.7408} = 0.5447.$$
Now, $C_u = 1.3498 \times 38 - 40 = 11.2924$ and $C_d = 0$. Thus,
$$C = e^{-rh}[p_u C_u + (1 - p_u)C_d] = e^{-0.07} \times 0.5447 \times 11.2924 = \$5.735 \blacksquare$$

The Jarrow and Rudd (lognormal) Binomial Tree
This tree is constructed based on the formulas
$$u = e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}}$$

$$d = e^{(r-\delta-0.5\sigma^2)h-\sigma\sqrt{h}}$$
$$p_u = \frac{e^{(r-\delta)h} - d}{u-d}.$$

Example 25.4
The stock prices of GS LLC can be modeled via a lognormal tree with 6 years as one time period. The annual continuously compounded risk-free interest rate is 0.08, and the stock pays dividends with an annual continuously compounded yield of 0.01. The prepaid forward price volatility is 0.43. The stock price is currently $454 per share. Find the stock price in 72 years if it goes up 7 times and down 5 times.

Solution.
We want to find u^7d^5S. But $u = e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}} = e^{(0.08-0.01-0.5\times0.43^2)\times6+0.43\sqrt{6}} = 2.5057$ and $d = e^{(r-\delta-0.5\sigma^2)h-\sigma\sqrt{h}} = e^{(0.08-0.01-0.5\times0.43^2)\times6-0.43\sqrt{6}} = 0.3048$. Thus, $u^7d^5S = 2.5057^7 \times 0.3048^5 \times 454 = \$741.19$ ■

Example 25.5
You are given the following information about a 1-year European call on a stock:
- The current price of the stock is $38.
- The strike price is $40.
- The stock price volatility is 30%
- The continuously compounded risk-free rate is 7%.
- The stock pays no dividends.

Use a one-period lognormal binomial tree to find the current price of the call.

Solution.
We have
$$u = e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}} = e^{(0.07-0.5(0.30)^2)+0.30} = 1.3840$$
and
$$d = e^{(r-\delta-0.5\sigma^2)h-\sigma\sqrt{h}} = e^{(0.07-0.5(0.30)^2)-0.30} = 0.7596$$
The risk-neutral probability is
$$p_u = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{0.07} - 0.7596}{1.3840 - 0.7596} = 0.5011.$$
Now, $C_u = 1.3840 \times 38 - 40 = 12.592$ and $C_d = 0$. Thus,
$$C = e^{-rh}[p_uC_u + (1-p_u)C_d] = e^{-0.07} \times 0.5011 \times 12.592 = \$5.8833$$ ■

Practice Problems

Problem 25.1
During the course of 34 time periods of 1 year each in the binomial model, two stocks that were identically priced at the beginning diverged in their prices. Stock A went up consistently, while Stock B went down consistently. The volatility of both stocks' prices is 0.09. Using CRR binomial tree, find the ratio of the price of Stock A to the price of Stock B.

Problem 25.2
Suppose that, in a CRR tree, you are given $h = 1$ and $\frac{u}{d} = 1.8221$. Find σ.

Problem 25.3
Consider a 1-year European call on a nondividend paying stock. The strike price is \$40. The current price of the stock is \$38 and the stock's price volatility is 30%. Suppose that the true probability of an upward movement is 0.661. Using a one-period CRR binomial model, find the expected rate of return α.

Problem 25.4
You are given the following information about a 1-year European put on a stock:
- The current price of the stock is \$38.
- The strike price is \$40.
- The stock price volatility is 30%
- The continuously compounded risk-free rate is 7%.
- The stock pays no dividends.

Use a one-period CRR binomial tree to find the current price of the put.

Problem 25.5
Given the following information about a three-month European call on a stock:
- The current price of the stock is \$100.
- The strike price is \$95.
- The prepaid forward price volatility is 30%
- The continuously compounded risk-free rate is 8%.
- The stock dividend yield is 5%.
- The three-month expected discount rate is 21.53%.

Use a one-period CRR binomial tree to find the true probability for an upward movement.

Problem 25.6
You are given the following information about a 1-year European put on a stock:
• The current price of the stock is \$38.
• The strike price is \$40.
• The stock price volatility is 30%
• The continuously compounded risk-free rate is 7%.
• The stock pays no dividends.
Use a one-period lognormal binomial tree to find the current price of the put.

Problem 25.7
You are given the following information about a 3-month European call on a stock:
• The current price of the stock is \$100.
• The strike price is \$95.
• The prepaid forward price volatility is 30%
• The continuously compounded risk-free rate is 8%.
• The continuously compounded dividend rate is 5%.
• The expected 3-month discount rate is 52.81%
Use a one-period lognormal binomial tree to find
(a) the true probability of an upward movement
(b) the expected rate of return α.

Problem 25.8
You are given the following information about a 6-month European call on a stock:
• The current price of the stock is \$58.
• The strike price is \$60.
• The prepaid forward price volatility is 27%
• The continuously compounded risk-free rate is 12%.
• The continuously compounded dividend rate is 4.5%.
Use the one-period lognormal binomial tree to calculate Δ, the number of shares in the replicating portfolio.

Problem 25.9
You are given the following information about a 6-month European call on a stock:
• The current price of the stock is \$100.

• The strike price is \$95.
• The stock price volatility is 30%
• The continuously compounded risk-free rate is 8%.
• The stock pays dividends at the continuously compound yield rate 5%.
Use the one-period CRR model to find the current price of the call.

Problem 25.10
You are given the following information about a 1-year European call on a stock:
• The current price of the stock is \$38.
• The strike price is \$40.
• The stock price volatility is 30%
• The continuously compounded risk-free rate is 7%.
• The stock pays no dividends.
Use a one-period lognormal binomial tree to find the current price of the call.

26 Estimating (Historical) Volatility

Recall that volatility is defined to be the annualized standard deviation of the continuously compounded stock returns. This parameter cannot usually be observed directly. In this section we examine a procedure for estimating volatility.

Suppose we are given a periodic data about the price of a stock. As usual, we let S_t denote the price of the stock at time t and S_{t+h} the price at time $t+h$. The continuously compounded return for the interval $[t, t+h]$ is

$$r_{t,t+h} = \ln\left(\frac{S_{t+h}}{S_t}\right).$$

Next, we describe how to estimate volatility, known as **historical volatility**. Suppose that we have data for n periods where each period is of length h. We compute the (historical) stock returns: $r_{t,t+h}, r_{t+h,t+2h}, \cdots, r_{t+(n-1)h,t+nh}$. We assume that the returns are independent and identically distributed. The next step is to calculate the average of the returns

$$\overline{r} = \frac{\sum_{i=1}^{n} r_{t+(i-1)h,t+ih}}{n}.$$

From probability theory, an estimate of the periodic standard deviation is given by the formula

$$\sigma_h = \sqrt{\frac{\sum_{i=1}^{n}(r_{t+(i-1)h,t+ih} - \overline{r})^2}{n-1}}.$$

Volatility, which is the annualized standard deviation, is then found using the formula

$$\sigma = \frac{\sigma_h}{\sqrt{h}}.$$

Volatility computed from historical stock returns is referred to as **historical volatility.** We illustrate the above procedure in the next example.

Example 26.1

The table below lists the closing price of a stock for 6 weeks. Estimate the annualized standard deviation of the stock's prices.

Date	Price
03/05/03	85
03/12/03	81
03/19/03	87
03/26/03	80
04/02/03	86
04/09/03	93

Solution.
The table below shows the historical weekly returns, the average of the returns, the squared deviations, and the sum of the squared deviations.

Date	Price	$r_t = \ln S_t/S_{t-1}$	$(r_t - \overline{r})^2$
03/05/03	85		
03/12/03	81	-0.048202	0.004381
03/19/03	87	0.071459	0.002859
03/26/03	80	-0.083881	0.010378
04/02/03	86	0.072321	0.002952
04/09/03	93	0.078252	0.003632
		$\overline{r} = 0.017990$	$\sum_{i=1}^{5}(r_i - \overline{r})^2 = 0.024201$

The estimate of the standard deviation of the weekly returns is

$$\sigma_{\frac{1}{52}} = \sqrt{\frac{0.024201}{4}} = 0.07784.$$

The historical volatility is

$$\sigma = \frac{\sigma_{\frac{1}{52}}}{\sqrt{\frac{1}{52}}} = 0.561 \blacksquare$$

Summarizing, the volatility needed for the binomial model can be estimated by computing the standard deviation of periodic continuously compounded returns and annualizing the result. Once the annualized standard deviation is found, we can use it to construct binomial trees. We multiply σ by $\sqrt{h}$ to adapt the annual standard deviation to any size binomial step.

Practice Problems

Problem 26.1
Weekly prices of a stock are given from 04/02/03 to 05/28/03. Estimate the value of σ.

Date	Price
04/02/03	77.73
04/09/03	75.18
04/16/03	82.00
04/23/03	81.55
04/30/03	81.46
05/07/03	78.71
05/14/03	82.88
05/21/03	85.75
05/28/03	84.90

Problem 26.2
Monthly prices of a stock are shown in the table below. Estimate the volatility σ.

Month	Price
1	100
2	110
3	112
4	105
5	113

Problem 26.3
The data of a stock's price for 7 months are given. Estimate the volatility σ.

Month	Price
1	85
2	81
3	87
4	93
5	102
6	104
7	100

Problem 26.4 ‡

You are to estimate a nondividend-paying stock's annualized volatility using its prices in the past nine months.

Month	Stock Price per share
1	80
2	64
3	80
4	64
5	80
6	100
7	80
8	64
9	80

Calculate the historical volatility for this stock over the period.

Problem 26.5

Suppose that the stock prices for six weeks were given and the estimate of the standard deviation of the weekly returns is 0.07784. Find the value of the sum of the squared deviations.

Problem 26.6

Stock prices for n months were given. Suppose that the estimated standard deviation for monthly returns is 0.059873 and that the sum of the squared deviations is 0.017924. Determine the value of n.

Problem 26.7

Suppose that the historical volatility is 0.82636 and the periodic standard deviation is 0.23855. Find the length of a period.

Problem 26.8

Monthly prices of a stock are shown in the table below.

Month	Price
1	100
2	110
3	112
4	105
5	X

Suppose that the average of the returns is 0.042185. Determine the value of X.

Problem 26.9
The data of a stock's price for 7 months are given.

Month	Price
1	85
2	81
3	87
4	93
5	102
6	X
7	$X - 4$

Suppose that the average of the returns is 0.027086. Determine the value of X.

The Black-Scholes Model

In this chapter we present the Black-Scholes formula for pricing European options and discuss various topics related to it.

27 The Black-Scholes Formulas for European Options

In this section we present the Black-Scholes formulas for European calls and puts options. Before doing that we examine a function that will appear in the formula.

The Cumulative Normal Distribution Function
Let X denote the standard normal random variable, that is, the distribution of X is normal with mean 0 and standard deviation 1. The probability density function[1] of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Let $N(x)$ denote the cumulative normal distribution function. That is,

$$N(x) = Pr(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt.$$

$N(x)$ is the area under the standard normal curve to the left of x. The SOA asks that you compute $N(x)$ using the standard normal distribution tables. However, in this book, we will find the value of $N(x)$ using Excel's NormSdist.

Example 27.1
Show that $N(x) + N(-x) = 1$.

Solution.
The function $N'(x) = f(x)$ is an even function so that $N'(-x) = N'(x)$ for all real numbers x. Integrating both sides we obtain $N(x) = -N(-x) + C$. Letting $x = 0$ we find $C = 2N(0) = 2(0.5) = 1$. Hence, $N(x) + N(-x) = 1$ ■

Before examining the Black-Scholes formulas, we list the following assumptions that were required in the derivation of the formulas:
• Continuously compounded returns on the stock are normally distributed and independent over time.

[1] See Section 46 for a further discussion of normal distributions.

• The volatility of continuously compounded returns is known and constant.
• Future dividends are known, either as a dollar amount or as a fixed dividend yield.
• The risk-free interest rate is known and constant.
• There are no transaction costs or taxes.
• It is possible to short-sell costlessly and to borrow at the risk-free rate.

Black-Scholes Formula for European Calls
For a European call on a stock that pays continuous dividends, the price is given by

$$C(S_t, K, \sigma, r, T-t, \delta) = S_t e^{-\delta(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2)$$

where

$$d_1 = \frac{\ln(S_t/K) + (r - \delta + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = \frac{\ln(S_t/K) + (r - \delta - 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.$$

As with the binomial model, the six inputs in the above formula are
• S_t is the stock price at time t.
• K is the strike price of the option.
• σ is the annual standard deviation of the rate of return on the stock or the prepaid forward price volatility.
• r is the annual continuously compounded risk-free interest rate.
• T is the time to expiration.
• δ is the annual continuously compounded dividend yield.

Example 27.2
The stock of XYZ currently sells for \$41 per share. The annual stock price volatility is 0.3, and the annual continuously compounded risk-free interest rate is 0.08. The stock pays no dividends.
(a) Find the values of $N(d_1)$ and $N(d_2)$ in the Black-Scholes formula for the price of a call option on the stock with strike price \$40 and time to expiration of 3 months.
(b) Find the Black-Scholes price of the call option.

Solution.
(a) With $t = 0$ we have

$$
\begin{aligned}
N(d_1) =& N\left(\frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}\right)\\
=& N\left(\frac{\ln\left(\frac{41}{40}\right) + \left(0.08 - 0 + \frac{0.3^2}{2}\right) \times 0.25}{0.3\sqrt{0.25}}\right)\\
=& N(0.3729507506) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.3729507506} e^{-\frac{x^2}{2}}dx\\
=& 0.645407
\end{aligned}
$$

and

$$N(d_2) = N(0.2229507506) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.2229507506} e^{-\frac{x^2}{2}}dx = 0.588213.$$

(b) The price of the call is

$$C(41, 40, 0.3, 0.08, 0.25, 0) = 41\times e^{-0\times 0.25}\times 0.645407 - 40e^{-0.08\times 0.25}\times 0.588213 = \$3.3993 \blacksquare$$

Black-Scholes Formula for European Puts
For a European put on a stock that pays continuous dividends, the price is given by

$$P(S_t, K, \sigma, r, T - t, \delta) = Ke^{-r(T-t)}N(-d_2) - S_te^{-\delta(T-t)}N(-d_1)$$

Proposition 27.1
The Black-Scholes formulas for calls and puts satisfy the put-call parity

$$C(S_t, K, \sigma, r, T - t, \delta) - P(S_t, K, \sigma, r, T - t, \delta) = S_te^{-\delta(T-t)} - Ke^{-r(T-t)}.$$

Proof.
We have

$$
\begin{aligned}
C(S_t, K, \sigma, r, T - t, \delta) - P(S_t, K, \sigma, r, T - t, \delta) =& S_te^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) - [Ke^{-r(T-}\\
&- S_te^{-\delta(T-t)}N(-d_1)]\\
=& S_te^{-\delta(T-t)}[(N(d_1) + N(-d_1)]\\
&- Ke^{-r(T-t)}[N(d_2) + N(-d_2)]\\
=& S_te^{-\delta(T-t)} - Ke^{-r(T-t)} \blacksquare
\end{aligned}
$$

Example 27.3
Using the same inputs as in Example 27.2, find the Black-Scholes price of a put option on the stock with strike price \$40 and time to expiration of 3 months.

Solution.
Using the put-call parity we find

$$P(41, 40, 0.3, 0.08, 0.25, 0) = 3.3993 + 40e^{-0.08\times 0.25} - 41e^{-0\times 0.25} = \$1.6073 \blacksquare$$

Example 27.4 ‡
You are asked to determine the price of a European put option on a stock. Assuming the Black-Scholes framework holds, you are given:
(i) The stock price is \$100.
(ii) The put option will expire in 6 months.
(iii) The strike price is \$98.
(iv) The continuously compounded risk-free interest rate is $r = 0.055$.
(v) $\delta = 0.01$
(vi) $\sigma = 0.50$
Calculate the price of this put option.

Solution.
With $t = 0$ we have

$$\begin{aligned} N(-d_1) =& N\left(\frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}\right) \\ =& N\left(-\frac{\ln\left(\frac{100}{98}\right) + \left(0.055 - 0.01 + \frac{0.5^2}{2}\right) \times 0.5}{0.5\sqrt{0.5}}\right) \\ =& N(-0.297558191) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-0.297558191} e^{-\frac{x^2}{2}}dx \\ =& 0.38302 \end{aligned}$$

and

$$N(-d_2) = N(-d_1 + \sigma\sqrt{T}) = N(0.0559951996) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.0559951996} e^{-\frac{x^2}{2}}dx = 0.522327.$$

The price of the put is

$$P(100, 98, 0.5, 0.055, 0.5, 0.01) = 98\times e^{-0.055\times 0.5}\times 0.522327 - 100e^{-0.01\times 0.5}\times 0.38302 = \$11.688 \blacksquare$$

Practice Problems

Problem 27.1
A European call option on XYZ stock has the following specifications: Strike price = \$45, current stock price = \$46, time to expiration = 3 months, annual continuously compounded interest rate = 0.08, dividend yield = 0.02, volatility=0.35. Calculate the Black-Scholes price of the option.

Problem 27.2
Using the same inputs as the previous problem, calculate the Black-Scholes price of a put option.

Problem 27.3
The stock of GS Co. currently sells for \$1500 per share. The prepaid forward price volatility is 0.2, and the annual continuously compounded risk-free interest rate is 0.05. The stock's annual continuously compounded dividend yield is 0.03. Within the Black-Scholes formula for the price of a put option on GS Co. stock with strike price \$1600 and time to expiration of 3 years, find the value of $N(-d_2)$.

Problem 27.4
You are asked to determine the price of a European call option on a stock. Assuming the Black-Scholes framework holds, you are given:
(i) The stock price is \$40.
(ii) The call option will expire in one year.
(iii) The strike price is \$35.
(iv) The continuously compounded risk-free interest rate is $r = 0.10$.
(v) $\delta = 0.02$
(vi) $\sigma = 0.30$
Calculate the price of this call option.

Problem 27.5
Show that, for $t = 0$, we have $d_1 = \frac{\ln\left(\frac{Se^{-\delta T}}{Ke^{-rT}}\right)+0.5\sigma^2 T}{\sigma\sqrt{T}}$.

Problem 27.6 ‡
Assume the Black-Scholes framework.
Eight months ago, an investor borrowed money at the risk-free interest rate to purchase a one-year 75-strike European call option on a nondividend-paying

stock. At that time, the price of the call option was 8.
Today, the stock price is 85. You are given:
(i) The continuously compounded risk-free rate interest rate is 5%.
(ii) The stock's volatility is 26%.
Find the current price of the call.

Problem 27.7 ‡
For a European call option on a stock within the Black-Scholes framework, you are given:
(i) The stock price is \$85.
(ii) The strike price is \$80.
(iii) The call option will expire in one year.
(iv) The continuously compound risk-free interest rate is 5.5%.
(v) $\sigma = 0.50$.
(vi) The stock pays no dividends.
Calculate the price of the call.

Problem 27.8 ‡
Assume the Black-Scholes framework. For a dividend-paying stock and a European option on the stock, you are given the following information:
- The current stock price is \$58.96.
- The strike price of the option is \$60.00.
- The expected annual return on the stock is 10%.
- The volatility is 20%.
- The continuously compounded risk-free rate is 6%.
- The continuously dividend yield is 5%.
- The expiration time is three months.

Calculate the price of the call.

Problem 27.9 ‡
Assume the Black-Scholes framework. For a nondividend-paying stock and a European option on the stock, you are given the following information:
- The current stock price is \$9.67.
- The strike price of the option is \$8.75.
- The volatility is 40%.
- The continuously compounded risk-free rate is 8%.
- The expiration time is three months.

Calculate the price of the put.

28 Applying the Black-Scholes Formula To Other Assets

In the previous section the options under consideration had stocks with continuous dividends as underlying assets. In this section we want to adapt the Black-Scholes formula to options with underlying assets consisting of stocks with discrete dividends, futures contracts, and currency contracts.
According to Problem 27.5, we can write

$$d_1 = \frac{\ln\left(\frac{Se^{-\delta T}}{Ke^{-rT}}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}}.$$

But the term $Se^{-\delta T}$ is the prepaid forward price for the stock and Ke^{-rT} is the prepaid forward price for the strike. Thus,

$$d_1 = \frac{\ln\left(\frac{F^P_{0,T}(S)}{F^P_{0,T}(K)}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}}.$$

Now, the Black-Scholes formulas can be written in terms of prepaid forward prices: for a call we have

$$C(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) = F^P_{0,T}(S)N(d_1) - F^P_{0,T}(K)N(d_2)$$

and for a put we have

$$P(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) = F^P_{0,T}(K)N(-d_2) - F^P_{0,T}(S)N(-d_1).$$

These formulas when written in terms of prepaid forward prices are useful when pricing options with underlying assets other than stocks with continuous dividends, namely, stocks with discrete dividends, futures, or currencies.

Example 28.1
Show that

$$P(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) = C(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) + F^P_{0,T}(K) - F^P_{0,T}(S).$$

Solution.
By the put-call parity we have

$$C(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) - P(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T) = Se^{-\delta T} - Ke^{-rT} = F^P_{0,T}(S) - F^P_{0,T}(K).$$

Now the result follows by solving this equation for $P(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, T)$ ■

Example 28.2 ‡
Let $S(t)$ denote the price at time t of a stock that pays no dividends. The Black-Scholes framework holds. Consider a European call option with exercise date $T, T > 0$, and exercise price $S(0)e^{rT}$, where r is the continuously compounded risk-free interest rate. You are given:
(i) $S(0) = \$100$
(ii) $T = 10$
(iii) $\mathrm{Var}[\ln S(t)] = 0.4t, t > 0$.
Determine the price of the call option.

Solution.
The variance over the interval $[0, t]$ is given by

$$\sigma_t^2 = \mathrm{Var}[\ln(S_t/S_0)] = \mathrm{Var}[\ln S_t] - \mathrm{Var}[\ln 100] = \mathrm{Var}[\ln S_t] = 0.4t.$$

Thus,

$$\sigma = \sigma_1 = \sqrt{0.4}.$$

We also have

$$d_1 = \frac{\ln\left(\frac{F_{0,T}^P(S)}{F_{0,T}^P(K)}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln(100/100) + 0.5 \times 0.4 \times 10}{2} = 1$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = 1 - 2 = -1.$$

Hence,

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1} e^{-\frac{x^2}{2}} dx = 0.841345$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1} e^{-\frac{x^2}{2}} dx = 0.158655.$$

The Black-Scholes price of the call option is

$$C = F_{0,T}^P(S)N(d_1) - F_{0,T}^P(K)N(d_2) = 100 \times 0.841345 - 100 \times 0.158655 = \$68.269 \blacksquare$$

• If the underlying asset is a stock with discrete dividends, then the prepaid forward price is $F_{0,T}^P(S) = S_0 - PV_{0,T}(\mathrm{Div})$.

Example 28.3
Consider a stock that pays dividends of \$40 in two years and \$32 in six years. The stock currently trades for \$221 per share. The annual continuously compounded risk-free interest rate is 5%, and the annual price volatility relevant for the Black-Scholes equation is 30%. Find the Black-Scholes price of a call option with strike price \$250 and expiration time of 8 years.

Solution.
The prepaid forward price of the stock is

$$F_{0,T}^P(S) = S_0 - PV_{0,T}(\text{Div}) = 221 - 40e^{-0.05\times 2} - 32e^{-0.05\times 6} = \$161.1003.$$

The prepaid forward on the strike is

$$F_{0,T}^P(K) = Ke^{-rT} = 250e^{-0.05\times 8} = \$167.58.$$

The values of d_1 and d_2 are

$$d_1 = \frac{\ln\left(\frac{F_{0,T}^P(S)}{F_{0,T}^P(K)}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln(161.1003/167.58) + 0.5\times 0.3^2\times 8}{0.3\sqrt{8}} = 0.3778$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = 0.3778 - 0.3\sqrt{8} = -0.4707.$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.3778} e^{-\frac{x^2}{2}}dx = 0.6472$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-0.4707} e^{-\frac{x^2}{2}}dx = 0.3189.$$

The Black-Scholes price of the call option is

$$\begin{aligned} C(F_{0,T}^P(S), F_{0,T}^P(K), \sigma, T) =& F_{0,T}^P(S)N(d_1) - F_{0,T}^P(K)N(d_2) \\ =& 161.1003\times 0.6472 - 167.58\times 0.3189 = \$50.8261 \blacksquare \end{aligned}$$

• If the underlying asset is a foreign currency, then $F_{0,T}^P(x) = x_0e^{-r_fT}$ where r_f is the foreign currency interest rate and x_0 is the current exchange rate (expressed as domestic currency per unit of foreign currency). In this case, the Black-Scholes price of a call option is given by

$$C(x_0, K, \sigma, r, T, r_f) = x_0e^{-r_fT}N(d_1) - Ke^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln(x_0/K) + (r - r_f + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}.$$

The above formula is also known as the **Garman-Kohlhagan** formula.
The Black-Scholes for a European currency put option is given by

$$P(x_0, K, \sigma, r, T, r_f) = Ke^{-rT}N(-d_2) - x_0e^{-r_fT}N(-d_1).$$

We can also get the put formula via put-call parity:

$$P(x_0, K, \sigma, r, T, r_f) = C(x_0, K, \sigma, r, T, r_f) + Ke^{-rT} - x_0e^{-r_fT}.$$

The options prices are in the domestic currency.

Example 28.4
One euro is currently trading for \$0.92. The dollar-denominated continuously compounded interest rate is 6% and the euro-denominated continuously compounded interest rate is 3.2%. Volatility is 10%.
(a) Find the Black-Scholes price of a 1-year dollar-denominated euro call with strike price of \$0.9/€.
(b) Find the Black-Scholes price of a 1-year dollar-denominated euro put with strike price of \$0.9/€.

Solution.
We first find d_1 and d_2. We have

$$d_1 = \frac{\ln(x_0/K) + (r - r_f + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(0.92/0.9) + (0.06 - 0.032 + 0.5 \times 0.10^2) \times 1}{0.10\sqrt{1}} = 0.549789$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = 0.549789 - 0.10 = 0.449789.$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.549789} e^{-\frac{x^2}{2}}dx = 0.708768$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.449789} e^{-\frac{x^2}{2}}dx = 0.673569.$$

(a) The Black-Scholes price of the call is

$$C = 0.92e^{-0.032} \times 0.708768 - 0.9e^{-0.06} \times 0.673569 = \$0.0606.$$

(b) The Black-Scholes price of the put is

$$P = C + Ke^{-rT} - x_0e^{-r_fT} = 0.0606 + 0.9e^{-0.06} - 0.92e^{-0.032} = \$0.0172 \blacksquare$$

• The prepaid forward price for a futures contract is just the present value of the futures price. Letting F denote the futures price, we have $F^P_{0,T}(F) = Fe^{-rT}$. The Black-Scholes formula for a call, also known as the **Black formula** is given by

$$C(F, K, \sigma, r, T, r) = Fe^{-rT}N(d_1) - Ke^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln(F/K) + 0.5\sigma^2T}{\sigma\sqrt{T}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}.$$

The put price is given by

$$P(F, K, \sigma, r, T, r) = Ke^{-rT}N(-d_2) - Fe^{-rT}N(-d_1)$$

which can be found via the put-call parity:

$$P(F, K, \sigma, r, T, r) = C(F, K, \sigma, r, T, r) + Ke^{-rT} - Fe^{-rT}.$$

Example 28.5
Futures contracts on natural gas currently trade for \$2.10 per MMBtu. The annual futures contract price volatility is 0.25, and the annual continuously compounded currency risk-free interest rate is 0.055.
(a) Find the Black-Scholes price of 1-year European call on natural gas futures contracts with strike price of \$2.10.
(b) Find the Black-Scholes price of 1-year European put on natural gas futures contracts with strike price of \$2.10.

Solution.
We first find d_1 and d_2. We have

$$d_1 = \frac{\ln(F/K) + 0.5\sigma^2T}{\sigma\sqrt{T}} = \frac{\ln(2.10/2.10) + 0.5 \times 0.25^2 \times 1}{0.25\sqrt{1}} = 0.125$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = 0.125 - 0.25 = -0.125.$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.125} e^{-\frac{x^2}{2}}\,dx = 0.549738$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-0.125} e^{-\frac{x^2}{2}}\,dx = 0.450262$$

(a) The price of the call is

$$C = Fe^{-rT}N(d_1) - Ke^{-rT}N(d_2) = 2.10e^{-0.055}\times 0.549738 - 2.10e^{-0.055}\times 0.450262 = \$0.19772.$$

(b) The price of the put is

$$P = C + Ke^{-rT} - Fe^{-rT} = C = \$0.19772 \blacksquare$$

Practice Problems

Problem 28.1 ‡

Which of the following is an assumption of the Black-Scholes option pricing model?
(A) Stock prices are normally distributed
(B) Stock price volatility is a constant
(C) Changes in stock price are lognormally distributed
(D) All transaction cost are included in stock returns
(E) The risk-free interest rate is a random variable.

Problem 28.2

Consider a stock that pays dividends of \$5 one month from now. The stock currently trades for \$46 per share. The annual continuously compounded risk-free interest rate is 8%, and the annual price volatility relevant for the Black-Scholes equation is 39.24%. Find the Black-Scholes price of a call option with strike price \$45 and expiration time of three months.

Problem 28.3

Consider a stock that pays dividends of \$40 in two years and \$32 in six years. The stock currently trades for \$221 per share. The annual continuously compounded risk-free interest rate is 5%, and the annual price volatility relevant for the Black-Scholes equation is 30%. Find the Black-Scholes price of a put option with strike price \$250 and expiration time of 8 years.

Problem 28.4

You are given the following information about a call option on a stock in the Black-Scholes framework:
• The annual continuously-compounded interest rate is 0.03.
• The annual price volatility is 0.03
• The current stock price is \$56
• The option's time to expiration is 2 years
• The price of the prepaid forward on the strike asset is \$42
• $d_1 = 0.4$
• The stock will pay a dividend in the amount of d dollars one year from today.
Find the size of the dividend.

Problem 28.5
Consider a stock that pays dividends of \$7 in 3 months and \$10 in 9 months. The stock currently trades for \$77 per share. The annual continuously compounded risk-free interest rate is 10%, and the annual price volatility relevant for the Black-Scholes equation is 27.43%. Find the Black-Scholes price of a put option with strike price \$73 and expiration time of 6 months.

Problem 28.6
One euro is currently trading for \$1.25. The dollar-denominated continuously compounded interest rate is 6% and the euro-denominated continuously compounded interest rate is 3.5%. Volatility relevant for the Black-Scholes equation is 11%.
(a) Find the Black-Scholes price of a 6-month dollar-denominated euro call with strike price of \$1.3/€.
(b) Find the Black-Scholes price of a 6-month dollar-denominated euro put with strike price of \$1.3/€.

Problem 28.7
One euro is currently trading for \$1.25. The dollar-denominated continuously compounded interest rate is 8% and the euro-denominated continuously compounded interest rate is 5%. Volatility relevant for the Black-Scholes equation is 10%. Find the Black-Scholes price of an at-the-money 1-year euro-denominated dollar put.

Problem 28.8
Futures contracts on superwidgets currently trade for \$444 per superwidget. The annual futures contract price volatility relevant for the Black-Scholes equation is 0.15, and the annual continuously compounded currency risk-free interest rate is 0.03.
(a) Find the Black-Scholes price of 2-year European call on superwidget futures contracts with strike price of \$454.
(b) Find the Black-Scholes price of 2-year European put on superwidget futures contracts with strike price of \$454.

Problem 28.9
The dividend yield in the Black-Scholes formula for stock option pricing is analogous to which of these variables in other related formulas?
(A) The risk-free interest rate in the Black-Scholes formula for stock option

pricing.
(B) The risk-free interest rate in the Black formula for futures contract option pricing.
(C) The domestic risk-free interest rate in the Garman-Kohlhagen formula for currency option pricing.
(D) The foreign risk-free interest rate in the Garman-Kohlhagen formula for currency option pricing.
(E) The volatility in the Black formula for futures contract option pricing.

Problem 28.10
A stock XYZ pays dividends at the continuously compounded rate of 5%. Currently the stock is trading for \$70. The continuously compounded risk-free interest rate is 9%. The volatility relevant for the Black-Scholes equation is 30%. Find the Black-Scholes of a European call on futures contracts on XYZ stock with strike price \$65 and expiration of six months.

Problem 28.11
A stock XYZ pays no dividends. Currently the stock is trading for \$100. The continuously compounded risk-free interest rate is 7%. The volatility relevant for the Black-Scholes equation is 35%. Find the Black-Scholes of a European put on futures contracts on XYZ stock with strike price \$105 and expiration of one year.

Problem 28.12 ‡
On January 1, 2007, the following currency information is given:
• Spot exchange rate: \$0.82/€
• Dollar interest rate= 5% compounded continuously
• Euro interest rate = 2.5% compounded continuously
• Exchange rate volatility relevant for the Black-Scholes equation = 0.10.
What is the price of 850 dollar-denominated euro call options with a strike exchange rate \$0.80/€that expire on January 1, 2008?

Problem 28.13 ‡
You are considering the purchase of 100 European call options on a stock, which pays dividends continuously at a rate proportional to its price. Assume that the Black- Scholes framework holds. You are given:
(i) The strike price is \$25.
(ii) The options expire in 3 months.

(iii) $\delta = 0.03$.
(iv) The stock is currently selling for \$20.
(v) The volatility relevant for the Black-Scholes equation $\sigma = 0.24$.
(vi) The continuously compounded risk-free interest rate is 5%.
Calculate the price of the block of 100 options.

Problem 28.14 ‡
For a six-month European put option on a stock, you are given:
(i) The strike price is \$50.00.
(ii) The current stock price is \$50.00.
(iii) The only dividend during this time period is \$1.50 to be paid in four months.
(iv) $\sigma_S = 0.30$
(v) The continuously compounded risk-free interest rate is 5%.
Under the Black-Scholes framework, calculate the price of the put option.

Problem 28.15 ‡
Consider a one-year 45-strike European put option on a stock S. You are given:
(i) The current stock price, $S(0)$, is 50.00.
(ii) The only dividend is 5.00 to be paid in nine months.
(iii) $\sigma_h^2 = \text{Var}[\ln F_{h,1}^P(S)] = 0.01h,\ 0 \leq h \leq 1$.
(iv) The continuously compounded risk-free interest rate is 12%.
Under the Black-Scholes framework, calculate the price of 100 units of the put option.

Problem 28.16 ‡
Company A is a U.S. international company, and Company B is a Japanese local company. Company A is negotiating with Company B to sell its operation in Tokyo to Company B. The deal will be settled in Japanese yen. To avoid a loss at the time when the deal is closed due to a sudden devaluation of yen relative to dollar, Company A has decided to buy at-the-money dollar-denominated yen put of the European type to hedge this risk.
You are given the following information:
(i) The deal will be closed 3 months from now.
(ii) The sale price of the Tokyo operation has been settled at 120 billion Japanese yen.
(iii) The continuously compounded risk-free interest rate in the U.S. is 3.5%.

(iv) The continuously compounded risk-free interest rate in Japan is 1.5%.
(v) The current exchange rate is 1 U.S. dollar = 120 Japanese yen.
(vi) The yen per dollar exchange rate and the dollar per yen exchange rate have the same daily volatility 0.261712%.
(vii) 1 year = 365 days; 3 months = $\frac{1}{4}$ year.
Assuming the Black-Scholes pricing framework, calculate Company A's option cost.

Problem 28.17 ‡
Assume the Black-Scholes framework. Consider a 9-month at-the-money European put option on a futures contract. You are given:
(i) The continuously compounded risk-free interest rate is 10%.
(ii) The strike price of the option is 20.
(iii) The price of the put option is 1.625.
If three months later the futures price is 17.7, what is the price of the put option at that time?

29 Option Greeks: Delta, Gamma, and Vega

In mathematical finance, the **Greeks** are the quantities representing the sensitivities of derivatives such as options to a change in underlying parameters on which the value of an instrument or portfolio of financial instruments is dependent. The name is used because the most common of these sensitivities are often denoted by Greek letters.
In practice, option values change on account of a number of factors: movements in the price of the underlying asset, passage of time, changes in volatility of the underlying asset, changes in the rate of interest, and changes in the dividend yield if the underlying asset pays dividends. There are formulas that measure the changes in the option price when only one parameter is being changed while leaving the remaining parameters fixed. Letters from the Greek alphabet are used to represent these derived measures:
• **Delta**(Δ) measures the change in the option price when the stock price increases by \$1.
• **Gamma** (Γ) measures the change in delta when the stock price increases by \$1.
• **Vega** (V) measures the change in the option price for an increase by 1% in volatility.
• **Theta**(θ) measures the change in the option price when time to maturity decreases by 1 day.
• **Rho** (ρ) measures the change in the option price when the risk-free interest rate increases by 1%.
• **Psi** (ψ) measures the change in the option price when the dividend yield increases by 1%.

Even though greek measures can be computed for options with any kind of underlying asset, we will focus our attention on stock options. We will be examining each greek measure in turn, for a purchased option. The Greek for a written option is opposite in sign to that for the same purchased option. In this section we will examine the first three Greek measures: Delta, gamma, and vega.
In what follows, the Black-Scholes value of a European call option at time t is given by the formula

$$C_t = Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2).$$

For a put option the formula is

$$P_t = Ke^{-r(T-t)}N(-d_2) - Se^{-\delta(T-t)}N(-d_1)$$

where

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

The Delta Measure

The option Greek Delta (Δ) measures the option price change when the stock price increases by \$1. More formally, the delta of an option is defined as the rate of change of the option value with respect to stock price:

$$\Delta = \frac{\partial V_{\text{option}}}{\partial S}$$

Proposition 29.1

We have

$$\Delta_{\text{Call}} = e^{-\delta(T-t)}N(d_1)$$

and

$$\Delta_{\text{Put}} = -e^{-\delta(T-t)}N(-d_1)$$

Proof.

Using Problem 29.1 we find

$$\begin{aligned}
\Delta_{\text{Call}} =& \frac{\partial C_t}{\partial S} = e^{-\delta(T-t)}N(d_1) + Se^{-\delta(T-t)}\frac{\partial N(d_1)}{\partial S} - Ke^{-r(T-t)}\frac{\partial N(d_2)}{\partial S}\\
=& e^{-\delta(T-t)}N(d_1) + Se^{-\delta(T-t)}\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial S}\\
=& e^{-\delta(T-t)}N(d_1) + e^{-\delta(T-t)}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\cdot\frac{1}{\sigma\sqrt{T-t}} - Ke^{-r(T-t)}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\cdot\frac{S}{K}\cdot\frac{e^{(r-\delta)(T-t)}}{S\sigma\sqrt{T-t}}\\
=& e^{-\delta(T-t)}N(d_1)
\end{aligned}$$

A similar argument for the put ■

According to the previous result, the delta of a call option is positive. Thus, an increase in stock price increases the option value. On the contrary, the delta of a put option is negative. Hence, an increase in the stock price decreases the value of the option.

The Black-Scholes formula tells us that a long call option on the stock can be replicated by buying $e^{-\delta(T-t)}N(d_1)$ shares of the stock and borrowing $Ke^{-r(T-t)}N(d_2)$. For a put option, the replicating portfolio consists of selling $e^{-\delta(T-t)}N(-d_1)$ shares and lending $Ke^{-r(T-t)}N(-d_2)$.

Next, we look at the range of values of delta:

• An option that is in-the-money will be more sensitive to price changes than an option that is out-of-the-money. For a call option in-the-money with a stock price high relative to the strike price (i.e. deep-in-the money), the option is most likely to be exercised. In this case, the option exhibits the price sensitivity of one full share. That is, Δ approaches 1 for a call option (-1 for a put option). If the option is out-of-the money, it is unlikely to be exercised and the option has a low price, behaving like a position with almost no shares. That is, Δ approaches 0. For an at-the-money option, the option may or may not be exercised leading to $0 < \Delta < 1$ for a call option and $-1 < \Delta < 0$ for a put option.

• As time of expiration increases, delta is less at high stock prices and greater at low stock prices. See Figure 29.1.[1] Indeed, for a greater time to expiration, the likelihood is greater for an out-of-the money option to become in-the-money option, and the likelihood is greater that an in-the-money option to become out-of-the money.

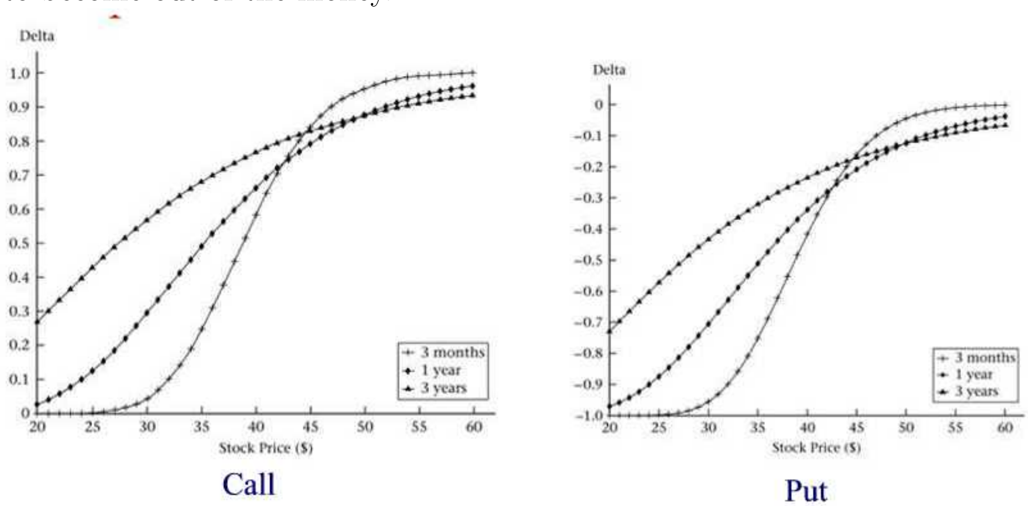


Figure 29.1

[1] See [1].

Example 29.1
The Black-Scholes price for a certain call option on GS stock is \$50. The stock currently trades for \$1000 per share, and it is known that \$452 must be borrowed in the replicating portfolio for this option. Find the delta of the option.

Solution.
We have $C = S\Delta - B$ with $B > 0$. Substituting we find $50 = 1000\Delta - 452$. Solving this equation we find $\Delta = 0.502$ ■

The Gamma Measure
The option Greek gamma (Γ) measures the change in delta when the stock price increases by \$1. More formally, the option Greek gamma is defined as the rate of change in the delta value with respect to stock price:

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V_{\text{option}}}{\partial S^2}.$$

Proposition 29.2
We have

$$\Gamma_{\text{Call}} = \frac{e^{-\delta(T-t)}}{S\sigma\sqrt{T-t}} N'(d_1)$$

and

$$\Gamma_{\text{Put}} = \Gamma_{\text{Call}}$$

where

$$N'(d_1) = \frac{\partial N(d_1)}{\partial d_1} = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} > 0.$$

Proof.
We show the first part. The second follows from the put-call parity. We have

$$\begin{aligned}
\Gamma_{\text{Call}} =& \frac{\frac{\partial C}{\partial S}}{\partial S} \\
=& \frac{\partial(e^{-\delta(T-t)} N(d_1))}{\partial S} \\
=& e^{-\delta(T-t)} \frac{\partial N(d_1)}{\partial d_1} \frac{\partial d_1}{\partial S} \\
=& \frac{e^{-\delta(T-t)}}{S\sigma\sqrt{T-t}} \frac{\partial N(d_1)}{\partial d_1}
\end{aligned}$$ ■

It follows from the above result that $\Gamma > 0$ for a purchased option. Thus, the call and put option prices are convex functions. Also, delta increases as the stock price increases. Hence, for a call, delta ($\Delta > 0$) approaches 1 as the stock price increases. For a put, delta ($\Delta < 0$) approaches 0 as the stock price increases.
Recall that Δ is close to 1 for deep in-the-money call options (-1 for put options). Thus, Δ cannot change much as the stock price increases (resp. decreases). Therefore, Γ is close to 0 for call. Similarly, for deep out-of-the money options, Γ is close to zero.

Example 29.2
Suppose that $\Gamma = 0.02$ and $\Delta = 0.5$. What is the new value of Δ if the stock price increases by \$3?

Solution.
The new value of Δ is $0.05 + 0.02 \times 3 = 0.56$ ■

The Vega Measure
Vega (the only greek that isn't represented by a real Greek letter) measures the change in the option price for an increase by 1% in volatility. More formally, it measures the change in the option price to changes in volatility:

$$V = \frac{\partial C_t}{\partial \sigma} \text{ or } V = \frac{\partial P_t}{\partial \sigma}$$

Proposition 29.3
We have

$$V_{\text{Call}} = Se^{-\delta(T-t)}\sqrt{T-t}N'(d_1) > 0$$

and

$$V_{\text{Put}} = V_{\text{Call}}.$$

Proof.
We have

$$V = \frac{\partial C_t}{\partial \sigma} = Se^{-\delta(T-t)}\frac{\partial N(d_1)}{\partial \sigma} - Ke^{-r(T-t)}\frac{\partial N(d_2)}{\partial \sigma} = Se^{-\delta(T-t)}\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial \sigma} - Ke^{-r(T-t)}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial \sigma}$$

$$= Se^{-\delta(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\left(\frac{\sigma^2(T-t)^{\frac{3}{2}} - \left[\ln(S/K) + (r-\delta+0.5\sigma^2)(T-t)\right](T-t)^{\frac{1}{2}}}{\sigma^2(T-t)}\right)$$

$$-Ke^{-r(T-t)}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\frac{S}{K}e^{(r-\delta)(T-t)}\right)\left(\frac{-\left[\ln(S/K) + (r-\delta+0.5\sigma^2)(T-t)\right](T-t)^{\frac{1}{2}}}{\sigma^2(T-t)}\right)$$

$$= Se^{-\delta(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\left(\frac{\sigma^2(T-t)^{\frac{3}{2}}}{\sigma^2(T-t)}\right) = Se^{-\delta(T-t)}\sqrt{T-t}N'(d_1)$$

The second result follows from the put-call parity ■
It follows that higher volatility means higher option prices. Figure 29.2 shows that 40-strike call vegas tends to be greater for at-the-money options, and greater for options with moderate than with short times to expiration.

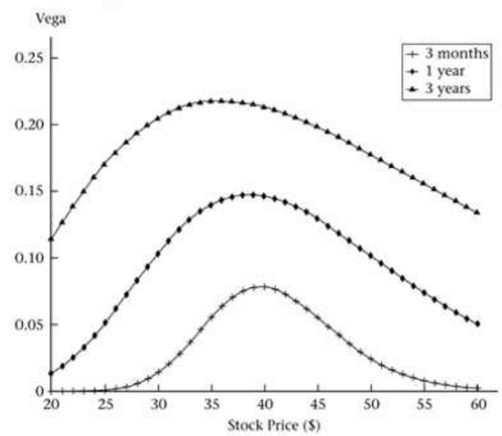


Figure 29.2

Remark 29.1
It is common to report vega as the change in the option price per percentage point change in volatility. This requires dividing the vega formula above by 100. We will follow this practice in the problems.

Example 29.3
The price of a call option on XYZ is currently \$2.00. Suppose that the vega

is 0.20 with the (prepaid forward) volatility of XYZ at 30%.
(a) If the volatility of XYZ rises to 31%, what will the price of the call option be?
(b) If the volatility of XYZ falls to 29%, what will the price of the call option be?

Solution.
(a) The value of the XYZ call will rise to \$2.20.
(b) The value of the XYZ call will drop to \$1.80 ■

Practice Problems

Problem 29.1
Show that
$$\frac{\partial N(d_2)}{\partial d_2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \frac{S}{K} e^{(r-\delta)(T-t)}.$$

Problem 29.2
Show that $\Delta_{\text{Put}} = \Delta_{\text{Call}} - e^{-\delta(T-t)}$.

Problem 29.3
Consider a call option on an a nondividend paying stock. Suppose that for $\Delta = 0.4$ the option is trading for \$33 an option. What is the new price of the option if the stock price increases by \$2?

Problem 29.4
A certain stock is currently trading for \$95 per share. The annual continuously compounded risk-free interest rate is 6%, and the stock pays dividends with an annual continuously compounded yield of 3%. The price volatility relevant for the Black-Scholes formula is 32%.
(a) Find the delta of a call option on the stock with strike price of \$101 and time to expiration of 3 years.
(b) Find the delta of a put option on the stock with strike price of \$101 and time to expiration of 3 years.

Problem 29.5
For otherwise equivalent call options on a particular stock, for which of these values of strike price (K) and time to expiration (T) would you expect delta to be the highest? The stock price at both T = 0.3 and T = 0.2 is \$50.
(A) K = \$43, T = 0.3
(B) K = \$43, T = 0.2
(C) K = \$55, T = 0.3
(D) K = \$55, T = 0.2
(E) K = \$50, T = 0.3
(F) K = \$50, T = 0.2

Problem 29.6
A certain stock is currently trading for \$86 per share. The annual continuously compounded risk-free interest rate is 9.5%, and the stock pays dividends

with an annual continuously compounded yield of 3%. The price volatility relevant for the Black-Scholes formula is 35%. Find the delta of a put option on the stock with strike price of \$90 and time to expiration of 9 months.

Problem 29.7
A stock currently trades for \$60 per share. For which of these otherwise equivalent options and strike prices (K) is the gamma the highest?
(A) Call, K = 2
(B) Put, K = 20
(C) Call, K = 45
(D) Put, K = 61
(E) Call, K = 98
(F) Put, K = 102

Problem 29.8
A call option on XYZ stock has a delta of 0.45, and a put option on XYZ stock with same strike and date to expiration has a delta of -0.55. The stock is currently trading for \$48.00. The gamma for both the call and put is 0.07.
(a) What is the value of Δ for the call and the put if the price of the stock moves up \$1?
(b) What is the value of Δ for the call and the put if the price of the stock drops \$1?

Problem 29.9
A stock has a price of \$567 and a volatility of 0.45. A certain put option on the stock has a price of \$78 and a vega of 0.23. Suddenly, volatility increases to 0.51. Find the new put option price.

Problem 29.10
The stock of GS Inc., has a price of \$567. For which of these strike prices (K) and times to expiration (T, in years) is the vega for one of these otherwise equivalent call options most likely to be the highest?
(A) K = 564, T = 0.2
(B) K = 564, T = 1
(C) K = 564, T = 30
(D) K = 598, T = 0.2
(E) K = 598, T = 1
(F) K = 598, T = 30

Problem 29.11
A stock is currently selling for \$40. The stock pays no dividends. Given that the volatility of the stock relevant for the Black-Scholes equation is 30% and the continuously compounded risk-free interest rate is 8%. Consider a \$45-strike purchased call on the stock with time to expiration in 6 months. What are the delta, gamma, and vega?

Problem 29.12 ‡
You are considering the purchase of a 3-month 41.5-strike American call option on a nondividend-paying stock. You are given:
(i) The Black-Scholes framework holds.
(ii) The stock is currently selling for 40.
(iii) The stock's volatility is 30%.
(iv) The current call option delta is 0.5.
Which of these expressions represents the price of this option?
(A) $20 - 20.453\int_{-\infty}^{0.15} e^{-\frac{x^2}{2}}dx$
(B) $20 - 16.138\int_{-\infty}^{0.15} e^{-\frac{x^2}{2}}dx$
(C) $20 - 40.453\int_{-\infty}^{0.15} e^{-\frac{x^2}{2}}dx$
(D) $16.138\int_{-\infty}^{0.15} e^{-\frac{x^2}{2}}dx - 20.453$
(E) $40.453\int_{-\infty}^{0.15} e^{-\frac{x^2}{2}}dx - 20.453$.

30 Option Greeks: Theta, Rho, and Psi

In this section we examine the three remaining option Greeks: Theta, Rho and Psi.

The Theta Measure

The option Greek theta (θ) measures the change in the option price when there is a decrease in the time to maturity of 1 day (also called **time decay**). More formally, it is defined as the rate of change of the option price with respect to the passage of time. We can write

$$\theta = \frac{\partial V}{\partial t}$$

where V is the option value and t is the passage of time or the time with $T-t$ periods to expiration.

Proposition 30.1

We have

$$\theta_{\text{Call}} = S\delta e^{-\delta(T-t)}N(d_1) - rKe^{-r(T-t)}N(d_2) - \frac{Se^{-\delta(T-t)}N'(d_1)\sigma}{2\sqrt{(T-t)}}$$

and

$$\theta_{\text{Put}} = rKe^{-r(T-t)}N(-d_2) - \delta Se^{-\delta(T-t)}N(-d_1) - \frac{Se^{-\delta(T-t)}N'(d_1)\sigma}{2\sqrt{(T-t)}}$$

Proof.

We consider the following version of the Black-Scholes formula

$$C_t = Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2).$$

We have

$$\begin{aligned}
\theta_{\text{Call}} =& \frac{\partial C}{\partial t} \\
=& S\delta e^{-\delta(T-t)}N(d_1) + Se^{-\delta(T-t)}\frac{\partial N(d_1)}{\partial t} - Kre^{-r(T-t)}N(d_2) - Ke^{-\delta(T-t)}\frac{\partial N(d_2)}{\partial t} \\
=& S\delta e^{-\delta(T-t)}N(d_1) + Se^{-\delta(T-t)}\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial t} - Kre^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}\frac{\partial N(d_2)}{\partial t}\frac{\partial d_2}{\partial t} \\
=& S\delta e^{-\delta(T-t)}N(d_1) + Se^{-\delta(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\left(\frac{\ln(S/K)}{2\sigma(T-t)^{\frac{3}{2}}} - \frac{r-\delta+0.5\sigma^2}{\sigma\sqrt{T-t}} + \frac{r-\delta+0.5\sigma^2}{2\sigma\sqrt{T-t}}\right) \\
&-Kre^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}\left(\frac{1}{\sqrt{2\pi}}\frac{S}{K}e^{-\frac{d_1^2}{2}}e^{(r-\delta)(T-t)}\right)\left(\frac{\ln(S/K)}{2\sigma(T-t)^{\frac{3}{2}}} - \frac{r-\delta-0.5\sigma^2}{\sigma\sqrt{T-t}} + \frac{r-\delta-0.5\sigma^2}{2\sigma\sqrt{T-t}}\right.
\end{aligned}$$

$$
\begin{aligned}
=&S\delta e^{-\delta(T-t)}N(d_1) - Kre^{-r(T-t)} + Se^{-\delta(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\left(-\frac{\sigma}{2\sqrt{T-t}}\right)\\
=&S\delta e^{-\delta(T-t)}N(d_1) - Kre^{-r(T-t)} - \frac{Se^{-\delta(T_t)}\sigma N'(d_1)}{2\sqrt{T-t}}
\end{aligned}
$$

The result for the put can be shown in a similar way ■

Remark 30.1
If time to expiration is measured in years, theta will be the annualized change in the option value. To obtain a per-day theta, divide by 365. This is the case we will consider in the problems.

The value of an option is the combination of time value and stock value. When time passes, the time value of the option decreases, that is, the option becomes less valuable. Thus, the rate of change of the option price with respect to the passage of time, theta, is usually negative. There are a few exceptions to this rule. Theta can be positive for deep in-the-money European calls when the underlying asset has a high dividend yield or for deep in-the-money European puts.
The theta for a purchased call and put at the same strike price and the same expiration time are not equal (See Problem 30.1). Data analysis shows that time decay is most rapid at expiration and that the theta for a call is highest (i.e. largest in absolute value) for at-the-money short-lived options, and is progressively lower (turns less and less negative) as options are in-the-money and out-of-the money.

The Measure Rho
The option Greek rho (ρ) measures the change in the option price when there is an increase in the interest rate of 1 percentage point. More formally, The rho of an option is defined as the rate of change of the option price with respect to the interest rate:

$$\rho = \frac{\partial V}{\partial r}$$

where V is the option value.

Proposition 30.2
We have

$$\rho_{\text{Call}} = (T-t)Ke^{-r(T-t)}N(d_2)$$

and

$$\rho_{\text{Put}} = -(T-t)Ke^{-r(T-t)}N(-d_2)$$

Proof.
We have

$$\begin{aligned}
\rho_{\text{Call}} =& \frac{\partial C_t}{\partial r} = Se^{-\delta(T-t)}\frac{\partial N(d_1)}{\partial r} + (T-t)Ke^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}\frac{\partial N(d_2)}{\partial r}\\
=& Se^{-\delta(T-t)}\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial r} + (T-t)Ke^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial r}\\
=& Se^{-\delta(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\left(\frac{\sqrt{T-t}}{\sigma}\right) + (T-t)Ke^{-r(T-t)}N(d_2)\\
&-Ke^{-r(T-t)}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\frac{S}{K}e^{(r-\delta)(T-t)}\right)\left(\frac{\sqrt{T-t}}{\sigma}\right)\\
=& Se^{-\delta(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\left(\frac{\sqrt{T-t}}{\sigma}\right) + (T-t)Ke^{-r(T-t)}N(d_2) - Se^{-\delta(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\left(\frac{\sqrt{T-t}}{\sigma}\right)\\
=& (T-t)Ke^{-r(T-t)}N(d_2)
\end{aligned}$$

A similar argument for the put option ■

Remark 30.2
We will use the above formula divided by 100. That is, the change of option value per 1% change in interest rate.

The rho for an ordinary stock call option should be positive because higher interest rate reduces the present value of the strike price which in turn increases the value of the call option. Similarly, the rho of an ordinary put option should be negative by the same reasoning. Figure 30.1 shows that for a European call, rho increases with time to maturity. This is also true for increases in the stock price.

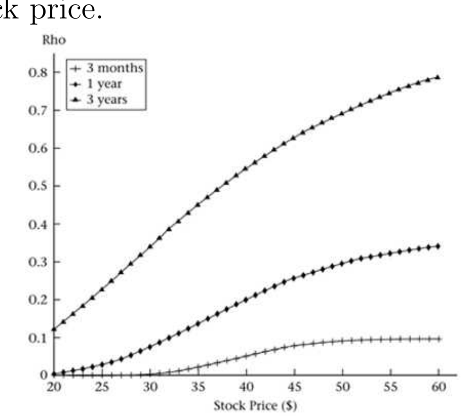


Figure 30.1

The Measure Psi

The option Greek psi (ψ) measures the change in the option price when there is an increase in the continuous dividend yield of 1%. More formally, the psi of an option is defined as the rate of change of the option price with respect to the dividend yield:

$$\psi = \frac{\partial V}{\partial \delta}$$

where V is the value of the option.

Proposition 30.3

We have

$$\psi_{\text{Call}} = -(T-t)Se^{-\delta(T-t)}N(d_1)$$

and

$$\psi_{\text{Put}} = (T-t)Se^{-\delta(T-t)}N(-d_1)$$

To interpret psi as a price change per percentage point change in the dividend yield, divide by 100. It follows that for call options, psi is negative. For put options, psi is positive. Data analysis shows that for a European call, psi decreases as time to maturity increases. For a European put, psi increases as time to maturity increases.

Remark 30.3

In the problems we will use the above formula divided by 100.

Example 30.1

Consider a stock with annual volatility of 30% . The stock pays no dividends and is currently selling for \$40. The strike price of a purchased call option is \$45 and the time to maturity is six months. The continuously compounded risk-free rate is 8%. What are θ, ρ, and ψ?

Solution.

We first find d_1 and d_2. We have

$$d_1 = \frac{\ln{(S/K)} + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln{(40/45)} + (0.08 - 0 + 0.5(0.3)^2)(0.5)}{0.3\sqrt{0.5}} = -0.260607$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t} = -0.260607 - 0.3\sqrt{0.5} = -0.47274.$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-0.260607} e^{-\frac{x^2}{2}}dx = 0.397198$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-0.47274} e^{-\frac{x^2}{2}}dx = 0.318199.$$

We have

$$\begin{aligned}
\theta_{\text{Call}} =& \frac{1}{365}[\delta e^{-\delta(T-t)}N(d_1) - rKe^{-r(T-t)}N(d_2) - \frac{Se^{-\delta(T-t)}N'(d_1)\sigma}{2\sqrt{(T-t)}}] \\
=& \frac{1}{365}[-0.08(45)e^{-0.08(0.5)}(0.318199) - \frac{40(0.3)}{2\sqrt{0.5}}\frac{e^{-\frac{0.260607^2}{2}}}{\sqrt{2\pi}}] = -0.0122 \\
\rho_{\text{Call}} =& \frac{1}{100}[(T-t)Ke^{-r(T-t)}N(d_2)] = \frac{1}{100}[0.5(45)e^{-0.08(0.5)}(0.318199)] = 0.0688 \\
\psi_{\text{Call}} =& \frac{1}{100}[-(T-t)Se^{-\delta(T-t)}N(d_1)] = \frac{1}{100}[-(0.5)(40)(0.397198)] = -0.079 \blacksquare
\end{aligned}$$

Greek Measures for Portfolios

The Greek measure of a portfolio is the sum of the Greeks of the individual portfolio components. For a portfolio containing n options with a single underlying stock, where the quantity of each option is given by $n_i, 1 \le i \le n$, we have

$$G = \sum_{i=1}^{n} n_i G_i$$

where the G and G_i's are the option Greeks. For example,

$$\Delta = \sum_{i=1}^{n} n_i \Delta_i$$

Example 30.2

A portfolio consists of 45 call options on Asset A (with $\Delta_A = 0.22$), 14 put options on Asset B (with $\Delta_B = -0.82$), 44 put options on Asset C (with $\Delta_C = -0.33$), and 784 call options on Asset D (with $\Delta_D = 0.01$). Find the delta of the entire portfolio.

Solution.
The delta of the entire portfolio is

$$\Delta = (45)(0.22) + (14)(-0.82) + (44)(-0.33) + (784)(0.01) = -8.26 \blacksquare$$

Practice Problems

Problem 30.1
Show that

$$\theta_{\text{Put}} = \theta_{\text{Call}} + rKe^{-r(T-t)} - \delta Se^{-\delta(T-t)}.$$

Problem 30.2
Prove Proposition 30.3.

Problem 30.3
The stock of GS Co. pays dividends at an annual continuously compounded yield of 0.12. The annual continuously compounded risk-free interest rate is 0.34. Certain call options on the stock of GS Co. have time to expiration of 99 days. The option currently trades for \$56.
(a) Suppose $\theta = -0.03$(per day). Find the price of the call option 65 days from expiration, all other things equal.
(b) Suppose $\rho = 0.11$. Find the price of the call option if the interest rate suddenly increases to 0.66, all other things equal.
(c) Suppose $\psi = -0.04$. Find the price of the call option if the stock's dividend yield suddenly decreases to 0.02, all other things equal.

Problem 30.4
A stock is currently selling for \$40. The stock pays no dividends. Given that the volatility relevant for the Black-Scholes equation is 30% and the continuously compounded risk-free interest rate is 8%. Consider a \$40-strike purchased call on the stock with time to expiration in 6 months. What are the theta, rho, and psi?

Problem 30.5
Consider a bull spread where you buy a 40-strike call and sell a 45-strike call. Suppose $S = \$40, \sigma = 30\%, r = 8\%, \delta = 0\%$, and $T = 0.5$. What are theta and rho for this portfolio?

Problem 30.6
A stock is currently selling for \$40. The stock pays no dividends. Given that the volatility relevant for the Black-Scholes equation is 30% and the continuously compounded risk-free interest rate is 8%. Consider a \$40-strike purchased put on the stock with time to expiration in 6 months. What are the delta, gamma, vega, theta, and rho?

Problem 30.7
A stock is currently selling for \$40. The stock pays no dividends. Given that the volatility is relevant for the Black-Scholes equation 30% and the continuously compounded risk-free interest rate is 8%. Consider a \$45-strike purchased put on the stock with time to expiration in 6 months. What are the delta, gamma, vega, theta, and rho?

Problem 30.8
Consider a bull spread where you buy a 40-strike put and sell a 45-strike put. Suppose $S = \$40, \sigma = 30\%, r = 8\%, \delta = 0$, and $T = 0.5$. What are delta, gamma, vega, theta, and rho?

Problem 30.9
Show that the delta of a $K_1 - K_2$ call bull spread is equal to the $K_1 - K_2$ put bull spread when the underlying stock pays no dividends. Here $K_1 < K_2$.

Problem 30.10 ‡
You compute the delta for a 50-60 bull spread with the following information:
(i) The continuously compounded risk-free rate is 5%.
(ii) The underlying stock pays no dividends.
(iii) The current stock price is \$50 per share.
(iv) The stock's volatility relevant for the Black-Scholes equation is 20%.
(v) The time to expiration is 3 months.
How much does the delta change after 1 month, if the stock price does not change?

31 Option Elasticity and Option Volatility

Suppose a stock price changes from S to S'. Let $\epsilon = S' - S$ denote the change in the stock price. Let $CV = V' - V$ denote the change in an option value. From the definition of the option Greek Δ, as the change in option value over the change in stock value we can write

$$CV = \epsilon\Delta.$$

That is, the change in option value is the change in stock value multiplied by delta.
The **option elasticity** Ω is defined as the percentage change in the option price relative to the percentage change in the stock price:

$$\Omega = \frac{\frac{V'-V}{V}}{\frac{S'-S}{S}} = \frac{\frac{\epsilon\Delta}{V}}{\frac{\epsilon}{S}} = \frac{S\Delta}{V}.$$

Thus, option elasticity gives the percentage change in the option value for a 1% change in the stock value. It tells us the risk of the option relative to the stock in percentage form.
For a call option we shall use the notation

$$\Omega_{\text{Call}} = \frac{S\Delta}{C}$$

and for a put option

$$\Omega_{\text{Put}} = \frac{S\Delta}{P}$$

For a call option we have $S\Delta = Se^{-\delta T}N(d_1) \geq Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2) = C$. That is, $\Omega_{\text{Call}} \geq 1$. For a put option $\Delta < 0$ so that $\Omega_{\text{Put}} \leq 0$.

Example 31.1
A certain stock is currently trading for \$41 per share with a stock price volatility of 0.3. Certain call options on the stock have a delta of 0.6911 and a price of \$6.961. Find the elasticity of such a call option.

Solution.
The call elasticity is

$$\Omega_C = \frac{S\Delta}{C} = \frac{41 \times 0.6911}{6.961} = 4.071.$$

This says that if the stock price goes up to \$41.41, the call option value goes up to $6.961 + 0.04071 \times 6.961 = \$7.24$ ■

Figure 31.1 displays the elasticity of a call option with strike price of \$35 with different maturities.

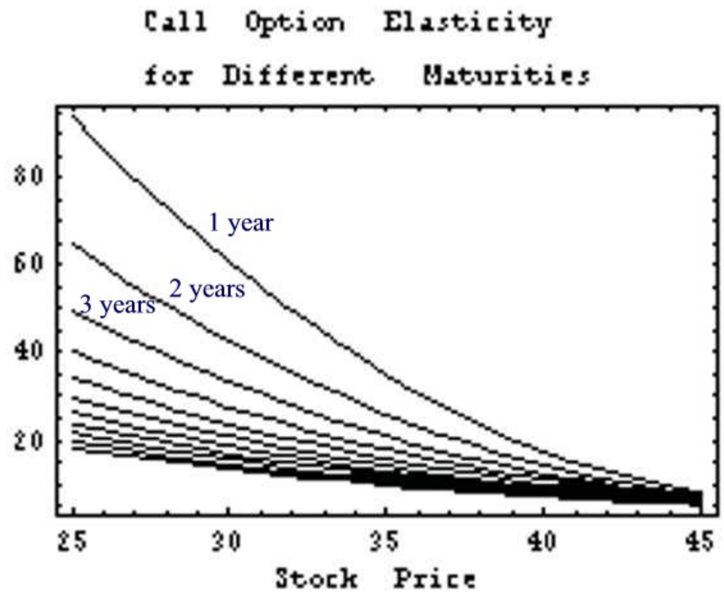


Figure 31.1

The following observations are in place:
- Ω_{Call} increases as stock price decreases.
- Ω_{Call} decreases as time to maturity increases.
- Ω_{Call} increases as the option becomes more out-of-the money. Ω_{Call} decreases as the option becomes more in-the-money.

Example 31.2
A European call option on XYZ stock has the following specifications: Strike price = \$45, current stock price = \$46, time to expiration = 3 months, annual continuously compounded interest rate = 0.08, dividend yield = 0.02, prepaid forward price volatility=0.35. Calculate the elasticity of the call.

Solution.
We first calculate d_1 and d_2. We have

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{46}{45}\right) + (0.08 - 0.02 + 0.35^2 \times 0.5)(0.25)}{0.35\sqrt{0.25}} = 0.29881$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = 0.29881 - 0.35\sqrt{0.25} = 0.12381.$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.29881} e^{-\frac{x^2}{2}} dx = 0.617457$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.12381} e^{-\frac{x^2}{2}} dx = 0.549267.$$

Thus, the price of the call is

$$C = 46e^{-0.02\times 0.25} \times 0.617457 - 45e^{-0.08\times 0.25} \times 0.549267 = \$4.0338.$$

Now,

$$\Delta_{\text{Call}} = e^{-\delta T} N(d_1) = e^{-0.02\times 0.25} \times 0.617457 = 0.6144.$$

Thus,

$$\Omega_{\text{Call}} = \frac{S\Delta}{C} = \frac{46 \times 0.6144}{4.0338} = 7.0064 \blacksquare$$

The volatility of an option can be found in terms of the option elasticity.

Proposition 31.1
The volatility of an option is the option elasticity times the volatility of the stock. That is,

$$\sigma_{\text{option}} = \sigma_{\text{stock}} \times |\Omega|.$$

Proof.
Consider a hedge portfolio consisting of purchased Δ shares of the stock and one short option. The portfolio value is $V - S\Delta$. One period later the portfolio will be worth

$$(V + \delta V) - \Delta(S + \delta S)$$

where δ means the change in S or V. If this portfolio is hedged, its value should grow at the risk-free rate. Hence, the following condition must hold:

$$(V - \Delta S)(1 + r) = (V + \delta V) - \Delta(S + \delta S).$$

This leads to the equation

$$\frac{\delta V}{V} = r + \left[\frac{\delta S}{S} - r\right] \Delta \frac{S}{V} = r + \left[\frac{\delta S}{S} - r\right] \Omega.$$

Using the expression created above for the return on the option as a function of the return on the stock, we take the variance of the option return:

$$\begin{aligned}\text{Var}\left[\frac{\delta V}{V}\right] =&\text{Var}\left(r + \left[\frac{\delta S}{S} - r\right]\Omega\right)\\ =&\text{Var}\left[\frac{\delta S}{S}\Omega\right]\\ =&\Omega^2\text{Var}\left(\frac{\delta S}{S}\right)\end{aligned}$$

Taking square root of both sides we obtain

$$\sigma_{\text{option}} = \sigma_{\text{stock}} \times |\Omega| \blacksquare$$

It follows that the option's risk is the risk of the stock times the risk of the option relative to the stock.

Example 31.3
A certain stock is currently trading for \$41 per share with a stock price volatility of 0.3. Certain call options on the stock have a delta of 0.6911 and a price of \$6.961. Find the volatility of such a call option.

Solution.
The answer is

$$\sigma_{\text{Call}} = \sigma_{\text{stock}} \times |\Omega_{\text{Call}}| = 0.3 \times 4.071 = 1.2213 \blacksquare$$

Example 31.4
Consider a 3-year call option on a stock that pays dividends at the continuously compounded yield of 3%. Suppose that the prepaid forward price volatility is 20% and the elasticity of the call is 3.9711. Find the volatility of the call option.

Solution.
We know from Remark 16.2 that

$$\sigma_F = \sigma_{\text{stock}} \times \frac{S}{F_{0,T}^P}.$$

But

$$\frac{S}{F_{0,T}^P} = \frac{S}{Se^{-\delta T}} = e^{\delta T}.$$

Thus,

$$\sigma_{\text{stock}} = \sigma_F e^{-\delta T} = 0.2e^{-0.03\times 3} = 0.1828.$$

Hence,

$$\sigma_{\text{Call}} = \sigma_{\text{stock}} \times |\Omega_{\text{Call}}| = 0.1828 \times 3.9711 = 0.7259 \blacksquare$$

Practice Problems

Problem 31.1
Since $C = \max\{0, S_T - K\}$, as the strike decreases the call becomes more in the money. Show that the elasticity of a call option decreases as the strike price decreases.

Problem 31.2
Which of the following statements is true?
(A) $\Omega_{\text{Put}} \leq -1$.
(B) $\Omega_{\text{Call}} \geq 0$.
(C) The elasticity of a call increases as the call becomes more out-of-the money.
(D) The elasticity of a call increases as stock price increases.

Problem 31.3
A certain stock is currently trading for \$41 per share with a stock volatility of 0.3. Certain put options on the stock have a delta of -0.3089 and a price of \$2.886. Find the elasticity of such a put option.

Problem 31.4
A European put option on XYZ stock has the following specifications: Strike price = \$45, current stock price = \$46, time to expiration = 3 months, annual continuously compounded interest rate = 0.08, dividend yield = 0.02, prepaid forward price volatility=0.35. Calculate the elasticity of such a put.

Problem 31.5
A European call option on XYZ stock has the following specifications: Strike price = \$45, current stock price = \$46, time to expiration = 3 months, annual continuously compounded interest rate = 0.08, dividend yield = 0.02, prepaid forward price volatility=0.35. Calculate the volatility of the call.

Problem 31.6
Given the following information about a 3-year call option on a certain stock:
• The current stock price is \$550.
• The prepaid forward price volatility (the volatility relevant for the Black-Scholes formula) is 0.2.
• The strike price is \$523.
• The stock pays dividends at an annual continuously compounded yield of

0.03.
• The annual continuously compounded interest rate is 0.07.
Find the elasticity of such a call option.

Problem 31.7
Find the call option volatility in the previous problem.

Problem 31.8
Given the following information about a 3-year put option on a certain stock:
• The current stock price is \$550.
• The prepaid forward price volatility (the volatility relevant for the Black-Scholes formula) is 0.2.
• The strike price is \$523.
• The stock pays dividends at an annual continuously compounded yield of 0.03.
• The annual continuously compounded interest rate is 0.07.
Find the elasticity of such a put option.

Problem 31.9 ‡
A call option is modeled using the Black-Scholes formula with the following parameters:
• $S = 25$
• $K = 24$
• $r = 4\%$
• $\delta = 0\%$
• $\sigma = 20\%$
• $T = 1$.
Calculate the call option elasticity.

Problem 31.10 ‡
For a European call option on a stock within the Black-Scholes framework, you are given:
(i) The stock price is \$85.
(ii) The strike price is \$80.
(iii) The call option will expire in one year.
(iv) The continuously compound risk-free interest rate is 5.5%.
(v) $\sigma = 0.50$
(vi) The stock pays no dividends.
Calculate the volatility of this call option.

Problem 31.11
For a European put option on a stock within the Black-Scholes framework, you are given:
(i) The stock price is \$50.
(ii) The strike price is \$55.
(iii) The put option will expire in one year.
(iv) The continuously compound risk-free interest rate is 3%.
(v) $\sigma = 0.35$
(vi) The stock pays no dividends.
Calculate the volatility of this put option.

32 The Risk Premium and Sharpe Ratio of an Option

When buying an asset you lose cash that can be invested at risk-free interest rate and you acquire the risk of owning the stock. On average you expect to earn interest as compensation for the time value of money and additional return as compensation for the risk of the stock. This additional return in known as the risk premium.

In finance, the **risk premium** on an asset is defined to be the excess return of the asset over the risk-free rate. Thus, if r denotes the risk-free rate and α the expected return on the asset, then the risk premium on the asset is the difference

$$\alpha - r.$$

Now, consider a call option on a stock. We know that the call can be replicated by buying Δ units of a share of the stock and borrowing B. A result in finance due to Brealey and Meyer states that the return on any portfolio is the weighted average of the returns on the assets in the portfolio. We apply this result to the above portfolio. Let γ be the expected return on the call, α the expected return on the stock and r the risk-free rate of return. Then

$$\gamma = \frac{S\Delta}{S\Delta - B}\alpha - \frac{B}{S\Delta - B}r.$$

But $C = S\Delta - B$ so that $-B = C - S\Delta$. Thus, the above equation can be written as

$$\gamma = \frac{S\Delta}{C}\alpha + \left(1 - \frac{S\Delta}{C}\right)r. \tag{32.1}$$

Since $\Omega = \frac{S\Delta}{C}$, the above equation reduces to

$$\gamma = \Omega\alpha + (1 - \Omega)r$$

or

$$\gamma - r = (\alpha - r) \times \Omega.$$

This says that the risk premium of the option is the risk premium of the stock multiplied by the option elasticity.

Remark 32.1

Note that in terms of the replicating portfolio, in Equation (32.1), $\frac{S\Delta}{C}$ is the percentage of the value of the option invested in the stock and $1 - \frac{S\Delta}{C}$ is the percentage of the value of the option for borrowing.

Remark 32.2
Suppose $\alpha - r > 0$. Since $\Omega_{\text{Call}} \geq 1$, we have $\gamma - r \geq \alpha - r$ or $\gamma \geq \alpha$. For a put, we know that $\Omega_{\text{Put}} \leq 0$ which leads to $\gamma - r = (\alpha - r) \times \Omega \leq 0 < \alpha - r$. That is, $\gamma < \alpha$.
We know from Section 31 that Ω_{Call} increases if either the stock price goes down or the strike price goes up. Therefore, the expected return on a call option increases if either the stock price goes down or the strike price goes up. That is, the expected return increases if the option is more out-of-the-money and decreases if the option is more in-the-money.

Example 32.1
You are given the following information of a call option on a stock:
• The expected return on the stock is 15% compounded continuously.
• The continuously compounded risk-free interest rate is 7%.
• The call elasticity is 4.5.
(a) Find the risk premium on the option.
(b) Find the expected annual continuously compounded return on the call option.

Solution.
(a) The risk premium on the option is $\gamma - r = (\alpha - r) \times \Omega = (0.15 - 0.07) \times 4.5 = 0.36$.
(b) The expected return on the option is $\gamma = 0.36 + 0.07 = 0.43$ ■

The Sharpe Ratio of an Option
The **Sharpe ratio** of an asset is the risk premium per unit of risk in an investment asset (defined as the standard deviation of the asset's excess return.) It is given by

$$\text{Sharpe ratio} = \frac{\alpha - r}{\sigma}.$$

The Sharpe ratio is used to characterize how well the return of an asset compensates the investor for the risk taken.
Applying this definition to a call option on a stock, we can define the Sharpe ratio of a call by

$$\text{Sharpe ratio for call} = \frac{(\alpha - r)\Omega}{\sigma\Omega} = \frac{\alpha - r}{\sigma}.$$

Thus, the Sharpe ratio of a call option is equal to the Sharpe ratio of the underlying stock.

The Sharpe ratio for a put option on a stock is

$$\text{Sharpe ratio for put} = \frac{(\alpha - r)\Omega}{\sigma \times |\Omega|} = \frac{(\alpha - r)\Omega}{\sigma \times (-\Omega)} = \frac{r - \alpha}{\sigma}.$$

Thus, the Sharpe ratio of a put is the opposite of the Sharpe ratio of the stock.

Example 32.2
You expect to get an annual continuously compounded return of 0.3 on the stock of GS Co. The stock has annual price volatility of 0.22. The annual continuously compounded risk-free interest rate is 0.02. A certain call option on GS Co. stock has elasticity of 2.3. Find the Sharpe ratio of the call option.

Solution.
The Sharpe ratio is

$$\frac{\alpha - r}{\sigma} = \frac{0.3 - 0.02}{0.22} = 1.2727 \blacksquare$$

The Elasticity and Risk Premium of a Portfolio of Options
Consider a portfolio comprising of n calls with the same underlying stock. For $1 \leq i \leq n$, we let C_i denote the value of the ith call and Δ_i the change in the value of the ith call for a \$1 change in the stock. Suppose that there are n_i units of the ith call in the portfolio. We define ω_i to be the percentage of the portfolio value invested in the ith call. That is

$$\omega_i = \frac{n_i C_i}{\sum_{j=1}^n n_j C_j},$$

where $\sum_{j=1}^n n_j C_j$ is the portfolio value.
Now, a \$1 change in the stock results in a change of the portfolio value by

$$\sum_{j=1}^n n_j \Delta_j.$$

Now, the elasticity of the portfolio is the percentage change in the portfolio (of calls) price divided by the percentage change in the stock price, or

$$\Omega_{\text{portfolio}} = \frac{\frac{\sum_{i=1}^n n_i \Delta_i}{\sum_{j=1}^n n_j C_j}}{\frac{1}{S}} = \sum_{i=1}^n \left(\frac{n_i C_i}{\sum_{j=1}^n n_j C_j} \right) \frac{S \Delta_i}{C_i} = \sum_{i=1}^n \omega_i \Omega_i.$$

Like the case of a single option, it is easy to establish that the risk premium of the portfolio is the elasticity times the risk premium of the underlying stock.

Example 32.3
Given the following information:

Call	Price	Elasticity
A	\$9.986	4.367
B	\$7.985	4.794
C	\$6.307	5.227

Find the elasticity of the portfolio consisting of buying one call option on stock A, one call option on stock B and selling one call option on stock C.

Solution.
The cost of the portfolio is

$$-9.986 - 7.985 + 6.307 = -\$11.664.$$

The elasticity of the portfolio is

$$\Omega_{\text{portfolio}} = \sum_{i=1}^{3} \omega_i \Omega_i = \frac{9.986}{11.664} \times 4.367 + \frac{7.985}{11.664} \times 4.794 - \frac{6.307}{11.664} \times 5.227 = 4.194 \blacksquare$$

Practice Problems

Problem 32.1
You are given the following information of a call option on a stock:
- The expected return on the stock is 30% compounded continuously.
- The continuously compounded risk-free interest rate is 2%.
- The call elasticity is 2.3.

(a) Find the risk premium on the option.
(b) Find the expected annual continuously compounded return on the call option.

Problem 32.2
A call option on a stock has elasticity of 3.4. The continuously compounded risk-free rate is 3.3% and the Black-Scholes price volatility of the stock is 0.11. Suppose that the risk premium on the stock is 77% of the stock volatility. Find the expected annual continuously compounded return on the option.

Problem 32.3
You expect an annual continuously compounded return of 0.145 on the stock of GS, Inc., and an annual continuously compounded return of 0.33 on a certain call option on that stock. The option elasticity is 4.44. Find the annual continuously compounded risk-free interest rate.

Problem 32.4
The price of a put option on a stock is \$4.15. The elasticity of the put is -5.35. Consider the replicating portfolio.
(a) What is the amount of money invested in the stock?
(b) What is the amount of money to be lent at the risk-free interest rate?

Problem 32.5
The stock of GS Co. has a Sharpe ratio of 0.77 and annual price volatility of 0.11. The annual continuously compounded risk-free interest rate is 0.033. For a certain call option on GS stock, the elasticity is 3.4. Find the expected annual continuously compounded return on the option.

Problem 32.6
A call option on a stock has option volatility of 1.45 and elasticity 4.377. The annual continuously compounded risk-free is 6% and the annual continuously compounded return on the stock is 15%. Find the Sharpe ratio of such a call option.

Problem 32.7
A stock has volatility of 25%. The annual continuously compounded risk-free is 4% and the annual continuously compounded return on the stock is 8%. What is the Sharpe ratio of a put on the stock?

Problem 32.8
Consider the following information:

Call	Price	Elasticity
A	\$9.986	4.367
B	\$7.985	4.794
C	\$6.307	5.227

Find the elasticity of a portfolio consisting of 444 options A, 334 options B, and 3434 options C.

Problem 32.9
Consider again the information of the previous problem. The expected annual continuously compounded return on the stock is 0.24, and the annual continuously compounded risk-free interest rate is 0.05. Find the risk premium on this option portfolio.

Problem 32.10
Consider the following information:

Call	Price	Elasticity
A	\$9.986	4.367
B	\$7.985	4.794
C	\$6.307	5.227

A portfolio consists of 444 of options A, 334 of options B and X units of option C. The elasticity of the portfolio is 5.0543. Find X to the nearest integer.

Problem 32.11
Consider the following information:

Call	Price	Elasticity
A	\$9.986	4.367
B	\$7.985	4.794
C	\$6.307	Ω_C

A portfolio consists of 444 of options A, 334 of options B and 3434 units of option C. The risk premium of the portfolio is 0.960317. The expected annual continuously compounded return on the stock is 0.24, and the annual continuously compounded risk-free interest rate is 0.05. Find Ω_C.

Problem 32.12
Consider a portfolio that consists of buying a call option on a stock and selling a put option. The stock pays continuous dividends at the yield rate of 5%. The options have a strike of \$62 and expire in six months. The current stock price is \$60 and the continuously compounded risk-free interest rate is 15%. Find the elasticity of this portfolio.

Problem 32.13
Given the following information

Option	Price	Elasticity
Call	\$10	Ω_C
Put	\$5	Ω_P

Consider two portfolios: Portfolio A has 2 call options and one put option. The elasticity of this portfolio is 2.82. Portfolio B consists of buying 4 call options and selling 5 put options. The delta of this portfolio is -3.50. The current value of the stock is \$86. Determine Ω_C and Ω_P.

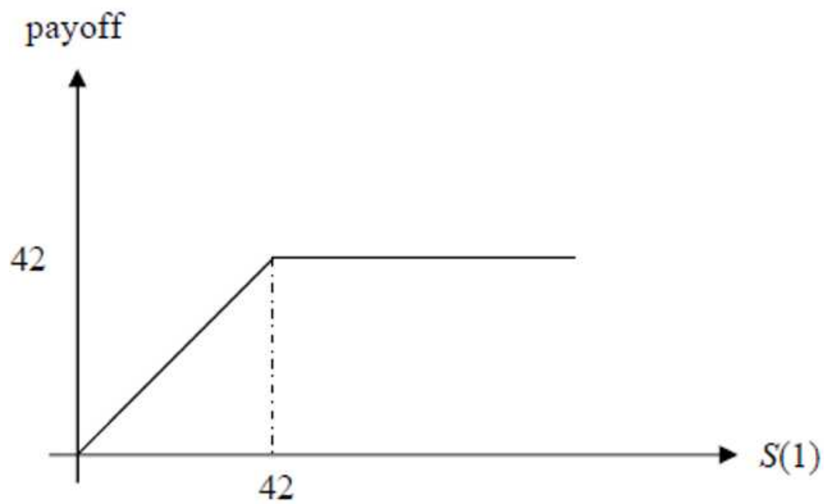
Problem 32.14 ‡
An investor is deciding whether to buy a given stock, or European call options on the stock. The value of the call option is modeled using the Black-Scholes formula and the following assumptions:
- Continuously compounded risk-free rate = 4%
- Continuously compounded dividend = 0%
- Expected return on the stock = 8%
- Current stock price = 37
- Strike price = 41
- Estimated stock volatility = 25%
- Time to expiration = 1 year.

Calculate the sharp ratio of the option.

Problem 32.15 ‡
Assume the Black-Scholes framework. Consider a stock, and a European call

option and a European put option on the stock. The current stock price, call price, and put price are 45.00, 4.45, and 1.90, respectively.
Investor A purchases two calls and one put. Investor B purchases two calls and writes three puts.
The current elasticity of Investor A's portfolio is 5.0. The current delta of Investor B's portfolio is -3.4.
Calculate the current put-option elasticity.

Problem 32.16 ‡
Assume the Black-Scholes framework. Consider a 1-year European contingent claim on a stock.
You are given:
(i) The time-0 stock price is 45.
(ii) The stock's volatility is 25%.
(iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.
(iv) The continuously compounded risk-free interest rate is 7%.
(v) The time-1 payoff of the contingent claim is as follows:



Calculate the time-0 contingent-claim elasticity.

33 Profit Before Maturity: Calendar Spreads

In this section we discuss the concepts of the holding period profit and calendar spreads.

The **holding period profit** of a position is defined to be the current value of the position minus the cost of the position, including interest. For example, consider a call option with maturity time T and strike price K. Suppose that at time $t = 0$ the price of the option is C. Let C_t be the price of the option $t < T$ years later. We want to find the holding period profit at time t. Using the Black-Scholes formula we have

$$C_t = Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2).$$

Thus, the holding period profit is

$$C_t - Ce^{rt}.$$

Example 33.1 ‡

Assume the Black-Scholes framework.

Eight months ago, an investor borrowed money at the risk-free interest rate to purchase a one-year 75-strike European call option on a nondividend-paying stock. At that time, the price of the call option was 8.

Today, the stock price is 85. The investor decides to close out all positions.

You are given:

(i) The continuously compounded risk-free rate interest rate is 5%.

(ii) The stock's volatility is 26%.

Calculate the eight-month holding profit.

Solution.

We first find the current value ($t = 8$) of the option. For that we need to find d_1 and d_2. We have

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(85/75) + (0.05 + 0.5(0.26)^2)(4/12)}{0.26\sqrt{\frac{4}{12}}} = 1.02$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = 1.02 - 0.26\sqrt{\frac{4}{12}} = 0.87.$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1.02} e^{-\frac{x^2}{2}} dx = 0.846136$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.87} e^{-\frac{x^2}{2}} dx = 0.80785.$$

Thus,

$$C_t = Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) = 85 \times 0.846136 - 75e^{-0.05\times\frac{4}{12}} \times 0.80785 = 12.3342.$$

Thus, the eight-month holding profit is

$$12.3342 - 8e^{0.05\times\frac{8}{12}} = \$4.0631 \blacksquare$$

Calendar Spread

If an investor believes that stock prices will be stable for a foreseeable period of time, he or she can attempt to profit from the declining time value of options by setting a calendar spread.

In simplest terms, a **calendar spread**, also called a **time spread** or **horizontal spread**, involves buying an option with a longer expiration and selling an option with the same strike price and a shorter expiration. The long call usually has higher premium than the short call.

Suppose that the stock price is very low at the expiry of the shorter-lived call. Then the call will not be exercised and therefore it is worthless. But also, the value of the longer-lived call is close to zero. So the investor loss is the cost of setting up the spread. If the stock price is very high, the shorter-lived call will be exercised and this costs the investor $S_T - K$ and the long-lived call has value close to $S_T - K$ so that the investor again incurs a loss which is the cost of setting the spread. If S_T is close to K, the short-lived call will not be exercised so it is worthless and costs the investor almost nothing but the long-lived call is still valuable and thus the investor incurs a net gain.

Example 33.2

The current price of XYZ stock is \$40. You sell a 40-strike call with three months to expiration for a premium of \$2.78 and at the same time you buy a 40-strike call with 1 year to expiration for a premium of \$6.28. The theta for the long call is -0.0104 and that for the short call is -0.0173. Assume that the continuously compounded risk-free rate is 8%, find the profit once the sold option has expired, if the stock price remains at \$40 and nothing else has changed.

Solution.
Three months from now, at a stock price of \$40, the written call option will be at-the-money and so will not be exercised. You keep the entire premium on the option− i.e., \$2.78, and this value could have accumulated interest over three months. Thus, your profit from selling the 3-month option is $2.78e^{0.08\times 0.25} = \$2.8362$.
You also lose some money on your purchased 1-year option because of time decay. Due to time decay, the option price changes by $90\theta = 90(-0.0104) = -0.936$.
You further lose some money in the form of the interest that could have accumulated on the \$6.28 premium you paid for the purchased call option, if instead you invested the money at the risk-free interest rate. That interest is $6.28(e^{0.08\times 0.25} - 1) = \$0.1269$. So your net gain from entering into this spread is

$$2.8362 - 0.936 - 0.1269 = \$1.7733 \blacksquare$$

Example 33.3
Suppose that the current month is April, 08 and the stock price is \$80. You buy a call option expiring on Dec, 08 and sell a call expiring on July, 08. Both calls have the same strike \$80. The premium for the long call is \$12 and for the short call \$7. On May, 08 the price of the stock drops to \$50 and the long-lived call is selling for \$1. In terms of premium only, what would be your profit in May?

Solution.
The cost on April, 08 from creating the spread is $12 - 7 = \$5$. On May 08, the short call is nearly worthless while the long-call can be sold for \$1. Thus, the loss from premiums is \$4 ■

A calendar spread may be implemented using either call options or put options, but never with calls and puts used together in the same trade.

Example 33.4
The current price of XYZ stock is \$40. You sell a 40-strike put with three months to expiration for a premium of \$2.78 and at the same time you buy a 40-strike put with 1 year to expiration for a premium of \$6.28. The theta for the long put is -0.002 and that for the short put is -0.008. Assume that the continuously compounded risk-free rate is 8%, find the profit at

the expiration date of the shorter put, if the stock price remains at \$40 and nothing else has changed.

Solution.
Three months from now, at a stock price of \$40, the written put option will be at-the-money and so will not be exercised. You keep the entire premium on the option− i.e., \$2.78, and this value could have accumulated interest over three months. Thus, your profit from selling the 3-month option is $2.78e^{0.08\times 0.25} = \$2.8362$.
You also lose some money on your purchased 1-year option because of time decay. Due to time decay, the option price changes by $90\theta = 90(-0.002) = -0.18$.
You further lose some money in the form of the interest that could have accumulated on the \$6.28 premium you paid for the purchased put option, if instead you invested the money at the risk-free interest rate. That interest is $6.28(e^{0.08\times 0.25} - 1) = \$0.1269$. So your net gain (after three months) from entering into this spread is

$$2.8362 - 0.18 - 0.1269 = \$2.53 \blacksquare$$

Practice Problems

Problem 33.1
Assume the Black-Scholes framework.
Eight months ago, an investor borrowed money at the risk-free interest rate to purchase a one-year 90-strike European put option on a nondividend-paying stock. At that time, the price of the put option was 7.
Today, the stock price is 80. The investor decides to close out all positions.
You are given:
(i) The continuously compounded risk-free rate interest rate is 6%.
(ii) The stock's volatility is 28%.
Calculate the eight-month holding profit.

Problem 33.2
Assume the Black-Scholes framework.
Eight months ago, an investor borrowed money at the risk-free interest rate to purchase a one-year 75-strike European call option on a nondividend-paying stock. Today, the stock price is 85.
You are given:
(i) The continuously compounded risk-free rate interest rate is 5%.
(ii) The stock's volatility is 26%.
(iii) The eight-month holding profit is \$4.0631.
Calculate the initial cost of the call.

Problem 33.3
Eight months ago the price of a put option on a nondividend-paying stock was \$7. Currently the price of the put is \$10.488. Find the continuously compounded risk-free interest rate if the eight-month holding period profit is \$3.7625.

Problem 33.4
You own a calendar spread on a the stock of GS Co., which you bought when the stock was priced at \$22. The spread consists of a written call option with a strike price of \$22 and a longer-lived purchased call option with a strike price of \$22. Upon the expiration of the shorter-lived option, at which stock price will you make the most money on the calendar spread?
(A) \$3 (B) \$12 (C) \$22 (D) \$23 (E) \$33

Problem 33.5
The current price of XYZ stock is \$60. You sell a 60-strike call with two months to expiration for a premium of \$3.45 and at the same time you buy a 60-strike call with 1 year to expiration for a premium of \$18.88. The theta for the long call is -0.05 and that for the short call is -0.04. Assume that the continuously compounded risk-free rate is 6%, find the profit once the sold option has expired, if the stock price remains at \$60 and nothing else has changed.

Problem 33.6
A **reverse calendar spread** is constructed by selling a long-term option and simultaneously buying a short-term option with the same strike price. Would you expect to profit if the stock price moves away in either direction from the strike price by a great deal?

Problem 33.7
Suppose that the current month is April, 08 and the stock price is \$80. You sell a call option expiring on Dec, 08 and buy a call price expiring on July, 08. Both calls have the same strike \$80. The premium for the long call is \$12 and for the short call \$7. On May, 08 the price of the stock drops to \$50 and the long-lived call is selling for \$1. In terms of premium only, what would be your profit in May?

Problem 33.8
Currently stock XYZ is selling for \$40. You believe that in the next three months the stock price will be almost the same. Assume that the continuously compounded risk-free rate is 8%. Create a calendar spread from the following options with the largest profit at the time of expiration of the short-lived call.
Call option A: $K = \$40, \theta = -0.017, T =$3months, Premium \$2.78.
Call option B: $K = \$40, \theta = -0.01, T =$1 year, Premium \$3.75.
call Option C: $K = \$40, \theta = -0.006, T =$3years, Premium \$7.05

Problem 33.9
Currently, stock XYZ is selling for \$40. You believe that in the next three months the stock price will be almost the same. Assume that the continuously compounded risk-free rate is 8%. Create a calendar spread from the following options with the largest profit at the time of expiration of the short-lived call.

Put option A: $K = \$40, \theta = -0.008, T =$3months, Premium \$2.78.
Put option B: $K = \$40, \theta = -0.002, T =$1 year, Premium \$6.28.
Put option C: $K = \$40, \theta = -0.0001, T =$3years, Premium \$9.75.

34 Implied Volatility

In Section 26 we described how to estimate volatility of the underlying asset using previously known returns. We called such a volatility historical volatility. The problem with this notion of volatility is that it uses the past to estimte future volatility and thus cannot be considered reliable. In this section we introduce a different type of volatility that depends on the observed market price of an option.
Implied volatility of the underlying asset is the volatility that, when used in a particular pricing model, yields a theoretical value for the option equal to the current market price of that option. Thus, historical volatility tells us how volatile an asset has been in the past. Implied volatility is the market's view on how volatile an asset will be in the future.
Now, if we assume that the option price can be modeled by the Black-Scholes formula, and the other variables $-$ stock price (S), strike price (K), annual continuously compounded risk-free interest rate (r), time to expiration (T), and annual continuously compounded dividend yield (δ) are known, the implied volatility $\hat{\sigma}$ is then the solution to the equation

$$\text{Market option price} = C(S, K, \hat{\sigma}, r, T, \delta). \tag{34.1}$$

There is no way to solve directly for implied volatility in (34.1). Instead, either iterative methods (such as Newton's method) or financial softwares are used.

Example 34.1
Suppose we observe a 40-strike 3-month European call option with a premium of \$2.78. The stock price is currently \$45, the interest rate is 8%, and the stock pays no dividends. Find the implied volatility.

Solution.
We want to find $\hat{\sigma}$ that satisfies the equation

$$2.78 = C(45, 40, \hat{\sigma}, 0.08, 0.25, 0).$$

Using an electronic device, we find $\hat{\sigma} \approx 36\%$ ∎

When using the Black-Scholes or the binomial model, it is possible to confine implied volatility within particular boundaries by calculating option prices for two different implied volatilities. If one of these prices is greater than the observed option price and the other is less than the observed option price, then we know that the implied volatility is somewhere between the two values for which calculations were made.

Remark 34.1
On the actuarial exam, you will be given several ranges within which implied volatility might fall. Test the extreme values of those ranges and see if the observed option price falls somewhere in between the prices calculated by considering each of these extreme values. If it does, then you have obtained the correct value of implied volatility.

Example 34.2
A one-year European call option is currently valued at \$60. The following parameters are given:
• The current stock price is \$300
• The continuously compounded risk-free rate is 9%
• The continuously compounded dividend yield is 3%
• The strike price is \$300.
Using a single-period binomial model and assuming the implied volatility of the stock to be at least 6%, determine the interval containing σ.
(A) less than 10%
(B) At least 10% but less than 20%
(C) At least 20% but less than 30%
(D) At least 30% but less than 40%
(E) At least 40%.

Solution.
We test the option prices that would be generated by volatilities of 0.06, 0.10, 0.20, 0.30, and 0.40. We have

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{0.06+\sigma} \text{ and } d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{0.06-\sigma}.$$

Because we know that $\sigma > 0.06$, we know that $d < e^{0.06-0.06} = 1$, so that $C_d = 0$. Thus, in calculating the option price we need to only apply the formula

$$\begin{aligned}
C =& e^{-rh}[p_uC_u + (1-p_u)C_d] = e^{-0.06}[p_uC_u + (1-p_u)0]\\
=& e^{-0.09}p_u[300u - 300]\\
=& e^{-0.09}\left(\frac{e^{0.06}-d}{u-d}\right)(300u-300)\\
=& 300e^{-0.09}\left(\frac{e^{0.06}-e^{0.06-\sigma}}{e^{0.06+\sigma}-e^{0.06-\sigma}}\right)(e^{0.06+\sigma}-1)
\end{aligned}$$

Now we can insert various values of σ to find corresponding values of $C(\sigma)$.

$$\begin{aligned}
C(0.06) =& 16.95\\
C(0.10) =& 22.60\\
C(0.20) =& 36.65\\
C(0.30) =& 50.56\\
C(0.40) =& 64.27
\end{aligned}$$

Thus, the implied volatility is between 0.30 and 0.40 so that the correct answer is (D) ■

European puts and calls on the same asset, strike, and maturity should have the same implied volatility to prevent arbitrage. That is, the call and put prices satisfy the put-parity relationship.

A Typical pattern for volatility with regards to strike price and maturity time occurs[1]:
• For fixed time to maturity, implied volatility tends to decrease as strike price increases.
• For fixed strike price, implied volatility tends to decrease as time to expiry increases.

Consider the graph of the implied volatility against the strike price:
• If the graph is a skew curvature then we refer to the graph as **volatility skew.** If the graph a right-skewed curve then the volatility is higher as the option goes more in-the-money and lower as the option goes out-of-the

[1]Since implied volatility changes with respect to time and expiry, the Black-Scholes model does not fully describe option prices. Nevertheless, it is widely used Benchmark

money. A similar observation for left-skewed curve. A right-skewed curve is also called **volatility smirk**.

• When both the in-the-money and out-of-the money options have a higher implied volatility than at-the-money options, we have a **volatility smile.**

• Rarely, both the in-the-money and out-of-the money options have a lower implied volatility than at-the-money options, and we have a **volatility frown.**

Implied volatility is important for a number of reasons. Namely,

(1) It allows pricing other options on the same stock for which there may not be observed market price. This is done by setting the Black-Scholes formula to the price of a traded option on the same stock and solving the equation for the volatility. This volatility can be used in conjunction with the Black-Scholes formula to price options on the same stock with no observed market price.

(2) It is a quick way to describe the level of option prices. That is, a range of option prices for the same asset can be assigned one single volatility.

(3) Volatility skew provides a measure of how good the Black-Scholes pricing model is. Any assets with options displaying a Volatility Skew is displaying inconsistency with the theories of the Black-Scholes Model in terms of constant volatility.

Practice Problems

Problem 34.1
A stock currently costs \$41 per share. In one year, it might increase to \$60. The annual continuously compounded risk-free interest rate is 0.08, and the stock pays dividends at an annual continuously compounded yield of 0.03. Find the implied volatility of this stock using a one-period binomial model.

Problem 34.2
The stock of GS LLC currently costs \$228 per share. The annual continuously compounded risk-free interest rate is 0.11, and the stock pays dividends at an annual continuously compounded yield of 0.03. The stock price will be \$281 next year if it increase. What will the stock price be next year if it decreases? Use a one-period binomial model.

Problem 34.3
For a 1-year European call, the following information are given:
• The current stock price is \$40
• The strike price is \$45
• The continuously compounded risk-free interest rate is 5%
• $\Delta = 0.5$. That is, for every increase of \$1 in the stock price, the option price increases by \$0.50.
Determine the implied volatility.

Problem 34.4
Consider a nondividend-paying stock with a current price of S, and which can go up to 160, or down to 120, one year from now. Find the implied volatility, σ, of the stock, assuming the binomial model framework.

Problem 34.5
Given the following information about a 1-year European put on a stock:
• The current price of the stock \$35
• The strike price is \$35
• The continuously compounded risk-free interest rate is 8%
• The continuously compounded dividend yield is 6%.
• The observed put option price is \$3.58.
Is $\sigma = 20\%$ the implied volatility of the stock under the Black-Scholes framework?

Problem 34.6 ‡
A one-year European call option is currently valued at 0.9645. The following parameters are given:
• The current stock price is 10
• The continuously compounded risk-free rate is 6%
• The continuously compounded dividend yield is 1%
• The strike price is 10.
Using a single-period binomial model and assuming the implied volatility of the stock to be at least 5%, determine the interval containing σ.
(A) less than 10%
(B) At least 10% but less than 20%
(C) At least 20% but less than 30%
(D) At least 30% but less than 40%
(E) At least 40%.

Problem 34.7 ‡
Assume the Black-Scholes framework. Consider a one-year at-the-money European put option on a nondividend-paying stock.
You are given:
(i) The ratio of the put option price to the stock price is less than 5%.
(ii) Delta of the put option is -0.4364.
(iii) The continuously compounded risk-free interest rate is 1.2%.
Determine the stock's volatility.

Option Hedging

In this chapter we explore the Black-Scholes framework by considering the market-maker perspective on options. More specifically, we look at the issues that a market professional encounters. In deriving the Black-Scholes formula, it is assumed that the market-makers are profit-maximizers who want to hedge (i.e., minimize) the risk of their option positions. On average, a competetive market-maker should expect to break even by hedging. It turns out that the break even price is just the Black-Scholes option price. Also, we will see that the hedging position can be expressed in terms of Greeks: delta, gamma, and theta.

We have been using the term "market-maker" in the introduction of this chapter. What do we mean by that term? By a **market-maker** we mean a market trader who sells assets or contracts to buyers and buys them from sellers. In other words, he/she is an intermediary between the buyers and sellers. Market-makers buy at the bid and sell at the ask and thus profit from the ask-bid spread. They do not speculate! A market-maker generates inventory as needed by short-selling.

In contrast to market-makers, **proprietary traders** are traders who buy and sell based on view of the market, if that view is correct, trading is profitable, if not leads to losses. Mostly speculators engage in proprietary trading.

35 Delta-Hedging

Market-makers have positions generated by fulfilling customer orders. They need to hedge the risk of these positions for otherwise an adverse price move has the potential to bankrupt the market-maker.
One way to control risk is by **delta-hedging** which is an option strategy that aims to reduce (hedge) the risk associated with price movements in the underlying asset by offsetting long and short positions. This requires computing the option Greek delta and this explains the use of the term "delta-hedged." For example, a long call position may be delta-hedged by shorting the underlying stock. For a written call, the position is hedged by buying shares of stock. The appropriate number of shares is determined by the number delta. The delta is chosen so that the portfolio is delta-neutral, that is, the sum of the deltas of the portfolio components is zero.
There is a cost for a delta-hedged position: The costs from the long and short positions are not the same and therefore the market-maker must invest capital to maintain a delta-hedged position. Delta-hedged positions should expect to earn the risk-free rate: the money invested to maintain a delta-hedged position is tied up and should earn a return on it and moreover it is riskless.
Next, we examine the effect of an unhedged position.

Example 35.1
Suppose that the current price of a stock is \$40. The stock has a volatility of 30% and pays no dividends. The continuously compounded risk-free interest rate is 8%. A customer buys a call option from the market-maker with strike price \$40 and time to maturity of 91 days. The call is written on 100 shares of the stock.
(a) Using the Black-Scholes framework, find C and Δ at day the time of the transaction.
(b) What is the risk for the market-maker?
(c) Suppose that the market-maker leaves his position unhedged. What is the realized profit if the stock increases to \$40.50 the next day?
(d) Suppose that the market maker-hedges his position by buying 0.58240 shares of the stock. What is his profit/loss in the next day when the price increases to \$40.50?
(e) The next day the value of delta has increased. What is the cost of keeping the portfolio hedged?

(f) Now on Day 2, the stock price goes down to \$39.25. What is the market-maker gain/loss on that day?

Solution.
(a) Using the Black-Scholes formula we find

$$C = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(40/40) + (0.08 - 0 + 0.5(0.3)^2) \times \frac{91}{365}}{0.3\sqrt{\frac{91}{365}}} = 0.2080477495.$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = 0.2080477495 - 0.3\sqrt{\frac{91}{365}} = 0.0582533699$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.2080477495} e^{-\frac{x^2}{2}}dx = 0.582404$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.0582533699} e^{-\frac{x^2}{2}}dx = 0.523227.$$

Hence,

$$C = 40 \times 0.582404 - 40e^{-0.08\times\frac{91}{365}} \times 0.523227 = \$2.7804$$

From the market-maker perspective we have $\Delta = -e^{-\delta T}N(d_1) = -0.5824$.
(b) The risk for the market-maker who has written the call option is that the stock price will rise.
(c) Suppose the stock rises to \$40.50 the next day. We now have

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(40.50/40) + (0.08 - 0 + 0.5(0.3)^2) \times \frac{90}{365}}{0.3\sqrt{\frac{90}{365}}} = 0.290291$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = 0.290291 - 0.3\sqrt{\frac{90}{365}} = 0.141322.$$

Thus,

$$N(d_1)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.290291}e^{-\frac{x^2}{2}}dx=0.614203$$

and

$$N(d_2)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.141322}e^{-\frac{x^2}{2}}dx=0.556192.$$

Hence,

$$C=40.5\times 0.614203-40e^{-0.08\times\frac{90}{365}}\times 0.556192=\$3.0621.$$

Thus, there is a one day gain on the premium so the market-maker profit is $2.7804e^{0.08\times\frac{1}{365}}-3.0621=-\$0.281$. That is a loss of about 28 cents a share or \$28 for 100 shares.
(d) On day 0, the hedged portfolio consists of buying 58.24 shares at \$40 for a total cost of $58.24\times 40=\$2329.60$. Since the market-maker received only $2.7804\times 100=\$278.04$, he/she must borrow $2329.60-278.04=\$2051.56$. Thus, the initial position from the market-maker perspective is $-2329.60+2051.56+278.04=\$0$.
The finance charge for a day on the loan is

$$2051.56(e^{0.08\times\frac{1}{365}}-1)=\$0.45$$

If the price of the stock goes up to \$40.50 the next day, then there is a gain on the stock, a loss on the option and a one-day finance charge on the loan. Thus, the market-maker realizes a profit of

$$58.24(40.50-40)+(278.04-306.21)-0.45=\$0.50.$$

This process is referred to as **marking-the-market.** Thus, there is an overnight profit of \$0.50.
(e) The delta for the price of \$40.50 is $\Delta=-N(d_1)=-0.6142$. In order, to delta-neutralize the portfolio, the market-maker needs to buy additional $61.42-58.24=3.18$ shares at a cost of $3.18\times 40.50=\$128.79$. This process is known as **rebalancing the portfolio**.
(f) **Marking-the-market:** The new call option price is \$2.3282 per share. In this case, there is a loss on the shares of $61.42\times(39.25-40.50)=-\$76.78$ and a gain on the option of about $306.21-232.82=\$73.39$, plus the finance charge. Thus, the Day 2 profit is

$$73.39-76.78-(2051.56+128.79)(e^{0.08\times\frac{1}{365}}-1)=-\$3.87.$$

That is, a loss of \$3.87 ■
The daily profit calculation over 5 days for the hedged portfolio is given next.

	Day					
	0	1	2	3	4	5
Stock (\$)	40.00	40.50	39.25	38.75	40.00	40.00
Call (\$)	278.04	306.21	232.82	205.46	271.04	269.27
Option Delta	0.5824	0.6142	0.5311	0.4956	0.5806	0.5801
Investment (\$)	2,051.58	2,181.30	1,851.65	1,715.12	2,051.35	2,051.29
Interest (\$)		−0.45	−0.48	−0.41	−0.38	−0.45
Capital Gain (\$)		0.95	−3.39	0.81	−3.62	1.77
Daily Profit (\$)		0.50	−3.87	0.40	−4.00	1.32

Table 35.1

From this table, we notice that the return on a delta-hedged position does not depend on the direction in which the stock price moves, but it does depend on the magnitude of the stock price move.
The three sources of cash flow into and out of the portfolio in the previous example are:
• **Borrowing:** The market-maker capacity to borrowing is limited by the market value of the securities in the portfolio. In practice, the market-maker can borrow only part of the funds required to buy the securities so he/she must have capital to make up the difference.
•**Purchase or Sale of shares** The market-maker must buy-sell shares in order to offset changes in the option price.
•**Interest:** The finance charges paid on borrowed money.

Remark 35.1
The calculation of the profits in the above example is referred to as **mark-to-market** profits. With positive mark-to-market profit the market-maker is allowed to take money out of the portfolio. In the case of negative mark-to-market profit the investor must put money into the portfolio. A hedged portfolio that never requires additional cash investments to remain hedged is called a **self-financing** portfolio. It can be shown that any portfolio for which the stock moves according to the binomial model is approximately self-financing. More specifically, the delta-hedged portfolio breaks even if the stock moves one standard deviation. See Problem 35.6.

Example 35.2
A stock is currently trading for \$40 per share. The stock will pay no dividends. The annual continuously compounded risk-free interest rate is 0.08, and the stock price volatility is 0.3. Suppose you sell a 40-strike put on 100 shares with 91 days to expiration.
(a) What is delta?
(b) What investment is required for a delta-hedged portfolio?
(c) What is your profit the next day if the stock falls to \$39?
(d) What if the stock goes up to \$40.50 instead?

Solution.
(a) Using the Black-Scholes formula we find

$$P = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$$

where

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(40/40) + (0.08 - 0 + 0.5(0.3)^2) \times \frac{91}{365}}{0.3\sqrt{\frac{91}{365}}} = 0.20805$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = 0.20805 - 0.3\sqrt{\frac{91}{365}} = 0.0582.$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.20805} e^{-\frac{x^2}{2}}dx = 0.582405$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.0582} e^{-\frac{x^2}{2}}dx = 0.523205.$$

Hence,

$$P = 40e^{-0.08 \times \frac{91}{365}}(1 - 0.523205) - 40(1 - 0.582405) = \$1.991$$

and

$$\Delta = -e^{-\delta T}N(-d_1) = -(1 - 0.582405) = -0.4176.$$

The value of delta from the market-maker perspective is $\Delta = 0.4176$.
(b) The market-maker sells 41.76 shares and deposit the amount

$$41.76 \times 40 + 199.1 = \$1869.50$$

at a savings account earning the risk-free interest rate. The initial position, from the perspective of the market-maker is

$$41.76 \times 40 + 199.1 - 1869.50 = \$0.$$

(c) Suppose the stock falls to \$39.00 the next day. We now have

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(39/40) + (0.08 - 0 + 0.5(0.3)^2) \times \frac{90}{365}}{0.3\sqrt{\frac{90}{365}}} = 0.03695$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = 0.03695 - 0.3\sqrt{\frac{90}{365}} = -0.1121.$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.03695} e^{-\frac{x^2}{2}}dx = 0.514738$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-0.1121} e^{-\frac{x^2}{2}}dx = 0.455372.$$

Hence,

$$P = 40e^{-0.08\times\frac{90}{365}}(1 - 0.455372) - 39(1 - 0.514738) = \$2.434.$$

Hence, there is a gain on the shares but a loss on the option and gain from the interest. The market-maker overnight profit is

$$41.76(40 - 39) - (243.40 - 199.10) + 1869.50(e^{0.08\times\frac{1}{365}} - 1) = -\$2.130.$$

(d) Suppose the stock rises to \$40.50 the next day. We now have

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(40.50/40) + (0.08 - 0 + 0.5(0.3)^2) \times \frac{90}{365}}{0.3\sqrt{\frac{90}{365}}} = 0.290291$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = 0.290291 - 0.3\sqrt{\frac{90}{365}} = 0.141322.$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.290291} e^{-\frac{x^2}{2}}dx = 0.614203$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.141322} e^{-\frac{x^2}{2}} dx = 0.556192.$$

Hence,

$$P = 40e^{-0.08\times\frac{90}{365}}(1 - 0.556192) - 40.50(1 - 0.614203) = \$1.7808.$$

Thus, there is a gain on the option but a loss on the shares and interest gain. The overnight profit is

$$(199.10 - 178.08) - 41.76 \times 0.50 + 1869.50(e^{0.08\times\frac{1}{365}} - 1) = \$0.55 \blacksquare$$

Practice Problems

Problem 35.1
A stock is currently trading for \$40 per share. The stock will pay no dividends. The annual continuously compounded risk-free interest rate is 0.08, and the stock price volatility is 0.3. Suppose you sell a 45-strike call on 100 shares with 91 days to expiration.
(a) What is delta?
(b) What investment is required for a delta-hedged portfolio?
(c) What is your profit the next day if the stock falls to \$39?
(d) What if the stock goes up to \$40.50?

Problem 35.2
A stock is currently trading for \$40 per share. The stock will pay no dividends. The annual continuously compounded risk-free interest rate is 0.08, and the stock price volatility is 0.3. Suppose you buy a 40-strike call and sell a 45-strike call both expiring in 91 days. Both calls are written on 100 shares.
(a) What investment is required for a delta-hedged portfolio?
(b) What is your profit the next day if the stock falls to \$39?
(c) What if the stock goes up to \$40.50 instead?

Problem 35.3
A stock is currently trading for \$60 per share. The stock will pay no dividends. The annual continuously compounded risk-free interest rate is 0.08, and the stock price volatility is 0.25. Suppose you sell a 60-strike call expiring in 91 days. What is you overnight profit if the stock goes up to \$60.35?

Problem 35.4
A stock is currently trading for \$40 per share. The stock will pay no dividends. The annual continuously compounded risk-free interest rate is 0.08, and the stock price volatility is 0.3. Suppose you buy a 45-strike put and sell two 40-strike puts both expiring in 91 days. That is, you enter into a put ratio spread. Both puts are written on 100 shares.
(a) What investment is required for a delta-hedged portfolio?
(b) What is your profit the next day if the stock falls to \$39?
(c) What if the stock goes up to \$40.50 instead?

Problem 35.5
Suppose that the stock moves up and down according to the binomial model with $h = \frac{1}{365}$. Assume that on Days 1 and 5 the stock price goes up and on Days 2, 3, and 4 it goes down. Find the magnitude move $\sigma\sqrt{h}$. Complete the following table. Assume $\sigma = 0.3$ and $r = 0.42\%$. Hint: Recall that $S_{t+h} = S_t e^{rh \pm \sigma\sqrt{h}}$.

Day	0	1	2	3	4	5
Stock Price	40					

Problem 35.6
Construct a table similar to Table 35.1 using the daily change in stock prices found in the previous problem.

Problem 35.7 ‡
A market-maker has sold 100 call options, each covering 100 shares of a dividend-paying stock, and has delta-hedged by purchasing the underlying stock.
You are given the following information about the market-maker's investment:
• The current stock price is \$40.
• The continuously compounded risk-free rate is 9%.
• The continuous dividend yield of the stock is 7%.
• The time to expiration of the options is 12 months.
• $N(d_1) = 0.5793$
• $N(d_2) = 0.5000$
The price of the stock quickly jumps to \$41 before the market-maker can react. This change the price of one call option to increase by \$56.08. Calculate the net profit on the market-maker investment associated with this price move. Note: Because of sudden jump interest on borrowing or lending is ignored.

Problem 35.8 ‡
Several months ago, an investor sold 100 units of a one-year European call option on a nondividend-paying stock. She immediately delta-hedged the commitment with shares of the stock, but has not ever re-balanced her portfolio. She now decides to close out all positions.
You are given the following information:
(i) The risk-free interest rate is constant.
(ii)

	Several Months Ago	Now
Stock Price	\$40	\$50
Call Option Price	\$8.88	\$14.42
Put Option Price	\$1.63	\$0.26
Call Option Delta	0.794	

The put option in the table above is a European option on the same stock and with the same strike price and expiration date as the call option.
Calculate her profit.

36 Option Price Approximations: Delta and Delta-Gamma Approximations

In this section we use finite Taylor approximations to estimate option price movements when the underlying stock price changes, particularly first and second order (delta and gamma.)
We start by recalling from calculus the Taylor series expansion given by

$$f(x+h) = f(x) + h\frac{df}{dx}(x) + \frac{h^2}{2}\frac{d^2f}{dx^2} + \frac{h^3}{6}\frac{d^3f}{dx^3}(x) + \text{higher order terms.}$$

Let $V(S_t)$ denote the option price when the stock price is S_t. Recall that

$$\Delta = \frac{\partial V}{\partial S}.$$

A linear approximation of the value of the option when the stock price is S_{t+h} can be found using Taylor series of first order given by

$$V(S_{t+h}) = V(S_t) + \epsilon\Delta(S_t) + \text{higher order terms}$$

where $\epsilon = S_{t+h} - S_t$ is the stock price change over a time interval of length h. We call the approximation

$$V(S_{t+h}) \approx V(S_t) + \epsilon\Delta(S_t)$$

the **delta approximation.** Notice that the delta approximation uses the value of delta at S_t.
Recall that the price functions (for a purchased option) are convex functions of the stock price (See Section 29). The delta approximation is a tangent line to the graph of the option price. Hence, the delta approximation is always an underestimate of the option price. See Figure 36.1.

Example 36.1
Consider a nondividend paying stock. The annual continuously compounded risk-free interest rate is 0.08, and the stock price volatility is 0.3. Consider a 40-strike call on 100 shares with 91 days to expiration.
(a) What is the option price today if the current stock price is \$40?
(b) What is the option price today if the stock price is \$40.75?
(c) Estimate the option price found in (b) using the delta approximation.

Solution.
(a) Using the Black-Scholes formula we find

$$C = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(40/40) + (0.08 - 0 + 0.5(0.3)^2) \times \frac{91}{365}}{0.3\sqrt{\frac{91}{365}}} = 0.2080477495.$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = 0.2080477495 - 0.3\sqrt{\frac{91}{365}} = 0.0582533699$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.2080477495} e^{-\frac{x^2}{2}}dx = 0.582404$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.0582533699} e^{-\frac{x^2}{2}}dx = 0.523227.$$

Hence,

$$C = 40 \times 0.582404 - 40e^{-0.08\times\frac{91}{365}} \times 0.523227 = \$2.7804$$

and

$$\Delta = e^{-\delta T}N(d_1) = 0.5824.$$

(b) Suppose that the option price goes up to \$40.75. Then

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(40.75/40) + (0.08 - 0 + 0.5(0.3)^2) \times \frac{90}{365}}{0.3\sqrt{\frac{91}{365}}} = 0.3320603167$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = 0.3320603167 - 0.3\sqrt{\frac{91}{365}} = 0.1822659371.$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.3320603167} e^{-\frac{x^2}{2}}dx = 0.630078$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.1822659371} e^{-\frac{x^2}{2}} dx = 0.572313.$$

Hence,

$$C(\$40.75) = 40.75 \times 0.630078 - 40e^{-0.08 \times \frac{91}{365}} \times 0.572313 = \$3.2352$$

(c) Using the delta approximation we find

$$C(\$40.75) = C(\$40) + 0.75 \times 0.5824 = 2.7804 + 0.75 \times 0.5824 = \$3.2172.$$

The error is $3.2172 - 3.2352 = -0.018$ ■

A second method of approximation involves both delta and gamma and uses Taylor approximation of order two:

$$V(S_{t+h}) = V(S_t) + \epsilon\Delta(S_t) + \frac{1}{2}\epsilon^2\Gamma(S_t) + \text{higher order terms}.$$

We define the **delta-gamma** approximation by

$$V(S_{t+h}) \approx V(S_t) + \epsilon\Delta(S_t) + \frac{1}{2}\epsilon^2\Gamma(S_t).$$

Example 36.2
Consider the information of the previous example.
(a) Find the option Greek gamma.
(b) Estimate the value of $C(\$40.75)$ using the delta-gamma approximation.

Solution.
(a) We have

$$\Gamma_{\text{call}} = \frac{e^{-\delta(T-t)}N'(d_1)}{S\sigma\sqrt{T-t}} = \frac{e^{-\frac{0.2080477495^2}{2}}}{40 \times 0.3\sqrt{2\pi} \times \sqrt{\frac{91}{365}}} = 0.0652.$$

(b) We have the estimate

$$C(\$40.75) = 2.7804 + 0.5824(0.75) + 0.5 \times 0.75^2 \times 0.0652 = \$3.2355.$$

In this case, the error is $3.2355 - 3.2352 = 0.0003$. Thus, the approximation is significantly closer to the true option price at \$40.75 than the delta approximation ■

Figure 36.1 shows the result of approximating the option price using the delta and delta-gamma approximations.

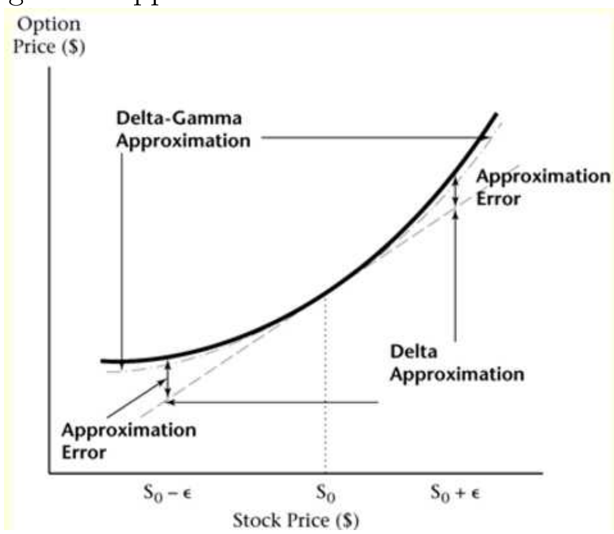


Figure 36.1

Practice Problems

Problem 36.1
For a stock price of \$40 the price of call option on the stock is \$2.7804. When the stock goes up to \$40.75 the price of the option estimated by the delta approximations is \$3.2172. Find $\Delta(40)$.

Problem 36.2
A stock is currently trading for \$45 per share. The stock will pay no dividends. The annual continuously compounded risk-free interest rate is 0.08, and the stock price volatility is 0.3. Consider a 45-strike call on 100 shares with 91 days to expiration. Use delta approximation to estimate $C(\$44.75)$.

Problem 36.3
A stock is currently trading for \$45 per share. The stock will pay no dividends. The annual continuously compounded risk-free interest rate is 0.08, and the stock price volatility is 0.3. Consider a 45-strike call on 100 shares with 91 days to expiration. Use delta-gamma approximation to estimate $C(\$44.75)$.

Problem 36.4
A stock is currently trading for \$40 per share. The stock will pay no dividends. The annual continuously compounded risk-free interest rate is 0.08, and the stock price volatility is 0.3. Consider a 40-strike put on 100 shares with 91 days to expiration. Using the delta approximation estimate $P(\$40.55)$.

Problem 36.5
A stock is currently trading for \$40 per share. The stock will pay no dividends. The annual continuously compounded risk-free interest rate is 0.08, and the stock price volatility is 0.3. Consider a 40-strike put on 100 shares with 91 days to expiration. Using the delta-gamma approximation estimate $P(\$40.55)$.

Problem 36.6
A stock is currently trading for \$1200 per share. A certain call option on the stock has a price of \$35, a delta of 0.72, and a certain value of gamma. When the stock price suddenly falls to \$1178, the call price falls to \$23. Using the delta-gamma approximation, what is the gamma of this call option?

Problem 36.7
A stock is currently trading for \$13 per share. A certain call option on the stock has a price of \$1.34, a gamma of 0.025, and a certain value of delta. When the stock price suddenly rises to \$19 per share, the call option price increases to \$5.67. Using the delta-gamma approximation, what is the original delta of this call option?

Problem 36.8
The stock of GS Co. currently trades for \$657 per share. A certain call option on the stock has a price of \$120, a delta of 0.47, and a gamma of 0.01. Use a delta-gamma approximation to find the price of the call option if, after 1 second, the stock of GS Co. suddenly begins trading at \$699 per share.

Problem 36.9
When the stock of GS Co. suddenly decreased in price by \$6 per share, a certain put option on the stock increased in price to \$5.99. The put option had an original Δ of -0.49 and a gamma of 0.002. Find the original put option price using the delta-gamma approximation.

Problem 36.10
A stock is currently trading for a price greater than \$75 per share. A certain put option on the stock has a price of \$5.92, a delta of -0.323, and a gamma of 0.015. Use a delta-gamma approximation to find the current price of the stock if, after 1 second, the put is valued at \$6.08 when the stock price is \$86.

Problem 36.11 ‡
Assume that the Black-Scholes framework holds. The price of a nondividend-paying stock is \$30.00. The price of a put option on this stock is \$4.00.
You are given:
(i) $\Delta = -0.28$
(ii) $\Gamma = 0.10$.
Using the delta-gamma approximation, determine the price of the put option if the stock price changes to \$31.50.

Problem 36.12 ‡
Assume that the Black-Scholes framework holds. Consider an option on a stock.
You are given the following information at time 0:

(i) The stock price is $S(0)$, which is greater than 80.
(ii) The option price is 2.34.
(iii) The option delta is -0.181.
(iv) The option gamma is 0.035.
The stock price changes to 86.00. Using the delta-gamma approximation, you find that the option price changes to 2.21.
Determine $S(0)$.

37 The Delta-Gamma-Theta Approximation and the Market-Maker's Profit

The delta-gamma approximation discussed in the previous section does not take into consideration the sensitivity of the option price change due to time passing. Recall that the Greek option theta measures the option's change in price due to time passing. Thus, the change in price over a period of length h is approximately $h\theta(S_t)$. Adding this term to the delta-gamma approximation formula we obtain

$$V(S_t, T-t-h) \approx V(S_t, T-t) + \epsilon\Delta(S_t, T-t) + \frac{1}{2}\epsilon^2\Gamma(S_t, T-t) + h\theta(S_t, T-t). \tag{37.1}$$

This formula is known as the **delta-gamma-theta approximation.**

Example 37.1
Yesterday, a nondividend paying stock was selling for \$40 and a call option on the stock was selling for \$2.7804 and has 91 days left to expiration. The call option on the stock had a delta of 0.5824, a gamma of 0.0652, and an annual theta of -6.3145. Today, the stock trades for \$40.75. The annual continuously compounded risk-free interest rate is 0.08. Find the new option price using the delta-gamma-theta approximation.

Solution.
Using the delta-gamma-theta approximation we have

$$\begin{aligned} C(40, \frac{90}{365}) \approx & C(40, \frac{91}{365}) + \epsilon\Delta(40, \frac{91}{365}) + \frac{1}{2}\epsilon^2\Gamma(40, \frac{91}{365}) + h\theta(40, \frac{91}{365}) \\ = & 2.7804 + 0.75(0.5824) + \frac{1}{2} \times (0.75)^2 0.0652 + \frac{1}{365}(-6.3145) = \$3.3637 \blacksquare \end{aligned}$$

We next examine the market-maker profit. Consider the case of long delta shares and short a call. The value of the market-maker investment is the cost of the stock plus the proceeds received from selling the call, that is,

$$C(S_t) - S_t\Delta(S_t) < 0.$$

This amount is borrowed at the risk-free interest rate r. Now, suppose that over the time interval h, the stock price changes from S_t to S_{t+h}. In this case, the market-maker profit is

$$(S_{t+h} - S_t)\Delta(S_t) - (C(S_{t+h}) - C(S_t)) - rh[S_t\Delta - C(S_t)].$$

But from equation (37.1), using $\epsilon = S_{t+h} - S_t$, we have

$$C(S_{t+h}) - C(S_t) = \epsilon\Delta(S_t) + \frac{1}{2}\epsilon^2\Gamma(S_t) + h\theta.$$

Thus, the market-maker profit can be expressed in the form

$$\begin{aligned}\text{Market-Maker Profit} =& \epsilon\Delta(S_t) - \left[\epsilon\Delta(S_t) + \frac{1}{2}\epsilon^2\Gamma(S_t) + h\theta\right] - rh[S_t\Delta(S_t) - C(S_t)] \\ =& -\left(\frac{1}{2}\epsilon^2\Gamma(S_t) + h\theta(S_t) + rh[S_t\Delta(S_t) - C(S_t)]\right).\end{aligned} \tag{37.2}$$

From this equation we observe the following:

• **Gamma factor:** Because the gamma of a purchased call is positive (so a stock price increase will increase the value of the call) and the market-maker is the writer of the call then the market-maker will lose money in proportion to the square of the stock price change ϵ. Thus, the larger the stock move, the greater is the loss.

• **Theta factor:** The theta for a purchased call is negative which means that the call value decreases with time passing. Since the market-maker is the writer of the call, he/she benefits from theta.

• **Interest cost:** The market-maker being the seller has a net investment which requires borrowing since the $\Delta(S_t)$ shares are more expensive than one short option. The net cost of borrowing for a time interval h is the term $-rh[S_t\Delta(S_t) - C(S_t)]$.

It follows that time passing benefits the market-maker, whereas interest and gamma work against him or her.

Example 37.2

22 days ago, GS stock traded for \$511 per share. A certain call option on the stock had a delta of 0.66, a gamma of 0.001, and an annual theta of -10.95. The option used to trade for \$59. Now the stock trades for \$556. The annual continuously compounded risk-free interest rate is 0.08. A hypothetical market-maker has purchased delta shares and short-sold one call. Find what a market-maker's profit on one such option would be using (37.2).

Solution.
Using (37.2) we find

$$\begin{aligned}\text{Market-Maker profit} =& -\left(\frac{1}{2}\epsilon^2\Gamma(S_t) + h\theta(S_t) + rh[S_t\Delta(S_t) - C(S_t)]\right)\\ =& -\left(\frac{1}{2}\times 45^2\times 0.001 + (-10.95)\times\frac{22}{365} + 0.08\times\frac{22}{365}[0.66\times 511 - 59]\right)\\ =& -\$1.694246849 \blacksquare\end{aligned}$$

Now, we examine the effect of ϵ^2 on the profit. Recall from Section 35 that the magnitude of ϵ^2- and not the direction of the stock price move$-$ affects the market-maker profit. Also, recall that the market-maker approximately breaks even for a one-standard deviation move in the stock. So if σ is measured annually, then a one-standard-deviation move over a period of length h is

$$\epsilon = S_{t+h} - S_t = S_t(1+\sigma\sqrt{h}) - S_t = \sigma S_t\sqrt{h}.$$

With this expression for ϵ, the market-maker profit when the stock moves one standard deviation is

$$\text{Market-Maker profit} = -\left(\frac{1}{2}\sigma^2 S_t^2\Gamma(S_t) + \theta(S_t) + r[S_t\Delta(S_t) - C(S_t)]\right)h.$$

Example 37.3
22 days ago, the stock of GS traded for \$511 per share. A certain call option on the stock had a delta of 0.66, a gamma of 0.001, and a daily theta of -0.03. The option used to trade for \$59. Now the stock trades for \$556. The annual continuously compounded risk-free interest rate is 0.08. A hypothetical market maker has purchased delta shares and short-sold the call. Assume a one-standard-deviation of stock price move, what is the annual standard deviation of the stock price movement?

Solution.
We have

$$\sigma = \left(\frac{\epsilon^2}{S_t^2 h}\right)^{\frac{1}{2}} = \left(\frac{45^2}{511^2\times\frac{22}{365}}\right)^{\frac{1}{2}} = 0.3586961419 \blacksquare$$

Greeks in the Binomial Model
We will use some of the relations of this section to compute the binomial

Greeks. Consider a binomial model with period of length h. At time $t = 0$ we have

$$\Delta(S,0) = e^{-\delta h}\frac{C_u - C_d}{uS - dS}.$$

That's the only Greek we can compute at that time. After one period, we have

$$\Delta(uS,h) = e^{-\delta h}\frac{C_{uu} - C_{ud}}{uuS - udS}$$

$$\Delta(dS,h) = e^{-\delta h}\frac{C_{ud} - C_{dd}}{udS - ddS}$$

$$\Gamma(S,0) \approx \Gamma(S_h,h) = \frac{\Delta(uS,h) - \Delta(dS,h)}{uS - dS}.$$

Now, letting

$$\epsilon = udS - S$$

we can use the delta-gamma-theta approximation to write

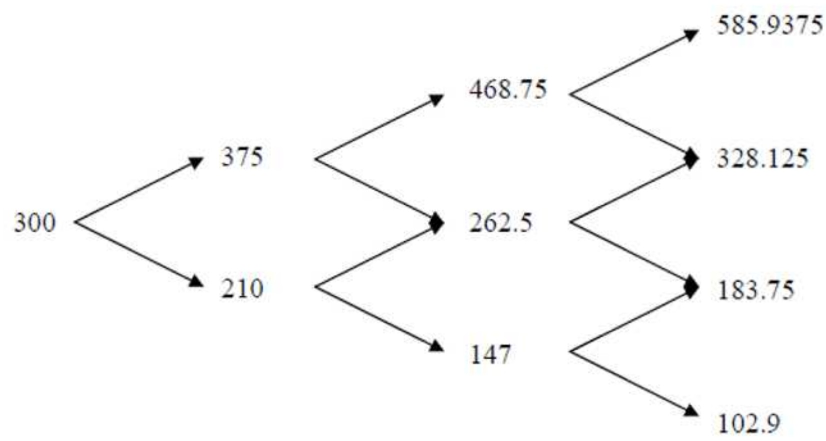
$$C(udS,2h) = C(S,0) + \epsilon\Delta(S,0) + \frac{1}{2}\epsilon^2\Gamma(S,0) + 2h\theta(S,0).$$

Solving this equation for θ we find

$$\theta(S,0) = \frac{C(udS,2h) - \epsilon\Delta(S,0) - 0.5\epsilon^2\Gamma(S,0) - C(S,0)}{2h}.$$

Here S is the stock price at time $t = 0$.

Example 37.4 ‡
Consider the following three-period binomial tree model for a stock that pays dividends continuously at a rate proportional to its price. The length of each period is 1 year, the continuously compounded risk-free interest rate is 10%, and the continuous dividend yield on the stock is 6.5%.



Approximate the value of gamma at time 0 for the 3-year at-the-money American put on the stock

Solution.
We first find u and d. We have $u = \frac{375}{300} = 1.25$ and $d = \frac{210}{300} = 0.70$. Thus,

$$p_u = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{300e^{0.10-0.065} - 210}{375 - 210} = 0.61022.$$

we have
Year 3, Stock Price $= u^3S = 585.9375$ Since we are at expiration, the option value is $P_{uuu} = 0$.
Year 3, Stock Price $= u^2dS = 328.125$ and $P_{uud} = 0$.
Year 3, Stock Price $= ud^2S = 183.75$ and $P_{udd} = P_{ddu} = 300 - 183.75 = 116.25$.
Year 3, Stock Price $= d^3S = 102.90$ and $P_{ddd} = 300 - 102.90 = 197.10$.
Year 2, Stock Price $= u^2S = 468.75$

$$\begin{aligned}P_{uu} =& \max\{300 - 468.75, e^{-0.10}[0.61022 \times 0 + (1 - 0.61022) \times 0]\}\\ =&0.\end{aligned}$$

Year 2, Stock Price $= udS = 262.50$

$$\begin{aligned}P_{ud} =& \max\{300 - 262.50, e^{-0.10}[0.61022 \times 0 + (1 - 0.61022) \times 116.25]\}\\ =&e^{-0.10}[0.61022 \times 0 + (1 - 0.61022) \times 116.25]\\ =&41.00\end{aligned}$$

Year 2, Stock Price $= d^2S = 147$

$$\begin{aligned}P_{dd} =& \max\{300 - 147, e^{-0.10}[0.61022 \times 116.25 + (1 - 0.61022) \times 197.10]\}\\=&300 - 147\\=&153.\end{aligned}$$

Hence,

$$\Delta(uS, h) = e^{-\delta h}\frac{P_{uu} - P_{ud}}{uuS - udS} = e^{-0.065\times 1}\frac{0 - 41}{468.75 - 262.50} = -0.1863$$

and

$$\Delta(dS, h) = e^{-\delta h}\frac{P_{ud} - P_{dd}}{udS - ddS} = e^{-0.065\times 1}\frac{41 - 153}{262.50 - 147} = -0.9087.$$

Thus,

$$\Gamma(S, 0) \approx \Gamma(S_h, h) = \frac{\Delta(uS, h) - \Delta(dS, h)}{uS - dS} = \frac{-0.1863 - (-0.9087)}{375 - 210} = 0.004378 \blacksquare$$

Practice Problems

Problem 37.1
Yesterday, a nondividend paying stock was selling for \$40 and a call option on the stock was selling for \$2.7804 and has 91 days left to expiration. The call option on the stock had a delta of 0.5824, a gamma of 0.0652, and a daily theta of -0.0173. Today, the stock trades for \$39.25. The annual continuously compounded risk-free interest rate is 0.08. Find the new option price using the delta-gamma-theta approximation.

Problem 37.2
A stock is currently trading for \$678 per share. 98 days ago, it traded for \$450 per share and a call option on the stock was selling for \$56. Now the call option trades for \$100. The option has a delta of 0.33 and a gamma of 0.006. What is the daily option theta? Use the delta-gamma-theta approximation.

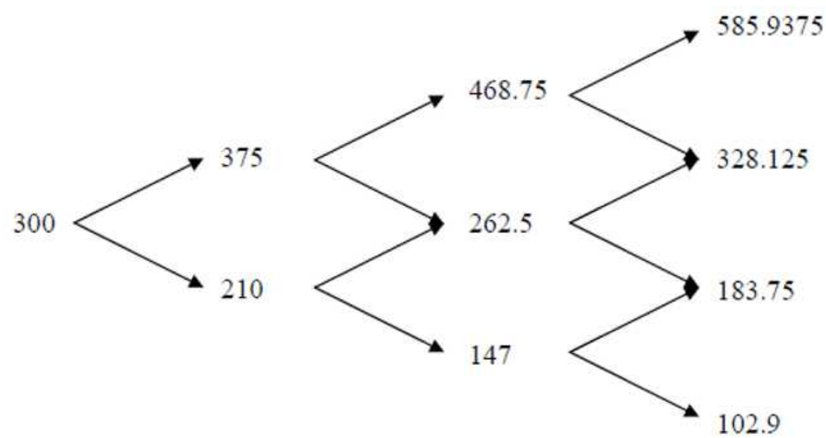
Problem 37.3
A call option on a nondividend-paying stock was valued \$6.13 yesterday when the stock price was \$54. Today, the stock price is \$56. Estimate the price of option using the delta-gamma-theta approximation. The option has a delta of 0.5910, a gamma of 0.0296 and an annualized theta of -14.0137.

Problem 37.4
GS stock has a price volatility of 0.55. A certain call option on the stock today costs \$71.80. The option has a delta of 0.32, a gamma of 0.001, and a daily theta of -0.06. The stock price today is \$3000 per share, and the annual continuously compounded risk-free interest rate is 0.1. Find what a market-maker's profit on one such option would be after 1 year using the delta-gamma-theta approximation. Assume stock price moves one standard deviation.

Problem 37.5 ‡
Which of the following are correct for option Greeks?
(I) If the gamma of a call is positive, by writing the call the market-maker will lose money in proportion to the stock price change.
(II) If the theta for a call is negative, the option writer benefits from theta.
(III) If, in order to hedge, the market-maker must purchase stock, then the net carrying cost is a component of the overall cost.

Problem 37.6
Yesterday, a stock traded for \$75 per share. A certain call option on the stock had a delta of 0.5910, a gamma of 0.0296, and an annual theta of -14.0317. The option used to trade for \$6.13. Now the stock trades for \$77. The annual continuously compounded risk-free interest rate is 0.10. A hypothetical market-maker has purchased delta shares and short-sold one call. Find what a market-maker's profit on one such option would be using (37.2).

Problem 37.7
Consider the following three-period binomial tree model for a stock that pays dividends continuously at a rate proportional to its price. The length of each period is 1 year, the continuously compounded risk-free interest rate is 10%, and the continuous dividend yield on the stock is 6.5%.



Compute the value of delta at time 0 for the 3-year at-the-money American put on the stock

Problem 37.8
Consider the same three-period binomial tree model as above for a stock that pays dividends continuously at a rate proportional to its price. The length of each period is 1 year, the continuously compounded risk-free interest rate is 10%, and the continuous dividend yield on the stock is 6.5%.
Estimate the value of theta at time 0 for the 3-year at-the-money American put on the stock

38 The Black-Scholes Analysis

In this section we describe the Black-Scholes analysis for pricing options which incorporate delta-hedging with pricing. We first start by listing the assumptions under which the Black-Scholes equation holds for options:

• The risk-free interest rate r and the stock volatility σ are constant over the life of the option.

• Both the stock and the option do not pay dividends during the life of the option.

• The stock price moves one standard deviation over a small time interval (i.e., binomial model is used).

Black and Scholes argued that the market-maker profit is zero for a one-standard-deviation of the price of a stock. But we know that the market-maker profit in this case is given by the formula

$$\text{Market-Maker profit} = -\left(\frac{1}{2}\sigma^2 S_t^2\Gamma(S_t) + \theta(S_t) + r[S_t\Delta(S_t) - C(S_t)]\right)h.$$

Thus, setting this equation to 0 and rearranging terms we find the differential equation

$$\frac{1}{2}\sigma^2 S_t^2\Gamma(S_t) + rS_t\Delta(S_t) + \theta = rC(S_t). \tag{38.1}$$

Since the Greeks Δ, Γ, and θ are partial derivatives, the previous equation is a partial differential equation of order 2. We call (38.1) the **Black-Scholes partial differential equation.** A similar equation holds for put options.

Example 38.1

A stock has a current price of \$30 per share, and the annual standard deviation of its price is 0.3. A certain call option on this stock has a delta of 0.4118, a gamma of 0.0866, and an annual theta of -4.3974. The annual continuously compounded risk-free interest rate is 0.08. What is the price of this call option, as found using the Black-Scholes equation?

Solution.

Using equation (38.1) we can write

$$\begin{aligned}C(S_t) =&\frac{1}{r}\left(\frac{1}{2}\sigma^2 S_t^2\Gamma(S_t) + rS_t\Delta(S_t) + \theta\right)\\ =&\frac{1}{0.08}\left(\frac{1}{2}\times 0.3^2\times 0.0866\times 30^2 + 0.08\times 0.4118\times 30 - 4.3974\right)\\ =&\$1.22775 \blacksquare\end{aligned}$$

Equation (38.1) holds for European options. The Black-Scholes equation applies to American options only when immediate exercise is not optimal. We examine this issue more closely. Consider an American call option where early exercise is optimal. Then the option price is $C = S - K$. Hence, $\Delta = \frac{\partial C}{\partial S} = 1, \Gamma = \frac{\partial^2 C}{\partial S^2} = 0$, and $\theta = \frac{\partial C}{\partial t} = 0$. In this case, equation (38.1) becomes

$$\frac{1}{2}\sigma^2 S_t^2 \times 0 + rS_t \times (1) + 0 = r(S - K).$$

This leads to $rK = 0$ which is impossible since $r > 0$ and $K > 0$.

Rehedging

Up to this point, we have assumed that market-makers maintain their hedged position every time there is a change in stock price. Because of transaction costs, this process becomes expensive. So what would the optimal frequency of rehedging be?

One approach to answer this question is to consider hedging at discrete intervals, rather than every time the stock price changes. For that purpose we introduce the Boyle-Emanuel framework:

• The market-maker shorts a call and delta-hedge it.

• Each discrete interval has length h with h in years. That is, rehedging occurring every h years.

• Let x_i be a standard random variable defined to be the number of standard deviations the stock price moves during time interval i. We assume that the x_is are independent random variables, that is, they are uncorrelated across time.

• Let $R_{h,i}$ denote the period−i return (not rate of return!). For x_i standard deviations the market-maker profit is given by

$$\text{Market-Maker profit} = -\left(\frac{1}{2}\sigma^2 S^2 x_i^2 \Gamma + \theta + r[S\Delta - C]\right)h.$$

Thus,

$$\begin{aligned}
R_{h,i} =& -\left(\frac{1}{2}\sigma^2 S^2 \Gamma + \theta + r[S\Delta - C]\right)h \\
& - \left\{-\left(\frac{1}{2}\sigma^2 S^2 x_i^2 \Gamma + \theta + r[S\Delta - C]\right)h\right\} \\
=& \frac{1}{2}\sigma^2 S^2 (x_i^2 - 1)\Gamma h
\end{aligned}$$

Thus, we can write

$$\text{Var}(R_{h,i}) = \left(\frac{1}{2}\sigma^2 S^2 \Gamma h\right)^2 \text{Var}(x_i^2 - 1).$$

But x_i^2 is a gamma distribution with $\alpha = 0.5$ and $\theta = 2$ so that $\text{Var}(x_i^2 - 1) = \text{Var}(x_i^2) = \alpha\theta^2 = 2$. Hence,

$$\text{Var}(R_{h,i}) = \frac{1}{2}\left(\sigma^2 S^2 \Gamma h\right)^2.$$

The annual variance of return is

$$\text{Annual variance of return} = \frac{1}{h}\text{Var}(R_{h,i}) = \frac{1}{2}\left(\sigma^2 S^2 \Gamma\right)^2 h.$$

This says that the annual variance of return is proportional to h. Thus, frequent hedging implies a smaller h and thus reduces the variance of the return. As an example, let's compare hedging once a day against hedging hourly. The daily variance of the return earned by the market-maker who hedges once a day is given by

$$\text{Var}(R_{\frac{1}{365},1}) = \frac{1}{2}\left(S^2\sigma^2\frac{\Gamma}{365}\right)^2.$$

Let $R_{\frac{1}{365\times 24},i}$ be the variance of the return of the market-maker who hedges hourly. But the daily return of the market-maker who hedges hourly is the sum of the hourly returns. Then, using independence, we have

$$\begin{aligned}
\text{Var}\left(\sum_{i=1}^{24} R_{h,i}\right) =& \sum_{i=1}^{24}\text{Var}(R_{\frac{1}{365\times 24},i})\\
=& \sum_{i=1}^{24}\frac{1}{2}[S^2\sigma^2\Gamma/(24\times 365)]^2\\
=& \frac{1}{24}\text{Var}(R_{\frac{1}{365},1})
\end{aligned}$$

Thus, by hedging hourly instead of daily the market-maker's variance daily return is reduced by a factor of 24.

Example 38.2
A delta-hedged market-maker has a short position in a call option on a certain stock. Readjusted hedges occur every 2 months. The stock has a price of \$45; the standard deviation of this price is 0.33. The gamma of the call option is 0.02. During a particular 2-month period, the stock price moves by 0.77 standard deviations. Find the variance of the return to the market-maker during this time period.

Solution.
Using Boyle-Emanuel formula we have

$$\begin{aligned}\mathrm{Var}(R_{\frac{1}{6},1}) =&\frac{1}{2}(S^2\sigma^2\Gamma/6)^2\\ =&\frac{1}{2}(45^2\times 0.33^2\times 0.02/6)^2 = 0.2701676278 \ \blacksquare\end{aligned}$$

Practice Problems

Problem 38.1
Show that the Black-Scholes equation does not hold for a put option with optimal early exercise.

Problem 38.2
A stock has a current price of \$30 per share, and the annual standard deviation of its price is 0.3. A certain put option on this stock has a delta of -0.5882, a gamma of 0.0866, and an annual theta of -1.8880. The annual continuously compounded risk-free interest rate is 0.08. What is the price of this put option, as found using the Black-Scholes equation?

Problem 38.3
A stock has a current price of \$50 per share, and the annual standard deviation of its price is 0.32. A certain European call option on this stock has a delta of 0.5901, a gamma of 0.0243, and an annual theta of -4.9231. The annual continuously compounded risk-free interest rate is 0.08. What is the price of a European put option on the stock? Both options have a strike price of \$53 and a 1-year maturity.

Problem 38.4
Assume that the Black-Scholes framework holds. The stock of GS has a price of \$93 per share, and the price has an annual standard deviation of 0.53. A certain European call option on the stock has a price of \$4, a delta of 0.53, and a gamma of 0.01. The annual continuously compounded risk-free interest rate is 0.02. What is the theta for this call option?

Problem 38.5
Assume that the Black-Scholes framework holds. The price of a stock has a standard deviation of 0.32. A certain put option on this stock has a price of \$2.59525, a delta of -0.5882, a gamma of 0.0866, and a theta of -1.8880. The annual continuously compounded risk-free interest rate is 0.08. What is the current price of one share of the stock? Round your answer to the nearest dollar.

Problem 38.6
Assume that the Black-Scholes framework holds. The stock of GS has a price of \$93 per share, and the price has an annual standard deviation of 0.53. A

certain European call option on the stock has a price of \$4, a delta of 0.53, and an annual theta of -13.0533205. The annual continuously compounded risk-free interest rate is 0.02. What is the gamma for this call option?

Problem 38.7
A stock has a price of \$30, and this price has a standard deviation of 0.3. A certain call option in this stock has a price of \$1.22775, a gamma of 0.0866, and a theta of -4.3974. The annual continuously compounded risk-free interest rate is 0.08. Find the delta of this call option.

Problem 38.8
A delta-hedged market-maker has a short position in a call option on a certain stock. Readjusted hedges occur every hour. The stock has a price of \$50; the standard deviation of this price is 0.30. The gamma of the call option is 0.0521. During a particular 1-hour period, the stock price moves by 0.57 standard deviations. Find the standard deviation of the hourly returns to the market-maker during this time period.

Problem 38.9
A delta-hedged market-maker has a long position in a call option on a certain stock. Readjusted hedges occur every 5 months. The stock has a price of \$500; the standard deviation of this price is 0.03. The gamma of the call option is 0.001. During a particular 5-month period, the stock price moves by 0.23 standard deviations. Find the return for the market-maker during this time period.

Problem 38.10
A delta-hedged market-maker has a short position in a call option on a certain stock. Readjusted hedges occur every 2 days. The stock has a price of \$80; the standard deviation of this price is 0.3. The gamma of the call option is 0.02058. Find the variance of the 2-day return if the market-maker hedges daily instead of every two days.

Problem 38.11 ‡
Consider the Black-Scholes framework. A market-maker, who delta-hedges, sells a three-month at-the-money European call option on a nondividend-paying stock.
You are given:
(i) The continuously compounded risk-free interest rate is 10%.

(ii) The current stock price is 50.
(iii) The current call option delta is 0.6179.
(iv) There are 365 days in the year.
If, after one day, the market-maker has zero profit or loss, determine the stock price move over the day.

39 Delta-Gamma Hedging

One of the major risk that a market-maker faces is the extreme moves of prices. There are at least four ways for a delta hedging market-maker to protect against extreme price moves.

(1)**Gamma-Neutrality:** Recall from the previous section that a delta-hedged portfolio with a negative gamma results in large losses for increasing move in the stock price. Thus, gamma is a factor that a delta-hedged market-maker needs to worry about. Since a delta-hedged portfolio maintains a delta-neutral position, it makes sense for the market-maker to adopt a gamma-neutral position. But the gamma of a stock is zero so the market-maker has to buy/sell options so as to offset the gamma of the existing position.

Example 39.1
Suppose that the current price of a stock is \$40. The stock has a volatility of 30% and pays no dividends. The continuously compounded risk-free interest rate is 8%. A customer buys a call option from the market-maker with strike price \$40 and time to maturity of three months. The call is written on 100 shares of the stock.
(a) Using the Black-Scholes framework, find C, Δ and Γ.
(b) Find C, Δ and Γ for a purchased 45-strike call with 4 months to expiration.
(c) Find the number of purchased 45-strike call needed for maintaining a gamma-hedged position.
(d) Find the number of shares of stock needed for maintaining a delta-hedged position.

Solution.
(a) Using the Black-Scholes formula we find

$$C = Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln(40/40) + (0.08 - 0 + 0.5(0.3)^2) \times \frac{3}{12}}{0.3\sqrt{\frac{3}{12}}} = 0.208333$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t} = 0.208333 - 0.3\sqrt{\frac{3}{12}} = 0.058333$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.208333} e^{-\frac{x^2}{2}}dx = 0.582516$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.058333} e^{-\frac{x^2}{2}}dx = 0.523258.$$

Hence,

$$C = 40 \times 0.582516 - 40e^{-0.08\times\frac{3}{12}} \times 0.523258 = \$2.7847,$$

$$\Delta = e^{-\delta(T-t)}N(d_1) = 0.5825,$$

and

$$\Gamma = \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \times \frac{1}{S\sigma\sqrt{T-t}} = 0.0651.$$

From the market-maker perspective we have $\Delta = -0.5825$ and $\Gamma = -0.0651$.
(b) Using the Black-Scholes formula as in (a), we find $C = \$1.3584, \Delta = 0.3285$, and $\Gamma = 0.0524$.
(c) Let n_1 be the number of K_1-strike written options and n_2 the number of K_2-strike purchased options such that the total gamma is 0. Then, $n_1\Gamma_1 + n_2\Gamma_2 = 0$ so that

$$n_2 = -\frac{n_1\Gamma_1}{\Gamma_2}.$$

In our case, $n_1 = 1, \Gamma_1 = -0.0651$ and $\Gamma_2 = 0.0524$. Thus,

$$n_2 = \frac{0.0651}{0.0524} = 1.2408.$$

Hence, to maintain a gamma-hedged position, the market-maker must buy 1.2408 of the 45-strike 4-month call. Indeed, we have

$$\Gamma_{40} + 1.2408\Gamma_{45} = -0.0651 + 1.2408 \times 0.3285 = 0.$$

(d) Since $\Delta_{40} + 1.2408\Delta_{45} = -0.5825 + 1.2408 \times 0.3285 = -0.1749$, the market-maker needs to buy 17.49 shares of stock to be delta-hedged (i.e., delta-neutral) ■

Figure 39.1 compares a delta-hedged position to a delta-gamma hedged position. Note that the loses from a large drop in stock prices for a delta-gamma hedged position is relatively small compared to the loses from a delta-hedged position. Moreover, for a large stock price increase, the delta-hedge causes loss whereas the delta-gamma hedged position causes gains.

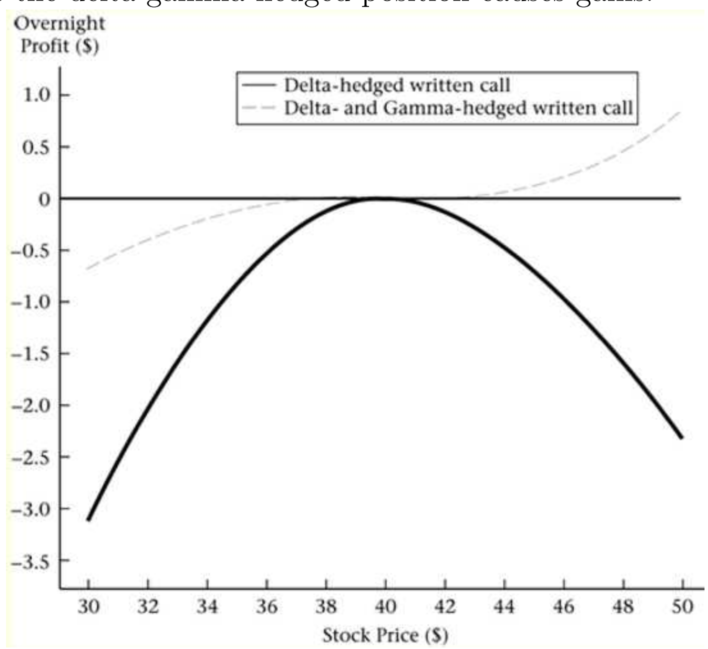


Figure 39.1

(2) **Static option replication strategy:** This strategy uses options to hedge options. It requires little if any rebalancing and thus the word "static" compared to "dynamic". For example, suppose that the market-maker sells a call on a share of stock. By the put-call parity relation we can write

$$C - P - S + Ke^{-rT} = 0.$$

To create a hedge that is both delta- and gamma-neutral, the market-maker purchases a put option with the same strike price and expiration, a share of stock, and borrow money to fund this position.
(3) Buy out of the money options as insurance− deep out of the money options are inexpensive but have positive gammas.
(4) Create a new financial product such as a variance swap in which a market-

maker makes a payment if the stock movement in either direction is small, but receives a payment if the stock movement in either direction is large.

Practice Problems

Problem 39.1
A hedged portfolio consists of selling 100 50-strike calls on a share of stock. The gamma of the call is -0.0521. Find the number of 55-strike call options that a market-maker must purchase in order to bring the hedged portfolio gamma to zero. The gamma of the purchased call is 0.0441.

Problem 39.2
A hedged portfolio consists of selling 100 50-strike calls on a stock. The gamma of the call is -0.0521 and the delta is -0.5824. A gamma-hedged portfolio is created by purchasing 118.14 55-strike call options. The gamma of the purchased call is 0.0441 and the delta is 0.3769. Find the number of shares of stock needed for maintaining a delta-hedged position.

Problem 39.3
A stock is currently trading for \$50. A hedged portfolio consists of selling 100 50-strike calls on a stock. The gamma of the call is -0.0521 and the delta is -0.5824. The price of the call is \$3.48. A gamma-hedged portfolio is created by purchasing 118.14 units of a 55-strike call option. The gamma of the purchased call is 0.0441 and the delta is 0.3769. The price of this call is \$2.05. Find the cost of establishing the delta-hedged position.

Problem 39.4
Consider the hedged portfolio of the previous problem. Suppose that in the next day the stock price goes up to \$51. For such a move, the price of the 50-strike call option goes up to \$4.06 while the 55-strike call goes up to \$2.43. Assume a continuously compounded risk-free rate of 8%, find the overnight profit of this position.

Problem 39.5 ‡
An investor has a portfolio consisting of 100 put options on stock A, with a strike price of 40, and 5 shares of stock A. The investor can write put options on stock A with strike price of 35. The deltas and the gammas of the options are listed below

	40-strike put	35-strike put
Delta	-0.05	-0.10
Gamma	0.25	0.50

What should be done to delta- and gamma-neutralize the investor's portfolio?

Problem 39.6 ‡
For two European call options, Call-I and Call-II, on a stock, you are given:

Greek	Call-I	Call-II
Delta	0.5825	0.7773
Gamma	0.0651	0.0746
Vega	0.0781	0.0596

Suppose you just sold 1000 units of Call-I. Determine the numbers of units of Call-II and stock you should buy or sell in order to both delta-hedge and gamma-hedge your position in Call-I.

An Introduction to Exotic Options

In this chapter we discuss exotic options. By an **exotic option** we mean an option which is created by altering the contractual terms of a standard option such as European and American options (i.e. **Vanilla option**). They permit hedging solutions tailored to specific problems and speculation tailored to particular views. The types of exotic options that we will discuss in this chapter are: Asian, barrier, compound, gap, and exchange options.

40 Asian Options

An **Asian option** is an option where the payoff is not determined by the underlying price at maturity but by the *average* underlying price over some pre-set period of time. Averages can be either arithmetic or geometric.
Suppose that a time interval $[0,T]$ is partitioned into N equal subintervals each of length $h=\frac{T}{N}$. Let S_{ih} denote the stock price at the end of the ith interval. The **arithmetic average** of the stock price is defined by

$$A(T)=\frac{1}{N}\sum_{i=1}^{N}S_{ih}.$$

This type of average is typically used. However, there are no simple pricing formulas for options based on the arithmetic average. A different type of average that is computationally easier than the arithmetic average, but less common in practice, is the **geometric average** defined as

$$G(T)=(S_{1h}\times S_{2h}\times\cdots\times S_{Nh})^{\frac{1}{N}}.$$

Simple pricing formulas exist for geometric average options.

Proposition 40.1
The geometric average is less than or equal to the arithmetic average. That is, $G(T)\le A(T)$.

Proof.
We know that $e^x\ge 1+x$ for all real number x. For $1\le i\le N$, let $x_i=\frac{S_{ih}}{A(T)}-1$. Thus, $\frac{S_{ih}}{A(T)}\le e^{\frac{S_{ih}}{A(T)}-1}$. Multiplying these inequalities we find

$$\frac{S_{1h}}{A(T)}\frac{S_{2h}}{A(T)}\cdots\frac{S_{Nh}}{A(T)}\le e^{\frac{S_{1h}+S_{2h}+\cdots+S_{Nh}}{A(T)}-N}=1.$$

Hence,

$$S_{1h}S_{2h}\cdots S_{Nh}\le(A(T))^N$$

or

$$G(T)=(S_{1h}S_{2h}\cdots S_{Nh})^{\frac{1}{N}}\le A(T)\ \blacksquare$$

Example 40.1
Suppose you observe the following prices $\{345,435,534,354\}$. What are the arithmetic and geometric averages?

Solution.
We have

$$A(T) = \frac{345 + 435 + 534 + 354}{4} = 417$$

and

$$G(T) = (345.435.534.354)^{\frac{1}{4}} = 410.405543 \blacksquare$$

The payoff at maturity of an Asian option can be computed using the average stock price either as the underlying asset price (an **average price option**) or as strike price (an **average strike option**). Hence, there are eight basic kinds of Asian options with payoffs listed next:

Type	Payoff
Arithmetic average price call	$\max\{0, A(T) - K\}$
Arithmetic average price put	$\max\{0, K - A(T)\}$
Arithmetic average strike call	$\max\{0, S_T - A(T)\}$
Arithmetic average strike put	$\max\{0, A(T) - S_T\}$
Geometric average price call	$\max\{0, G(T) - K\}$
Geometric average price put	$\max\{0, K - G(T)\}$
Geometric average strike call	$\max\{0, S_T - G(T)\}$
Geometric average strike put	$\max\{0, G(T) - S_T\}$

Example 40.2
At the end of each of the past four months, Stock GS had the following prices: \$345, \$435, \$534, and \$354. A certain 4-month Asian geometric average price call on this stock has a strike price of \$400. It expires today, and its payoff is computed on the basis of the geometric average of the stock prices given above. What is the payoff of this option?

Solution.
The payoff is

$$\max\{0, G(T) - K\} = \max\{0, 410.405543 - 400\} = \$10.405543 \blacksquare$$

Example 40.3
Consider the following information about a European call on a stock:
- The strike price is \$100
- The current stock price is \$100
- The time to expiration is one year
- The stock price volatility is 30%

• The annual continuously-compounded risk-free rate is 8%
• The stock pays no dividends
• The price is calculated using two-step binomial model where each step is 6 months in length.
(a) Construct the binomial stock price tree including all possible arithmetic and geometric averages after one year.
(b) What is the price of an Asian arithmetic average price call?
(c) What is the price of an Asian geometric average price call?

Solution.
(a) We first find u and d. We have

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{0.08\times 0.5+0.3\sqrt{0.5}} = 1.2868$$

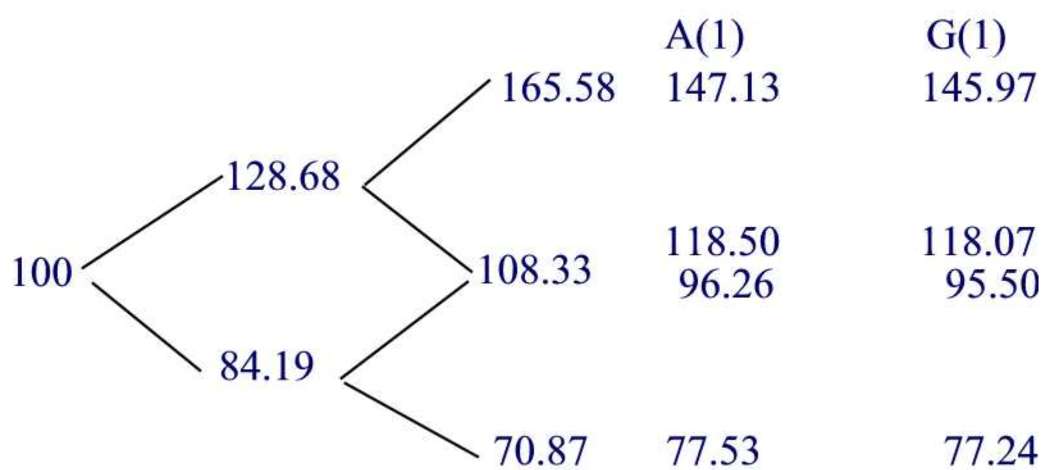
and

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{0.08\times 0.5-0.3\sqrt{0.5}} = 0.84187.$$

Thus, the risk-neutral probability for an up move is

$$p_u = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{0.08(0.5)} - 0.84187}{1.2868 - 0.84187} = 0.44716.$$

The binomial stock price tree with all the possible arithmetic and geometric averages is shown in the tree below.



(b) The price of an Asian arithmetic average price call is

$$e^{-0.08(0.5)}p_u(e^{-0.08(0.5)}[p_u(47.13) + (1-p_u)(18.50)]) = \$12.921.$$

(c) The price of an Asian geometric average price call is

$$e^{-0.08(0.5)}[p_u(e^{-0.05(0.5)}[p_u(45.97)+(1-p_u)(18.07)])] = \$12.607 \blacksquare$$

Remark 40.1
An Asian option is an example of **path-dependent option**, which means that the payoff at expiration depends upon the path by which the stock reached its final price. In the previous example, the payoff of an Asian call at expiration for the stock price \$108.33, using arithmetic average price, is either \$18.50 or \$0 depending on the path leading to the price \$108.33.

Example 40.4
Show that the price of an arithmetic average price Asian call is greater or equal to a geometric average price Asian call.

Solution.
By Proposition 40.1 we have $G(T) \leq A(T)$ so that $G(T) - K \leq A(T) - K$. Thus, $\max\{0, G(T) - K\} \leq \max\{0, A(T) - K\}$ $\blacksquare$

Next, consider geometric average price and geometric average strike call and put options with $S = K = \$40, r = 0.08, \sigma = 0.3, \delta = 0$ and $T = 1$. The table below gives the premiums of these Asian options.

	Average Price (\$)		Average Strike (\$)	
N	Call	Put	Call	Put
1	6.285	3.209	0.000	0.000
2	4.708	2.645	2.225	1.213
3	4.209	2.445	2.748	1.436
5	3.819	2.281	3.148	1.610
10	3.530	2.155	3.440	1.740
50	3.302	2.052	3.668	1.843
1000	3.248	2.027	3.722	1.868
∞	3.246	2.026	3.725	1.869

We observe the following:
- The value of an average price Asian option decreases as N increases (because the more samples, the less volatile the average). For Asian options that average the stock price, averaging reduces the volatility of the value of the underlying asset. Thus, the price of the option at issuance is less than otherwise equivalent standard option.
- The value of an average strike Asian option increases as N increases.

Practice Problems

Problem 40.1
At the end of each of the past four months, a certain stock had the following prices: \$345, \$435, \$534, and \$354. A certain 4-month Asian arithmetic average price call on this stock has a strike price of \$400. It expires today, and its payoff is computed on the basis of the arithmetic average of the stock prices given above. What is the payoff of this option?

Problem 40.2
A stock price is \$1 at the end of month 1 and increases by \$1 every month without exception. A 99-month Asian arithmetic average price put on the stock has a strike price of \$56 and a payoff that is computed based on an average of monthly prices. The option expires at the end of month 99. What is the payoff of this option? Recall that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Problem 40.3
A stock price is \$1 at the end of month 1 and increases by \$1 every month without exception. A 10-month Asian geometric average price call on the stock has a strike price of \$2 and a payoff that is computed based on an average of monthly prices. The option expires at the end of month 10. What is the payoff of this option?

Problem 40.4
Consider the information of Example 40.3.
(a) What is the price of an Asian arithmetic average strike call?
(b) What is the price of an Asian geometric average strike call?

Problem 40.5
Consider the information of Example 40.3.
(a) What is the price of an Asian arithmetic average price put?
(b) What is the price of an Asian geometric average price put?

Problem 40.6
Consider the information of Example 40.3.
(a) What is the price of an Asian arithmetic average strike put?
(b) What is the price of an Asian geometric average strike put?

Problem 40.7
(a) Show that the price of an arithmetic average price Asian put is less than or equal to a geometric average price Asian put.
(b) Show that the price of an arithmetic average strike Asian call is less than or equal to a geometric average strike Asian call.
(c) Show that the price of an arithmetic average strike Asian put is greater than or equal to a geometric average strike Asian put.

Problem 40.8 ‡
You have observed the following monthly closing prices for stock XYZ:

Date	Stock Price
January 31, 2008	105
February 29, 2008	120
March 31, 2008	115
April 30, 2008	110
May 31, 2008	115
June 30, 2008	110
July 31, 2008	100
August 31, 2008	90
September 30, 2008	105
October 31, 2008	125
November 30, 2008	110
December 31, 2008	115

Calculate the payoff of an arithmetic average Asian call option (the average is calculated based on monthly closing stock prices) with a strike of 100 and expiration of 1 year.

Problem 40.9 ‡
At the beginning of the year, a speculator purchases a six-month geometric average price call option on a company's stock. The strike price is 3.5. The payoff is based on an evaluation of the stock price at each month end.

Date	Stock Price
January 31	1.27
February 28	4.11
March 31	5.10
Apricl	5.50
May 31	5.13
June 30	4.70

Based on the above stock prices, calculate the payoff of the option.

41 European Barrier Options

The second type of exotic options is the so-called barrier option. By a **barrier option** we mean an option whose payoff depends on whether the underlying asset price ever reaches a specified level− or **barrier**− during the life of the option. Thus, a barrier option is another example of a path-dependent option. If the barrier's level is the strike price, then the barrier option is equivalent to a standard option.
Barrier options either come to existence or go out of existence the first time the asset price reaches the barrier. Thus, there are three types of barrier options:
• **Knock-out options:** Options go out of existence when the asset price reaches the barrier before option maturity. These options are called **down-and-out** when the asset price has to decline to reach the barrier. They are called **up-and-out** when the price has to increase to reach the barrier. "Out" options start their lives active and become null and void when the barrier price is reached.
• **Knock-in options:** Options come into existence when asset price reaches the barrier before option maturity. These options are called **down-and-in** when the asset price has to decline to reach the barrier. They are called **up-and-in** when the price has to increase to reach the barrier. "In" options start their lives worthless and only become active when barrier price is reached.
• **Rebate options:** Options make a fixed payment if the asset price reaches the barrier. The payment can be made at the time the barrier is reached or at the time the option expires. If the latter is true, then the option is a **deferred rebate. Up rebate** options occur when the barrier is above the current asset price. **Down rebate** options occur when the barrier is below the current price.
The in-out parity of barrier options is given by

$$\text{“knock-in” option} + \text{“knock-out” option} = \text{standard option}$$

For example, for otherwise equivalent options, we can write

$$\begin{aligned}
C_{\text{up-and-in}} + C_{\text{up-and-out}} =& C \\
C_{\text{down-and-in}} + C_{\text{down-and-out}} =& C \\
P_{\text{up-and-in put}} + P_{\text{up-and-out}} =& P \\
P_{\text{down-and-in}} + P_{\text{down-and-out}} =& P
\end{aligned}$$

Since barrier options premiums are nonnegative, the in-out parity shows that barrier options have lower premiums than standard options.

Example 41.1
An ordinary call option on a certain stock has a strike price of \$50 and time to expiration of 1 year trades for \$4. An otherwise identical up-and-in call option on the same stock with a barrier of \$55 trades for \$2.77. Find the price of an up-and-out call option on the stock with a barrier of \$55,a strike price of \$50, and time to expiration of 1 year.

Solution.
Using the in-out parity relation we can write

$$2.77 + \text{up-and-out call} = 4.$$

Thus, up-and-out call premium $= 4 - 2.77 = \$1.23$ ■

Example 41.2
The stock of Tradable Co. once traded for \$100 per share. Several barrier option contracts were then written on the stock. Suddenly, the stock price increased to \$130 per share − which is the barrier for the options. Find the payoff of
(a) An up-and-out call option with a strike of \$120.
(b) An up-and-in call option with a strike of \$120.
(c) An up-and-out put option with a strike of \$120.
(d) An up-and-in put option with a strike of \$120.
(e) A rebate option that pays a rebate of \$12

Solution.
(a) The barrier 130 is reached so the option is knocked-out. Therefore, the payoff of the option is zero.
(b) The barrier 130 is reached so the option is knocked-in. Therefore, the payoff of the option is

$$\max\{0, 130 - 120\} = \$10.$$

(c) The barrier 130 is reached so the option is knocked-out. Therefore, the payoff of the option is zero.

(d) The barrier 130 is reached so the option is knocked-in. Therefore, the payoff of the option is

$$\max\{0, 120 - 130\} = \$0.$$

(e) Payoff is \$12 ■

Example 41.3
When is a barrier option worthless? Assume that the stock price is greater than the barrier at issuance for a down-and-out and less than the barrier for up-and-out.

Solution.
Consider an up-and-out call option with barrier level H. For this option to have a positive payoff the final stock price must exceed the strike price. If $H < K$ then the final stock price exceeds also the barrier H and hence is knocked-out. Consider a down-and-out put. For this option to have a positive payoff, the final stock price must be less than the strike price. Suppose $H > K$, since the stock price at issuance is greater than H, for the final stock price to be less than K means that the stock price exceeded H and hence the option is knocked-out ■

Example 41.4
You are given the following information about a European call option: $S = \$70, K = \$75, \sigma = 0.30, r = 0.08, T = 1$, and $\delta = 0$.
(a) Using the Black-Scholes framework, find the price of the call.
(b) What is the price of a knock-in call with a barrier of \$74?
(c) What is the price of a knock-out call with a barrier of \$74?

Solution.
(a) We have

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma)T}{\sigma\sqrt{T}} = \frac{\ln(70/75) + (0.08 + 0.5 \times 0.3^2)}{0.3} = 0.1867$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = 0.1867 - 0.3 = -0.1133.$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.1867} e^{-\frac{x^2}{2}} dx = 0.574052$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-0.1133} e^{-\frac{x^2}{2}} dx = 0.454896.$$

Hence,

$$C = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) = 70 \times 0.574052 - 75e^{-0.08} \times 0.454896 = \$8.69.$$

(b) (b) For a standard call to ever be in the money, it must pass through the barrier. Therefore, the knock-in call option has the same price as the standard call of \$8.69.
(c) Using the in-out parity for barrier options we find that the price of a knock-out call is zero ■

Practice problems

Problem 41.1
When is a barrier option the same as an ordinary option? Assume that the stock price has not reached the barrier.

Problem 41.2
Consider a European up-and-out barrier call option with strike price \$75 and barrier \$72. What is the payoff of such a call option?

Problem 41.3
Consider a European call option on a stock with strike price of \$50 and time to expiration of 1 year. An otherwise identical knock-in and knock-out call options with a barrier of \$57 trade for \$10.35 and \$5.15 respectively. Find the price of the standard call option.

Problem 41.4
Consider a European call option on a stock with strike price of \$50 and time to expiration of 1 year. An otherwise identical knock-in and knock-out call options with a barrier of \$57 trade for \$10.35 and \$5.15 respectively. An otherwise identical knock-in call option with a barrier of \$54 has a price of \$6.14. Find the price of an otherwise identical knock-out option with a barrier of \$54.

Problem 41.5
GS owns a portfolio of the following options on the stock of XYZ:
- An up-and-in call with strike of \$63 and barrier of \$66
- An up-and-out put with strike of \$78 and barrier of \$65
- An up rebate option with rebate of \$14 and barrier of \$61
- An up-and-out call with strike of \$35 and barrier of \$61

Originally, the stock traded at \$59 per share. Right before the options expired, the stock began to trade at \$67 per share, its record high. What is GS's total payoff on this portfolio?

Problem 41.6
The stock of GS pays no dividends. The stock currently trades at \$54 per share. An up-and-in call with strike \$55 and barrier \$60 has a price of \$3.04, and an up-and-out call with strike \$55 and barrier \$60 has a price of \$1.32. The options expire in 2 months, and the annual continuously compounded interest rate is 0.03. Find the price of one ordinary put option on the stock of GS with strike price of \$55 and time to expiration of 2 months.

Problem 41.7
You are given the following information about a call option: $S = \$40, K = \$45, \sigma = 0.30, r = 0.08, T = 1$,and $\delta = 0$. Find the standard price of the option using one-period binomial model.

Problem 41.8 ‡
You have observed the following monthly closing prices for stock XYZ:

Date	Stock Price
January 31, 2008	105
February 29, 2008	120
March 31, 2008	115
April 30, 2008	110
May 31, 2008	115
June 30, 2008	110
July 31, 2008	100
August 31, 2008	90
September 30, 2008	105
October 31, 2008	125
November 30, 2008	110
December 31, 2008	115

The following are one-year European options on stock XYZ. The options were issued on December 31, 2007.
(i) An arithmetic average Asian call option (the average is calculated based on monthly closing stock prices) with a strike of 100.
(ii) An up-and-out call option with a barrier of 125 and a strike of 120.
(iii) An up-and-in call option with a barrier of 120 and a strike of 110.
Calculate the difference in payoffs between the option with the largest payoff and the option with the smallest payoff.

Problem 41.9 ‡
Barrier option prices are shown in the table below. Each option has the same underlying asset and the same strike price.

Type of option	Price	Barrier
down-and-out	\$25	30,000
up-and-out	\$15	50,000
down-and-in	\$30	30,000
up-and-in	\$X	50,000
down rebate	\$25	30,000
up rebate	\$20	50,000

Calculate X, the price of the up-and-in option.

Problem 41.10 ‡
Prices for 6-month 60-strike European up-and-out call options on a stock S are available. Below is a table of option prices with respect to various H, the level of the barrier. Here, $S(0) = 50$.

H	Price of up-and-out call
60	0
70	0.1294
80	0.7583
90	1.6616
∞	4.0861

Consider a special 6-month 60-strike European "knock-in, partial knock-out" call option that knocks in at $H_1 = 70$, and "partially" knocks out at $H_2 = 80$. The strike price of the option is 60. The following table summarizes the payoff at the exercise date:

	H_1 Hit	
H_1 Not Hit	H_2 Not Hit	H_2 Hit
0	$2\max\{S(0.5) - 60, 0\}$	$\max\{S(0.5) - 60, 0\}$

Calculate the price of the option.

42 Compound Options

In this section we discuss a third type of exotic options. A **compound option** is an option with underyling asset another option. Thus, a compound option has two strikes and two expirations associated with it− one for the underlying option and one for the compound option itself.

There are four types of compound options: a call on a call (option to buy a call), a call on a put (option to buy a put), a put on a call (option to sell a call), and a put on a put (option to sell a put). We will assume that the compound options as well as the underlying options are of European type unless otherwise stated. For example, consider a call on a call. The owner of the compound option can exercise this option only at the expiration time of the compound option. If exercised, the underlying call option can only be exercised at its expiration time.

Let K_u be the strike price of the underlying call option and T_u be the maturity date. Let K_c be the strike price of the compound call option and $T_c < T_u$ be the maturity time. Because we are dealing with European options, the owner of the compound option cannot exercise the option before time T_c. At time T_c, either the owner will pay K_c and receives the underlying call or do nothing and let the option expires worthless. The owner will exercise if the price of the underlying call is greater than K_c. In this case, the payoff is

$$\max\{C(S_{T_c}, K_u, T_u - T_c) - K_c, 0\}$$

where $C(S_{T_c}, K_u, T_u - T_c)$ is the price of the underlying call at time T_c.

Now, let S^* be the underlying asset price for which the price of the underlying call is the cost of acquiring it. That is, $C(S^*, K_u, T_u - T_c) = K_c$. Then for $S_{T_c} > S^*$ we expect that $C(S_{T_c}, K_u, T_u - T_c) > K_c$ so that the compound option is exercised. It follows that two conditions must exist for a compound CallOnCall option to be valuable ultimately: The first is that $S_{T_c} > S^*$ and the second is that $S_{T_u} > K_u$. Thus, the pricing of a CallOnCall requires a bivariate normal cummulative probability distribution as opposed to the univariate distribution in the Black-Scholes formula.

Even though the CallOnCall option looks conceptually like an ordinary option on a stock, it is mathematically different. That is, the Black-Scholes pricing formula cannot be applied. Even though the underyling asset of the call is lognormally distributed, the price of the underyling asset is not.

It can be shown that the pricing of a call on a call satisfies the following parity relationship:

$$\text{CallOnCall} - \text{PutOnCall} + K_c e^{-rT_c} = \text{BSCall}$$

where BSCall is the price of the underlying option according to the Black-Scholes model and the two compound options have the same strike price and maturity time. For a put, the parity relationship is

$$\text{CallOnPut} - \text{PutOnPut} + K_c e^{-rT_c} = \text{BSPut}.$$

Example 42.1
The Black-Scholes price of Call Option $Q-$ which expires is 2 years$-$ is \$44. The annual continuously-compounded risk-free interest rate is 0.03. The price of a PutOnCall option on Option Q with a strike price of \$50 and expiring in 1 year is \$10. What is the price of a CallOnCall option on Option Q with a strike price of \$50 and expiring in 1 year?

Solution.
Using the put-call parity for compound options we can write

$$\text{CallOnCall} - \text{PutOnCall} + K_c e^{-rT_c} = \text{BSCall}.$$

We are given $\text{PutOnCall} = 10, \text{BSCall} = 44, r = 0.03, T_c = 1, K_c = 50$. Thus,

$$\text{CallOnCall} - 10 + 50e^{-0.03} = 44.$$

Solving this equation we find CallOnCall = \$5.477723323 ∎

Valuing American Options on Dividend-Paying Stocks
Next, we examine the use of the compound option model to price an American call option on a stock where the stock pays a single dividend of D at time T_c (the expiration time of the compound option). We can either exercise the option at the cum-divident price[1] $S_{T_c} + D$, or we can hold the option until expiration. In the latter case, the underlying option will be priced on the basis of the stock price S_{T_c} after the dividend is paid. The value of call option at T_c is

$$\max\{C(S_{T_c}, T_u - T_c), S_{T_c} + D - K_u\}.$$

[1] The **ex-dividend date** is the first date when buying a stock does not entitle the new buyer to the declared dividend, as the transfer of stock ownership cannot be completed before the company initiates dividend payment. Before this date, the stock trades **cum-dividend.**

By the put-call parity, the value of the unexercised call is

$$C(S_{T_c}, T_u - T_c) = P(S_{T_c}, T_u - T_c) + S_{T_c} - K_u e^{-r(T_u - T_c)}.$$

The value of the call option at time T_c can thus be written as

$$S_{T_c} + D - K_u + \max\{P(S_{T_c}, T_u - T_c) - [D - K_u(1 - e^{-r(T_u - T_c)})], 0\}. \quad (42.1)$$

The time 0 value of the American call option is the present value of this amount. Note that $\max\{P(S_{T_c}, T_u - T_c) - [D - K_u(1 - e^{-r(T_u - T_c)})], 0\}$ is the payoff of a call on a put option with strike price $D - K_u(1 - e^{-r(T_u - T_c)})$ and maturity date T_c permitting the owner to buy a put option with strike K_u and maturity date T_u. Letting $S_0 = PV_{0,T_c}(S_{T_c} + D)$ we obtain

$$C_{\text{Amer}} = S_0 - K_u e^{-rT_c} + \text{CallOnPut}.$$

Remark 42.1
Note that in the above compound option we assume that the strike $D - K(1 - e^{-r(T_u - T_c)})$ is positive for otherwise the interest on the strike (over the life of the option from the ex-dividend date to expiration) would exceed the value of the dividend and early exercise would never be optimal.

Remark 42.2
The compound option that is implicit in the early exercise decision gives the right to acquire a put option *after* the dividend is paid. (The put option is acquired if we do not exercise the call. If the call is unexercised after the dividend, all subsequent valuation will be with respect to the ex-dividend value of the stock.) Thus, for purposes of valuing the compound option, the underlying asset is really the stock without the dividend, which is the prepaid forward. In obtaining this put, the call owner gives up the dividend and earns interest on the strike. Remember that *exercising the compound option is equivalent to keeping the option on the stock unexercised.*

Example 42.2
Suppose that a stock will pay a dividend of \$2 in 4 months. A call option on the stock has a strike price of \$33 and will expire in 6 months. Four months from now, the stock price is expected to be \$35 and the price of the call will be \$3 after the dividend is paid. What will be the value of the call option in four months?

Solution.
The value of the call option in 4 months is given by

$$\max\{C(S_{T_c}, T_u - T_c), S_{T_c} + D - K_u\}.$$

We are given that $S_{T_c} = 35, T_c = \frac{1}{3}, T_u = 0.5, K_u = 33, D = 2$, and $C(S_{T_c}, T_u - T_c) = 3$. Thus,

$$\max\{3, 35 + 2 - 33\} = \$4 \blacksquare$$

Example 42.3
You have perfect knowledge that 3 months from now, a stock will pay a dividend of \$5 per share. Right after it pays the dividend, the stock will be worth \$80 per share. A certain put option on this stock has a strike price of \$83, time to expiration of 6 months. The annual continuously compounded risk-free interest rate r is 0.07.
(a) What is the strike price of the CallOnPut compound option (after the dividend is paid)?
(b) Suppose that three months from now, the CallOnPut compound option has a price of \$4.04. What is the price of the American call?

Solution.
(a) The strike price of the CallOnPut compound option is

$$D - K_u(1 - e^{-r(T_u - T_c)}) = 5 - 83(1 - e^{-0.07(0.5-0.25)}) = \$3.56.$$

(b) The price of the American call today is

$$C_{\text{Amer}} = S_0 - K_u e^{-rT_c} + \text{CallOnPut} = 80 - 83e^{-0.07\times 0.25} + 4.04 = \$2.48 \blacksquare$$

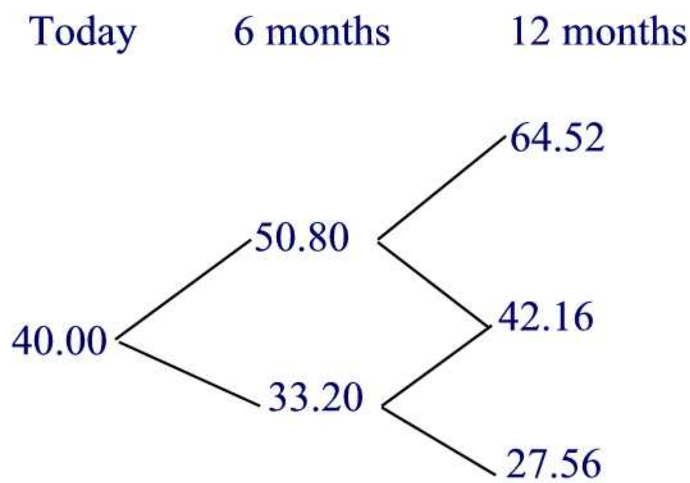
Practice Problems

Problem 42.1
The annual continuously-compounded risk-free interest rate is 0.12. A CallOnCall option on Call Option A has a price of \$53. A PutOnCall option on Option A has a price of \$44. Both compound options have a strike price of \$100 and time to expiration of 2 years. Find the Black-Scholes price of the underlying Option.

Problem 42.2
The Black-Scholes price of Call Option A is \$12. A CallOnCall option on Call Option A has a price of \$5. A PutOnCall option on Option A has a price of \$3.33. Both compound options have a strike price of \$14 and expire in 3 years. Find the annual continuously-compounded interest rate.

Problem 42.3
The Black-Scholes price of put Option $Q-$ which expires is 2 years$-$ is \$6.51. The annual continuously-compounded risk-free interest rate is 0.05. The price of a CallOnPut option on Option Q with a strike price of \$3 and expiring in 1 year is \$4.71. What is the price of a PutOnPut option on Option Q with a strike price of \$3 and expiring in 1 year?

Problem 42.4 ‡
You are given the following information on a CallOnPut option:
• The continuously compounded risk-free rate is 5%
• The strike price of the underlying option is 43
• The strike price of the compound option is 3
• The compound option expires in 6 month
• The underlying option expires six months after te compound option.
• The underlying option is American.



Based on the above binomial stock price tree, calculate the value of the compound option.

Problem 42.5
Suppose that a stock will pay a dividend of $2 in 4 months. A call option on the stock has a strike price of $33 and will expire in 6 months. Four months from now, the stock price is expected to be $35 and the price of the call will be $3 after the dividend is paid. The annual continuously compounded risk-free interest rate r is 0.03. If the call option is unexercised, what will be the value 4 months from now of a put option on the stock with a strike price of $33 and time to expiration of 6 months?

Problem 42.6
You have perfect knowledge that 1 year from now, the stock of GS will pay a dividend of $10 per share. Right after it pays the dividend, the stock will be worth $100 per share. A certain put option on this stock has a strike price of $102, time to expiration of 2 years, and will have a price of $12 in 1 year if unexercised. The annual continuously compounded risk-free interest rate r is 0.08. What will be the value in 1 year (after the dividend is paid) of a call option on this stock with a strike price of $102 and time to expiration of 2 years?

Problem 42.7
You have perfect knowledge that 1 year from now, the stock of GS will pay a dividend of $10 per share. Right after it pays the dividend, the stock will

be worth \$100 per share. A certain put option on this stock has a strike price of \$102, time to expiration of 2 years, and will have a price of \$12 in 1 year if unexercised. The annual continuously compounded risk-free interest rate r is 0.08. You can use a compound CallOnPut option on the GS put to determine the price in 1 year (after the dividend is paid) of a call option on this stock with a strike price of \$102 and time to expiration of 2 years. What is the strike price of such a CallOnPut option?

Problem 42.8

You have perfect knowledge that 3 months from now, a stock will pay a dividend of \$6 per share. Right after it pays the dividend, the stock will be worth \$65 per share. A certain put option on this stock has a strike price of \$65, time to expiration of one. The annual continuously compounded risk-free interest rate r is 0.08.

(a) What is the strike price of the CallOnPut compound option (after the dividend is paid)?

(b) Suppose that the current American call price is \$4.49. What is the price of the CallOnPut compound option?

43 Chooser and Forward Start Options

A **chooser option** is exactly what its name suggests− it is an option which allows the holder to choose whether his/her option is a call or a put at a predetermined time t. Both options have the same strike price K.
The payoff at the choice time t is given by:

$$\text{Payoff} = \max\{C(S_t, t, T), P(S_t, t, T)\}$$

where $C(S_t, t, T)$ and $P(S_t, t, T)$ denote the respective time-t price of a European call and a put at the choice date t.
Note that if $T = t$, that is the chooser option and the underlying options expire simultaneously, then the chooser payoff is the payoff of a call if $S_T > K$ and the payoff of a put if $S_T < K$. That is, the chooser option is equivalent to a straddle with strike price K and expiration time T.
Now, suppose that the chooser must be exercised at choice time t, using call-put parity at t, the value of a chooser option at t can be expressed as:

$$\begin{aligned}\max\{C(S_t, t, T), P(S_t, t, T)\} =& C(S_t, t, T) + \max\{0, P(S_t, t, T) - C(S_t, t, T)\}\\ =& C(S_t, t, T) + \max\{0, Ke^{-r(T-t)} - S_te^{-\delta(T-t)}\}\\ =& C(S_t, t, T) + e^{-\delta(T-t)}\max\{0, Ke^{-(r-\delta)(T-t)} - S_t\}\end{aligned}$$

This says that the chooser option is equivalent to a call option with strike price K and maturity T and $e^{-\delta(T-t)}$ put options with strike price $Ke^{-(r-\delta)(T-t)}$ and maturity t.

Example 43.1
Consider a chooser option (also known as an as-you-like-it option) on a nondividend-paying stock. At time 2, its holder will choose whether it becomes a European call option or a European put option, each of which will expire at time 3 with a strike price of \$90. The time-0 price of the call option expiring at time 3 is \$30. The chooser option price is \$42 at time $t = 0$. The stock price is \$100 at time $t = 0$. Find the time-0 price of a European call option with maturity at time 2 and strike price of \$90. The risk-free interest rate is 0.

Solution.
The time-0 price of the chooser is given by

$$\text{chooser option price} = C(S_t, t, T) + e^{-\delta(T-t)}\max\{0, Ke^{-(r-\delta)(T-t)} - S_t\}.$$

We are given that $r = \delta = 0, S_0 = 100, C(100, 2, 3) = 30, t = 2, T = 3, K = 90$. Thus, at time $t = 0$ we have

$$42 = 30 + P(90, 2).$$

Solving this equation we find $P(90, 2) = 12$. Using the put-call parity of European options we can write

$$C(90, 2) = P(90, 2) + S_0 e^{-2\delta} - Ke^{-2r} = 12 + 100 - 90 = \$22 \blacksquare$$

Example 43.2
Consider a chooser option on a stock. The stock currently trades for \$50 and pays dividend at the continuously compounded yield of 4%. The choice date is six months from now. The underlying European options expire 9 months from now and have a strike price of \$55. The time-0 price of the European call option is \$4. The continuously compounded risk-free rate is 10% and the volatility of the prepaid forward price of the stock is 30%. Find the time-0 chooser option price.

Solution.
The chooser option time-0 price is the time-0 price of a European call with strike \$55 and maturity date of 9 months and the time-0 price of $e^{-\delta(T-t)} = e^{-0.04\times 0.25}$ put options with strike price of $Ke^{-(r-\delta)(T-t)} = 55e^{-(0.10-0.04)\times 0.25} =$ \$54.1812 and expiration of 6 months. To find the price of such a put option, we use the Black-Scholes framework. We have

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(50/54.1812) + (0.10 - 0.04 + 0.5 \times 0.3^2) \times 0.5}{0.3\sqrt{0.5}} = -0.1$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = -0.1311 - 0.3\sqrt{0.5} = -0.3432.$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-0.1311} e^{-\frac{x^2}{2}} dx = 0.447848$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-0.3432} e^{-\frac{x^2}{2}} dx = 0.365724.$$

The price of the European put with strike \$54.1812 and maturity of six months is

$$P(54.1812, 0.5) = 54.1812e^{-0.10\times 0.5}(1-0.365724) - 50e^{-0.04\times 0.5}(1-0.447848) = \$5.6289.$$

The time-0 price of the chooser option is

$$4 + e^{-0.04\times 0.25}\times 5.6289 = \$9.5729 \blacksquare$$

Forward Start Option

A **Forward start** option is an option whose strike will be determined at some future date. Like a standard option, a forward-start option is paid for in the present; however, the strike price is not fully determined until an intermediate date before expiration.

Consider a forward start option with maturity T. Suppose that at time $0 < t < T$ the forward start option will give you one call with $T - t$ periods to expiration and strike αS_t. The time-t price of the forward start option is

$$C = S_t e^{-\delta(T-t)}N(d_1) - \alpha S_t e^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{-\ln\alpha + (r - \delta + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Now, the time-0 price of the forward start option is $e^{-rt}C$.

Example 43.3

A nondividend-paying stock currently trades for \$100. A forward start option with maturity of nine months will give you, six months from today, a 3-month at-the-money call option. The expected stock volatility is 30% and the continuously compounded risk-free rate is 8%. Assume that r, σ, and δ will remain the same over the next six months.

(a) Find the value of the forward start option six months from today. What fraction of the stock price does the option cost?

(b) What investment today would guarantee that you had the money in 6 months to purchase an at-the-money call option?

(c) Find the time-0 price of the forward start option today.

Solution.

(a) After six months, the price of forward start option is found as follows:

$$d_1 = \frac{-\ln\alpha + (r - \delta + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{-\ln 1 + (0.08 - 0 + 0.5(0.3)^2)(0.25)}{0.3\sqrt{0.25}} = 0.2083$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t} = 0.2083 - 0.3\sqrt{0.25} = 0.0583.$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.2083} e^{-\frac{x^2}{2}}dx = 0.582503$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.0583} e^{-\frac{x^2}{2}}dx = 0.523245.$$

Hence, the price of the forward start six months from now is

$$C = S_te^{-\delta(T-t)}N(d_1) - S_te^{-r(T-t)}N(d_2) = S_t[0.582503 - e^{-0.08\times 0.25}\times 0.523245] = 0.069618S_t.$$

(b) In six months we will need $0.069618S_{0.5}$ dollars to buy the call. Thus, we must have 0.069618 shares of the stock today.
(c) The time-0 price of the forward start option must be 0.069618 multiplied by the time-0 price of a security that gives S_t as payoff at time $t = 0.5$ years, i.e., multiplied by 100 or \$6.9618 ■

Example 43.4
A dividend-paying stock currently trades for \$50. The continuously-compound dividend yield is 2%. A forward start option with maturity of two years will give you, one year from today, a 1-year at-the-money put option. The expected stock volatility is 30% and the continuously compounded risk-free rate is 10%. Assume that r, σ, and δ will remain the same over the year.
(a) Find the value of the forward start option one year from today. What fraction of the stock price does the option cost?
(b) What investment today would guarantee that you had the money in one year to purchase an at-the-money put option?
(c) Find the time-0 price of the forward start option today.

Solution.
(a) After one year, the price of forward start option is found as follows:

$$d_1 = \frac{-\ln\alpha + (r - \delta + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{-\ln 1 + (0.10 - 0.02 + 0.5(0.3)^2)(1)}{0.3\sqrt{1}} = 0.41667$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t} = 0.41667 - 0.3\sqrt{1} = 0.11667.$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.41667} e^{-\frac{x^2}{2}} dx = 0.66154$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.11667} e^{-\frac{x^2}{2}} dx = 0.546439.$$

Hence, the price of the forward start one year from now is

$$P = S_t e^{-r(T-t)} N(-d_2) - S_t e^{-\delta(T-t)} N(-d_1) = S_t[(1-0.546439)e^{-0.10} - e^{-0.02} \times (1-0.66154)] = 0.0786 S_t.$$

(b) In one year we will need $0.0786S_1$ dollars to buy the put. Thus, we must have 0.0786 shares of the stock today.
(c) The time-0 price of the forward start option must be 0.0786 multiplied by the time-0 price of a security that gives S_t as payoff at time $t = 1$ year, i.e., multiplied by the prepaid forward price $F_{0,1}^P(S_t)$. Hence, the time-0 price of the forward start option is

$$0.0786 F_{0,1}^P(S_t) = 0.0786 e^{-0.02} \times 50 = 3.85128 \blacksquare$$

Practice Problems

Problem 43.1
Consider a chooser option on a stock. The stock currently trades for \$50 and pays dividend at the continuously compounded yield of 8%. The choice date is two years from now. The underlying European options expire in four years from now and have a strike price of \$45. The continuously compounded risk-free rate is 5% and the volatility of the prepaid forward price of the stock is 30%. Find the delta of the European call with strike price of \$45 and maturity of 4 years.

Problem 43.2
Consider a chooser option on a stock. The stock currently trades for \$50 and pays dividend at the continuously compounded yield of 8%. The choice date is two years from now. The underlying European options expire in four years from now and have a strike price of \$45. The continuously compounded risk-free rate is 5% and the volatility of the prepaid forward price of the stock is 30%. Find the delta of the European put with strike price of \$47.7826 and maturity of 2 years.

Problem 43.3
Consider a chooser option on a stock. The stock currently trades for \$50 and pays dividend at the continuously compounded yield of 8%. The choice date is two years from now. The underlying European options expire in four years from now and have a strike price of \$45. The continuously compounded risk-free rate is 5% and the volatility of the prepaid forward price of the stock is 30%. Find the delta of the chooser option.

Problem 43.4
Consider a chooser option on a stock that pays dividend at the continuously compounded yield of 5%. After one year, its holder will choose whether it becomes a European call option or a European put option, each of which will expire in 3 years with a strike price of \$100. The time-0 price of a call option expiring in one year is \$9.20. The stock price is \$95 at time $t = 0$. The time-0 price of the chooser option is \$28.32. The continuously compounded risk-free interest rate is 5%. Find the time-0 price of the European option with strike price of \$100 and maturity of 3 years.

Problem 43.5 ‡
Consider a chooser option (also known as an as-you-like-it option) on a nondividend-paying stock. At time 1, its holder will choose whether it becomes a European call option or a European put option, each of which will expire at time 3 with a strike price of \$100. The chooser option price is \$20 at time $t = 0$. The stock price is \$95 at time t = 0. Let $C(T)$ denote the price of a European call option at time $t = 0$ on the stock expiring at time $T, T > 0$, with a strike price of \$100. You are given:
(i) The risk-free interest rate is 0.
(ii) $C(1) = \$4$
Determine $C(3)$.

Problem 43.6
A dividend-paying stock currently trades for \$50. The continuously-compound dividend yield is 2%. A forward start option with maturity of two years will give you, one year from today, a 1-year at-the-money call option. The expected stock volatility is 30% and the continuously compounded risk-free rate is 10%. Assume that r, σ, and δ will remain the same over the year.
(a) Find the value of the forward start option one year from today. What fraction of the stock price does the option cost?
(b) What investment today would guarantee that you had the money in one year to purchase an at-the-money call option?
(c) Find the time-0 price of the forward start option today.

Problem 43.7
A nondividend-paying stock currently trades for \$100. A forward start option with maturity of nine months will give you, six months from today, a 3-month call option with strike price 105% of the current stock price. The expected stock volatility is 30% and the continuously compounded risk-free rate is 8%. Assume that r, σ, and δ will remain the same over the next six months.
(a) Find the value of the forward start option six months from today. What fraction of the stock price does the option cost?
(b) What investment today would guarantee that you had the money in 6 months to purchase a 3-month call option?
(c) Find the time-0 price of the forward start option today.

Problem 43.8 ‡
Consider a forward start option which, 1 year from today, will give its owner

a 1-year European call option with a strike price equal to the stock price at that time.
You are given:
(i) The European call option is on a stock that pays no dividends.
(ii) The stock's volatility is 30%.
(iii) The forward price for delivery of 1 share of the stock 1 year from today is \$100.
(iv) The continuously compounded risk-free interest rate is 8%.
Under the Black-Scholes framework, determine the price today of the forward start option.

Problem 43.9 ‡
You own one share of a nondividend-paying stock. Because you worry that its price may drop over the next year, you decide to employ a *rolling insurance strategy,* which entails obtaining one 3-month European put option on the stock every three months, with the first one being bought immediately.
You are given:
(i) The continuously compounded risk-free interest rate is 8%.
(ii) The stock's volatility is 30%.
(iii) The current stock price is 45.
(iv) The strike price for each option is 90% of the then-current stock price.
Your broker will sell you the four options but will charge you for their total cost now.
Under the Black-Scholes framework, how much do you now pay your broker?

44 Gap Options

We know by now that the payoff of a standard option is the result of comparing the stock price with the strike price. In the case of a call, we obtain a nonzero payoff when the stock price is greater than the strike price. In the case of a put, we obtain a nonzero payoff when the stock price is less than the strike price. For a gap option, we obtain exactly these payoffs but by comparing the stock price to a price different than the strike price.
A **gap option** involves a **strike price** K_1 and a **trigger price** K_2. We assume that K_1 and K_2 are different.[1] The trigger price determines whether the option has a nonnegative payoff. For example, the payoff of a gap call option is given by

$$\text{Gap call option payoff} = \begin{cases} S_T - K_1 & \text{if } S_T > K_2 \\ 0 & \text{if } S_T \leq K_2 \end{cases}$$

The diagram of the gap call option's payoff is shown in Figure 44.1.

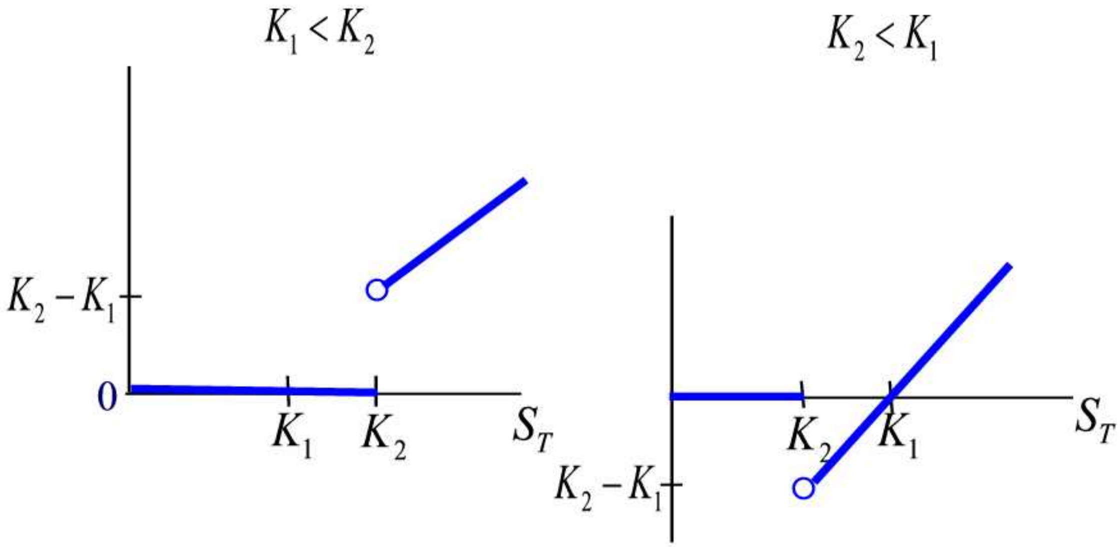


Figure 44.1

Note that the payoffs of a gap call can be either positive or negative. Also, note the gap in the graph when $S_T = K_2$.

[1]If $K_1 = K_2$ a gap option is just an ordinary option.

Likewise, we define the payoff of a gap put option by

$$\text{Gap put option payoff} = \begin{cases} K_1 - S_T & \text{if } S_T < K_2 \\ 0 & \text{if } S_T \geq K_2 \end{cases}$$

The diagram of the gap put option's payoff is shown in Figure 44.2.

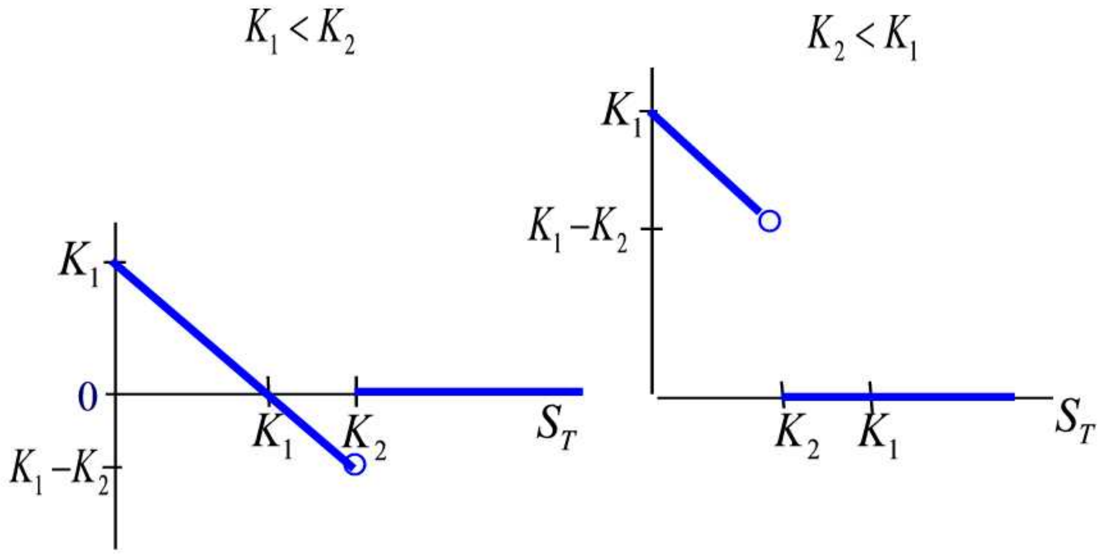


Figure 44.2

The pricing formulas of gap options are a modification of the Black-Scholes formulas where the strike price K in the primary formulas is being replaced by K_1 and the strike price in d_1 is being replaced by the trigger price K_2. Thus, the price of a gap call option is given by

$$C = Se^{-\delta T}N(d_1) - K_1 e^{-rT}N(d_2)$$

and that for a gap put option is

$$P = K_1 e^{-rT}N(-d_2) - Se^{-rT}N(-d_1)$$

where

$$d_1 = \frac{\ln(S/K_2) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}.$$

It follows that the deltas of an ordinary option and otherwise an identical gap option are equal.

Example 44.1
Assume that the Black-Scholes framework holds. A gap call option on a stock has a trigger price of \$55, a strike price of \$50, and a time to expiration of 2 years. The stock currently trades for \$53 per share and pays dividends with a continuously compounded annual yield of 0.03. The annual continuously compounded risk-free interest rate is 0.09, and the relevant price volatility for the Black-Scholes formula is 0.33. Find the Black-Scholes price of this gap call.

Solution.
The values of d_1 and d_2 are

$$d_1 = \frac{\ln(S/K_2) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(53/55) + (0.09 - 0.03 + 0.5(0.33)^2 \times 2}{0.33\sqrt{2}} = -0.3603$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = -0.3603 - 0.33\sqrt{2} = -0.8269.$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-0.3603} e^{-\frac{x^2}{2}}dx = 0.359311$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-0.8269} e^{-\frac{x^2}{2}}dx = 0.204147$$

Hence,

$$C = Se^{-\delta T}N(d_1) - K_1e^{-rT}N(d_2) = 53e^{-0.03\times 2}\times 0.359311 - 50e^{-0.09\times 2}\times 0.204147 = \$9.409 \blacksquare$$

Because negative payoffs are possible, gap options can have negative premiums.

Example 44.2
Assume that the Black-Scholes framework holds. A gap call option on a stock has a trigger price of \$52, a strike price of \$65, and a time to expiration of 3 months. The stock currently trades for \$50 per share and pays dividends with a continuously compounded annual yield of 0.04. The annual continuously compounded risk-free interest rate is 0.07, and the relevant price volatility for the Black-Scholes formula is 0.40. Find the Black-Scholes price of this gap call.

Solution.
The values of d_1 and d_2 are

$$d_1 = \frac{\ln(S/K_2) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(50/52) + (0.07 - 0.04 + 0.5(0.40)^2 \times 0.25}{0.40\sqrt{0.252}} = -0.0586$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = -0.05860 - 0.40\sqrt{0.252} = -0.25860.$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-0.05860} e^{-\frac{x^2}{2}} dx = 0.476635$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-0.25860} e^{-\frac{x^2}{2}} dx = 0.397972$$

Hence,

$$C = Se^{-\delta T}N(d_1) - K_1 e^{-rT}N(d_2) = 50e^{-0.04\times 0.25}\times 0.476635 - 65e^{-0.07\times 0.25}\times 0.397972 = -\$1.825$$

Practice Problems

Problem 44.1
True or False: When the strike equal the trigger, the premium for a gap option is the same as for an ordinary option with the same strike.

Problem 44.2
True or False: For a fixed K_1, if $K_2 > K_1$ for a gap put option then increasing the trigger price reduces the premium.

Problem 44.3
True or False: For a fixed K_2, increasing K_1 for a gap put option then increasing will reduce the premium.

Problem 44.4
True or False: For any strike $K_1 > 0$, there is a K_2 such that $P_{\text{Gap}} = 0$.

Problem 44.5
Assume that the Black-Scholes framework holds. A gap call option on a stock has a trigger price of \$31, a strike price of \$29, and a time to expiration of 6 months. The stock currently trades for \$30 per share and pays dividends with a continuously compounded annual yield of 0.03. The annual continuously compounded risk-free interest rate is 0.06, and the relevant price volatility for the Black-Scholes formula is 0.30. Find the Black-Scholes price of this gap call.

Problem 44.6 ‡
Let $S(t)$ denote the price at time t of a stock that pays dividends continuously at a rate proportional to its price. Consider a European gap option with expiration date $T, T > 0$.
If the stock price at time T is greater than \$100, the payoff is

$$S(T) - 90;$$

otherwise, the payoff is zero.
You are given:
(i) $S(0) = \$80$
(ii) The price of a European call option with expiration date T and strike price \$100 is \$4.
(iii) The delta of the call option in (ii) is 0.2.
Calculate the price of the gap option.

Problem 44.7 ‡
Which one of the following statements is true about exotic options?
(A) Asian options are worth more than European options
(B) Barrier options have a lower premium than standard options
(C) Gap options cannot be priced with the Black-Scholes formula
(D) Compound options can be priced with the Black-Scholes formula
(E) Asian options are path-independent options.

Problem 44.8
Consider three gap calls A, B, and C with underlying asset a certain stock. The strike price of call A is \$48, that of call B is \$56 and call C is \$50. Suppose that $C_A = \$4.98, C_B = \$1.90$. All three options have the same trigger price and time to expiration. Find C_C.

Problem 44.9 ‡
A market-maker sells 1,000 1-year European gap call options, and delta-hedges the position with shares.
You are given:
(i) Each gap call option is written on 1 share of a nondividend-paying stock.
(ii) The current price of the stock is 100.
(iii) The stock's volatility is 100%.
(iv) Each gap call option has a strike price of 130.
(v) Each gap call option has a payment trigger of 100.
(vi) The risk-free interest rate is 0%.
Under the Black-Scholes framework, determine the initial number of shares in the delta-hedge.

45 Exchange Options

A **exchange option,** also known as an **outperformance option**, is an option that gives the owner the right to exchange an underlying asset with a strike asset known as the **benchmark.** Such an option pays off only when the underyling asset outperforms the strike asset. For example, a call option on a stock is an exchange option which entails the owner to exchange the stock with cash when the stock outperforms cash.
The pricing formula for an exchange option is a variant of the Black-Scholes formula. The exchange call option price is

$$\text{ExchangeCallPrice} = Se^{-\delta_S T}N(d_1) - Ke^{-\delta_K T}N(d_2)$$

and that for a put is

$$\text{ExchangePutPrice} = Ke^{-\delta_K T}N(-d_2) - Se^{-\delta_S T}N(-d_1)$$

where

$$d_1 = \frac{\ln\left(\frac{Se^{-\delta_S T}}{Ke^{-\delta_K T}}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}.$$

Here, σ denotes the volatility of $\ln(S/K)$ over the life of the option. Hence,

$$\begin{aligned}\text{Var}[\ln(S/K)] =&\text{Var}[\ln(S)] + \text{Var}[\ln(K)] - 2\text{Cov}[\ln(S), \ln(K)]\\ =&\sigma_S^2 + \sigma_K^2 - 2\rho\sigma_S\sigma_K\end{aligned}$$

where ρ is the correlation between the continuously compounded returns on the two assets.

Example 45.1
One share of Stock A is used as the underlying asset on an exchange option, for which the benchmark asset is one share of a Stock B. Currently, Stock A trades for \$321 per share, and Stock B trades for \$300 per share. Stock A has an annual price volatility of 0.34 and pays dividends at an annual continuously compounded yield of 0.22. Stock B has an annual price volatility of 0.66 and pays dividends at an annual continuously compounded yield of 0.02. The correlation between the continuously compounded returns on the two assets is 0.84. The exchange option expires in 4 years. Find the Black-Scholes price of this option.

Solution.
The underlying asset is a share of Stock A and the strike asset is one share of Stock B. Therefore, we have $S = 321$ and $K = 300$. The volatility of $\ln(S/K)$ is

$$\sigma = \sqrt{\sigma_S^2 + \sigma_K^2 - 2\rho\sigma_S\sigma_K} = \sqrt{0.34^2 + 0.66^2 - 2 \times 0.84 \times 0.34 \times 0.66} = 0.4173823187.$$

The values of d_1 and d_2 are

$$\begin{aligned} d_1 =& \frac{\ln\left(\frac{Se^{-\delta_S T}}{Ke^{-\delta_K T}}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} \\ =& \frac{\ln\left(\frac{321e^{-0.22\times 4}}{300e^{-0.02\times 4}}\right) + 0.5(0.4173823187)^2(4)}{0.4173823187 \times 2} \\ =& -0.4599204785 \end{aligned}$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = -0.4599204785 - 0.4173823187 \times 2 = -1.294685116.$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-0.4599204785} e^{-\frac{x^2}{2}} dx = 0.32278665$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1.294685116} e^{-\frac{x^2}{2}} dx = 0.097714438.$$

The value of the exchange call is

$$\begin{aligned} \text{ExchangeCallPrice} =& Se^{-\delta_S T} N(d_1) - Ke^{-\delta_K T} N(d_2) \\ =& 321e^{-0.22\times 4} \times 0.32278665 - 300e^{-0.02\times 4} \times 0.097714438 = \$15.91699158 \blacksquare \end{aligned}$$

Example 45.2
An exchange call option with expiration of nine months allows the owner to acquire one share of a stock A for three shares of a stock B. The price of the option is \$17.97. Stock A pays dividends at the continuously compounded yield of 4% and that of stock B is 3%. Stock A currently trades for \$100 and stock B trades for \$30. Find the value of an exchange option that allows the owner to acquire one share of stock B by giving up $\frac{1}{3}$ shares of stock A.

Solution.
Let $P(S_t, K_t, T-t)$ be the price of an option that allows the owner to give up one share of stock A for three shares of stock B. Then by the put-call parity we have

$$C(S_t, K_t, T-t) - P(S_t, K_t, T-t) = F^P_{t,T}(S) - F^P_{t,T}(K)$$

or

$$17.97 - P(S_t, K_t, T-t) = 100e^{-0.04\times 0.75} - 3(30)e^{-0.03\times 0.75}.$$

Thus, $P(S_t, K_t, T-t) = \$8.923$. Hence, the value of an option allowing its owner to give up $\frac{1}{3}$ shares of stock A in exchange of 1 share of stock B is $8.923/3 = \$2.97$ ■

Example 45.3
Assume the Black-Scholes framework. Consider two nondividend-paying stocks whose time$-t$ prices are denoted by $S_1(t)$ and $S_2(t)$, respectively.
You are given:
(i) $S_1(0) = 20$ and $S_2(0) = 11$.
(ii) Stock 1's volatility is 0.15.
(iii) Stock 2's volatility is 0.20.
(iv) The correlation between the continuously compounded returns of the two stocks is -0.30.
(v) The continuously compounded risk-free interest rate is 6%.
(vi) A one-year European option with payoff $\max\{22-\max\{S_1(1), 2S_2(1)\}, 0\}$ has a current (time-0) price of 0.29.
Consider a European option that gives its holder the right to buy either one share of Stock 1 or two shares of Stock 2 at a price of 22 one year from now. Calculate the current (time-0) price of this option.

Solution.
Suppose that $X = \max\{S_1(1), 2S_2(1)\}$ is the payoff of a certain asset A. Using (vi) we have the payoff of a European put option on asset A

$$\max\{22 - \max\{S_1(1), 2S_2(1)\}, 0\} = \max\{22 - X, 0\} = 0.29.$$

We are asked to find the time-0 price of a European call option with payoff, one year from today, given by

$$\max\{\max\{S_1(1), 2S_2(1)\} - 22, 0\} = \max\{X - 22, 0\}.$$

Using, the put-call parity

$$C - P = S - Ke^{-rT}$$

we find $C = 0.29 + A_0 - 22e^{-0.06\times 1}$ where A_0 is the time-0 price of asset A. It follows that in order to find C we must find the value of A_0. Since

$$\max\{S_1(1), 2S_2(1)\} = 2S_2(1) + \max\{S_1(1) - 2S_2(1), 0\}$$

the time-0 value of A is twice the time-0 value of stock 2 plus the time-0 value of an exchange call option that allows the owner to give two shares of stock 2 (strike asset) for one share of stock 1 (underlying asset). Hence,

$$A_0 = 2S_2(0) + \text{ExchangeCallPrice}$$

We next find the time-0 value of the exchange call option. We have

$$\sigma = \sqrt{\sigma_S^2 + \sigma_K^2 - 2\rho\sigma_S\sigma_K} = \sqrt{0.15^2 + 0.20^2 - 2(-0.30)(0.15)(0.20)} = 0.28373$$

$$\begin{aligned} d_1 =& \frac{\ln\left(\frac{Se^{-\delta_S T}}{Ke^{-\delta_K T}}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} \\ =& \frac{\ln\left(\frac{(20)e^{-0\times 1}}{2\times 11 e^{-0\times 1}}\right) + 0.5(0.28373)^2(1)}{0.28373\times 1} \\ =& -0.19406 \end{aligned}$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = -0.19406 - 0.28373\times 1 = -0.47779.$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-0.19406} e^{-\frac{x^2}{2}}dx = 0.423064$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-0.47779} e^{-\frac{x^2}{2}}dx = 0.3164.$$

The value of the exchange call is

$$\begin{aligned} \text{ExchangeCallPrice} =& Se^{-\delta_S T}N(d_1) - Ke^{-\delta_K T}N(d_2) \\ =& 20\times 0.423064 - 2\times 11\times 0.3164 = 1.5\$ \end{aligned}$$

Hence,

$$A_0 = 2 \times 11 + 1.5 = \$23.5$$

and

$$C = 0.29 + 23.5 - 22e^{-0.06} = 3.07 \blacksquare$$

Practice Problems

Problem 45.1
One share of Stock A is used as the underlying asset on an exchange option, for which the benchmark asset is four shares of a Stock B. Currently, Stock A trades for \$42 per share, and Stock B trades for \$10 per share. Stock A has an annual price volatility of 0.3 and pays no dividends. Stock B has an annual price volatility of 0.5 and pays dividends at an annual continuously compounded yield of 0.04. The correlation between the continuously compounded returns on the two assets is 0.4. The exchange option expires in 1 year. Find the Black-Scholes price of this call option.

Problem 45.2
One share of Stock A is used as the underlying asset on an exchange option, for which the benchmark asset is one share of a Stock B. Currently, Stock A trades for \$75 per share, and Stock B trades for \$75 per share. Stock A has an annual price volatility of 0.3 and pays no dividends. Stock B has an annual price volatility of 0.25 and pays no dividends. The correlation between the continuously compounded returns on the two assets is 1. The exchange option expires in 1 year. Find the Black-Scholes price of this call option.

Problem 45.3
An exchange call option with expiration of one year allows the owner to acquire one share of a stock A for one share of a stock B. The price of the option is \$2.16. Stock A pays dividends at the continuously compounded yield of 7%. Stock B pays no dividends. Stock A currently trades for \$50 and stock B trades for \$55. Find the value of an exchange option that allows the owner to give up one share of stock A for one share of stock B.

Problem 45.4
One share of Stock A is used as the underlying asset on an exchange option, for which the benchmark asset is four shares of a Stock B. Currently, Stock A trades for \$42 per share, and Stock B trades for \$10 per share. Stock A has an annual price volatility of 0.4 and pays dividends at the continuously compounded yield rate of 2%. Stock B has an annual price volatility of 0.3 and pays no dividends. The correlation between the continuously compounded returns on the two assets is 0.5. The exchange option expires in 1 year. Find the Black-Scholes price of this call option.

Problem 45.5 ‡
Assume the Black-Scholes framework. Consider two nondividend-paying stocks whose time$-t$ prices are denoted by $S_1(t)$ and $S_2(t)$, respectively.
You are given:
(i) $S_1(0) = 10$ and $S_2(0) = 20$.
(ii) Stock 1's volatility is 0.18.
(iii) Stock 2's volatility is 0.25.
(iv) The correlation between the continuously compounded returns of the two stocks is -0.40.
(v) The continuously compounded risk-free interest rate is 5%.
(vi) A one-year European option with payoff $\max\{\min\{2S_1(1), S_2(1)\} - 17, 0\}$ has a current (time-0) price of 1.632.
Consider a European option that gives its holder the right to sell either two shares of Stock 1 or one share of Stock 2 at a price of 17 one year from now. Calculate the current (time-0) price of this option.

The Lognormal Stock Pricing Model

The Black-Scholes pricing model assumes that the asset prices are lognormally distributed. The purpose of this chapter is to examine in more details the lognormal distribution.

46 The Normal Distribution

A **normal** random variable with parameters μ and σ^2 has a pdf

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

This density function is a bell-shaped curve that is symmetric about μ (See Figure 46.1).

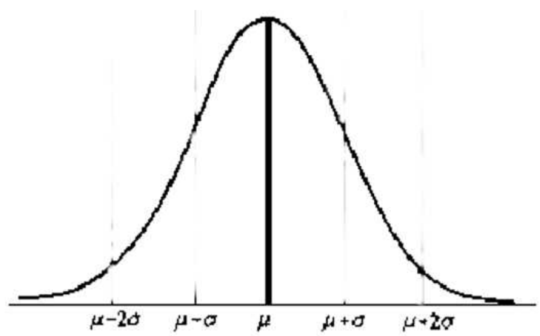


Figure 46.1

To prove that the given $f(x)$ is indeed a pdf we must show that the area under the normal curve is 1. That is,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1.$$

First note that using the substitution $y = \frac{x-\mu}{\sigma}$ we have

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy.$$

Toward this end, let $I = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy$. Then

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dxdy \\ &= \int_{0}^{\infty} \int_{0}^{2\pi} e^{-\frac{r^2}{2}} rdrd\theta = 2\pi \int_{0}^{\infty} re^{-\frac{r^2}{2}} dr = 2\pi \end{aligned}$$

Thus, $I = \sqrt{2\pi}$ and the result is proved. Note that in the process above, we used the polar substitution $x = r\cos\theta, y = r\sin\theta$, and $dydx = rdrd\theta$.

Example 46.1
The width of a bolt of fabric is normally distributed with mean 950mm and standard deviation 10 mm. What is the probability that a randomly chosen bolt has a width between 947 and 950 mm?

Solution.
Let X be the width (in mm) of a randomly chosen bolt. Then X is normally distributed with mean 950 mm and variation 100 mm. Thus,

$$Pr(947 \leq X \leq 950) = \frac{1}{10\sqrt{2\pi}} \int_{947}^{950} e^{-\frac{(x-950)^2}{200}} dx \approx 0.118 \blacksquare$$

Theorem 46.1
If X is a normal distribution with parameters (μ, σ^2) then $Y = aX + b$ is a normal distribution with parmaters $(a\mu + b, a^2\sigma^2)$.

Proof.
We prove the result when $a > 0$. The proof is similar for $a < 0$. Let F_Y denote the cdf of Y. Then

$$\begin{aligned} F_Y(x) =& Pr(Y \leq x) = Pr(aX + b \leq x) \\ =& Pr\left(X \leq \frac{x-b}{a}\right) = F_X\left(\frac{x-b}{a}\right) \end{aligned}$$

Differentiating both sides to obtain

$$\begin{aligned} f_Y(x) =& \frac{1}{a} f_X\left(\frac{x-b}{a}\right) \\ =& \frac{1}{\sqrt{2\pi}a\sigma} exp\left[-(\frac{x-b}{a} - \mu)^2/(2\sigma^2)\right] \\ =& \frac{1}{\sqrt{2\pi}a\sigma} exp\left[-(x-(a\mu+b))^2/2(a\sigma)^2\right] \end{aligned}$$

which shows that Y is normal with parameters $(a\mu + b, a^2\sigma^2)$ $\blacksquare$
Note that if $Z = \frac{X-\mu}{\sigma}$ then this is a normal distribution with parameters (0,1). Such a random variable is called the **standard** normal random variable.

Theorem 46.2
If X is a normal random variable with parameters (μ, σ^2) then
(a) $E(X) = \mu$
(b) $\text{Var}(X) = \sigma^2$.

Proof.
(a) Let $Z = \frac{X-\mu}{\sigma}$ be the standard normal distribution. Then

$E(Z) = \int_{-\infty}^{\infty} x f_Z(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} = 0$

Thus,

$$E(X) = E(\sigma Z + \mu) = \sigma E(Z) + \mu = \mu.$$

(b)

$$\mathrm{Var}(Z) = E(Z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx.$$

Using integration by parts with $u = x$ and $dv = xe^{-\frac{x^2}{2}}$ we find

$\mathrm{Var(Z)} = \frac{1}{\sqrt{2\pi}} \left[-xe^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = 1.$

Thus,

$$\mathrm{Var}(X) = \mathrm{Var}(\sigma Z + \mu) = \sigma^2 \mathrm{Var}(Z) = \sigma^2 \blacksquare$$

Figure 46.2 shows different normal curves with the same μ and different σ.

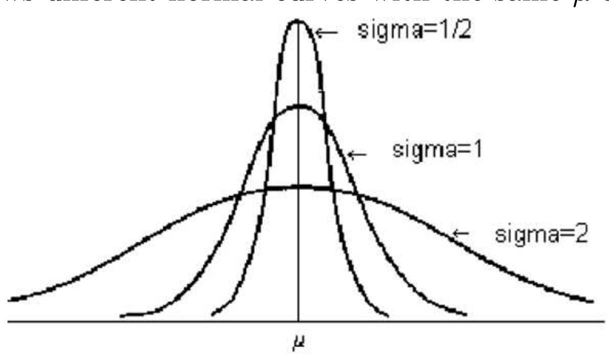


Figure 46.2

It is traditional to denote the cdf of Z by $\Phi(x)$. However, to be consistent with the Black-Scholes notation, we will use the letter N. That is,

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy.$$

Now, since $f_Z(x) = N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $f_Z(x)$ is an even function. This implies that $N'(-x) = N'(x)$. Integrating we find that $N(x) = -N(-x) + C$. Letting

$x = 0$ we find that $C = 2N(0) = 2(0.5) = 1$. Thus,

$$N(x) = 1 - N(-x), \quad -\infty < x < \infty. \tag{46.1}$$

This implies that

$$Pr(Z \leq -x) = Pr(Z > x).$$

Now, $N(x)$ is the area under the standard curve to the left of x. The values of $N(x)$ for $x \geq 0$ are given in the table at the end of the section. Equation 46.1 is used for $x < 0$.

Example 46.2
On May 5, in a certain city, temperatures have been found to be normally distributed with mean $\mu = 24°$C and variance $\sigma^2 = 9$. The record temperature on that day is 27°C.
(a) What is the probability that the record of 27°C will be broken next May 5 ?
(b) What is the probability that the record of 27°C will be broken at least 3 times during the next 5 years on May 5 ? (Assume that the temperatures during the next 5 years on May 5 are independent.)
(c) How high must the temperature be to place it among the top 5% of all temperatures recorded on May 5?

Solution.
(a) Let X be the temperature on May 5. Then X has a normal distribution with $\mu = 24$ and $\sigma = 3$. The desired probability is given by

$$\begin{aligned} Pr(X > 27) =& Pr\left(\frac{X-24}{3} > \frac{27-24}{3}\right) = Pr(Z > 1) \\ =& 1 - Pr(Z \leq 1) = 1 - N(1) = 1 - 0.8413 = 0.1587 \end{aligned}$$

(b) Let Y be the number of times with broken records during the next 5 years on May 5. Then, Y has a binomial distribution with $n = 5$ and $p = 0.1587$. So, the desired probability is

$$\begin{aligned} Pr(Y \geq 3) =& Pr(Y = 3) + Pr(Y = 4) + Pr(Y = 5) \\ =& C(5,3)(0.1587)^3(0.8413)^2 + C(5,4)(0.1587)^4(0.8413)^1 \\ &+ C(5,5)(0.1587)^5(0.8413)^0 \\ \approx& 0.03106 \end{aligned}$$

(c) Let x be the desired temperature. We must have $Pr(X > x) = 0.05$ or equivalently $Pr(X \leq x) = 0.95$. Note that

$$Pr(X \leq x) = Pr\left(\frac{X-24}{3} < \frac{x-24}{3}\right) = Pr\left(Z < \frac{x-24}{3}\right) = 0.95$$

From the Z-score Table (at the end of this section) we find $Pr(Z \leq 1.65) = 0.95$. Thus, we set $\frac{x-24}{3} = 1.65$ and solve for x we find $x = 28.95°C$ ■

Next, we point out that probabilities involving normal random variables are reduced to the ones involving standard normal variable. For example

$$Pr(X \leq a) = Pr\left(\frac{X-\mu}{\sigma} \leq \frac{a-\mu}{\sigma}\right) = N\left(\frac{a-\mu}{\sigma}\right).$$

Example 46.3
Let X be a normal random variable with parameters μ and σ^2. Find
(a)$Pr(\mu - \sigma \leq X \leq \mu + \sigma)$.
(b)$Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$.
(c)$Pr(\mu - 3\sigma \leq X \leq \mu + 3\sigma)$.

Solution.
(a) We have

$$\begin{aligned} Pr(\mu - \sigma \leq X \leq \mu + \sigma) =& Pr(-1 \leq Z \leq 1) \\ =& N(1) - N(-1) \\ =& 2(0.8413) - 1 = 0.6826. \end{aligned}$$

Thus, 68.26% of all possible observations lie within one standard deviation to either side of the mean.
(b) We have

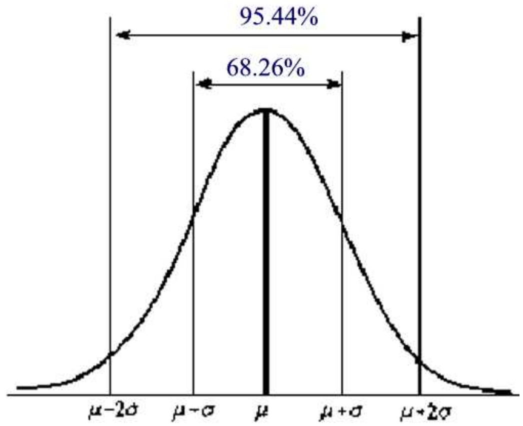
$$\begin{aligned} Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma) =& Pr(-2 \leq Z \leq 2) \\ =& N(2) - N(-2) \\ =& 2(0.9772) - 1 = 0.9544. \end{aligned}$$

Thus, 95.44% of all possible observations lie within two standard deviations to either side of the mean.

(c) We have

$$\begin{aligned}Pr(\mu-3\sigma\leq X\leq \mu+3\sigma)=&Pr(-3\leq Z\leq 3)\\=&N(3)-N(-3)\\=&2(0.9987)-1=0.9974.\end{aligned}$$

Thus, 99.74% of all possible observations lie within three standard deviations to either side of the mean ■



Sum of Normal Random Variables[1] and the Central Limit Theorem

Suppose that X and Y are two jointly normally distributed[2]random variables. For any real numbers a and b, the sum $aX + bY$ is also a normal random variable[3] with variance

$$\begin{aligned}\mathrm{Var}(aX+bY)=&\mathrm{Cov}(aX+bY,aX+bY)=a^2\mathrm{Cov}(X,X)+2ab\mathrm{Cov}(X,Y)+b^2\mathrm{Cov}(Y,Y)\\=&a^2\sigma_X^2+b^2\sigma_Y^2+2ab\rho\sigma_X\sigma_Y\end{aligned}$$

where

$$\rho=\frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

is the correlation coefficient that measures the degree of linearity between X and Y and $\mathrm{Cov}(X,Y)$ is the covariance of X and Y.

[1]See Section 36 of [3]

[2]See Section 29 of [3].

[3]In general, the sum of two normal random variables is not normal. However, the sum of two independent normal random variables is normal.

Now, by the linearity of the mean we have $E(aX + bY) = aE(X) + bE(Y)$. The normal distribution arises so frequently in applications due the amazing result known as the **central limit theorem** which states that the sum of large number of independent identically distributed random variables is well-approximated by a normal random variable.

Example 46.4
Let X have a standard normal distribution. Let Z be a random variable independent from X such that $Pr(Z = 1) = Pr(Z = -1) = \frac{1}{2}$. Define $Y = XZ$.
(a) Show that Y has the standard normal distribution.
(b) Show that $X + Y$ is not normally distributed.

Solution.
(a) We have

$$F_Y(x) = Pr(Y \leq x) = Pr(XZ \leq x) = \begin{cases} F_X(x) & \text{if } Z = 1 \\ 1 - F_X(-x) & \text{if } Z = -1 \end{cases}$$

Taking the derivative of both sides we find $f_Y(x) = f_X(x)$.
(b) This follows from the fact that $Pr(X+Y = 0) = Pr(Z = -1) = \frac{1}{2} \neq 1$ ■

Example 46.5
Suppose that X_1 is a random variable with mean 1 and variance 5 and X_2 is a normal random variable independent from X_1 with mean -2 and variance 2. The covariance between X_1 and X_2 is 1.3. What is the distribution $X_1 - X_2$?

Solution.
Since the random variables are normal and independent, the difference is a random variable with mean $E(X_1 - X_2) = E(X_2) - E(X_2) = 1 - (-2) = 3$ and variance

$$\text{Var}(X_1 - X_2) = 5^2 + 2^2 - 2(1.3) = 26.4 \blacksquare$$

Remark 46.1
In general, the product of two normal random variables needs not be normal.

Area under the Standard Normal Curve from $-\infty$ to x

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Practice Problems

Problem 46.1
Scores for a particular standardized test are normally distributed with a mean of 80 and a standard deviation of 14. Find the probability that a randomly chosen score is
(a) no greater than 70
(b) at least 95
(c) between 70 and 95.
(d) A student was told that her percentile score on this exam is 72%. Approximately what is her raw score?

Problem 46.2
Suppose that egg shell thickness is normally distributed with a mean of 0.381 mm and a standard deviation of 0.031 mm.
(a) Find the proportion of eggs with shell thickness more than 0.36 mm.
(b) Find the proportion of eggs with shell thickness within 0.05 mm of the mean.
(c) Find the proportion of eggs with shell thickness more than 0.07 mm from the mean.

Problem 46.3
Assume the time required for a certain distance runner to run a mile follows a normal distribution with mean 4 minutes and variance 4 seconds.
(a) What is the probability that this athlete will run the mile in less than 4 minutes?
(b) What is the probability that this athlete will run the mile in between 3 min55sec and 4min5sec?

Problem 46.4
You work in Quality Control for GE. Light bulb life has a normal distribution with $\mu = 2000$ hours and $\sigma = 200$ hours. What's the probability that a bulb will last
(a) between 2000 and 2400 hours?
(b) less than 1470 hours?

Problem 46.5
Human intelligence (IQ) has been shown to be normally distributed with mean 100 and standard deviation 15. What fraction of people have IQ greater than 130 ("the gifted cutoff"), given that $\Phi(2) = .9772$?

Problem 46.6
Let X represent the lifetime of a randomly chosen battery. Suppose X is a normal random variable with parameters $(50, 25)$.
(a) Find the probability that the battery lasts at least 42 hours.
(b) Find the probability that the battery will lasts between 45 to 60 hours.

Problem 46.7
For Company A there is a 60% chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean 10,000 and standard deviation 2,000 .
For Company B there is a 70% chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean 9,000 and standard deviation 2,000 .
Assuming that the total claim amounts of the two companies are independent, what is the probability that, in the coming year, Company B's total claim amount will exceed Company A's total claim amount?

Problem 46.8
If for a certain normal random variable X, $Pr(X < 500) = 0.5$ and $Pr(X > 650) = 0.0227$, find the standard deviation of X.

Problem 46.9
Suppose that X is a normal random variable with parameters $\mu = 5, \sigma^2 = 49$. Using the table of the normal distribution , compute: (a) $Pr(X > 5.5)$, (b) $Pr(4 < X < 6.5)$, (c) $Pr(X < 8)$, (d) $Pr(|X - 7| \geq 4)$.

Problem 46.10
A company wants to buy boards of length 2 meters and is willing to accept lengths that are off by as much as 0.04 meters. The board manufacturer produces boards of length normally distributed with mean 2.01 meters and standard deviation σ.
If the probability that a board is too long is 0.01, what is σ?

Problem 46.11
Let X be a normal random variable with mean 1 and variance 4. Find $Pr(X^2 - 2X \leq 8)$.

Problem 46.12
Scores on a standardized exam are normally distributed with mean 1000 and

standard deviation 160.
(a) What proportion of students score under 850 on the exam?
(b) They wish to calibrate the exam so that 1400 represents the 98th percentile. What should they set the mean to? (without changing the standard deviation)

Problem 46.13
The daily number of arrivals to a rural emergency room is a Poisson random variable with a mean of 100 people per day. Use the normal approximation to the Poisson distribution to obtain the approximate probability that 112 or more people arrive in a day.

Problem 46.14
A machine is used to automatically fill 355ml pop bottles. The actual amount put into each bottle is a normal random variable with mean 360ml and standard deviation of 4ml.
(a) What proportion of bottles are filled with less than 355ml of pop?
(b) Suppose that the mean fill can be adjusted. To what value should it be set so that only 2.5% of bottles are filled with less than 355ml?

Problem 46.15
Suppose that your journey time from home to campus is normally distributed with mean equal to 30 minutes and standard deviation equal to 5 minutes. What is the latest time that you should leave home if you want to be over 99% sure of arriving in time for a class at 2pm?

Problem 46.16
Suppose that the current measurements in a strip of wire are assumed to follow a normal distribution with mean of 12 milliamperes and a standard deviation of 3 (milliamperes).
(a) What is the probability that a measurement will exceed 14 milliamperes?
(b) What is the probability that a current measurement is between 9 and 16 milliamperes?
(c) Determine the value for which the probability that a current measurement is below this value is 0.95.

Problem 46.17
Suppose that X_1 is a random variable with mean 2 and variance 0.5 and X_2 is a normal random variable independent from X_1 with mean 8 and variance 14. The correlation coefficient is -0.3. What is the distribution $X_1 + X_2$?

47 The Lognormal Distribution

If X is a normal random variable with parameters μ and σ then the random variable $Y = e^X$ is called **lognormal** random variable with parameters μ and σ. In other words, Y is lognormal if and only if $\ln Y$ is normal. Equivalently, we can write $X = \ln Y$.
The lognormal density function is given by

$$f_Y(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2}, \ x > 0$$

and 0 otherwise. To see this, we have for $x > 0$

$$F_Y(x) = Pr(Y \le x) = Pr(X \le \ln x) = F_X(\ln x).$$

Differentiating both sides with respect to x we find

$$f_Y(x) = f_X(\ln x) \cdot \frac{1}{x} = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2}$$

where F stands for the cumulative distribution function. Notice that the graph of a lognormal distribution is nonnegative and skewed to the right as shown in Figure 47.1.

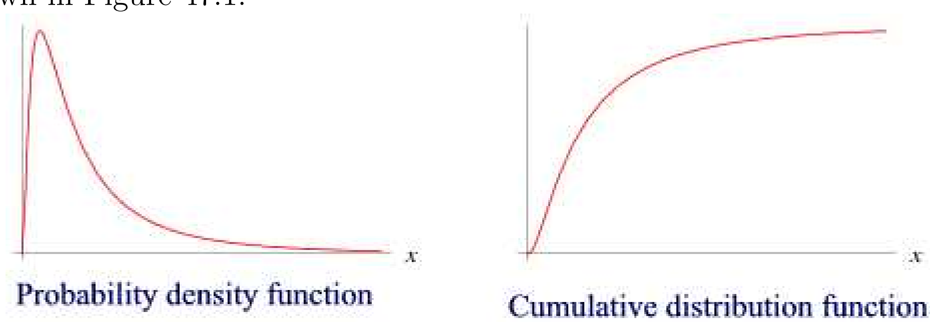


Figure 47.1

The mean of Y is

$$E(Y) = e^{\mu + \frac{1}{2}\sigma^2}.$$

Indeed, we have

$$\begin{aligned}
E(Y) =&E(e^X) = \int_{-\infty}^{\infty} e^x f_X(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}[(x-\mu)^2-2\sigma^2 x]}dx\\
=&e^{\mu+\frac{1}{2}\sigma^2}\int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-(\mu+\sigma^2)}{\sigma}\right)^2}dx\\
=&e^{\mu+\frac{1}{2}\sigma^2}
\end{aligned}$$

since the integral on the right is the total area under a normal density with mean $\mu+\sigma^2$ and variance σ^2 so its value is 1.
Similarly,

$$\begin{aligned}
E(Y^2) =&E(e^{2X}) = \int_{-\infty}^{\infty} e^{2x} f_X(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}[(x-\mu)^2-4\sigma^2 x]}dx\\
=&e^{2\mu+2\sigma^2}\int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-(\mu+2\sigma^2)}{\sigma}\right)^2}dx\\
=&e^{2\mu+2\sigma^2}
\end{aligned}$$

Hence, the variance of Y is

$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}.$$

Example 47.1
Show that the product of two independent lognormal variables is lognormal.[1]

Solution.
Suppose that $Y_1 = e^{X_1}$ and $Y_2 = e^{X_2}$ are two independent lognormal random variables. Then X_1 and X_2 are independent normal random variables. Hence, by the previous section, $X_1 + X_2$ is normal. But $Y_1Y_2 = e^{X_1+X_2}$ so that Y_1Y_2 is lognormal ■

Example 47.2
Lifetimes of a certain component are lognormally distributed with parameters $\mu = 1$ day and $\sigma = 0.5$ days. Find the probability that a component lasts longer than four days.

[1] The sum and the product of any two lognormally distributed random variables need not be lognormally distributed.

Solution.
Let Y represent the lifetime of a randomly chosen component. We need to find $Pr(Y > 4)$. We have

$$Pr(Y > 4) = Pr(e^X > 4) = Pr(X > \ln 4) = Pr(X > 1.386)$$

where X is a normal random variable with mean 1 and variance 0.5^2. Using the z-score, we have

$$z = \frac{1.386 - 1}{0.5} = 0.77.$$

Using the z-table from the previous section we find $Pr(X < 1.386) = Pr(\frac{X-1}{0.5} < 0.77) = 0.7794$. Hence,

$$Pr(Y > 4) = 1 - Pr(X < 1.386) = 1 - 0.7794 = 0.2206 \blacksquare$$

Example 47.3
Suppose that Y is lognormal with parameters μ and σ^2. Show that for $a > 0$, aY is lognormal with parameters $\ln a + \mu$ and σ^2.

Solution.
We have for $x > 0$

$$F_{aY}(x) = Pr(aY \le x) = Pr(Y \le \frac{x}{a}) = F_Y(x/a).$$

Taking the derivative of both sides we find

$$f_{aY}(x) = \frac{1}{a} f_Y(x/a) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln x - (\ln a + \mu)}{\sigma}\right)^2}, \ x > 0$$

and 0 otherwise. Thus, aY is lognormal with parameters $\ln a + \mu$ and σ^2 $\blacksquare$

Example 47.4
Find the median of a lognormal random variable Y with parameters μ and σ^2.

Solution.
The median is a number $\alpha > 0$ such that $Pr(Y \le \alpha) = 0.5$. That is $N(Z \le \frac{\ln \alpha - \mu}{\sigma}) = 0.5$. From the z-score table we find that

$$\frac{\ln \alpha - \mu}{\sigma} = 0$$

Solving for α we find $\alpha = e^{\mu}$ $\blacksquare$

Example 47.5
Find the mode of a lognormal random variable Y with parameters μ and σ^2.

Solution.
The mode is the point of global maximum of the pdf function. It is found by solving the equation $(f_Y(x))' = 0$. But

$$\frac{df_Y}{dx}(x) = \frac{-1}{x^2\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2} \left[1 + \frac{\ln x - \mu}{\sigma^2}\right]$$

Setting this equation to 0 and solving for x we find

$$x = e^{\mu - \sigma^2} \blacksquare$$

Practice Problems

Problem 47.1
Which of the following statements best characterizes the relationship between normal and lognormal distribution?
(A) The lognormal distribution is logarithm of the normal distribution
(B) If $\ln X$ is lognormally distributed then X is normally distributed
(C) If X is lognormally distributed then $\ln X$ is normally distributed
(D) The two distributions have nothing in common.

Problem 47.2
Which of the following statements are true?
(A) The sum of two independent normal random variables is normal
(B) The product of two normal random variables is normal
(C) The sum of two lognormal random variables is lognormal
(D) The product of two independent lognormal random variables is lognormal.

Problem 47.3
Suppose that X is a normal random variable with mean 1 and standard variation 0.5. Find the mean and the variance of the random variable $Y = e^X$.

Problem 47.4
For a lognormal variable X, we know that $\ln X$ has a normal distribution with mean 0 and standard deviation of 0.2. What is the expected value of X?

Problem 47.5
Suppose Y is lognormal with parameters μ and σ^2. Show that $\frac{1}{Y}$ is lognormal with parameters $-\mu$ and σ^2.

Problem 47.6
The lifetime of a semiconductor laser has a lognormal distribution with paramters $\mu = 10$ hours and $\sigma = 1.5$ hours. What is the probability that the lifetime exceeds 10,000 hours?

Problem 47.7
The lifetime of a semiconductor laser has a lognormal distribution with paramters $\mu = 10$ hours and $\sigma = 1.5$ hours. What lifetime is exceeded by 99% of lasers?

Problem 47.8
The lifetime of a semiconductor laser has a lognormal distribution with paramters $\mu = 10$ hours and $\sigma = 1.5$ hours. Determine the mean and the standard deviation of the lifetime of the laser.

Problem 47.9
The time (in hours) to fail for 4 light bulbs are:115, 155, 183, and 217. It is known that the time to fail of the light bulbs is lognormally distributed. Find the parameters of this random variable.

Problem 47.10
The time (in hours) to fail for 4 light bulbs are:115, 155, 183, and 217. It is known that the time to fail of the light bulbs is lognormally distributed. Find the probability that the time to fail is greater than 100 hours.

Problem 47.11
Concentration of pollution produced by chemical plants historically are known to exhibit behavior that resembles a lognormal distribution. Suppose that it is assumed that the concentration of a certain pollutant, in parts per million, has a lognormal distribution with parameters $\mu = 3.2$ and $\sigma^2 = 1$. What is the probability that the concentration exceeds 8 parts per million?

Problem 47.12
The life, in thousands per mile, of a certain electronic control for locomotives has a lognormal distribution with parameters $\mu = 5.149$ and $\sigma = 0.737$. Find the 5th percentile of the life of such a locomotive.

Problem 47.13
Let X be a normal random variable with mean 2 and variance 5. What is $E(e^X)$? What is the median of e^X?

48 A Lognormal Model of Stock Prices

Let S_t denote the price of a stock at time t. Consider the **relative price** of an asset between periods 0 and t defined by

$$\frac{S_t}{S_0} = 1 + R_{0,t}$$

where $R_{0,t}$ is the **holding period return.** For example, if $S_0 = \$30$ and $S_1 = \$34.5$ then $\frac{S_1}{S_0} = 1.15$ so that $R_{0,t} = 15\%$.
We define the continuously compounded return from time 0 to t by

$$r_{0,t} = \ln\left(\frac{S_t}{S_0}\right) = \ln(1 + R_{0,t}).$$

For the above example, $r_{0,1} = \ln 1.5 = 13.97\%$ which is lower than the holding period return. Note that

$$\frac{S_t}{S_0} = \frac{S_t}{S_{t-1}} \times \frac{S_{t-1}}{S_{t-2}} \times \cdots \times \frac{S_1}{S_0} \longrightarrow r_{0,t} = r_{t-1,t} + \cdots + r_{0,1}.$$

Thus, the continuously compounded return from 0 to t is the sum of one-period continuously compounded returns.

Example 48.1
Given the following: $S_0 = \$100, r_{0,1} = 15\%, r_{1,2} = 3\%$. Find the stock price after 2 year.

Solution.
We have $S_2 = S_0e^{r_{0,2}} = S_0e^{r_{0,1}+r_{1,2}} = 100e^{0.15+0.03} = 100e^{0.18} = \$119.7222$ ■

It is commonly assumed in investments that return are independent and identically distributed over time. This means that investors can not predict future returns based on past returns and the distribution of returns is stationary. It follows that if the one-period continuously compounded returns are normally distributed, their sum will also be normal. Even if they are not, then by the Central Limit Theorem, their sum will be normal. From this, we conclude that the relative price of the stock is lognormally distributed random variable.

Now, take the period of time from 0 to t and divide it into n equal subintervals each of length $h = \frac{t}{n}$. Since the continuously compounded returns are identically distributed, we can let $E(r_{(i-1)h,ih}) = \mu_h$ and $\text{Var}(r_{(i-1)h,ih}) = \sigma_h^2$ for $1 \le i \le n$. Thus,

$$E(r_{0,t}) = \sum_{i=1}^{n} E(r_{(i-1)h,ih}) = n\mu_h = \frac{t}{h}\mu_h$$

and

$$\text{Var}(r_{0,t}) = \sum_{i=1}^{n} \text{Var}(r_{(i-1)h,ih}) = n\sigma_h^2 = \frac{t}{h}\sigma_h^2.$$

Hence, the mean and the variance of the continuously compounded returns are proportional to time.
In what follows t will be expressed in years. For $t = 1$, we let $E(r_{0,1}) = \alpha$ and $\text{Var}(r_{0,1}) = \sigma^2$ be the annual mean and variance respectively. Let δ be the continuously compounded annual yield on the stock. We examine a lognormal model of the stock price as follows: We assume that the continuously compounded return from time 0 to time t, $\ln\left(\frac{S_t}{S_0}\right)$, is normally distributed with mean $(\alpha - \delta - 0.5\sigma^2)t$ and variance $\sigma^2 t$. Using the $Z-$score we can write

$$\ln\left(\frac{S_t}{S_0}\right) = (\alpha - \delta - 0.5\sigma^2)t + \sigma\sqrt{t}Z$$

and solving for S_t we find

$$S_t = S_0 e^{(\alpha-\delta-0.5\sigma^2)t+\sigma\sqrt{t}Z}. \tag{48.1}$$

Example 48.2
Find the expected stock price S_t.

Solution.
First recall that if X is a normal random variable with parameters μ and σ then $E(e^X) = e^{\mu+\frac{1}{2}\sigma^2}$. Let $X = \ln\left(\frac{S_t}{S_0}\right)$. Then X is a normal random variable with parameters $(\alpha - \delta - 0.5\sigma^2)t$ and $\sigma^2 t$. Hence,

$$E(e^X) = e^{(\alpha-\delta-0.5\sigma^2)t+\frac{1}{2}\sigma^2 t}.$$

But $e^X = \frac{S_t}{S_0}$ so that

$$\frac{1}{S_0}E(S_t) = e^{(\alpha-\delta-0.5\sigma^2)t+\frac{1}{2}\sigma^2 t}.$$

Hence,

$$E(S_t) = S_0 e^{(\alpha-\delta)t}.$$

We call the difference $\alpha - \delta$ the **expected continuously rate of appreciation** on the stock ■

Example 48.3
Find the median stock price. That is the price where 50% of the time prices will be above or below that value.

Solution.
For the median we want to find the stock price such that

$$Pr\left(Z < \frac{\ln(S_t/S_0) - (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}}\right) = 0.5.$$

This happens when

$$\frac{\ln(S_t/S_0) - (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}} = 0.$$

Solving we find

$$S_t = S_0 e^{(\alpha-\delta-0.5\sigma^2)t} = E(S_t)e^{-\frac{1}{2}\sigma^2 t} \blacksquare$$

Now, since $e^{-\frac{1}{2}\sigma^2 t} < 1$, we use the last equation to write $S_t < E(S_t)$. That is, the median is below the mean. This says that more than 50% of the time, a lognormally distributed stock price will earn below its expected return. Note that if σ is such that $(\alpha - \delta - 0.5\sigma^2)t < 0$ then $S_t < S_0 < E(S_t)$. In this case, a lognormally distributed stock will lose money more than half the time.

Example 48.4
A nondividend-paying stock is currently trading for \$100. The annual expected rate of return on the stock is $\alpha = 10\%$ and the annual standard deviation is $\sigma = 30\%$. Assume the stock price is lognormally distributed.
(a) Using the above model, find an expression for the stock price after 2 years.
(b) What is the expected stock price two years from now?
(c) What is the median stock price?

Solution.
(a) The stock price is

$$S_2 = S_0 e^{(\alpha-\delta-0.5\sigma^2)\times 2+\sigma\sqrt{2}Z} = 100e^{(0.1-\frac{1}{2}\times 0.3^2)\times 2+0.3\sqrt{2}Z}.$$

(b) We have

$$E(S_2) = 100e^{0.1\times 2} = \$122.14.$$

(c) The median stock price is

$$E(S_2)e^{-\frac{1}{2}\sigma^2 t} = 122.14e^{-\frac{1}{2}(0.3)^2(2)} = 122.14e^{-0.3^2} = \$111.63 \blacksquare$$

Now, since

$$Z = \frac{\ln(S_t/S_0) - E(\ln(S_t/S_0))}{\operatorname{Var}(\ln(S_t/S_0))}$$

we define a **one standard deviation move up** in the stock price by letting $Z = 1$ and a **one standard deviation down** by letting $Z = -1$

Example 48.5
Using the same information in the prvious example, find the stock price over 2 years for a one standard deviation up and a one standard deviation down.

Solution.
A one standard deviation move up over 2 years is given by

$$S_2 = 100e^{(0.1-\frac{1}{2}(0.3)^2)\times 2+0.3\sqrt{2}} = \$170.62$$

and a one standard deviation move down over 2 years is given by

$$S_2 = 100e^{(0.1-\frac{1}{2}(0.3)^2)\times 2-0.3\sqrt{2}} = \$73.03 \blacksquare$$

Estimating the Parameters of the Lognormal Pricing Model
We next see how to estimate the mean and the variance of lognormally distributed price data.
When stocks are lognormally distributed, if S_{t-h} is the stock price at the beginning of a period then the price at the end of the period is found using the formula

$$S_t = S_{t-h}e^{(\alpha-\delta-0.5\sigma^2)h+\sigma\sqrt{h}Z}.$$

From this it follows that $\ln(S_t/S_{t-h})$ is normally distributed with mean $(\alpha - \delta - 0.5\sigma^2)h$ and variance $\sigma^2 h$. That is, by using the log of ratio prices at adjacent points in time, we can compute the continuously compounded mean and variance.

Example 48.6
The table below contains seven weekly stock price observations. The stock pays no dividends.

Week	1	2	3	4	5	6	7
Price	100	105.04	105.76	108.93	102.50	104.80	104.13

(a) Estimate the continuously compounded mean and variance.
(b) Estimate the annual expected rate of return.

Solution.
(a) We compute the weekly continuous compounded returns

Week	1	2	3	4	5	6	7
Price	100	105.04	105.76	108.93	102.50	104.80	104.13
$\ln(S_t/S_{t-1})$	–	0.0492	0.0068	0.0295	-0.0608	0.0222	-0.0064

The weekly continuously compounded mean is

$$\overline{X} = \frac{0.0492 + 0.0068 + 0.0295 - 0.0608 + 0.0222 - 0.0064}{6} = 0.00675.$$

The annual continuously compounded mean is $0.00675 \times 52 = 0.351$.
The weekly continuously compounded variance is

$$\frac{\sum_{t=1}^{6}(\ln(S_t/S_{t-1}) - \overline{X})^2}{5} = 0.0382^2.$$

The annualized standard deviation is

$$\sigma_{\text{Annual}} = 0.0382 \times \sqrt{52} = 0.274.$$

(b) The annual expected rate of return is the annual mean plus one-half the variance. That is,

$$\alpha = 0.00675 \times 52 + 0.5 \times 0.274^2 = 0.3885 \blacksquare$$

Remark 48.1
The above data are hypothetical data. Statistical theory tells us that the observed mean is determined by the difference between the beginning and ending stock price. What happens in between is irrelevant. Thus, having frequent observations is not helpful in estimating the mean returns. Observations taken over a longer period of time are more likely to reflect the true mean and therefore they increase the precision of the estimate.
Unlike the mean, one can increase the precision of the estimate of the standard deviation by making more frequent observations.

Practice Problems

Problem 48.1
Suppose that $S_0 = \$100$ and $S_2 = \$170.62$. Find the relative stock price from time 0 to time 2. What is the holding period return over the 2 years?

Problem 48.2
The continuously compounded return from time 0 to time 2 is 18%. Find the holding period return over the 2-year period.

Problem 48.3
Suppose the stock price is initially \$100 and the continuously compounded return on a stock is 15% in one year and $r_{1,2}$ the next year. What is $r_{1,2}$ if the stock price after 2 years is \$119.722?

Problem 48.4
Suppose the stock price is initially \$100 and the continuously compounded return on a stock is 15% in one year and $r_{1,2}$ the next year. What is $R_{1,2}$ if the stock price after 2 years is \$119.722?

Problem 48.5
Suppose that the continuously compounded return from time 0 to time t is normal with mean $(\alpha - \delta - \frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$. The stock pays dividend at the annual continuously compounded yield of 3%. The annual variance is 0.09. Initially, the stock was trading for \$75. Suppose that the mean for the continuously compounded 2-year return is 0.07. Find the the annual expected rate of return α.

Problem 48.6
Suppose that the continuously compounded return from time 0 to time t is normal with mean $(\alpha - \delta - \frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$. The stock pays dividend at the annual continuously compounded yield of 3%. The annual variance is 0.09. Initially, the stock was trading for \$75. Suppose that the mean for the continuously compounded 2-year return is 0.07. Find the expected stock price after 4 years.

Problem 48.7
Suppose that the continuously compounded return from time 0 to time t is normal with mean $(\alpha - \delta - \frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$. The stock pays dividend

at the annual continuously compounded yield of 3%. The annual variance is 0.09. Initially, the stock was trading for \$75. Suppose that the mean for the continuously compounded 2-year return is 0.07. Find the median price at time 4 years.

Problem 48.8
Suppose that the continuously compounded return from time 0 to time t is normal with mean $(\alpha - \delta - \frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$. The stock pays dividend at the annual continuously compounded yield of 3%. The annual variance is 0.09. Initially, the stock was trading for \$75. Suppose that the mean for the continuously compounded 2-year return is 0.07. Also, suppose that the dividends are reinvested in the stock. Find the median of the investor's position at time 4 years.

Problem 48.9
A nondividend-paying stock is currently trading for \$100. The annual expected rate of return $\alpha = 10\%$ and the annual standard deviation is $\sigma = 60\%$. Assume the stock price is lognormally distributed.
(a) Using the above model, find an expression for the stock price after 2 years.
(b) What is the expected stock price two years from now?
(c) What is the median stock price?

Problem 48.10
Consider a stock with annual variance of 0.16. Suppose that the expected stock price in five years is \$135. Find the median of the stock price at time 5 years.

Problem 48.11
A nondividend-paying stock is currently trading for \$100. The annual expected rate of return on the stock is $\alpha = 13\%$ and the annual standard deviation is $\sigma = 25\%$. Assume the stock price is lognormally distributed. Find the stock price over 6 months for a one standard deviation up and a one standard deviation down.

Problem 48.12
A stock is currently trading for \$100. The annual expected rate of return on the stock is $\alpha = 13\%$ and the annual standard deviation is $\sigma = 25\%$. The stock pays dividends at the continuously compounded yield of 2%. Assume

the stock price is lognormally distributed. Find the stock price over 6 months for a two standard deviation up.

Problem 48.13
The table below contains five monthly stock price observations. The stock pays no dividends.

Monthly	1	2	3	4	5
Price	100	104	97	95	103

(a) Estimate the annual continuously compounded mean and standard deviation.
(b) Estimate the annual expected rate of return.

Problem 48.14 ‡
Assume the Black−Scholes framework.
The price of a nondividend-paying stock in seven consecutive months is:

Month	Price
1	54
2	56
3	48
4	55
5	60
6	58
7	62

Estimate the continuously compounded expected rate of return on the stock.

49 Lognormal Probability Calculations

In this section we use the lognormal stock pricing model discussed in the previous section to compute a number of probabilities and expectations. Among the questions we consider are the question of the probability of an option to expire in the money, and given that the option expires in the money, what will be the expected stock price.

Let K be an arbitrary positive number. K can be the strike price of an option. We would like to find the probability that a stock price is less than K. Since

$$\frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln(S_t/S_0)-(\alpha-\delta-0.5\sigma^2)t}{\sigma\sqrt{t}}\right)^2} = \frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln S_t-\ln S_0-(\alpha-\delta-0.5\sigma^2)t}{\sigma\sqrt{t}}\right)^2}$$

the random variable $\ln S_t$ is normally distributed with parameters $\ln S_0 + (\alpha-\delta-0.5\sigma^2)t$ and $\sigma^2 t$.

Now, we have

$$\begin{aligned}
Pr(S_t < K) =& Pr(\ln S_t < \ln K)\\
=& Pr\left(\frac{\ln S_t - \ln S_0 - (\alpha-\delta-0.5\sigma^2)t}{\sigma\sqrt{t}} < \frac{\ln K - \ln S_0 - (\alpha-\delta-0.5\sigma^2)t}{\sigma\sqrt{t}}\right)\\
=& Pr\left(Z < \frac{\ln K - \ln S_0 - (\alpha-\delta-0.5\sigma^2)t}{\sigma\sqrt{t}}\right)\\
=& N(-\hat{d}_2)
\end{aligned}$$

where

$$\hat{d}_2 = \frac{\ln S_0 - \ln K + (\alpha-\delta-0.5\sigma^2)t}{\sigma\sqrt{t}}.$$

Now, using the property that $N(\hat{d}_2) + N(-\hat{d}_2) = 1$ we obtain

$$Pr(S_t \geq K) = 1 - Pr(S_t < K) = N(\hat{d}_2).$$

Remark 49.1

We have found that the probability for a call option to expire in the money is $N(\hat{d}_2)$ and that for a put option is $N(-\hat{d}_2)$. These probabilities involve the true expected return on the stock. If α is being replaced by the risk-free rate r then we obtain the risk-neutral probability that an option expires in the money.

Example 49.1
You are given the following information:$S_0 = 50, \alpha = 0.10, \sigma = 0.25$, and $\delta = 0$. What is $Pr(S_2 > \$65)$?

Solution.
We are asked to find $Pr(S_2 > 65) = N(\hat{d}_2)$ where

$$\hat{d}_2 = \frac{\ln S_0 - \ln K + (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}} = \frac{\ln 50 - \ln 65 + (0.10 - 0 - 0.5 \times 0.25^2)(2)}{0.25\sqrt{2}} = -0.3532.$$

Thus,

$$Pr(S_t > 65) = N(\hat{d}_2) = N(-0.3532) = 0.361969 \blacksquare$$

Next, suppose we are interested in knowing the range of prices such that there is 95% probability that the stock price will be in that range after a certain time. This requires generating a 95% confidence interval. Using the lognormal distribution, we will discuss how to generate confidence intervals. Suppose that we want to be $(1-p)\%$ confident that a stock price is in the range $[S_t^L, S_t^U]$. This is equivalent to saying that we want to be $p\%$ confident that either $S_t < S_t^L$ or $S_t > S_t^U$. We split this probability into two, half for each tail. In other words, we want

$$Pr(S_t < S_t^L) = \tfrac{p}{2} \text{ and } Pr(S_t > S_t^U) = \tfrac{p}{2}.$$

But $Pr(S_t < S_t^L) = \frac{p}{2}$ implies $N(-\hat{d}_2) = p/2$ so that $\hat{d}_2 = -N^{-1}(p/2)$. Hence,

$$N^{-1}(p/2) = -\frac{\ln S_t^L - \ln S_0 + (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}}.$$

Solving this equation for S_t^L we find

$$S_t^L = S_0 e^{(\alpha-\delta-0.5\sigma^2)t + \sigma\sqrt{t}N^{-1}(p/2)}.$$

Likewise, we solve for the S_t^U such that

$$N^{-1}(1-p/2) = \hat{d}_2$$

This gives us

$$S_t^U = S_0 e^{(\alpha-\delta-0.5\sigma^2)t + \sigma\sqrt{t}N^{-1}(1-p/2)}.$$

Example 49.2
Use the Z-score table to derive the 95% confidence interval if $S_0 = \$100, t = 2, \delta = 0, \alpha = 0.10$, and $\sigma = 0.30$.

Solution.
We have $p = 5\%$ so that $N(-\hat{d}_2) = 0.025$ and $N(\hat{d}_2) = 0.975$. Using the $Z-$score table we find $N^{-1}(0.025) = -1.96$ and $N^{-1}(0.975) = 1.96$. Hence,

$$S_t^L = S_0 e^{(\alpha-\delta-0.5\sigma^2)t+\sigma\sqrt{t}N^{-1}(p/2)} = 100e^{(0.10-0.5\times0.3^2)\times2-0.30\sqrt{2}\times1.96} = \$48.599$$

and

$$S_t^U = S_0 e^{(\alpha-\delta-0.5\sigma^2)t+\sigma\sqrt{t}N^{-1}(1-p/2)} = 100e^{(0.10-0.5\times0.3^2)\times2+0.30\sqrt{2}\times1.96} = \$256.40$$

There is 95% probability that in two years the stock price will be between \$48.599 and \$256.40 ■

Portfolio Insurance in the Long Run
Consider a portfolio of stocks. Historical return statistics has indicated that the rate of return from investing in stocks over a long time has outperformed that from investing in risk-free bonds. That is, one is led to believe that if a stock is held for sufficiently long period of time, the investment is as safe as the free-risk bond.
To guarantee that investing in stocks is safe in the long run, it was suggested that a put option is to be bought today *insuring* that after T years the stock portfolio would worth at least as much as if one had instead invested in a zero-coupon bond. If the initial stock price is S_0 then one has to set the strike price of the put option to $K_T = S_0e^{rT}$. Using the Black-Scholes model, it was shown that in the absence of arbitrage the premium of the insurance increases with time to maturity T.
We can use the results of this section to show that the probability of the payoff of the put option increases with the time to maturity. Suppose that the stock price is lognormally distributed using the model of the previous section. Assume that the stock pays no dividends. Then the probability that the bond will outperform the stock is given by

$$Pr(S_T < K_T) = N\left(\frac{\frac{1}{2}\sigma^2 - (\alpha - r)}{\sigma}\sqrt{T}\right).$$

In the Black-Scholes analysis, the put price depends in part on the risk-neutral probability that the stock will underperform bonds. We can obtain this probability by setting $\alpha = r$ in the previous equation obtaining

$$Pr^*(S_T < K_T) = N\left(\frac{1}{2}\sigma\sqrt{T}\right).$$

This says that the risk-neutral probability that the put will pay off is increasing with time.

Example 49.3
Consider a European put option on a stock with initial price S_0. Suppose that the risk-premium is greater than $0.5\sigma^2$. Show that the probability that the option expires in the money is less than 50%.

Solution.
The probability that the option expires in the money is

$$Pr(S_T < S_0e^{rT}) = N(-\hat{d}_2)$$

where

$$\hat{d}_2 = \frac{\ln S_0 - \ln K + (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}} = \frac{[(\alpha - r) - 0.5\sigma^2]\sqrt{T}}{\sigma}.$$

Hence,

$$Pr(S_T < S_0e^{rT}) = N\left(\frac{[0.5\sigma^2 - (\alpha - r)]\sqrt{T}}{\sigma}\right)$$

Since $\alpha - r > 0.5\sigma^2$, we find $0.5\sigma^2 - (\alpha - r) < 0$ and therefore $-\hat{d}_2 < 0$. This shows that $Pr(S_T < S_0e^{rT}) < 50\%$ ■

Practice Problems

Problem 49.1
A nondividend-paying stock that is lognormally distributed is currently selling for \$50. Given that the annual expected rate of return is 15% and the annual standard deviation is 30%. Find the monthly continuously compounded mean return and the monthly standard deviation.

Problem 49.2
A nondividend-paying stock that is lognormally distributed is currently selling for \$50. Given that the annual expected rate of return is 15% and the annual standard deviation is 30%. It is known that for the standard normal distribution, there is 68.27% probability of drawing a number in the interval $(-1, 1)$ and a 95.45% probability of drawing a number in the interval $(-2, 2)$.
(a) Find r^L and r^U so that there is 68.27% that the monthly continuously compounded return on the stock will be between r^L and r^U.
(b) Find r^L and r^U so that there is 95.45% that the monthly continuously compounded return on the stock will be between r^L and r^U.

Problem 49.3
A nondividend-paying stock that is lognormally distributed is currently selling for \$50. Given that the annual expected rate of return is 15% and the annual standard deviation is 30%. It is known that for the standard normal distribution, there is 68.27% probability of drawing a number in the interval $(-1, 1)$ and a 95.45% probability of drawing a number in the interval $(-2, 2)$.
(a) Find S^L and S^U so that there is 68.27% that the monthly stock price will be between S^L and S^U.
(b) Find S^L and S^U so that there is 95.45% that the monthly stock price will be between S^L and S^U.

Problem 49.4
You are given the following information:$S_0 = 50, \alpha = 0.10, \sigma = 0.25$, and $\delta = 0$. What is $P(S_2 < \$65)$?

Problem 49.5
You are given: $S_0 = 75, \alpha = 0.16, \delta = 0, \sigma = 0.30, t = 0.5$ and $p = 0.10$. Find S_t^U.

Problem 49.6
Let $P(S_t < S_0)$ be the probability that the stock price at the end of t years is less than the original stock price. You are given:$\alpha = 0.10, \delta = 0$, and $\sigma = 0.4$ Show $P(S_t < S_0) < 50\%$. Hint: Assume the contrary and get a contradiction.

Problem 49.7
Consider a European put option on a stock with initial price S_0. Suppose that the risk-premium is greater than $0.5\sigma^2$. Show that the probability that the option expires in the money decreases as the time to expiration increases.

Problem 49.8
A stock is currently selling for \$100. Given that annual rate of return on the stock is 10%. The annual continuously compounded risk-free rate of 8%. Suppose that T years from now the probability that the stock will outperform a zero-coupon bond is 50%. Find the volatility σ.

Problem 49.9
A nondividend-paying stock that is lognormally distributed is currently selling for \$75. Given that the annual expected rate of return is 12% and the annual standard deviation is 25%. Find the biannual continuously compounded mean return and the biannual standard deviation.

Problem 49.10
A nondividend-paying stock that is lognormally distributed is currently selling for \$75. Given that the annual expected rate of return is 12% and the annual standard deviation is 25%. It is known that for the standard normal distribution, there is 68.27% probability of drawing a number in the interval $(-1, 1)$ and a 95.45% probability of drawing a number in the interval $(-2, 2)$.
(a) Find r^L and r^U so that there is 68.27% that the biannual continuously compounded return on the stock will be between r^L and r^U.
(b) Find r^L and r^U so that there is 95.45% that the biannual continuously compounded return on the stock will be between r^L and r^U.

Problem 49.11
A nondividend-paying stock that is lognormally distributed is currently selling for \$75. Given that the annual expected rate of return is 12% and the annual standard deviation is 25%. It is known that for the standard normal distribution, there is 68.27% probability of drawing a number in the interval $(-1, 1)$ and a 95.45% probability of drawing a number in the interval $(-2, 2)$.

(a) Find S^L and S^U so that there is 68.27% that the biannual stock price will be between S^L and S^U.
(b) Find S^L and S^U so that there is 95.45% that the biannual stock price will be between S^L and S^U.

Problem 49.12
A stock is currently selling for \$100. The stock pays dividends at the annual continuously compounded yield of 3%. The annual rate of return on the stock is 15% and the annual standard deviation is 30%. Find the probability that a call on the stock with strike price of \$125 will expire in the money 2 years from today.

Problem 49.13 ‡
You are given the following information about a nondividend-paying stock:
(i) The current stock price is 100.
(ii) The stock-price is lognormally distributed.
(iii) The continuously compounded expected return on the stock is 10%.
(iv) The stock's volatility is 30%.
Consider a nine-month 125-strike European call option on the stock. Calculate the probability that the call will be exercised.

Problem 49.14 ‡
Assume the Black-Scholes framework.
You are given the following information for a stock that pays dividends continuously at a rate proportional to its price.
(i) The current stock price is 0.25.
(ii) The stock's volatility is 0.35.
(iii) The continuously compounded expected rate of stock-price appreciation is 15%.
Calculate the upper limit of the 90% lognormal confidence interval for the price of the stock in 6 months.

50 Conditional Expected Price and a Derivation of Black-Scholes Formula

Given that an option expires in the money, what is the expected stock price? More specifically, suppose that a call option with a strike K expires in the money, the **expected stock price** is the conditional stock price $E(S_T|S_T > K)$ which is defined to be the ratio of the **partial expectation** over the probability that $S_T > K$. That is,

$$E(S_T|S_T > K) = \frac{PE(S_t|S_t > K)}{Pr(S_t > K)}$$

where PE stands for partial expectation which is given by

$$PE(S_t|S_t > K) = \int_K^\infty S_t g(S_t, S_0)dS_t$$

where

$$g(S_t, S_0) = \frac{1}{S_t\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2}\left(\frac{\ln S_t - \ln S_0 - (\alpha-\delta-0.5\sigma^2)t}{\sigma\sqrt{t}}\right)^2}.$$

Now, we have

$$\begin{aligned}
PE(S_t|S_t > K) =& \int_K^\infty S_t g(S_t, S_0)dS_t\\
=&\frac{1}{\sqrt{2\pi\sigma^2 t}}\int_K^\infty e^{-\frac{1}{2}\left(\frac{\ln S_t - \ln S_0 - (\alpha-\delta-0.5\sigma^2)t}{\sigma\sqrt{t}}\right)^2} dS_t\\
=&\frac{1}{\sqrt{2\pi}}\int_{\frac{\ln K - \ln S_0 - (\alpha-\delta-0.5\sigma^2)t}{\sigma\sqrt{t}}}^\infty e^{\ln S_0 + (\alpha-\delta-0.5\sigma^2)t + \sigma\sqrt{t}w_t - 0.5w_t^2} dw_t\\
=&e^{\ln S_0 + (\alpha-\delta-0.5\sigma^2)t + 0.5\sigma^2 t}\frac{1}{\sqrt{2\pi}}\int_{\frac{\ln K - \ln S_0 - (\alpha-\delta-0.5\sigma^2)t}{\sigma\sqrt{t}}}^\infty e^{-\frac{1}{2}(w_t - \sigma\sqrt{t})^2} dw_t\\
=&S_0 e^{(\alpha-\delta)t}\frac{1}{\sqrt{2\pi}}\int_{\frac{\ln K - \ln S_0 - (\alpha-\delta+0.5\sigma^2)t}{\sigma\sqrt{t}}}^\infty e^{-\frac{1}{2}w_t^2} dw_t\\
=&S_0 e^{(\alpha-\delta)t} N\left(\frac{\ln S_0 - \ln K + (\alpha-\delta+0.5\sigma^2)t}{\sigma\sqrt{t}}\right)\\
=&S_0 e^{(\alpha-\delta)t} N(\hat{d}_1)
\end{aligned}$$

where

$$\hat{d}_1 = \frac{\ln S_0 - \ln K + (\alpha - \delta + 0.5\sigma^2)t}{\sigma\sqrt{t}}.$$

Hence,

$$E(S_t|S_t > K) = \frac{S_0 e^{(\alpha-\delta)t} N(\hat{d}_1)}{P(S_t > K)} = S_0 e^{(\alpha-\delta)t} \frac{N(\hat{d}_1)}{N(\hat{d}_2)}.$$

For a put, See Problem 50.1, we find

$$E(S_t|S_t < K) = S_0 e^{(\alpha-\delta)t} \frac{N(-\hat{d}_1)}{N(-\hat{d}_2)}.$$

Example 50.1
You are given the following information:$S_0 = 50, \alpha = 0.10, \sigma = 0.25$, and $\delta = 0$. What is $E(S_2|S_2 > \$65)$?

Solution.
We are asked to find $E(S_2|S_2 > 65) = S_0 e^{(\alpha-\delta)t} \frac{N(\hat{d}_1)}{N(\hat{d}_2)}$ where

$$\hat{d}_1 = \frac{\ln S_0 - \ln K + (\alpha - \delta + 0.5\sigma^2)t}{\sigma\sqrt{t}} = \frac{\ln 50 - \ln 65 + (0.10 - 0 + 0.5 \times 0.25^2)(2)}{0.25\sqrt{2}} = 0.0004$$

and

$$\hat{d}_2 = \frac{\ln S_0 - \ln K + (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}} = \frac{\ln 50 - \ln 65 + (0.10 - 0 - 0.5 \times 0.25^2)(2)}{0.25\sqrt{2}} = -0.3532.$$

Thus, using Excel spreadsheet we find

$$N(\hat{d}_1) = N(0.0004) = 0.50016 \text{ and } N(\hat{d}_2) = N(-0.3532) = 0.361969.$$

Hence,

$$E(S_2|S_2 > 65) = 50e^{0.10\times 2} \times \frac{0.50016}{0.361969} = \$84.385 \blacksquare$$

Example 50.2
A stock is currently selling for \$100. The stock pays dividends at the continuously compounded yield of 3%. Consider a European call on the stock with strike price of \$100 and time to maturity T. Suppose that the annual rate of return is 3%. Show that the partial expectation of S_T conditioned on $S_T > 100$ increases as the time to maturity increases.

Solution.
We have

$$PE(S_t|S_t > K) = S_0e^{(\alpha-\delta)t}N(\hat{d}_1) = 100N(0.5\sigma\sqrt{T}).$$

It follows that as T increases $N(0.5\sigma\sqrt{T})$ increases and therefore the partial expectation increases ■

Derivation of the Black-Scholes Formula
We next examine a derivation of the Black-Scholes formula using the results established above. We use risk-neutral probability where $\alpha = r$ in the formulas above. We let g^* denote the risk-neutral pdf, E^* denote the expectation with respect to risk-neutral probabilities, and Pr^* the risk-neutral probability.
For a European call option with strike K, the price is defined as the expected value of $e^{-rt}\max\{0, S_t - K\}$. That is,

$$C(S, K, \sigma, r, t, \delta) = E[e^{-rt}\max\{0, S_t - K\}]$$

Now we proceed to establish the Black-Scholes formula based on risk-neutral probability:

$$\begin{aligned}
C(S, K, \sigma, r, t, \delta) =& E[e^{-rt}\max\{0, S_t - K\}] \\
=& e^{-rt}\int_K^\infty (S_t - K)g^*(S_t, S_0)dS_t \\
=& e^{-rt}E^*(S_t - K|S_t > K)Pr^*(S_t > K) \\
=& e^{-rt}E^*(S_t|S_t > K)Pr^*(S_t > K) - e^{-rt}E^*(K|S_t > K)Pr^*(S_t > K) \\
=& e^{-\delta t}S_0N(d_1) - Ke^{-rt}N(d_2)
\end{aligned}$$

which is the celebrated Black-Scholes formula.
A similar argument, shows that the price of a put option is given by

$$P(S, K, \sigma, r, t, \delta) = Ke^{-rt}N(-d_2) - S_0e^{-\delta t}N(-d_1).$$

Practice Problems

Problem 50.1
The **partial expectation** of S_t, conditioned on $S_t < K$, is given by

$$PE(S_t|S_t < K) = \int_0^K S_t g(S_t, S_0) dS_t$$

where

$$g(S_t, S_0) = \frac{1}{S_t\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2}\left(\frac{\ln S_t - \ln S_0 - (\alpha-\delta-0.5\sigma^2)t}{\sigma\sqrt{t}}\right)^2}.$$

What is the expected stock price conditioned on a put option with strike price K expiring in the money?

Problem 50.2
Derive the Black-Scholes formula for a put:

$$P(S, K, \sigma, r, t, \delta) = Ke^{-rt}N(-d_2) - S_0e^{-\delta t}N(-d_1).$$

Problem 50.3
A stock is currently selling for \$100. The stock pays dividends at the continuously compounded yield of 3%. Consider a European put on the stock with strike price of \$100 and time to maturity T. Suppose that the annual rate of return is 3%. Show that the expectation of S_T conditioned on $S_T < 100$ decreases as the time to maturity increases.

Problem 50.4
A stock is currently selling for \$100. The stock pays dividends at the continuously compounded yield of 3%. Consider a European put on the stock with strike price of \$100 and time to maturity T. Suppose that the annual rate of return is 3%. Show that the partial expectation of S_T conditioned on $S_T < 100$ decreases as the time to maturity increases.

Problem 50.5
A stock is currently selling for \$100. The stock pays no dividends and has annual standard deviation of 25%. Consider a European call on the stock with strike price of K and time to maturity in 2 years. Suppose that the annual rate of return is 10%. It is found that the partial expectation of S_t, conditioned on $S_t > K$ is $50e^{0.2}$. Determine the value of K.

Problem 50.6
A stock is currently selling for \$50. The stock pays dividends at the continuously compounded yield of 4%. The annual standard deviation is 30%. Consider a European call on the stock with strike price of \$48 and time to maturity in six months. Suppose that the annual rate of return is 13%. Find the partial expectation of S_t if the call expires in the money.

Problem 50.7
A stock is currently selling for \$50. The stock pays dividends at the continuously compounded yield of 4%. The annual standard deviation is 30%. Consider a European call on the stock with strike price of \$48 and time to maturity in six months. Suppose that the annual rate of return is 13%. Find the conditional expectation that the call expires in the money.

Problem 50.8
A stock is currently selling for S_0. The stock pays dividends at the continuously compounded yield of 3%. Consider a European call on the stock with strike price S_0 and time to maturity of 3 years. The annual standard deviation on the stock price is 25%. The annual return on the stock is 9%. The partial expectation that the stock will expire in the money in three years is 85.88. That is, $PE(S_3|S_3 > S_0) = 85.88$. Find the value of S_0.

Problem 50.9
Consider a binomial model in which a put strike price is \$50, and the stock price at expiration can be \$20,\$40,\$60, and \$80 with probabilities 1/8,3/8,3/8, and 1/8.
(a) Find the partial expectation for a put to be in the money.
(b) Find the conditonal expectation.

Option Pricing Via Monte Carlo Simulation

A **Monte Carlo option model** uses Monte Carlo methods to calculate the price of an option with complicated features. In general, the technique is to generate several thousand possible (but random) price paths for the underlying asset via simulation, and to then calculate the associated payoff of the option for each path. These payoffs are then averaged and discounted to today, and this result is the value of the option today.

51 Option Valuation as a Discounted Expected Value

In this section, we examine an option pricing that is performed by discounting expected payoff values at the risk-free rate. We will see that Monte Carlo valuation model exploits the ideas of this section.

Consider a stock that is currently trading for \$41. A European call option on the stock has a strike price of \$40 and expiration of one year. The stock pays no dividends and its price volatility is 30%. The continuously compounded risk-free interest rate is 8%.

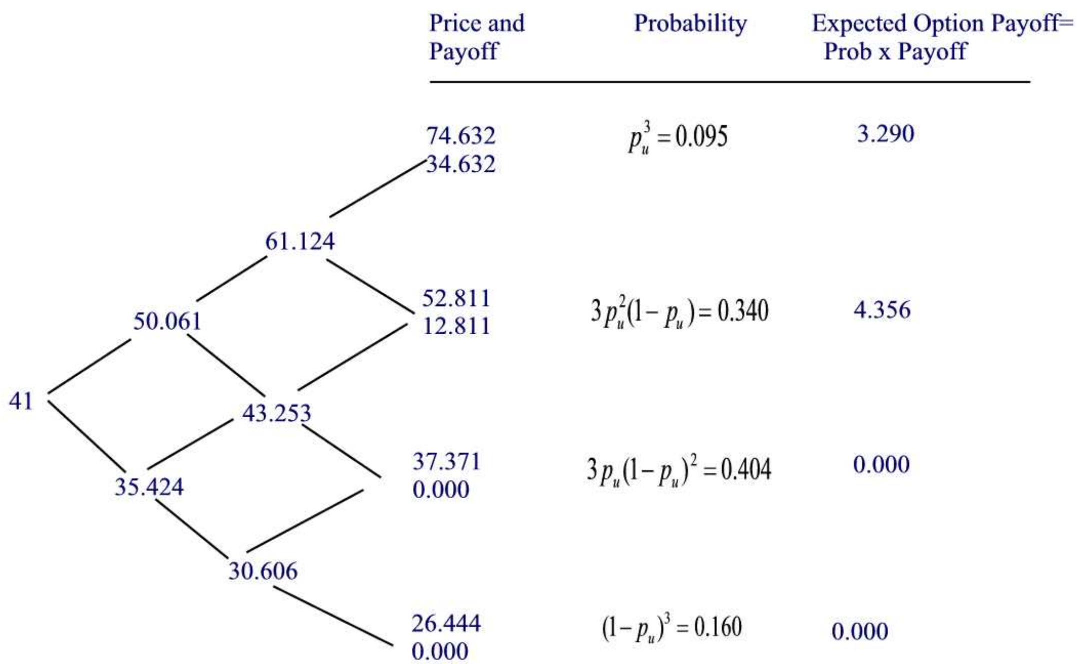


Figure 51.1

We consider a three-period binomial tree. The values of u and d are

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{0.08\times\frac{1}{3}+0.3\sqrt{\frac{1}{3}}} = 1.221$$

and

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{0.08\times\frac{1}{3}-0.3\sqrt{\frac{1}{3}}} = 0.864.$$

The risk-neutral probability of an up movement is

$$p_u = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{0.08\times\frac{1}{3}} - 0.864}{1.221 - 0.864} = 0.4566.$$

The tree of stock prices, option payoffs, the associated probabilities, and the expected payoff values are shown in Figure 51.1. The option price can be found be discounting the expected payoff values obtaining:

$$C = e^{-0.08}(3.290 + 4.356) = \$7.058$$

which is the same price obtained by using the binomial pricing model as discussed in Section 17.
We can use Figure 51.1 to illustrate Monte Carlo estimation of the option price. Imagine a gambling wheel divided into four unequal sections: A (corresponds to the payoff \$34.362), B (\$12.811), C (\$0.00), and D (\$0.00). Suppose we spin the wheel 100 times with the following number of occurrence: A (30), B (22), C (38), and D (10). Then an estimate of the expected payoff is

$$\frac{30\times 34.362 + 22\times 12.811 + 38\times 0 + 10\times 0}{100} = \$13.13.$$

An estimate to the call option price is

$$C \approx e^{-0.08}\times 13.13 = \$12.12.$$

The procedure we used to price the call option discussed earlier does not work if risk-neutral probabilities are replaced by true probabilities. The reason is that the discount rate is not the same at all nodes of the tree. Suppose that the expected rate of return is 15%. Then the true probability of an upward move is

$$p = \frac{e^{(\alpha-\delta)h} - d}{u-d} = \frac{e^{(0.15-0)\times\frac{1}{3}} - 0.864}{1.221 - 0.864} = 0.5246.$$

We next find the discount rate γ at each node of the tree:
Node with Stock price = \$74.632 $\gamma = \text{N/A}$.
Node with Stock price = \$52.811 $\gamma = \text{N/A}$.
Node with Stock price = \$37.371 $\gamma = \text{N/A}$.
Node with Stock price = \$26.444 $\gamma = \text{N/A}$.
Node with Stock price = \$61.124 We have

$$e^{-\gamma\times h}[pC_{uuu} + (1-p)C_{uud}] = e^{-rh}[p_uC_{uuu} + (1-p_u)C_{uud}$$

or

$$e^{-\gamma\times\frac{1}{3}}[0.5246\times 34.632+(1-0.5246)\times 12.811] = e^{-0.08\times\frac{1}{3}}[0.4566\times 34.632+(1-0.4566)\times 12.811].$$

Solving this equation we find $\gamma = 0.269$. Likewise, we find
Node with Stock price = \$43.253 $\gamma = 0.495$
Node with Stock price = \$30.606 $\gamma =$N/A
Node with Stock price = \$50.061 $\gamma = 0.324$
Node with Stock price = \$35.424 $\gamma = 0.496$
Node with Stock price = \$41.00 $\gamma = 0.357$.
With true probabilities, the unique discount rate we used for the risk-neutral probability is now being replaced by the discount rate for each path. The discount rate of each path is defined to be the average of the discount rate at each node along the path. In Figure 51.1, there are eight possible paths for the stock price but only four have a positive option payff, namely the paths: uuu, uud, udu, and duu. We next find the discounting payoff for each path:

- Along the uuu path, the path discount rate is $(0.357 + 0.324 + 0.269)/3 =$ 31.67%. The probability of the path is $p^3 = 0.1444$ with payoff of \$34.632. Hence, the discounted expected value of the path is $e^{-0.3167} \times 0.1444 \times 34.632 = \$3.643$.
- Along the uud path, the path discount rate is $(0.357 + 0.324 + 0.269)/3 =$ 31.67%. The probability of the path is $p^2(1 - p) = 0.1308$ with payoff of \$12.811. Hence, the discounted expected value of the path is $e^{-0.3167} \times 0.1308 \times 12.811 = \$1.221$.
- Along the udu path, the path discount rate is $(0.357 + 0.324 + 0.496)/3 =$ 39.23%. The probability of the path is $p^2(1 - p) = 0.1308$ with payoff of \$12.811. Hence, the discounted expected value of the path is $e^{-0.3923} \times 0.1308 \times 12.811 = \$1.132$.
- Along the duu path, the path discount rate is $(0.357 + 0.496 + 0.495)/3 =$ 44.93%. The probability of the path is $p^2(1 - p) = 0.1308$ with payoff of \$12.811. Hence, the discounted expected value of the path is $e^{-0.4493} \times 0.1308 \times 12.811 = \$1.069$.

It follows that the call option price is

$$3.643 + 1.221 + 1.132 + 1.069 = \$7.065$$

Remark 51.1
In remainder of the chapter it is assumed that the world is risk-neutral.

Practice Problems

Problem 51.1
Consider a stock that is currently trading for \$50. A European put option on the stock has a strike price of \$47 and expiration of one year. The continuously compounded dividend yield is 0.05. The price volatility is 30%. The continuously compounded risk-free interest rate is 6%. Using a two-period binomial model, find the risk-neutral probability of two ups, one up and one down, and two downs.

Problem 51.2
Using the information of the previous problem, construct a figure similar to Figure 51.1

Problem 51.3
Using a gambling wheel with unequal sections, where each section has a probability to one of the option payoffs in the previous problem, the following occurrence are recorded for 140 spins of the wheel: 31 (for the uu path), 70 (for the ud or du path) and 39 (for the dd path). Estimate the put option price.

Problem 51.4
Consider a stock that is currently trading for \$50. A European call option on the stock has a strike price of \$47 and expiration of one year. The continuously compounded dividend yield is 0.05. The price volatility is 30%. The continuously compounded risk-free interest rate is 6%. Using a two-period binomial model, construct a tree showing stock prices, option payoffs, the associated probabilities, and the expected payoff values.

Problem 51.5
Using a gambling wheel with unequal sections, where each section has a probability to one of the option payoffs in the previous problem, the following occurrence are recorded for 140 spins of the wheel: 31 (for the uu path), 70 (for the ud or du path) and 39 (for the dd path). Estimate the call option price.

Problem 51.6
Given the following information about a 1-year European call option on a stock:

- The strike price is \$47.
- The current price of the stock is \$50.
- The expected rate of return is 10%.
- The continuously compounded yield is 5%.
- The continuously compounded risk-free rate is 6%.
- Volatility is 30%.

Find the discount rate at each node in a two-period binomial tree using actual probabilities.

Problem 51.7

Estimate the price of the option in the previous exercise using the discounted expected value approach.

52 Computing Normal Random Numbers

In this section we discuss a couple of techniques to compute normally distributed random numbers required for Monte Carlo valuation. Since both techniques involve uniformly distributed random variables, we start by a discussion of this concept.
A continuous random variable X is said to be **uniformly** distributed over the interval $a \leq x \leq b$ if its pdf is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Since $F(x) = \int_{-\infty}^{x} f(t)dt$, the cdf is given by

$$F(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a < x < b \\ 1 & \text{if } x \geq b \end{cases}$$

If $a = 0$ and $b = 1$ then X is called the **standard uniform** random variable.

Remark 52.1
The values at the two boundaries a and b are usually unimportant because they do not alter the value of the integral of $f(x)$ over any interval. Sometimes they are chosen to be zero, and sometimes chosen to be $\frac{1}{b-a}$. Our definition above assumes that $f(a) = f(b) = f(x) = \frac{1}{b-a}$. In the case $f(a) = f(b) = 0$ then the pdf becomes

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

Example 52.1
You are the production manager of a soft drink bottling company. You believe that when a machine is set to dispense 12 oz., it really dispenses 11.5 to 12.5 oz. inclusive. Suppose the amount dispensed has a uniform distribution. What is the probability that less than 11.8 oz. is dispensed?

Solution.
Since $f(x) = \frac{1}{12.5-11.5} = 1$,

$$Pr(11.5 \leq X \leq 11.8) = \text{area of rectangle of base 0.3 and height } 1 = 0.3 \blacksquare$$

Example 52.2
Suppose that X has a uniform distribution on the interval $(0, a)$, where $a > 0$. Find $P(X > X^2)$.

Solution.
If $a \leq 1$ then $Pr(X > X^2) = \int_0^a \frac{1}{a}dx = 1$. If $a > 1$ then $Pr(X > X^2) = \int_0^1 \frac{1}{a}dx = \frac{1}{a}$. Thus, $Pr(X > X^2) = \min\{1, \frac{1}{a}\}$ ■

The expected value of X is

$$\begin{aligned} E(X) =& \int_a^b xf(x) \\ =& \int_a^b \frac{x}{b-a}dx \\ =& \left.\frac{x^2}{2(b-a)}\right|_a^b \\ =& \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2} \end{aligned}$$

and so the expected value of a uniform random variable is halfway between a and b. Because

$$\begin{aligned} E(X^2) =& \int_a^b \frac{x^2}{b-a}dx \\ =& \left.\frac{x^3}{3(b-a)}\right|_a^b \\ =& \frac{b^3-a^3}{3(b-a)} \\ =& \frac{a^2+b^2+ab}{3} \end{aligned}$$

then

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{a^2+b^2+ab}{3} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}.$$

Example 52.3
You arrive at a bus stop 10:00 am, knowing that the bus will arrive at some

time uniformly distributed between 10:00 and 10:30 am. Let X be your wait time for the bus. Then X is uniformly distributed in $(0, 30)$. Find $E(X)$ and $\text{Var}(X)$.

Solution.
We have $a = 0$ and $b = 30$. Thus, $E(X) = \frac{a+b}{2} = 15$ and $\text{Var}(X) = \frac{(b-a)^2}{12} = \frac{30^2}{12} = 75$ ■

The first technique to compute normally distributed random variables consists of summing 12 uniformly distributed random variables on $(0, 1)$ and subtracting 6. That is, considering the random variable

$$\hat{Z} = \sum_{i=1}^{12} U_i - 6$$

where the U_i are uniformly distributed random variables on $(0, 1)$. The variance of each U_i is $1/12$ and the mean is $1/2$. Thus, the mean of $\hat{Z}$ is

$$E(\hat{Z}) = \sum_{i=1}^{12} E(U_i) - 6 = 12 \times \frac{1}{2} - 6 = 0$$

and the variance is

$$\text{Var}(\hat{Z}) = \sum_{i=1}^{12} \text{Var}(U_i) = \sum_{i=1}^{12} \text{Var}\left(U_i - \frac{1}{2}\right) = \sum_{i=1}^{12} \text{Var}(U_i) = 1.$$

What we obtain is technically not normally distributed but is symmetric with mean zero and standard deviation of 1.0, which are three properties associated with the normal distribution. We can use the central limit theorem and look at $\hat{Z}$ as a close approximation to a standard normal random variable.

Example 52.4
Using the function RAND() that generates unifromly distributed random numbers on (0,1) we draw the set of numbers $\{0.123, 0.456, 0.013, 0.222, 0.781, 0.345, 0.908, 0.111, 0.415, 0$
Find the standard normal estimate of this draw.

Solution.
The sum of these numbers is

$$0.123+0.456+0.013+0.222+0.781+0.345+0.908+0.111+0.415+0.567+0.768+0.777 = 5.486.$$

Hence,

$$\hat{Z} = 5.486 - 6 = -0.514 \blacksquare$$

The second technique that we consider uses the inverse of the cumulative standard normal distribution function $N(x)$. The idea is to convert a single uniformly distributed random number to a normally distributed random number. Let u be a uniformly distributed random number in $(0,1)$. The idea is to interpret $F(u)$ as a quantile.[1] Thus, if $F(u) = 0.5$, we interpret it as 50% quantile. We then use the inverse distribution function, $N^{-1}(u)$, to find the value from the standard normal distribution corresponding to that quantile.

Remark 52.2
The above process simulate draws from the standard normal distribution. Exponentiating these draws we simulate a lognormal random variable. The above procedure of using the inverse cumulative distribution function works for any distribution for which the cdf has an inverse. That is, suppose that D is a CDF such that D^{-1} exists. Let u be a number from the uniform distribution on $(0,1)$. To find the estimate of this draw in the distribution of D we solve the equation $D(d) = F(u)$ for d or $d = D^{-1}(F(u))$.

Example 52.5
Find the 30% quantile of the standard normal random variable.

Solution.
We want to find z such that $Pr(Z \le z) \ge 0.30$ or equivalently $N(z) \ge 0.30$. Considering the equation $N(z) = 0.30$ we find $z = N^{-1}(0.30) = -0.524$ using *NormSInv* in Excel. Using a table, we have $N(-z) = 1 - N(z) = 1 - 0.30 = 0.70 \approx 0.7019$ so that $-z = 0.53$ or $z = -0.53 \blacksquare$

Example 52.6
A draw from a uniformly distributed random variable on $(3,5)$ is 4.6. Find the corresponding single draw from the standard normal distribution.

Solution.
We have

$$F(4.6) = \frac{4.6 - 3}{5 - 3} = 0.8.$$

[1] The q^{th} **quantile** of a random variable X is the smallest number such that $F_X(x) = P(X \le x) \ge q$. Quantiles for any distribution are uniformly distributed which means that any quantile is equally likely to be drawn.

Next, we want to find the 0.8 quantile of the standard normal variable. That is, we want to find z such that $N(z) = 0.8$. Using the inverse cumulative standard normal distribution we find $z = N^{-1}(0.8) = 0.842$ using Excel spreadsheet. Using the table we will have $z = N^{-1}(80) = N^{-1}(0.7995) = 0.84$■

Practice Problems

Problem 52.1
The total time to process a loan application is uniformly distributed between 3 and 7 days.
(a) Let X denote the time to process a loan application. Give the mathematical expression for the probability density function.
(b) What is the probability that a loan application will be processed in fewer than 3 days ?
(c) What is the probability that a loan application will be processed in 5 days or less ?

Problem 52.2
Customers at Santos are charged for the amount of salad they take. Sampling suggests that the amount of salad taken is uniformly distributed between 5 ounces and 15 ounces. Let X = salad plate filling weight
(a) Find the probability density function of X.
(b) What is the probability that a customer will take between 12 and 15 ounces of salad?
(c) Find $E(X)$ and Var(X).

Problem 52.3
Suppose thst X has a uniform distribution over the interval (0, 1). (a) Find $F(x)$.
(b) Show that $P(a \leq X \leq a+b)$ for $a, b \geq 0,\ a+b \leq 1$ depends only on b.

Problem 52.4
Using the function RAND() that generates unifromly distributed random numbers on (0,1) we draw the set of numbers $\{0.126, 0.205, 0.080, 0.303, 0.992, 0.481, 0.162, 0.786$
Find the standard normal estimate of this draw.

Problem 52.5
Find the 10%quantile of the standard normal random variable.

Problem 52.6
Consider an exponentially distributed random variable X with CDF given by $F(x) = 1 - e^{-0.5x}$. Let 0.21072 be drawn from this distribution. Find the corresponding single draw from the standard normal distribution.

Problem 52.7

Consider the following three draws from the uniform distribution in $(0,1)$: $0.209, 0.881$, and 0.025. Find the corresponding draws from the standard normal distribution.

53 Simulating Lognormal Stock Prices

Recall that for a standard normal distribution Z, a lognormal stock price is given by

$$S_t = S_0 e^{(\alpha-\delta-0.5\sigma^2)t+\sigma\sqrt{t}Z}. \tag{53.1}$$

Suppose we want to draw random stock prices for t years from today. We can do that by randomly drawing a set of standard Z's and substituting the results into the equation above.

Example 53.1
Given the following: $S_0 = 50, \alpha = 0.12, \delta = 0, \sigma = 0.30$, and $T = 3$. Find the random set of lognormally distributed stock prices over 3 years using the set of uniform random numbers on $(0, 1) : 0.209, 0.881, 0.025$.

Solution.
We have $N(Z_1) = F(0.209) = 0.209$ so that $Z_1 = N^{-1}(0.209) = -0.8099$. Similarly, $Z_2 = N^{-1}(0.881) = 1.18$ and $Z_3 = N^{-1}(0.025) = -1.95996$. Thus, the new stock prices are

$$\begin{aligned}
S_3 =& 50e^{(0.12-0-0.5\times0.3^2)(3)+0.3\sqrt{3}\times(-0.8099)} = \$41.11\\
S_3 =& 50e^{(0.12-0-0.5\times0.3^2)(3)+0.3\sqrt{3}\times(1.18)} = \$115.60\\
S_3 =& 50e^{(0.12-0-0.5\times0.3^2)(3)+0.3\sqrt{3}\times(-1.96)} = \$22.61 \blacksquare
\end{aligned}$$

Now, if we want to simulate the path of the stock price over t years (which is useful for pricing path-dependent options) then we can do so by splitting t into n equal intervals each of length h, that is, $n = \frac{t}{h}$. In this case, we find the following stock prices:

$$\begin{aligned}
S_h =& S_0 e^{(\alpha-\delta-0.5\sigma^2)h+\sigma\sqrt{h}Z(1)}\\
S_{2h} =& S_h e^{(\alpha-\delta-0.5\sigma^2)h+\sigma\sqrt{h}Z(2)}\\
\vdots =& \vdots\\
S_{nh} =& S_{(n-1)h} e^{(\alpha-\delta-0.5\sigma^2)h+\sigma\sqrt{h}Z(n)}
\end{aligned}$$

Note that

$$S_t = S_0 e^{(\alpha-\delta-0.5\sigma^2)t+\sigma\sqrt{t}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z(i)\right]}.$$

Since $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}Z(i)$[1] is a normal random variable on $(0,1)$, we get the same distribution at time t with the above equation as if we had drawn a single normal random variable on $(0,1)$ as in equation (53.1). The advantage of splitting up the problem into n draws is to create a simulation of the path taken by the stock price.

Example 53.2
Given the following information about a stock: $T = 1, S_0 = 100, \alpha = 0.10, \delta = 0$, and $\sigma = 0.30$. Find the simulated stock price one year from today based on the drawing of the following two random numbers from the uniform distribution on $(0,1)$: 0.15 and 0.65.

Solution.
We first find the corresponding draws from the standard normal distribution. We have $N(z_1) = F(0.15) = 0.15$ so that $z_1 = N^{-1}(0.15) = -1.0363$. Likewise, $z_2 = N^{-1}(0.65) = 0.38532$. Hence,

$$\begin{aligned}
S_T =& S_0 e^{(\alpha-\delta-0.5\sigma^2)T+\sigma\sqrt{T}\left[\frac{z_1+z_2}{\sqrt{2}}\right]}\\
=&100e^{(0.10-0-0.5\times0.3^2)\times1+0.3\sqrt{1}\times\frac{-1.0363+0.38532}{\sqrt{2}}}\\
=&\$112.16 \blacksquare
\end{aligned}$$

Example 53.3
The price of a stock is to be estimated using simulation. It is known that:
(i) The time-t stock price, S_t, follows the lognormal distribution: $\ln(S_t/S_{t-1})$ is normal with mean $(\alpha - 0.5\sigma^2)t$ and variance $\sigma^2 t$.
(ii) $S_0 = 50, \alpha = 0.12$, and $\sigma = 0.30$.
The following are three uniform (0, 1) random numbers:

$$0.209, 0.881, 0.025.$$

Use each of these three numbers to simulate a time-3 stock price. Calculate the mean of the three simulated prices.

Solution.
We have $Z(1) = N^{-1}(0.209) = -0.81, Z(2) = N^{-1}(0.881) = 1.18$, and

[1] Note that the Z_i's are independent.

$Z(3) = N^{-1}(0.025) = -1.96$. Thus,

$$S_1^1 = 50e^{(0.12-0.5\times 0.3^2)\times 3+0.3\times\sqrt{3}\times(-0.81)} = 41.11$$
$$S_1^2 = 50e^{(0.12-0.5\times 0.3^2)\times 3+0.3\times\sqrt{3}\times 1.18} = 115.60$$
$$S_1^3 = 50e^{(0.12-0.5\times 0.3^2)\times 3+0.3\times\sqrt{3}\times(-1.96)} = 22.61$$

Thus, the mean of the three simulated prices is

$$\frac{41.11 + 115.60 + 22.61}{3} = \$59.77 \blacksquare$$

Practice Problems

Problem 53.1
A stock is currently selling for \$100. The continuously compounded risk-free rate is 11%. The continuously compounded dividend yield is 3%. The volatility of the stock according to the Black-Scholes framework is 30%. Find the new stock prices along a path based on the uniformly distributed random numbers: 0.12, 0.87, and 0.50, given that $T = 1$ year. That is, find $S_{\frac{1}{3}}, S_{\frac{2}{3}}$, and S_1.

Problem 53.2
A nondividend-paying stock is currently selling for \$40. The continuously compounded risk-free rate is 8%. The volatility of the stock is 30%. Find a simulation of the stock prices along a path based on the uniformly distributed random numbers: 0.038, 0.424, 0.697, and 0.797, given that $T = 1$ year.

Problem 53.3 ‡
The price of a stock is to be estimated using simulation. It is known that:
(i) The time-t stock price, S_t, follows the lognormal distribution: $\ln{(S_t/S_{t-1})}$ is normal with mean $(\alpha - 0.5\sigma^2)t$ and variance $\sigma^2 t$.
(ii) $S_0 = 50, \alpha = 0.15$, and $\sigma = 0.30$.
The following are three uniform (0, 1) random numbers:

$$0.9830, 0.0384, 0.7794.$$

Use each of these three numbers to simulate a time-2 stock price. Calculate the mean of the three simulated prices.

54 Monte Carlo Valuation for European Options

In this section we discuss the pricing of standard European options using Monte Carlo simulation.

Consider an option on a stock with strike price K and expiration time T. Let $S_T^1, S_T^2, \cdots, S_T^n$ be n randomly drawn time-T stock prices. Let $V(S,T)$ denote the time-T option payoff. We define the time-0 Monte Carlo price of the option by

$$V(S_0,0) = \frac{1}{n}e^{-rT}\sum_{i=1}^{n} V(S_T^i,T) \tag{54.1}$$

where r is the risk-free interest rate. For the case of a call option we have $V(S_T^i,T) = \max\{0, S_T^i - K\}$. For a put, we have $V(S_T^i,T) = \max\{0, K - S_T^i\}$. Note that equation (54.1) uses simulated lognormal stock prices to approximate the lognormal stock price distribution. Also, note that the Monte Carlo valuation uses the risk-neutral probabilities so that the simulated stock price is given by the formula

$$S_T = S_0 e^{(r-\delta-0.5\sigma^2)T+\sigma\sqrt{T}Z}.$$

In the next example, we price a European call option using both the Black-Scholes pricing model and the Monte Carlo pricing model so that we can assess the performance of Monte Carlo Valuation.

Example 54.1

Consider a nondividend paying stock. The annual continuously compounded risk-free interest rate is 0.08, and the stock price volatility is 0.3. Consider a 40-strike call on the stock with 91 days to expiration.

(a) Within the Black-Scholes framework, what is the option price today if the current stock price is \$40?

(b) Using the Monte Carlo, what is the option price today if the current stock price is \$40?

Solution.

(a) Using the Black-Scholes formula we find

$$C = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(40/40) + (0.08 - 0 + 0.5(0.3)^2) \times \frac{91}{365}}{0.3\sqrt{\frac{91}{365}}} = 0.2080477495.$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = 0.2080477495 - 0.3\sqrt{\frac{91}{365}} = 0.0582533699$$

Thus,

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.2080477495} e^{-\frac{x^2}{2}} dx = 0.582404$$

and

$$N(d_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.0582533699} e^{-\frac{x^2}{2}} dx = 0.523227.$$

Hence,

$$C = 40 \times 0.582404 - 40e^{-0.08 \times \frac{91}{365}} \times 0.523227 = \$2.7804$$

(b) We draw random 3-month stock prices using the formula

$$S_{\text{3months}} = 40e^{(0.08 - 0.5(0.3)^2) \times 0.25 + 0.3\sqrt{0.25}Z}.$$

For each stock price, we compute the option payoff

$$\text{Option payoff} = \max\{0, S_{\text{3months}} - 40\}.$$

The following table shows the results of Monte Carlo valuation of the call option using 5 trials where each trial uses 500 random draws.

Trial	Computed Call Price
1	2.98
2	2.75
3	2.63
4	2.75
5	2.91

Averaging the prices in the table we find \$2.804 versus the Black-Scholes price \$2.7804 ■

Example 54.2
A stock is currently selling for \$100. The annual continuously compounded yield is 0.02. The annual continuously compounded risk-free interest rate is 0.07, and the stock price volatility is 0.25. Consider a \$102-strike put with one year to expiration. Using the four draws from the uniform distribution on (0,1): 0.90, 0.74, 0.21, 0.48, compute the price of the put using the Monte Carlo valuation.

Solution.
First we find the corresponding draws from the normal distribution on (0,1):

$$N(Z_1) = F(0.90) \Rightarrow Z_1 = N^{-1}(0.90) = 1.282$$

$$N(Z_2) = F(0.74) \Rightarrow Z_1 = N^{-1}(0.74) = 0.64$$
$$N(Z_3) = F(0.21) \Rightarrow Z_1 = N^{-1}(0.21) = -0.806$$
$$N(Z_4) = F(0.48) \Rightarrow Z_1 = N^{-1}(0.48) = -0.05$$

The stock prices are found using the formula

$$S_T = S_0 e^{(r-\delta-0.5\sigma^2)T+\sigma\sqrt{T}Z}.$$

The four simulated stock prices with their payoffs are shown in the table below

S_1^i	$V(S_1^i, 1)$
140.39	0
119.57	0
83.30	18.70
100.63	1.37

Thus, the Monte Carlo time-0 price of the put is

$$V(100, 0) = \frac{e^{-0.07}}{4}[0 + 0 + 18.70 + 1.37] = \$4.68 \blacksquare$$

Error Analysis of Monte Carlo Methods
The Monte Carlo valuation is simple but relatively inefficient. A key question in error analysis is how many simulated stock prices needed in order to achieve a certain desired degree of accuracy. One approach is to run the simulation different times and see how much variability there is in the result.

For example, the table of Example 54.1 shows the results from running five Monte Carlo valuations, each consisting of 500 stock price draws. It took 2500 draws to get a close answer to the correct answer of \$2.7804.
The option price estimate obtained from a Monte Carlo simulation is a sample average. Thus an elementary principle of statistics states that the standard deviation of the estimate is the standard deviation of the sample (i.e., of the individual price estimates) divided by the square root of the sample size. Consequently, the error reduces at the rate of 1 over the square root of the sample size. Thus, the accuracy of the estimate is increased by increasing the sample size. To see this, suppose σ is the standard deviation of the sample. We first conduct a Monte Carlo simulation using n_1 random drawings. Since the option value is a sample mean, the standard deviation of our estimate of the option value is $\sigma/\sqrt{n_1}$. Now suppose we wanted to reduce that standard deviation of the estimate in half. How much larger must the sample be? Let this new sample size be n_2. Then its standard deviation of the estimate of the option price is $\sigma/\sqrt{n_2}$. But then

$$\frac{\sigma}{\sqrt{n_2}} = \frac{1}{2}\frac{\sigma}{\sqrt{n_1}} \text{ if and only if } n_2 = 4n_1.$$

Thus, to achieve a 50% reduction in error, i.e., a 50% increase in accuracy, we must quadruple the number of random drawings. That is, the standard error reduces only at the rate of the square root of the sample size, not at the rate of the sample size itself.

Example 54.3
The standard deviation of the 2500 price estimates of Example 54.1 is \$4.05. Find the standard deviation of a sample of 500 draws. What percentage of the correct option price is that?

Solution.
For 500 draws, the standard deviation is

$$\frac{\sigma}{\sqrt{n}} = \frac{4.05}{\sqrt{500}} = \$0.18$$

which is $0.18/2.78 = 6.5\%$ of the correct option price ■

Practice Problems

Problem 54.1
A stock is currently selling for \$100. The annual continuously compounded yield is 0.03. The annual continuously compounded risk-free interest rate is 0.11, and the stock price volatility is 0.30. Consider a \$102-strike call with six months to expiration. Using the two draws from the uniform distribution on (0,1): 0.15 and 0.65 compute the price of the call using the Monte Carlo valuation.

Problem 54.2
A stock is currently selling for \$100. The annual continuously compounded yield is 0.03. The annual continuously compounded risk-free interest rate is 0.11, and the stock price volatility is 0.30. Consider a \$102-strike put with one year to expiration. Using the three draws from the uniform distribution on (0,1): 0.12, 0.87, and 0.50, compute the price of the put using the Monte Carlo valuation.

Problem 54.3
How many draws are required in Example 54.1 so that the standard deviation of the estimate is 1% of the correct price?

Problem 54.4
In a Monte Carlo estimate of a European call, the standard deviation of the individual price estimates is found to be 3.46. What is the standard deviation of the estimate in a sample of 10000 draws?

Problem 54.5
In a Monte Carlo estimate of a European call, the standard deviation of the individual price estimates is found to be 3.46. The correct price is \$2.74. How many draws are required so that the standard deviation of the estimate is 2% of the correct price?

55 Monte Carlo Valuation of Asian Options

In this section, we consider the Monte Carlo valuation of an average (either arithmetic or geometric) price[1] Asian option where the payoff is based on an average stock price that replaces the actual price at expiration. Similar discussion holds for average strike Asian options.
Consider an Asian call option with strike price K and time to expiration T. Split T into n equal periods each of length $h = \frac{T}{n}$. Let $Z(i), 1 \leq i \leq n$, be random draws from the normal distribution on $(0,1)$. We can find the simulated stock prices as follows:

$$\begin{aligned} S_h =& S_0 e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}Z(1)} \\ S_{2h} =& S_h e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}Z(2)} \\ \vdots =& \vdots \\ S_{nh} =& S_{(n-1)h} e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}Z(n)} \end{aligned}$$

For an average price Asian option the payoff is given by

$$\max\{0, \overline{S} - K\}$$

where

$$\overline{S} = \frac{S_h + S_{2h} + \cdots + S_{nh}}{n}$$

in the case of an arithmetic average or

$$\overline{S} = (S_h \times S_{2h} \times \cdots \times S_{nh})^{\frac{1}{n}}$$

for a geometric average.

Example 55.1
A stock price is currently selling for \$100. The continuously compounded risk-free rate is 11%. The continuously compounded yield is 3%. The stock price volatility is 30%. Consider an arithmetic average price Asian put option with time to maturity of 3 years and strike price of \$100. The stock price at the end of years 1, 2 , and 3 are simulated using the following (0,1) uniform random numbers: 0.12, 0.87, and 0.50. Find the payoff of this put at expiration.

[1]The advantage of averaging stock prices is that it reduces the likelihood of large gains or losses.

Solution.
Using NormSInv in Excel spreadsheet we find $Z(1) = N^{-1}(0.12) = -1.175$, $Z(2) = N^{-1}(0.87) = 1.126$, and $Z(3) = N^{-1}(0.50) = 0.000$. Thus the yearly stock prices are

$$\begin{aligned}
S_1 =& S_0 e^{(r-\delta-0.5\sigma^2)T+\sigma\sqrt{T}Z(1)} = 100e^{(0.11-0.03-0.5(0.3)^2)+0.3\sqrt{1}\times(-1.175)} = \$71.36 \\
S_2 =& S_1 e^{(r-\delta-0.5\sigma^2)T+\sigma\sqrt{T}Z(2)} = 71.36e^{(0.11-0.03-0.5(0.3)^2)+0.3\sqrt{1}\times(1.126)} = \$101.04 \\
S_3 =& S_2 e^{(r-\delta-0.5\sigma^2)T+\sigma\sqrt{T}Z(3)} = 101.04e^{(0.11-0.03-0.5(0.3)^2)+0.3\sqrt{1}\times(0)} = \$102.57
\end{aligned}$$

Thus, the payoff of the put option is

$$\max\{0, 100 - (71.36 + 101.04 + 102.57)/3\} = \$8.34 \blacksquare$$

Example 55.2
A stock price is currently selling for \$100. The continuously compounded risk-free rate is 11%. The continuously compounded yield is 3%. The stock price volatility is 30%. Consider a geometric average strike Asian call option with time to maturity of 1 year. The stock price at the end of four months, eight months, and 12 months are simulated using the following (0,1) uniform random numbers: 0.12, 0.87, and 0.50. Find the payoff of this call at expiration.

Solution.
Using NormSInv in Excel spreadsheet we find $Z(1) = N^{-1}(0.12) = -1.175$, $Z(2) = N^{-1}(0.87) = 1.126$, and $Z(3) = N^{-1}(0.50) = 0.000$. Thus, the simulated stock prices are

$$\begin{aligned}
S_{\frac{1}{3}} =& S_0 e^{(r-\delta-0.5\sigma^2)T+\sigma\sqrt{T}Z(1)} = 100e^{(0.11-0.03-0.5(0.3)^2)\times\frac{1}{3}+0.3\sqrt{\frac{1}{3}}\times(-1.175)} = \$82.54 \\
S_{\frac{2}{3}} =& S_{\frac{1}{3}} e^{(r-\delta-0.5\sigma^2)T+\sigma\sqrt{T}Z(2)} = 82.54e^{(0.11-0.03-0.5(0.3)^2)\times\frac{1}{3}+0.3\sqrt{\frac{1}{3}}\times(1.126)} = \$101.49 \\
S_1 =& S_{\frac{2}{3}} e^{(r-\delta-0.5\sigma^2)T+\sigma\sqrt{T}Z(3)} = 101.49e^{(0.11-0.03-0.5(0.3)^2)\times\frac{1}{3}+0.3\sqrt{\frac{1}{3}}\times(0)} = \$102.68
\end{aligned}$$

Thus, the payoff of the call option is

$$\max\{0, 102.68 - (82.54 \times 101.49 \times 102.68)^{\frac{1}{3}}\} = \$7.58 \blacksquare$$

Example 55.3
A stock price is currently selling for \$100. The continuously compounded

risk-free rate is 11%. The continuously compounded yield is 3%. The stock price volatility is 30%. Consider a geometric average strike Asian call option with time to maturity of 1 year. Find the payoff of this call at expiration if he stock price at the end of four months, eight months, and 12 months are simulated using the following (0,1) uniform random numbers:
(a) 0.12, 0.87, and 0.50.
(b) 0.341, 0.7701, and 0.541
(c) 0.6751, 0.111, and 0.078.
Estimate the Monte Carlo valuation of this call option.

Solution.
(a) From the previous example, we found the payoff of the call option to be

$$\max\{0, 102.68 - (82.54 \times 101.49 \times 102.68)^{\frac{1}{3}}\} = \$7.58.$$

(b) Using NormSInv in Excel spreadsheet we find $Z(1) = N^{-1}(0.341) = -0.41, Z(2) = N^{-1}(0.7701) = 0.739$, and $Z(3) = N^{-1}(0.541) = 0.103$. Thus, the simulated stock prices are

$$\begin{aligned}
S_{\frac{1}{3}} =& S_0e^{(r-\delta-0.5\sigma^2)T+\sigma\sqrt{T}Z(1)} = 100e^{(0.11-0.03-0.5(0.3)^2)\times\frac{1}{3}+0.3\sqrt{\frac{1}{3}}\times(-0.41)} = \$94.24\\
S_{\frac{2}{3}} =& S_{\frac{1}{3}}e^{(r-\delta-0.5\sigma^2)T+\sigma\sqrt{T}Z(2)} = 94.24e^{(0.11-0.03-0.5(0.3)^2)\times\frac{1}{3}+0.3\sqrt{\frac{1}{3}}\times(0.739)} = \$108.37\\
S_1 =& S_{\frac{2}{3}}e^{(r-\delta-0.5\sigma^2)T+\sigma\sqrt{T}Z(3)} = 108.37e^{(0.11-0.03-0.5(0.3)^2)\times\frac{1}{3}+0.3\sqrt{\frac{1}{3}}\times(0.103)} = \$111.62
\end{aligned}$$

Thus, the payoff of the call option is

$$\max\{0, 111.62 - (94.24 \times 108.37 \times 111.62)^{\frac{1}{3}}\} = \$7.16.$$

(c) Using NormSInv in Excel spreadsheet we find $Z(1) = N^{-1}(0.6751) = 0.454, Z(2) = N^{-1}(0.111) = -1.22$, and $Z(3) = N^{-1}(0.078) = -1.41$. Thus, the simulated stock prices are

$$\begin{aligned}
S_{\frac{1}{3}} =& S_0e^{(r-\delta-0.5\sigma^2)T+\sigma\sqrt{T}Z(1)} = 100e^{(0.11-0.03-0.5(0.3)^2)\times\frac{1}{3}+0.3\sqrt{\frac{1}{3}}\times(0.454)} = \$109.45\\
S_{\frac{2}{3}} =& S_{\frac{1}{3}}1e^{(r-\delta-0.5\sigma^2)T+\sigma\sqrt{T}Z(2)} = 109.45e^{(0.11-0.03-0.5(0.3)^2)\times\frac{1}{3}+0.3\sqrt{\frac{1}{3}}\times(-1.22)} = \$89.64\\
S_1 =& S_{\frac{2}{3}}e^{(r-\delta-0.5\sigma^2)T+\sigma\sqrt{T}Z(3)} = 89.64e^{(0.11-0.03-0.5(0.3)^2)\times\frac{1}{3}+0.3\sqrt{\frac{1}{3}}\times(-1.41)} = \$71.04
\end{aligned}$$

Thus, the payoff of the call option is

$$\max\{0, 71.04 - (109.45 \times 89.64 \times 71.04)^{\frac{1}{3}}\} = \$0.00$$

The Monte Carlo estimate of the call option price is

$$C_{\text{Asian}} = \frac{e^{-0.11}}{3}[7.58 + 7.16 + 0.00] = \$0.90 \blacksquare$$

Practice Problems

Problem 55.1
A stock price is currently selling for \$50. The continuously compounded risk-free rate is 8%. The stock pays no dividends. The stock price volatility is 30%. Consider an arithmetic average price Asian call option with time to maturity of 3 years and strike price of \$50. The stock price at the end of years 1, 2 , and 3 are simulated using the following (0,1) uniform random numbers: 0.983, 0.0384, and 0.7794. Find the payoff of this call at expiration.

Problem 55.2
The price of a stock is to be estimated using simulation. It is known that:
(i) The time-t stock price, S_t, follows the lognormal distribution: $\ln(S_t/S_{t-1})$ is normal with mean $(\alpha - 0.5\sigma^2)t$ and variance $\sigma^2 t$.
(ii) $S_0 = 50, \alpha = 0.15$, and $\sigma = 0.30$.
The following are three uniform (0, 1) random numbers:

$$0.9830, 0.0384, 0.7794.$$

Use each of these three numbers to simulate stock prices at the end of four months, eight months, and 12 motnhs. Calculate the arithmetic mean and the geometric mean of the three simulated prices.

Problem 55.3
You are given the following:
(i) The time-t stock price, S_t, follows the lognormal distribution: $\ln(S_t/S_{t-1})$ is normal with mean $(\alpha - 0.5\sigma^2)t$ and variance $\sigma^2 t$.
(ii) $S_0 = 50, \alpha = 0.15$, and $\sigma = 0.30$.
The following are three uniform (0, 1) random numbers:

$$0.9830, 0.0384, 0.7794.$$

Find the payoff at expiration of an Asian call option with strike price of \$50 based on the arithmetic average of the simulated prices of Problem 55.2

Problem 55.4
You are given the following:
(i) The time-t stock price, S_t, follows the lognormal distribution: $\ln(S_t/S_{t-1})$ is normal with mean $(\alpha - 0.5\sigma^2)t$ and variance $\sigma^2 t$.

(ii) $S_0 = 50, \alpha = 0.15$, and $\sigma = 0.30$.
The following are three uniform (0, 1) random numbers:

$$0.9830, 0.0384, 0.7794.$$

Find the payoff at expiration of an Asian put option with strike price of \$70 based on the arithmetic average of the simulated prices of Problem 55.2

Problem 55.5
You are given the following:
(i) The time-t stock price, S_t, follows the lognormal distribution: $\ln(S_t/S_{t-1})$ is normal with mean $(\alpha - 0.5\sigma^2)t$ and variance $\sigma^2 t$.
(ii) $S_0 = 50, \alpha = 0.15$, and $\sigma = 0.30$.
The following are three uniform (0, 1) random numbers:

$$0.9830, 0.0384, 0.7794.$$

Find the payoff at expiration of an arithmetic average strike Asian call option based on the simulated prices of Problem 55.2

Problem 55.6
You are given the following:
(i) The time-t stock price, S_t, follows the lognormal distribution: $\ln(S_t/S_{t-1})$ is normal with mean $(\alpha - 0.5\sigma^2)t$ and variance $\sigma^2 t$.
(ii) $S_0 = 50, \alpha = 0.15$, and $\sigma = 0.30$.
The following are three uniform (0, 1) random numbers:

$$0.9830, 0.0384, 0.7794.$$

Find the payoff at expiration of an arithmetic average strike Asian put option based on the simulated prices of Problem 55.2

Problem 55.7
You are given the following:
(i) The time-t stock price, S_t, follows the lognormal distribution: $\ln(S_t/S_{t-1})$ is normal with mean $(\alpha - 0.5\sigma^2)t$ and variance $\sigma^2 t$.
(ii) $S_0 = 50, \alpha = 0.15$, and $\sigma = 0.30$.
The following are three uniform (0, 1) random numbers:

$$0.9830, 0.0384, 0.7794.$$

Find the payoff at expiration of an Asian call option with strike price of \$50 based on the geometric average of the simulated prices of Problem 55.2

Problem 55.8
You are given the following:
(i) The time-t stock price, S_t, follows the lognormal distribution: $\ln(S_t/S_{t-1})$ is normal with mean $(\alpha - 0.5\sigma^2)t$ and variance $\sigma^2 t$.
(ii) $S_0 = 50, \alpha = 0.15$, and $\sigma = 0.30$.
The following are three uniform (0, 1) random numbers:

$$0.9830, 0.0384, 0.7794.$$

Find the payoff at expiration of an Asian put option with strike price of \$70 based on the geometric average of the simulated prices of Problem 55.2

Problem 55.9
You are given the following:
(i) The time-t stock price, S_t, follows the lognormal distribution: $\ln(S_t/S_{t-1})$ is normal with mean $(\alpha - 0.5\sigma^2)t$ and variance $\sigma^2 t$.
(ii) $S_0 = 50, \alpha = 0.15$, and $\sigma = 0.30$.
The following are three uniform (0, 1) random numbers:

$$0.9830, 0.0384, 0.7794.$$

Find the payoff at expiration of a geometric average strike Asian call option based on the simulated prices of Problem 55.2

Problem 55.10
You are given the following:
(i) The time-t stock price, S_t, follows the lognormal distribution: $\ln(S_t/S_{t-1})$ is normal with mean $(\alpha - 0.5\sigma^2)t$ and variance $\sigma^2 t$.
(ii) $S_0 = 50, \alpha = 0.15$, and $\sigma = 0.30$.
The following are three uniform (0, 1) random numbers:

$$0.9830, 0.0384, 0.7794.$$

Find the payoff at expiration of a geometric average strike Asian put option based on the simulated prices of Problem 55.2

Problem 55.11
The table below lists the arithemtic and geometric averages of simulated stock prices as well as the stock price at expiration (i.e., end of one year).

Trial	Arithmetic Ave.	Geometric Ave.	Stock Price at Exp.
1	39.13	37.42	32.43
2	60.77	59.68	47.26
3	61.85	61.35	60.11
4	53.44	53.43	52.44

The continuously compounded risk-free rate is 10%.
(a) Find the Monte Carlo estimate of an arithmetic average strike of an Asian put option that matures in one year.
(b) Find the Monte Carlo estimate of a geometric average strike of an Asian put option that matures in one year.

56 Control Variate Method

In this section, we discuss a method, the **control variate method**, that increases the accuracy of Monte Carlo valuations.
A control variate in this context is a somewhat similar option to the option we are trying to price but whose true value is known. We then obtain a simulated value of that option. The difference between the true value of the control variate and its simulated value is then added to the simulated value of the option we are trying to price. In this manner, the error in the control variate is added to the simulated value of the option of interest. Let us see how this method works.
Let C_s be the simulated price of the option we are trying to price. Let V_t be the true value of another similar option (the control variate) and V_s be its simulated value. Our control variate estimate is then found as

$$C^* = C_s + (V_t - V_s). \tag{56.1}$$

Since Monte Carlo valuation provides an unbiased estimate, we have $E(V_s) = V_t$ and $E(C_s) = C_t$. Thus, $E(C^*) = E(C_s) = C_t$. For the variance of C^* we have the following result from probability theory

$$\text{Var}(C^*) = \text{Var}(C_s - V_s) = \text{Var}(C_s) + \text{Var}(V_s) - 2\text{Cov}(C_s, V_s).$$

This will be less than $\text{Var}(C_s)$ if $\text{Var}(V_s) < 2\text{Cov}(C_s, V_s)$, meaning that the control variate method relies on the assumption of a large covariance between C_s and V_s. The control variate chosen should be one that is very highly correlated with the option we are pricing.
The **reduction in the variance** is

$$\text{Var}(C_s) - \text{Var}(C^*).$$

Example 56.1
Consider the pricing of an arithmetic average strike Asian put. A control variate is a geometric average strike Asian put since we have a formula for such an option.[1] Suppose that for a given set of random stock prices the following results were found

[1] See Appendix 19.A of [1].

Trial	Arithmetic Average	Geometric Average	Stock Price at Exp.	Arithmetic Option payoff	Geometric Option payoff
1	39.13	37.42	32.43	6.70	4.99
2	60.77	59.68	47.26	13.51	12.42
3	61.85	61.35	60.11	1.74	1.24
4	53.44	53.43	52.44	1.00	0.99

Suppose that the continuously compounded risk-free rate is 10%. Find the control variate estimate of the arithmetic average strike Asian put.

Solution.
The Monte Carlo estimate for the arithmetic average Asian put is

$$A_s = \frac{e^{-0.10}}{4}[6.70 + 13.51 + 1.74 + 1.00] = \$5.19$$

The Monte Carlo estimate for the geometric average Asian put is

$$G_s = \frac{e^{-0.10}}{4}[4.99 + 12.42 + 1.24 + 0.99] = \$4.44.$$

The true price of the geometric option is \$2.02. Hence, the control variate estimate of the arithmetic average strike option is

$$A^* = A_s + (G_t - G_s) = 5.19 + (2.02 - 4.44) = \$2.77 \blacksquare$$

A better estimate of the option price is to replace (56.1) with the equation

$$C^* = C_s + \beta(V_t - V_s).$$

In this case, $E(C^*) = C_t$ and

$$\begin{aligned}\text{Var}(C^*) =& \text{Var}(C_s) + \beta^2\text{Var}(V_s) - 2\beta\text{Cov}(C_s, V_s)\\ =& \text{Var}(C_s) - \left(\frac{\text{Cov}^2(C_s, V_s)}{\text{Var}(V_s)}\right) + \text{Var}(V_s)\left(\beta - \frac{\text{Cov}(C_s, V_s)}{\text{Var}(V_s)}\right)^2.\end{aligned}$$

The variance $\text{Var}(C^*)$ is minimized when

$$\beta = \frac{\text{Cov}(C_s, V_s)}{\text{Var}(V_s)}.$$

The minimal variance is

$$\mathrm{Var}(C^*) = \mathrm{Var}(C_s) - \left(\frac{\mathrm{Cov}^2(C_s, V_s)}{\mathrm{Var}(V_s)}\right).$$

The **variance reduction factor** is

$$\frac{\mathrm{Var}(C^*)}{\mathrm{Var}(C_s)} = 1 - \rho^2_{C_s,V_s}$$

where ρ_{C_s,V_s} is the correlation coefficient between C_s and V_s.

Estimating Cov$(X,Y)/$**Var**(X)
Let $\{x_1, x_2, \cdots, x_n\}$ be a set of observations drawn from the probability distribution of a random variable X. We define the sample mean by

$$\overline{X} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

The sample variance estimation is given by

$$\text{Var}(X) = \frac{\sum_{i=1}^n (x_i - \overline{X})^2}{n-1}.$$

If $\{y_1, y_2, \cdots, y_n\}$ is a set of observations drawn from the probability distribution of a random variable Y, then an estimation of the covariance is given by

$$\text{Cov}(X,Y) = \frac{\sum_{i=1}^n (x_i - \overline{X})(y_i - \overline{Y})}{n-1}.$$

Thus, an estimation of $\beta = \text{Cov}(X,Y)/\text{Var}(X)$ is

$$\beta = \frac{\sum_{i=1}^n (x_i - \overline{X})(y_i - \overline{Y})}{\sum_{i=1}^n (x_i - \overline{X})^2}. \tag{56.2}$$

Example 56.2
Show that $\beta = \frac{\sum_{i=1}^n x_i y_i - n\overline{XY}}{\sum_{i=1}^n x_i^2 - n\overline{X}^2}$.

Soltution.
This follows from

$$\begin{aligned}
\sum_{i=1}^n (x_i - \overline{X})^2 &= \sum_{i=1}^n (x_i^2 - 2\overline{X}x_i + \overline{X}^2) \\
&= \sum_{i=1}^n x_i^2 - 2\overline{X}\sum_{i=1}^n x_i + n\overline{X}^2 = \sum_{i=1}^n x_i^2 - 2n\overline{X}^2 + n\overline{X}^2 \\
&= \sum_{i=1}^n x_i^2 - n\overline{X}^2.
\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{n}(x_i-\overline{X})(y_i-\overline{Y}) &= \sum_{i=1}^{n}x_iy_i-\overline{Y}\sum_{i=1}^{n}x_i-\overline{X}\sum_{i=1}^{n}y_i+n\overline{XY}\\ &= \sum_{i=1}^{n}x_iy_i-n\overline{XY}-n\overline{XY}+n\overline{XY}\\ &= \sum_{i=1}^{n}x_iy_i-n\overline{XY}\ \blacksquare\end{aligned}$$

Example 56.3
Let $C(K)$ denote the Black-Scholes price for a 3-month K-strike European call option on a nondividend-paying stock.
Let $C_s(K)$ denote the Monte Carlo price for a 3-month K-strike European call option on the stock, calculated by using 5 random 3-month stock prices simulated under the risk-neutral probability measure.
You are to estimate the price of a 3-month 52-strike European call option on the stock using the formula

$$C^*(52) = C_s(52) + \beta[C(50) - C_s(50)],$$

where the coefficient β is such that the variance of $C^*(52)$ is minimized.
You are given: (i) The continuously compounded risk-free interest rate is 10%.
(ii) $C(50) = \$2.67$.
(iii) Both Monte Carlo prices, $C_(50)$ and $C_s(52)$, are calculated using the following 5 random 3-month stock prices:

$$43.30, 47.30, 52.00, 54.00, 58.90$$

(a) Based on the 5 simulated stock prices, estimate β.
(b) Compute $C^*(52)$.

Solution.
(a) We know that

$$\beta = \frac{\text{Cov}(C_s, V_s)}{\text{Var}(V_s)}.$$

We use the β of the previous example with x_i and y_i are simulated payoffs of the 50-strike and 52-strike calls respectively corresponding to the ith simulated price where $1 \le i \le 5$. Note that the 50-strike call is our control variate.

We do not need to discount the payoffs because the effect of discounting is canceled in the formula of β. Thus, we have the following table:

Simulated $S_{0.25}$	$\max\{0, S_{0.25} - 50\}$	$\max\{0, S_{0.25} - 52\}$
43.30	0.00	0.00
47.30	0.00	0.00
52.00	2.00	0.00
54.00	4.00	2.00
58.90	8.90	6.90

Hence,

$$\overline{X} = \frac{2 + 4 + 8.90}{5} = 2.98$$

$$\overline{Y} = \frac{2 + 6.90}{5} = 1.78$$

$$\sum_{i=1}^{5} x_i^2 = 2^2 + 4^2 + 8.90^2 = 99.21$$

$$\sum_{i=1}^{5} x_i y_i = 4 \times 2 + 8.90 \times 6.90 = 69.41.$$

The estimate of β is

$$\beta = \frac{69.41 - 5 \times 2.98 \times 1.78}{99.21 - 5 \times 2.98^2} = 0.78251.$$

(b) We have

$$C_s(50) = \frac{e^{-0.10 \times 0.25}}{5} \times (2 + 4 + 8.90) = \$2.91$$

and

$$C_s(52) = \frac{e^{-0.10 \times 0.25}}{5} \times (2 + 6.90) = \$1.736.$$

Hence,

$$C^*(52) = C_s(52) + \beta[C(50) - C_s(50)] = 1.736 + 0.78251(2.67 - 2.91) = \$1.548 \blacksquare$$

Practice Problems

Problem 56.1
For the control variate method, you are given the following parameters: $\rho_{C_s,V_s} = 0.8, \text{Var}(C_s) = 3$, and $\text{Var}(V_s) = 5$. Calculate
(a) The covariance of C_s and V_s.
(b) The variance of of the control variate estimate.
(c) The variance reduction.

Problem 56.2
For the control variate method with beta, you are given the following: $\text{Var}(C^*) = 0.875, \text{Var}(C_s) = 2$. Find the variance reduction factor.

Problem 56.3
For the control variate method with beta, you are given the following: $\text{Var}(C^*) = 0.875, \text{Var}(C_s) = 2$. Find the coefficient of correlation between C_s and V_s.

Problem 56.4
For the control variate method with beta, you are given the following: $\text{Var}(C_s) = 2$ and $\rho_{C_s,V_s} = 0.75$. Claculate the variance of the control variate estimate.

Problem 56.5 ‡
Let $C(K)$ denote the Black-Scholes price for a 3-month K-strike European call option on a nondividend-paying stock.
Let $C_s(K)$ denote the Monte Carlo price for a 3-month K-strike European call option on the stock, calculated by using 5 random 3-month stock prices simulated under the risk-neutral probability measure.
You are to estimate the price of a 3-month 42-strike European call option on the stock using the formula

$$C^*(42) = C_s(42) + \beta[C(40) - C_s(40)],$$

where the coefficient β is such that the variance of $C^*(42)$ is minimized.
You are given:
(i) The continuously compounded risk-free interest rate is 8%.
(ii) $C(40) = \$2.7847$.
(iii) Both Monte Carlo prices, $C_s(40)$ and $C_s(42)$, are calculated using the following 5 random 3-month stock prices:

$$33.29, 37.30, 40.35, 43.65, 48.90$$

(a) Based on the 5 simulated stock prices, estimate β.
(b) Based on the 5 simulated stock prices, compute $C^*(42)$.

Problem 56.6
Let G denote the Black-Scholes price for a 3-month geometric average strike put option on a nondividend-paying stock.
Let G_s denote the Monte Carlo price for a 3-month geometric average strike put option on the stock, calculated by using the following geoemtric average strike prices

$$37.423, 59.675, 61.353, 53.425$$

simulated under the risk-neutral probability measure.
The corresponding simulated arithmetic average strike prices are

$$39.133, 60.767, 61.847, 53.440.$$

You are to estimate the price of a 3-month arithmetic average strike put option on the stock using the formula

$$A^* = A_s + \beta[G - G_s],$$

where the coefficient β is such that the variance of A^* is minimized.
You are given:
(i) The continuously compounded risk-free interest rate is 10%.
(ii) $G = \$2.02$.
(iii) The 3-month simulated stock prices are

$$32.43, 47.26, 70.11, 52.44$$

(a) Estimate β.
(b) Compute the control variate estimate A^*.

57 Antithetic Variate Method and Stratified Sampling

A second variance reduction technique for improving the efficiency of Monte Carlo valuation is the **antithetic variate method.** The fundamental idea behind the method is to bring in negative correlation between two estimates. Remember that in the standard Monte Carlo simulation, we are generating observations of a standard normal random variable. The standard normal random variable is distributed with a mean of zero, a variance of 1.0, and is symmetric. Thus, for each value we draw, there is an equally likely chance of having drawn the observed value times -1. Consequently, for each value of x we draw, we can legitimately create an artificially observed companion observation of $-x$. This is the antithetic variate. This procedure automatically doubles our sample size without having increased the number of random drawings.

Let $Z(1), Z(2), \cdots, Z(n)$ be n standard normal numbers that simulate stock prices and let $C_s^{(1)}$ be the corresponding Monte Carlo estimate. Using, the standard normal numbers $-Z(1), -Z(2), \cdots, -Z(n)$, we simulate an additional set of n stock prices and obtain a corresponding Monte Carlo estimate $V_s^{(2)}$.

We next show that the variance of the antithetic estimate is reduced based on the fact that the estimates $V_s^{(1)}$ and $V_s^{(2)}$ are negatively correlated:

$$\begin{aligned}\text{Var}(V_s^{(1)} + V_s^{(2)}) =&\text{Var}(V_s^{(1)}) + \text{Var}(V_s^{(2)}) + 2\text{Cov}(V_s^{(1)}, V_s^{(2)})\\ =&2\text{Var}(V_s^{(1)}) + 2\text{Cov}(V_s^{(1)}, V_s^{(2)})\\ =&2\text{Var}(V_s^{(1)})(1+\rho)\end{aligned}$$

where ρ is the coefficient of correlation between $V_s^{(1)}$ and $V_s^{(2)}$. Since $\rho < 0$ we conclude that

$$\text{Var}(V_s^{(1)} + V_s^{(2)}) < 2\text{Var}(V_s^{(1)}) = 2\frac{\text{Var}(V^{(1)})}{n}$$

where $\text{Var}(V^{(1)})$ is the variance of the sample $\{V(Z(1)), V(Z(2)), \cdots, V(Z(n))\}$. Hence, for $n \geq 2$, we have $\text{Var}(V_s^{(1)} + V_s^{(2)}) < \text{Var}(V_s^{(1)})$.

Example 57.1

A stock is currently selling for \$40. The continuously compounded risk-free rate is 0.08. The stock pays no dividends. The stock price volatility is 0.35. Six months stock prices are simulated using the standard normal numbers:

$$-0.52, 0.13, -0.25, -1.28.$$

Find the antithetic control estimate of a call option with strike price \$40 and time to maturity of six months.

Solution.
The simulated prices are found by means of the following formula:

$$S_T = S_0 e^{(r-\delta-0.5\sigma^2)T+\sigma\sqrt{T}Z}.$$

Thus, we obtain the following chart

Z	$S_{0.5}^Z$	Option Payoff
-0.52	35.50	0.00
0.13	41.70	1.70
-0.25	37.95	0.00
-1.28	29.41	0.00
0.52	45.92	5.92
-0.13	39.10	0.00
0.25	42.96	2.96
1.28	55.43	15.43

Hence, the antithetic variate estimate of the price of the option is

$$C_s = \frac{e^{-0.08\times 0.5}}{8} \times (0 + 1.70 + 0 + 0 + 5.92 + 0 + 2.96 + 15.43) = \$3.12 \blacksquare$$

Stratified Sampling
A third variance reduction technique is the **stratified sampling method.** The idea behind this method is to generate random numbers in such a way to improve the Monte Carlo estimate.
Given uniform random numbers $U_1, U_2, \cdots, U_n$ on (0,1). Divide the interval $(0,1)$ into $k \leq n$ equal subintervals. We can use the given uniform random numbers to generate uniform random in each of the k intervals by using the formula

$$\hat{U}_i = \begin{cases} \frac{i-1+U_i}{k} & \text{if } i \leq k \\ \frac{(i-k)-1+U_i}{k} & \text{if } i > k \end{cases}$$

Thus, $\hat{U}_1$ is a random number in the interval $(0, \frac{1}{k})$, $\hat{U}_2$ belongs to the interval $(\frac{1}{k}, \frac{2}{k})$ and so on. Moreover, the numbers $\hat{U}_1, \hat{U}_{1+k}, \hat{U}_{1+2k}, \cdots$ are uniformly distributed in the interval $(0, \frac{1}{k})$, the numbers $\hat{U}_2, \hat{U}_{2+k}, \hat{U}_{2+2k}, \cdots$ are uniformly distributed in $(\frac{1}{k}, \frac{2}{k})$ and so on.

Example 57.2
Consider the following 8 uniform random numbers in (0,1):

i	1	2	3	4	5	6	7	8
U_i	0.4880	0.7894	0.8628	0.4482	0.3172	0.8944	0.5013	0.3015

Use the given numbers, to generate uniform random numbers in each quartile, i.e. $k = 4$.

Solution.
We have

$$\begin{aligned}
\hat{U}_1 =& \frac{1-1+U_1}{4} = 0.122\\
\hat{U}_2 =& \frac{2-1+U_2}{4} = 0.44735\\
\hat{U}_3 =& \frac{3-1+U_3}{4} = 0.7157\\
\hat{U}_4 =& \frac{4-1+U_4}{4} = 0.86205\\
\hat{U}_5 =& \frac{(5-4)-1+U_5}{4} = 0.0793\\
\hat{U}_6 =& \frac{(6-4)-1+U_6}{4} = 0.4736\\
\hat{U}_7 =& \frac{(7-4)-1+U_7}{4} = 0.625325\\
\hat{U}_8 =& \frac{(8-4)-1+U_8}{4} = 0.825375 \blacksquare
\end{aligned}$$

Practice Problems

Problem 57.1
A stock is currently selling for \$40. The continuously compounded risk-free rate is 0.08. The stock pays no dividends. The stock price volatility is 0.35. Six months stock prices are simulated using the standard normal numbers:

$$-0.52, 0.13, -0.25, -1.28.$$

Find the antithetic control estimate of a put option with strike price \$40 and time to maturity of six months.

Problem 57.2
Consider the following 8 uniform random numbers in (0,1):

i	1	2	3	4	5	6	7	8
U_i	0.4880	0.7894	0.8628	0.4482	0.3172	0.8944	0.5013	0.3015

Find the difference between the largest and the smallest simulated uniform random variates generated by the stratified sampling method.

Problem 57.3
Consider the following 8 uniform random numbers in (0,1):

i	1	2	3	4	5	6	7	8
U_i	0.4880	0.7894	0.8628	0.4482	0.3172	0.8944	0.5013	0.3015

Compute 8 standard normal random variates by $Z_i = N^{-1}(\hat{U}_i)$, where N^{-1} is the inverse of the cumulative standard normal distribution function.

Problem 57.4 ‡
Michael uses the following method to simulate 8 standard normal random variates:
Step 1: Simulate 8 uniform (0, 1) random numbers $U_1, U_2, \cdots, U_8$.
Step 2: Apply the stratified sampling method to the random numbers so that U_i and U_{i+4} are transformed to random numbers $\hat{U}_i$ and $\hat{U}_{i+4}$ that are uniformly distributed over the interval $((i-1)/4, i/4), i = 1, 2, 3, 4$.
Step 3: Compute 8 standard normal random variates by $Z_i = N^{-1}(\hat{U}_i)$, where N^{-1} is the inverse of the cumulative standard normal distribution function.
Michael draws the following 8 uniform (0,1) random numbers:

i	1	2	3	4	5	6	7	8
U_i	0.4880	0.7894	0.8628	0.4482	0.3172	0.8944	0.5013	0.3015

Find the difference between the largest and the smallest simulated normal random variates.

Problem 57.5
Given the following: $S_0 = 40, \alpha = 0.08, \delta = 0, \sigma = 0.30$, and $T = 1$. Also given the following uniform random numbers in $(0,1)$.

i	1	2	3	4
U_i	0.152	0.696	0.788	0.188

(a) Apply the stratified sampling method with $k = 4$ to find the uniformly distributed numbers, $\hat{U}_1, \hat{U}_2, \hat{U}_3, \hat{U}_4$.
(b) Compute thet 4 standard normal random numbers $Z_i = N^{-1}(\hat{U}_i), 1 \leq i \leq 4$.
(c) Find the random set of lognormally distributed stock prices over the one year period.

Problem 57.6
Using the information of the previous problem, simulate the stock prices at the end of three months, six months, nine months, and one year.

Problem 57.7
Using the simulated stock prices of the previous problem, find the payoff of an arithmetic average price Asian call that expires in one year.

Brownian Motion

Brownian motion is the basic building block for standard derivatives pricing models. For example, the Black-Scholes option pricing model assumes that the price of the underlying asset follows a geometric Brownian motion. In this chapter we explain what this means.

58 Brownian Motion

A **Brownian motion**[1] can be thought of as a random walk where a coin is flipped infinitely fast and infinitesimally small steps (backward or forward based on the flip) are taken at each point. That is, a Brownian motion is a random walk occurring in continuous time with movements that are continuous rather than discrete. The motion is characterized by a family of random variables $Z = \{Z(t)\}$ indexed by time t where $Z(t)$ represents the random walk− the cumulative sum of all moves− after t periods. Such a family is called a **stochastic process.**
For a Brownian motion that starts from z, the process $Z(t)$ has the following characteristics:
(1) $Z(0) = z$.
(2) $Z(t+s) - Z(t)$ is normally distributed with mean 0 and variance s.
(3) $Z(t+s_1) - Z(t)$ is independent of $Z(t) - Z(t-s_2)$ where $s_1, s_2 > 0$. That is, nonoverlapping increments are independently distributed. In other words, $Z(t+s_1) - Z(t)$ and $Z(t) - Z(t-s_2)$ are independent random variables.
(4) $Z(t)$ is a continuous function of time t. (Continuous means you can draw the motion without lifting your pen from the paper.)
If $z = 0$ then the Brownian motion is called a **standard** or a **pure** Brownian motion. Also, a standard Brownian motion is known as **Wiener** motion.

Example 58.1
What is the variance of $Z(t) - Z(s)$? Here, $0 \le s < t$.

Solution.
We have $Z(t) - Z(s) = Z(s + (t-s)) - Z(s)$ so that by (2), the variance is $t - s$ ■

Example 58.2
Show that $E[Z(t+s)|Z(t)] = Z(t)$.

Solution.
For any random variables X, Y, and Z we know that $E(X+Y|Z) = E(X|Z) + E(Y|Z)$ and $E(X|X) = X$. Using these properties, We have

$$\begin{aligned}
E[Z(t+s)|Z(t)] =& E[Z(t+s) - Z(t) + Z(t)|Z(t)]\\
=& E[Z(t+s) - Z(t)|Z(t)] + E[Z(t)|Z(t)]\\
=& 0 + Z(t) = Z(t)
\end{aligned}$$

[1]Named after the Scottish botanist Robert Brown.

where we used Property (2). The above result implies that $Z(t)$ is a **martingale**[2] ■

Example 58.3
Suppose that $Z(2) = 4$. Compute $E[Z(5)|Z(2)]$.

Solution.
We have $E[Z(5)|Z(2)] = E[Z(2+3)|Z(2)] = Z(2) = 4$ ■

Example 58.4
Let Z be a standard Brownian motion. Show that $E(Z(t)Z(s)) = \min\{t, s\}$ where $t, s \geq 0$.

Solution.
Assume $t \geq s$. Since $Z(t)$ is a standard Brownian motion, $Z(0) = 0$ so that by Example 58.1, we have $E[Z(t)] = E[Z(t) - Z(0)] = 0$ and $\text{Var}[Z(t)] = \text{Var}[Z(t) - Z(0)] = t - 0 = t$. But $\text{Var}[Z(t)] = E[Z(t)^2] - (E[Z(t)])^2 = E[Z(t)^2] - 0 = E[Z(t)^2] = t$. Now,

$$\begin{aligned}E(Z(t)Z(s)) =& E[(Z(s) + Z(t) - Z(s))Z(s))] = E[(Z(s))^2] + E[(Z(t) - Z(s))Z(s)]\\ =& s + E[Z(t) - Z(s)]E[Z(s)] = s + 0 = \min\{t, s\}\end{aligned}$$

since $Z(t) - Z(s)$ and $Z(s) = Z(s) - Z(0)$ are independent (by (3)) ■

We next show that a standard Brownian motion Z can be approximated by a sum of independent binomial random variables. By the continuity of Z, for a small time period h we can estimate the change in Z from t to $t + h$ by the equation

$$Z(t + h) - Z(t) = Y(t + h)\sqrt{h}$$

where $Y(t)$ is a random draw from a binomial distribution[3]. Now, take the interval $[0, T]$. Divide $[0, T]$ into n equal subintervals each of length $h = \frac{T}{n}$. Then we have

$$\begin{aligned}Z(T) =& \sum_{i=1}^{n}[Z(ih) - Z((i-1)h)] = \sum_{i=1}^{n} Y(ih)\sqrt{h}\\ =& \sqrt{T}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Y(ih)\right] \qquad (58.1)\end{aligned}$$

[2]A stochastic process $Z(t)$ for which $E[Z(t + s)|Z(t)] = Z(t)$ is called a **martingale**.
[3]$Y(t) = \pm 1$ with probability 0.5. Also, $E(Y(t)) = 0$ and $\text{Var}(Y(t)) = 1$.

Since $E(Y(ih)) = 0$, we have

$$E\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Y(ih)\right] = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} E(Y(ih)) = 0.$$

Likewise, since $\text{Var}(Y(ih)) = 1$ we have

$$\text{Var}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Y(ih)\right] = \frac{1}{n}\sum_{i=1}^{n} 1 = 1.$$

Hence, we can think of a standard Brownian motion as being approximately generated from the sum of independent binomial draws with mean 0 and variance h.
Now, by the Central Limit Theorem[4] we have that the limit

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\sum_{i=1}^{n} Y(ih)$$

approaches a standard normal distribution, say W. Hence,

$$Z(T) = \sqrt{T}W(T).$$

It follows that $Z(T)$ is approximated by a normal random variable with mean 0 and variance T.
In Stochastic calculus, $Z(t)$ can be represented in integral form

$$Z(T) = \int_0^T dZ(t).$$

The integral on the right is called a **stochastic integral.**

Example 58.5
Let Z be a standard Brownian motion. Calculate $E(Z(4)Z(5))$.

Solution.
Note that $Z(4) = Z(4) - Z(0)$ and $Z(5) - Z(4)$ are independent random variables. Thus,

$$\begin{aligned}
E(Z(4)Z(5)) =& E[Z(4)\{Z(5) - Z(4) + Z(4)\}] = E[Z(4)^2] + E[Z(4)(Z(5) - Z(4))] \\
=& E[Z(4)^2] + E[Z(4)]E[Z(5) - Z(4)] \\
=& E[Z(4)^2] - [E(Z(4))]^2 + E[Z(4)]E[Z(5)] \\
=& \text{Var}(Z(4)) = 4 \blacksquare
\end{aligned}$$

[4] See Theorem 40.2 in [3].

Now, renaming h as dt and the change in Z as $dZ(t)$ we can write the differential form[5]

$$dZ(t) = W(t)\sqrt{dt}.$$

In words, this equation says, that over small periods of time, changes in the value of the process are normally distributed with a variance that is proportional to the length of the time period.

[5]Differential forms of Brownian motions are called **stochastic differential equations**

Additional Properties of $Z(t)$

We next discuss two important properties of Brownian motion that prove to be extremely important. We continue to use the binomial approximation to the Brownian process. We first introduce the following definition: Partition an interval $[a, b]$ into n equal subintervals. The **quadratic variation** of a stochastic process $\{Z(t)\}_{a \le t \le b}$ is defined to be

$$\lim_{n\to\infty}\sum_{i=1}^{n}[Z(t_i) - Z(t_{i-1})]^2 = \int_a^b [dZ(t)]^2$$

if the limit exists (under convergence in probability). In the case of a standard Brownian motion as defined above we have

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}[Z(ih) - Z((i-1)h)]^2 &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(Y_{ih}\sqrt{h}\right)^2\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}Y_{ih}^2 h = T < \infty.\end{aligned}$$

An important implication of the fact that the quadratic variation of a Brownian process is finite is that higher-order variations are all zero.

Example 58.6

Let Z be a standard Brownian motion. Show that $\lim_{n\to\infty}\sum_{i=1}^{n}[Z(ih) - Z((i-1)h)]^4 = 0$.

Solution.

We have

$$\begin{aligned}\left|\sum_{i=1}^{n}[Z(ih) - Z((i-1)h)]^4\right| &= \left|\sum_{i=1}^{n}\left(Y_{ih}\sqrt{h}\right)^4\right|\\ &= \left|\sum_{i=1}^{n}Y_{ih}^4 h^2\right|\\ &\le \sum_{i=1}^{n}h^2 = \frac{T^2}{n^2}.\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\sum_{i=1}^{n}[Z(ih) - Z((i-1)h)]^4 = 0 \blacksquare$$

The **total variation** of the standard Brownian process is

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}|Z(t_i)-Z(t_{i-1})| &= \lim_{n\to\infty}\sum_{i=1}^{n}|Y_{ih}|\sqrt{h}\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{h}=\sqrt{T}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{n}}\\ &=\sqrt{T}\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{n}=\infty\end{aligned}$$

This means that the Brownian path moves up and down very rapidly in the interval $[0,T]$. That is, the path will cross its starting point an infinite number of times in the interval $[0,T]$.

Practice Problems

Problem 58.1
Given that Z follows a Brownian motion. Find the variance of $Z(17) - Z(5)$.

Problem 58.2
Let Z be a Brownian motion. Suppose that $0 \leq s_1 < t_1 \leq s_2 < t_2$. Given that $E(Z(t_1) - Z(s_1)) = 4$ and $E(Z(t_2) - Z(s_2)) = 5$. Find $E[(Z(t_1) - Z(s_1))(Z(t_2) - Z(s_2))]$.

Problem 58.3
Let Z represent a Brownian motion. Show that for any constant α we have $\lim_{t\to 0} Z(t + \alpha) = Z(\alpha)$.

Problem 58.4
Let Z represent a standard Brownian motion. Show that $\{Z(t+\alpha)-Z(\alpha)\}_{t\geq 0}$ is also a standard Brownian motion for a fixed $\alpha > 0$.

Problem 58.5
Let $\{Z(t)\}_{0\leq t\leq 1}$ represent a standard Brownian motion. Show that $\{Z(1) - Z(1 - t)\}_{0\leq t\leq 1}$ is also a standard Brownian motion.

Problem 58.6
Let $\{Z(t)\}_{t\geq 0}$ represent a standard Brownian motion. Show that $\{s^{-\frac{1}{2}}Z(st)\}_{t\geq 0}$ is a standard Brownian motion, where $s > 0$ is fixed.

Problem 58.7 ‡
Let Z be a standard Brownian motion. Show that $\lim_{n\to\infty} \sum_{i=1}^{n} [Z(ih) - Z((i - 1)h)]^3 = 0$.

59 Arithmetic Brownian Motion

With a standard Brownian motion, $dZ(t)$ has an mean of 0 and variance of 1 per unit time. We can generalize this to allow nonzero mean and an arbitrary variance. Define $X(t)$ by

$$X(t+h) - X(t) = \alpha h + \sigma Y(t+h)\sqrt{h}.$$

where $Y(t)$ is a random draw from a binomial distribution. We call αh the **drift term** and $\sigma\sqrt{h}$ the **noise term**. Let $T > 0$. Partition $[0, T]$ into n subintervals each of length $h = \frac{T}{n}$. Then

$$\begin{aligned}X(T) - X(0) =& \sum_{i=1}^{n}\left(\alpha\frac{T}{n} + \sigma Y(ih)\sqrt{\frac{T}{n}}\right)\\ =&\alpha T + \sigma\left(\sqrt{T}\sum_{i=1}^{n}\frac{Y(ih)}{\sqrt{n}}\right)\end{aligned}$$

By the Central Limit Theorem $\sqrt{T}\sum_{i=1}^{n}\frac{Y(ih)}{\sqrt{n}}$ approaches a normal distribution with mean 0 and variance T. Hence, we can write

$$X(T) - X(0) = \alpha T + \sigma Z(t)$$

where Z is the standard Brownian motion. The stochastic differential form of this expression is

$$dX(t) = \alpha dt + \sigma dZ(t). \tag{59.1}$$

A stochastic process $\{X(t)\}_{t\geq 0}$ that satisfies (59.1) is called an **arithmetic Brownian motion.** Note that $E(X(t) - X(0)) = \alpha t$ and $\text{Var}(X(T) - X(0)) = \text{Var}(\alpha t + \sigma Z(t)) = \sigma^2 t$. We call α the **instantaneous mean per unit time** or the **drift factor** and σ^2 the **instantaneous variance per unit time.** Note that $X(t) - X(0)$ is normally distributed with expected mean of αt and variance $\sigma^2 t$. Hence, $X(t)$ is normally distributed with mean $X(0) + \alpha t$ and variance $\alpha^2 t$.

Example 59.1
Let $\{X(t)\}_{t\geq 0}$ be an arithmetic Brownian motion with drift factor α and volatility σ. Show that $X(t) = X(a) + \alpha(t-a) + \sigma\sqrt{t-a}W(t)$ where W is a standard normal random variable.

Solution.
Using (59.1) we can write $X(t)-X(a)=\alpha(t-a)+\sigma\sqrt{t-a}W(t)$. Thus, $X(t)$ is normally distributed with mean $X(a)+\alpha(t-a)$ and variance $\sigma^2(t-a)$ ■

Example 59.2
$\{X(t)\}_{t\geq 0}$ follows an arithmetic Brownian motion such that $X(30)=2$. The drift factor of this Brownian motion is 0.435, and the volatility is 0.75. What is the probability that $X(34)<0$?

Solution.
The mean of the normal distribution $X(34)$ is $X(30)+\alpha(34-30)=2+0.435\times 4=3.74$. The standard deviation is $\sigma\sqrt{t-a}=0.75\sqrt{34-30}=1.5$. Thus,

$$Pr(X(34)<0)=Pr\left(Z<\frac{0-3.74}{1.5}\right)=N(Z<-2.49)=0.006387\ ■$$

Example 59.3
A particular arithmetic Brownian motion is as follows:

$$dX(t)=0.4dt+0.8dZ(t).$$

(a) What is the drift factor of this Brownian motion?
(b) What is the instantaneous variance per unit time?

Solution.
(a) The drift factor is the instantaneous mean per unit time $\alpha=0.4$.
(b) The instantaneous variance per unit time is $\sigma^2=0.8^2=0.64$ ■

The following observations about equation (59.1) are worth mentioning:
• $X(t)$ is normally distributed.
• The random term $dZ(t)$ is multiplied by a scale factor that enables us to change the variance.
• The αdt term introduces a **nonrandom drift** into the process. Adding αdt has the effect of adding α per unit time to $X(0)$.

Example 59.4
Let $\{X(t)\}_{t\geq 0}$ be an arithmetic Brownian motion starting from 0 with $\alpha=0.2$ and $\sigma^2=0.125$. Calculate the probability that $X(2)$ is between 0.1 and 0.5.

Solution.
We have that $X(2)$ is normally distributed with mean $\alpha T = 0.4$ and variance $\sigma^2 T = 0.125 \times 2 = 0.25$. Thus,

$$Pr(0.1 < X(2) < 0.5) = N\left(\frac{0.5 - 0.4}{0.5}\right) - N\left(\frac{0.1 - 0.4}{0.5}\right) = 0.57926 - 0.274253 = 0.305007 \blacksquare$$

The Ornestein-Uhlenbeck Process
When modeling commodity prices, one notices that these prices exhibit a tendency to revert to the mean. That is, if a value departs from the mean in either direction, in the long run, it will tend to revert back to the mean. Thus, mean-reversion model has more economic logic than the arithmetic Brownian process described above. We can incorporate mean reversion by modifying the drift term in (59.1) and obtain:

$$dX(t) = \lambda(\alpha - X(t))dt + \sigma dZ(t) \tag{59.2}$$

where α is the long-run equilibrium level (or the long-run mean value which $X(t)$ tends to revert); σ is the volatility factor, λ is the speed of reversion or the reversion factor, and $Z(t)$ is the standard Brownian motion. Equation (59.2) is known as the **Ornstein-Uhlenbeck** process.

Example 59.5 ‡
Solve equation (59.2).

Solution.
We introduce the change of variables $Y(t) = X(t) - \alpha$. In this case, we obtain the differential form

$$dY(t) = -\lambda Y(t)dt + \sigma dZ(t)$$

or

$$dY(t) + \lambda Y(t)dt = \sigma dZ(t).$$

This can be written as

$$d[e^{\lambda t}Y(t)] = e^{\lambda t}\sigma dZ(t).$$

Hence, integrating from 0 to t we obtain

$$e^{\lambda t}Y(t) - Y(0) = \sigma \int_0^t e^{\lambda s} dZ(s)$$

or

$$Y(t) = e^{-\lambda t}Y(0) + \sigma \int_0^t e^{-\lambda(t-s)} dZ(s).$$

Writing the answer in terms of X we find

$$X(t) = X(0)e^{-\lambda t} + \alpha(1 - e^{-\lambda t}) + \sigma \int_0^t e^{-\lambda(t-s)} dZ(s) \blacksquare$$

Practice Problems

Problem 59.1
Let $\{X(t)\}_{t\geq 0}$ be an arithmetic Brownian motion with $X(0)=0$. Show that the random variable $\frac{X(t)-\alpha t}{\sigma}$ is normally distributed. Find the mean and the variance.

Problem 59.2
Let $\{X(t)\}_{t\geq 0}$ be an arithmetic Brownian motion with $X(0)=0$. Show that the random variable $\frac{X(t)-\alpha t}{\sigma\sqrt{t}}$ has the standard normal distribution.

Problem 59.3
Let $\{X(t)\}_{t\geq 0}$ be an arithmetic Brownian motion with $X(0)=0, \alpha=0.1$ and $\sigma^2=0.125$. Calculate the probability that $X(2)$ is between 0.1 and 0.3.

Problem 59.4
Consider the Ornstein-Uhlenbeck process:

$$\frac{dX(t)}{X(t)}=-0.4dt+0.20\frac{dZ(t)}{X(t)}.$$

Determine the parameters α, λ, and σ.

Problem 59.5
Consider the Ornstein-Uhlenbeck process:

$$dX(t)=0.048dt+0.30dZ(t)-0.40X(t)dt.$$

Determine the parameters α, λ, and σ.

Problem 59.6
Solve the equation

$$dX(t)=0.4[0.12-X(t)]dt+0.30dZ(t)$$

subject to $X(0)=0.24$.

Problem 59.7
The price of Stock R follows a mean reversion process, where the instantaneous mean per unit time is 54, and the volatility factor is 0.46. The reversion factor for this process is 3.5. At some time t, you know that $dt=0.24, dZ(t)=0.94$, and the stock price is \$46 per share. What is the instantaneous change in the stock price, i.e., $dX(t)$?

Problem 59.8
The price of Stock S follows a mean reversion process, but the mean is currently unknown. You do know that at time $t, dt = 0.125, dZ(t) = 0.3125$, and $dX(t)$, the instantaneous change in the stock price, is 0.634. Also, $X(t)$, the stock price, is 613, the volatility is 0.53, and the reversion factor for this process is 0.1531. What is the instantaneous mean per unit time for this process?

Problem 59.9
The price of Stock Q follows an Ornstein-Uhlenbeck process. The reversion factor for this process is 0.0906, and the volatility factor is 0.63. At some time $t, dt = 1, dZ(t) = 0.9356$, and $dX(t)$, the instantaneous change in the stock price, is 0.0215. The long-run equlibrium level α is zero. What is the price of Stock Q at time t?

Problem 59.10
$\{X(t)\}_{t\geq 0}$ follows an arithmetic Brownian motion with a drift factor of 0.35 and a volatility of 0.43. Given that $X(4) = 2$.
(a) Find the mean of the normal distribution $X(13)$.
(b) Find the standard deviation of the normal distribution $X(13)$.
(c) What is the probability that $X(13) > 9$?

Problem 59.11
$\{X(t)\}_{t\geq 0}$ follows an arithmetic Brownian motion such that $X(45) = 41$. The drift factor of this Brownian motion is 0.153, and the volatility is 0.98. What is the probability that $X(61) < 50$?

60 Geometric Brownian Motion

The arithmetic Brownian motion has many drawbacks, namely,
• $X(t)$ can be negative and therefore the arithmetic Brownian process is a poor model for stock prices.
• The mean and variance of changes in dollar terms are independent of the level of the stock price. In practice, if a stock price doubles, we would expect both the dollar expected return and the dollar standard deviation of returns to approximately double.
These drawbacks and others can be eliminated with geometric Brownian motion which we discuss next.
When the drift factor and the volatility in the arithmetic Brownian motion are functions of $X(t)$, then the stochastic differential form

$$dX(t) = \alpha(X(t))dt + \sigma(X(t))dZ(t)$$

is called an **Itô process.** In particular, if $\alpha(X(t)) = \alpha X(t)$ and $\sigma(X(t)) = \sigma X(t)$ then the previous equation becomes

$$dX(t) = \alpha X(t)dt + \sigma X(t)dZ(t)$$

or

$$\frac{dX(t)}{X(t)} = \alpha dt + \sigma dZ(t). \tag{60.1}$$

The process in the previous equation is known as **geometric Brownian motion.** Note that equation (60.1) says that the the percentage change in the asset value is normally distributed with instantaneous mean α and instantaneous variance σ^2.

Example 60.1
A given geometric Brownian motion can be expressed as follows:

$$\frac{dX(t)}{X(t)} = 0.215dt + 0.342dZ(t).$$

(a) What is the instantaneous mean of the percentage change in the asset value?
(b) What is the instantaneous variance of the percentage change in the asset value?

Solution.
(a) The instantaneous mean of the percentage change in the asset value is $\alpha = 0.215$.
(b) The instantaneous variance of the percentage change in the asset value is $\sigma^2 = 0.342^2 = 0.116964$ ■
For an arbitrary initial value $X(0)$ equation (60.1) has the analytic solution[1]

$$X(t) = X(0)e^{(\alpha - 0.5\sigma^2)t + \sigma\sqrt{t}Z}$$

where Z is a normal random variable with parameters 0 and 1. Alternatively, we can write

$$X(t) = X(0)e^{(\alpha - 0.5\sigma^2)t + \sigma Z(t)}$$

where Z is a normal random variable with parameters 0 and t. Note that $X(t)$ is an exponential and therefore is not normal. However, $X(t)$ is lognormally distributed with mean $E(X(t)) = X(0)e^{\alpha t}$ (See Example 48.2) and variance $\text{Var}(X(t)) = e^{2\alpha t}X(0)^2(e^{\sigma^2 t} - 1)$. See Section 47. It follows that $\ln(X(t))$ is normally distributed with mean $\ln(X(0)) + (\alpha - 0.5\sigma^2)t$ and variance $\sigma^2 t$. That is,

$$\ln X(t) = \ln X(0) + (\alpha - 0.5\sigma^2)t + \sigma Z(t).$$

It follows that when X follows a geometric Brownian motion its logarithm follows an arithmetic Brownian motion[2], that is,

$$d[\ln X(t)] = (\alpha - 0.5\sigma^2)dt + \sigma dZ(t).$$

Example 60.2
The current price of a stock is 100. The stock price follows a geometric Brownian motion with drift rate of 10% per year and variance rate of 9% per year. Calculate the probability that two years from now the price of the stock will exceed 200.

Solution.
We want

$$Pr(S(2) > 200) = Pr\left(\ln\left(\frac{S(2)}{S(0)}\right) > \ln 2\right).$$

[1]The correctness of the solution can be verified using Itô's lemma to be discussed in Section 64.
[2]This follows from Itô's lemma.

But $\ln\left(\frac{S(2)}{S(0)}\right)$ is normally distributed with mean $(\alpha - 0.5\sigma^2)t = (0.10 - 0.5(0.09) \times 2 = 0.11$ and variance $\sigma^2 t = 0.09 \times 2 = 0.18$. Therefore,

$$Pr(S(2) > 200) = Pr\left(Z > \frac{\ln 2 - 0.11}{\sqrt{0.18}}\right) = 1-N(1.37) = 1-0.915 = 0.085 \blacksquare$$

Example 60.3
Given a geometric Brownian motion with drift factor 0.10. For $h = \frac{1}{365}$, suppose that the ratio of the noise term to the drift term is found to be 22.926. Find the standard deviation σ.

Solution.
We are given that $\frac{\sigma\sqrt{h}}{\alpha h} = 22.926$ or $\frac{\sigma}{0.10\sqrt{\frac{1}{365}}} = 22.926$. Solving for σ we find $\sigma = 0.12 \blacksquare$

Practice Problems

Problem 60.1
Stock A follows a geometric Brownian motion where the drift factor is 0.93 and the variance factor is 0.55. At some particular time t, it is known that $dt = 0.035$, and $dZ(t) = 0.43$. At time t, the stock trades for \$2354 per share. What is the instantaneous change in the price of stock A?

Problem 60.2
Stock B follows a geometric Brownian motion where the drift factor is 0.0234 and the variance factor is 0.953. At some particular time t, it is known that $dt = 0.531, dZ(t) = 0.136$, and $dX(t)-$ the instantaneous change in the price of the stock$-$ is 0.245. What is the stock price at time t?

Problem 60.3
Stock C follows a geometric Brownian motion where the variance factor is 0.35. At some particular time t, it is known that $dt = 0.0143, dZ(t) = 0.0154$, and $dX(t)-$ the instantaneous change in the price of the stock$-$ is 0.353153. The price of Stock C at time t is \$31.23. What is the drift factor for this Brownian motion?

Problem 60.4
Given a geometric Brownian motion with drift factor 0.09. For $h = \frac{1}{365}$, suppose that the ratio of the noise term to the drift term is found to be 13.428. Find the standard deviation σ.

Problem 60.5
$\{X(t)\}_{t\geq 0}$ follows a geometric Brownian motion with a drift factor of 0.35 and a volatility of 0.43. We know that $X(0) = 2$.
(a) What is the mean of the normal distribution $\ln\left(\frac{X(13)}{X(0)}\right)$?
(b) What is the standard deviation of the normal distribution $\ln\left(\frac{X(13)}{X(0)}\right)$?
(c) Find $Pr(X(13) > 9)$.

Problem 60.6 ‡
At time $t = 0$, Jane invests the amount of $W(0)$ in a mutual fund. The mutual fund employs a proportional investment strategy: There is a fixed real number ϕ, such that, at every point of time, $100\phi\%$ of the funds assets are invested in a nondividend paying stock and $100(1-\phi)\%$ in a risk-free

asset.
You are given:
(i) The continuously compounded rate of return on the risk-free asset is r.
(ii) The price of the stock, $S(t)$, follows a geometric Brownian motion,

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dZ(t),\ \ t \geq 0$$

where $\{Z(t)\}$ is a standard Brownian motion.
Let $W(t)$ denote the Janet's fund value at time $t, t \geq 0$. Show that

$$W(t) = W(0)\left[\frac{S(t)}{S(0)}\right]^{\phi} e^{(1-\phi)(r+0.5\phi\sigma^2)t}.$$

Problem 60.7 ‡
Consider the Black-Scholes framework. You are given the following three statements on variances, conditional on knowing $S(t)$, the stock price at time t. Determine the ones that are true.
(i) $\text{Var}[\ln S(t+h)|S(t)] = \sigma^2 h$
(ii) $\text{Var}\left[\frac{dS(t)}{S(t)}|S(t)\right] = \sigma^2 dt$
(iii) $\text{Var}[S(t+dt)|S(t)] = S^2\sigma^2 dt$.

61 Ito Process Multiplication Rules

In many problems involving Itô processes and geometric Brownian motion, you will encounter situations where you need to multiply dZ by itself, dt by itself, or dZ by dt. Rules for these products are known as **multiplication rules.** But first we define the following: For $\alpha > 1$ we will define $(dt)^\alpha = 0$. This, makes senses since dt represents a very small number.

Example 61.1
Show that $(dt)^2 = 0$.

Solution.
This follows from our definition of $(dt)^\alpha = 0$ with $\alpha = 2$ ■

Example 61.2
(a) Show that $E[dZ \times dt] = 0$.
(b) Show that $E[(dZ \times dt)^2] = 0$.
(c) Show that $\text{Var}[dZ \times dt] = 0$.
(d) Show that $dZ \times dt = 0$.

Solution.
(a) We have $E[dZ \times dt] = dtE[Z(t)] = 0$ since dt is a constant.
(b) We have $E[(dZ \times dt)^2] = (dt)^2 E[(dZ)^2] = (dt)^2 \text{Var}(dZ) = (dt)^2(dt) = (dt)^3 = 0$.
(c) We have $\text{Var}[dZ \times dt] = E[(dZ \times dt)^2] - (E[dZ \times dt])^2 = 0 - 0 = 0$.
(d) This follows from (a) and (c) ■

Example 61.3
(a) Show that $E[(dZ)^2] = dt$.
(b) Show that $\text{Var}[(dZ)^2] = 0$.
(c) Show that $(dZ)^2 = dt$.

Solution.
(a) We know that $E[dZ] = 0$ and $\text{Var}(dZ) = dt$. Thus, $E[(dZ)^2] = E[(dZ)^2] - [E(dZ)]^2 = \text{Var}(dZ) = dt$.
(b) We have

$$\begin{aligned}
\text{Var}[(dZ)^2] =& E[(dZ)^4] - \left(E[(dZ)^2]\right)^2 \\
=& E[Y^4(t)(dt)^2] - (dt)^2 \\
=& (dt)^2 - (dt)^2 = 0.
\end{aligned}$$

(c) It follows from (b) that $(dZ)^2$ is a constant. Thus, using (a) we find that $(dZ)^2 = dt$ ■

Example 61.4
Let Z and Z' be two standard Brownian motions. Show that $dZ \times dZ' = \rho dt$ where $\rho = E[Y(t)Y'(t)]$. ρ is also known as the correlation of the underlying assets driven by the different Brownian motions.

Solution.
We know that $dZ(t) = Y(t)\sqrt{dt}$ and $dZ'(t) = Y'(t)\sqrt{dt}$. Thus, $E(dZ \times dZ') = dtE[Y(t)Y'(t)] = \rho dt$. We also have,

$$\begin{aligned} \mathrm{Var}(dZ \times dZ') =& E[(dZ \times dZ')^2] - (E[dZ \times dZ'])^2 \\ notag =& (dt)^2 E[(YY')^2] - \rho^2 (dt)^2 = 0. \end{aligned} \tag{61.1}$$

Hence, $dZ \times dZ'$ must be a constant. Since $E(dZ \times dZ') = \rho dt$ we conclude that $dZ \times dZ' = \rho dt$ ■.

Summarizing, the multiplication rules are

$$\begin{aligned} (dt)^2 =& 0 \\ dZ \times dt =& 0 \\ (dZ)^2 =& dt \\ dZ \times dZ' =& \rho dt. \end{aligned}$$

Example 61.5
A geometric Brownian motion is described by the stochastic form

$$dX(t) = 0.07X(t)dt + 0.03X(t)dZ(t).$$

Calculate $(dX(t))^2$.

Solution.
We have

$$\begin{aligned} (dX(t))^2 =& [0.07X(t)dt + 0.03X(t)dZ(t)]^2 \\ =& (0.07X(t))^2(dt)^2 + 2(0.07)(0.03)dtdZ(t) + (0.03X(t))^2(dZ(t))^2 \\ =& 0 + 0 + 0.0009X^2(t)dt \blacksquare \end{aligned}$$

Example 61.6
Let X be a Brownian motion characterized by $dX(t) = (\alpha - \delta - 0.5\sigma^2)dt + \sigma dZ(t)$. For $T > 0$, split the interval $[0, T]$ into n equal subintervals each of length $h = \frac{T}{n}$. Evaluate

$$\lim_{n\to\infty} \sum_{i=1}^{n} [X(ih) - X((i-1)h)]^2.$$

Solution.
We have (See Problem 61.5)

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} [X(ih) - X((i-1)h)]^2 =& \int_0^T [dX(t)]^2\\ =& \int_0^T [(\alpha - \delta - 0.5\sigma^2)^2(dt)^2 + 2\sigma(\alpha - \delta - 0.5\sigma^2)dtdZ(t)\\ &+\sigma^2(dZ(t))^2]\\ =& \int_0^T \sigma^2 dt = \sigma^2 T \blacksquare\end{aligned}$$

Practice Problems

Problem 61.1
Let X be an arithmetic Brownian motion with drift factor α and volatility σ. Find formulas for $(dX)^2$ and $dX \times dt$.

Problem 61.2
Find the simplest possible equivalent expression to $(34dZ+45dt)(42dZ+3dt)$.

Problem 61.3
Given that the assets have a correlation $\rho = 0.365$. Find the simplest possible equivalent expression to $(95dZ + 424dZ')(2dZ + 241dt)$.

Problem 61.4
Let Z be a standard Brownian motion. Show that the stochastic integral $\int_0^T [dZ(t)]^4$ is equal to 0.

Problem 61.5
Let Z be a standard Brownian motion and $T > 0$. Partition the interval $[0, T]$ into n equal subintervals each of length $h = \frac{T}{n}$. For any positive integer k we define

$$\lim_{n\to\infty}\sum_{i=1}^{n}[Z(ih) - Z((i-1)h)]^k = \int_0^T [dZ(t)]^k.$$

Using the multiplication rules, simplify

$$X = \lim_{n\to\infty}\sum_{i=1}^{n}[Z(ih)-Z((i-1)h)] + \lim_{n\to\infty}\sum_{i=1}^{n}[Z(ih)-Z((i-1)h)]^2 + \lim_{n\to\infty}\sum_{i=1}^{n}[Z(ih)-Z((i-1)h)]^3.$$

Problem 61.6
Find the mean and the variance of the random variable of the previous problems.

Problem 61.7
Given the following Brownian motion

$$dX(t) = (0.05 - X(t))dt + 0.10dZ(t).$$

Find $(dX(t))^2$.

Problem 61.8 ‡
Consider the Black-Scholes framework. Let $S(t)$ be the stock price at time $t, t \geq 0$. Define $X(t) = \ln[S(t)]$.
You are given the following three statements concerning $X(t)$.
(i) $\{X(t), t \geq 0\}$ is an arithmetic Brownian motion.
(ii) $\mathrm{Var}[X(t+h) - X(t)] = \sigma^2 h, t \geq 0, h > 0$.
(iii) $\lim_{n\to\infty} \sum_{i=1}^{n} [X(ih) - X((i-1)h)]^2 = \sigma^2 T$.
Which of these statements are true?

Problem 61.9 ‡
Define
(i) $W(t) = t^2$.
(ii) $X(t) = [t]$, where $[t]$ is the greatest integer part of t; for example, $[3.14] = 3, [9.99] = 9$, and $[4] = 4$.
(iii) $Y(t) = 2t + 0.9Z(t)$, where $\{Z(t) : t \geq 0\}$ is a standard Brownian motion.
Let $V_T^2(U)$ denote the quadratic variation of a process U over the time interval $[0, T]$.
Rank the quadratic variations of W, X and Y over the time interval $[0, 2.4]$.

62 Sharpe ratios of Assets that Follow Geometric Brownian Motions

We have first encountered the concept of Sharpe ratio in Section 32: In finance, the **risk premium** on an asset is defined to be the excess return of the asset over the risk-free rate. Thus, if r denotes the risk-free rate and α the expected return on the asset, then the risk premium on the asset is the difference

$$\text{Risk premium} = \alpha - r.$$

The **Sharpe ratio** of an asset is the risk premium of a stock divided by its volatility, or, standard deviation. It is given by

$$\textbf{Sharpe ratio} = \phi = \frac{\alpha - r}{\sigma}.$$

The Sharpe ratio is used to characterize how well the return of an asset compensates the investor for the risk taken.

Consider two nondividend paying stocks that are **perfectly positively correlated.**[1] That is, their stock prices are driven by the same $dZ(t)$. In this case, we can write

$$dS_1 = \alpha_1 S_1 dt + \sigma_1 S_1 dZ$$
$$dS_2 = \alpha_2 S_2 dt + \sigma_2 S_2 dZ$$

It is important to remember that a stock price is assumed to follow a geometric Brownian motion. We claim that $\phi_1 = \phi_2$. Suppose not. Without loss of generality, we can assume $\phi_1 > \phi_2$. Buy $\frac{1}{\sigma_1 S_1}$ shares of asset 1 (at a cost of $\frac{1}{\sigma_1}$) and sell $\frac{1}{\sigma_2 S_2}$ shares of asset 2 (at a cost of $\frac{1}{\sigma_2}$). Invest the cost $\frac{1}{\sigma_2} - \frac{1}{\sigma_1}$ in the risk-free bond, which has the rate of return rdt. The return on this portfolio (i.e. the arbitrage profit) after a short time dt is

$$\begin{aligned}\frac{dS_1}{\sigma_1 S_1} - \frac{dS_2}{\sigma_2 S_2} + \left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1}\right) rdt =& \frac{\alpha_1 S_1 dt + \sigma_1 S_1 dZ}{\sigma_1 S_1} - \frac{\alpha_2 S_2 dt + \sigma_2 S_2 dZ}{\sigma_2 S_2} + \left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1}\right) rdt \\ =&(\phi_1 - \phi_2)dt.\end{aligned}$$

Thus, we have constructed a zero-investment portfolio with a positive risk-free return. Therefore, to preclude arbitrage, the two stocks must have equal Sharpe ratio.

[1]Two stocks are **perfectly negatively correlated** if one stock prices move in one direction, the other stock prices move in the opposite direction. For such stocks, $dS_1 = \alpha_1 S_1 dt + \sigma_1 S_1 dZ$ and $dS_2 = \alpha_2 S_2 dt - \sigma_2 S_2 dZ$. See Problem 62.10

Example 62.1
A nondividend paying stock follows a geometric Brownian motion given by

$$dS(t) = 0.10Sdt + 0.12SdZ.$$

Find the risk-free interest rate r if the Sharpe ratio of the stock is 0.10.

Solution.
We are given $\alpha = 0.10, \sigma = 0.12$, and $\phi = 0.10$. Hence, $0.10 = \frac{0.10-r}{0.12}$. Solving this equation we find $r = 8.8\%$ ■

Example 62.2
Nondividend-paying stocks S_1 and S_2 are perfectly correlated and follow these geometric Brownian motions:

$$\frac{dS_1}{S_1} = Adt + 0.884dZ$$
$$\frac{dS_2}{S_2} = 0.3567dt + 0.643dZ.$$

The annual continuously compounded risk-free interest rate is 0.1. What is A?

Solution.
We know that

$$\frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\sigma_2},$$

where $\alpha_1 = A, r = 0.1, \sigma_1 = 0.884, \alpha_2 = 0.3567$, and $\sigma_2 = 0.643$. Thus,

$$\frac{A - 0.1}{0.884} = \frac{0.3567 - 0.1}{0.643}.$$

Solving this equation we find $A = 0.453$ ■

Example 62.3
Nondividend-paying stocks S_1 and S_2 are perfectly correlated and follow these geometric Brownian motions:

$$\frac{dS_1}{S_1} = 0.08dt + 0.23dZ$$
$$\frac{dS_2}{S_2} = 0.07dt + 0.25dZ.$$

The annual continuously compounded risk-free interest rate is 0.04. Demonstrate an arbitrage opportunity.

Solution.
The Sharpe ratio of the first stock is $\phi_1 = \frac{0.08-0.04}{0.23} = 0.174$. For the second stock, we have $\phi_2 = \frac{0.07-0.04}{0.25} = 0.12$. Since $\phi_1 > \phi_2$, an arbitrage opportunity occurs by buying $\frac{1}{0.23S_1}$ shares of stock 1, selling $\frac{1}{0.25S_2}$ of stock 2, and borrowing $\frac{1}{0.23} - \frac{1}{0.25}$. The arbitrage profit is

$$(\phi_1 - \phi_2)dt = (0.174 - 0.12)dt = 0.054dt \blacksquare$$

Example 62.4 ‡
Consider two nondividend-paying assets X and Y. There is a single source of uncertainty which is captured by a standard Brownian motion $\{Z(t)\}$. The prices of the assets satisfy the stochastic differential equations

$$\frac{dX(t)}{X(t)} = 0.07dt + 0.12dZ(t)$$

and

$$\frac{dY(t)}{Y(t)} = Adt + BdZ(t),$$

where A and B are constants. You are also given:
(i) $d[\ln Y(t)] = \mu dt + 0.085dZ(t)$;
(ii) The continuously compounded risk-free interest rate is 4%.
Determine A.

Solution.
Since Y follows a geometric Brownian motion, we have $d[\ln Y(t)] = (A - 0.5B^2)dt + BdZ(t) = \mu dt + 0.085dZ(t)$. Hence, $A - 0.5B^2 = \mu$ and $B = 0.085$. Since X and Y have the same underlying source of risk Z, they have equal Sharpe ratios. That is,

$$\frac{\alpha_X - r}{\sigma_X} = \frac{\alpha_Y - r}{\sigma_Y}.$$

or

$$\frac{0.07 - 0.04}{0.12} = \frac{A - 0.04}{0.085}.$$

Solving this equation for A we find $A = 0.06125$ ■

Sharpe Ratio Under Capital Asset Pricing Model
In CAPM model, the risk-premium of an asset is defined to be

$$\alpha_S - r = \beta_{SM}(\alpha_M - r)$$

with

$$\beta_{SM} = \frac{\rho_{SM}\sigma_S}{\sigma_M}$$

where ρ_{SM} is the correlation of asset with the market. It follows from this definition that

$$\phi = \frac{\alpha_S - r}{\sigma_S} = \rho_{SM}\phi_M.$$

Example 62.5
A nondividend-paying Stock A has annual price volatility of 0.20. Its correlation with the market is 0.75. A nondividend-paying Stock B has a market correlation equals to 0.4 and Sharpe ratio equals 0.0625. Find the risk-premium of Stock A.

Solution.
We are given that $\rho_{BM} = 0.4$ and $\phi_B = 0.0625$ so that $0.0625 = 0.4 \times \frac{\alpha_M - r}{\sigma_M}$. Hence, $\phi_M = \frac{\alpha_M - r}{\sigma_M} = 0.15625$. Thus, the Sharpe ratio of Stock A is

$$\phi_A = \rho_{AM}\phi_M = 0.75 \times 0.15625 = 0.1171875$$

and the risk-premium of Stock A is

$$\alpha_A - r = \sigma_A \times \phi_A = 0.20 \times 0.1171875 = 0.0234375 \blacksquare$$

Practice Problems

Problem 62.1
A nondividend paying stock follows a geometric Brownian motion given by

$$dS(t) = 0.08Sdt + 0.12SdZ.$$

Find the risk-free interest rate r if the Sharpe ratio of the stock is 0.31.

Problem 62.2
A nondividend-paying stock follows a geometric Brownian motion such that $\frac{dS}{S} = 0.5dt + 0.354dZ$. The annual continuously compounded risk-free interest rate is 0.034. What is the Sharpe ratio of the stock?

Problem 62.3
Nondividend-paying stocks S_1 and S_2 are perfectly correlated and follow these geometric Brownian motions:

$$\frac{dS_1}{S_1} = 0.453dt + 0.884dZ$$

$$\frac{dS_2}{S_2} = 0.3567dt + AdZ.$$

The annual continuously compounded risk-free interest rate is 0.1. What is A?

Problem 62.4
Nondividend-paying stocks S_1 and S_2 are perfectly correlated and follow these geometric Brownian motions:

$$\frac{dS_1}{S_1} = 0.152dt + 0.251dZ$$

$$\frac{dS_2}{S_2} = 0.2dt + 0.351dZ.$$

What is the annual continuously compounded risk-free interest rate?

Problem 62.5
Nondividend-paying stocks S_1 and S_2 are perfectly correlated and follow these geometric Brownian motions:

$$\frac{dS_1}{S_1} = 0.07dt + 0.12dZ$$

$$\frac{dS_2}{S_2} = 0.05dt + 0.11dZ.$$

The annual continuously compounded risk-free interest rate is 0.03. Demonstrate an arbitrage opportunity.

Problem 62.6
A nondividend-paying Stock A has annual price volatility of 0.28 and Sharpe ratio equals to 0.15. The beta of the stock with the market is 0.7. Find the market risk-premium.

Problem 62.7
A nondividend-paying Stock A has annual price volatility of 0.28 and Sharpe ratio equals to 0.15. The beta of the stock with the market is 0.7. A nondividend-paying stock has price volatility of 0.555 and beta 1.3. Find the Sharpe ratio of Stock B.

Problem 62.8 ‡
Consider an arbitrage-free securities market model, in which the risk-free interest rate is constant. There are two nondividend-paying stocks whose price processes are

$$S_1(t) = S_1(0)e^{0.1t+0.2Z(t)}$$
$$S_2(t) = S_2(0)e^{0.125t+0.3Z(t)}$$

where $Z(t)$ is a standard Brownian motion and $t \geq 0$.
Determine the continuously compounded risk-free interest rate.

Problem 62.9 ‡
Consider two nondividend-paying assets X and Y, whose prices are driven by the same Brownian motion Z. You are given that the assets X and Y satisfy the stochastic differential equations:

$$\frac{dX(t)}{X(t)} = 0.07dt + 0.12dZ(t)$$

and

$$\frac{dY(t)}{Y(t)} = Gdt + HdZ(t),$$

where G and H are constants.
You are also given:
(i) $d[\ln Y(t)] = 0.06dt + \sigma dZ(t)$;
(ii) The continuously compounded risk-free interest rate is 4%.
(iii) $\sigma < 0.25$.
Determine G.

Problem 62.10 ‡
The prices of two nondividend-paying stocks are governed by the following stochastic differential equations:

$$\frac{dS_1(t)}{S_1(t)} = 0.06dt + 0.02dZ(t)$$

$$\frac{dS_2(t)}{S_2(t)} = 0.03dt + kdZ(t)$$

where $Z(t)$ is a standard Brownian motion and k is a constant.
The current stock prices are $S_1(0) = 100$ and $S_2(0) = 50$. The continuously compounded risk-free interest rate is 4%.
You now want to construct a zero-investment, risk-free portfolio with the two stocks and risk-free bonds.
If there is exactly one share of Stock 1 in the portfolio, determine the number of shares of Stock 2 that you are now to buy. (A negative number means shorting Stock 2.)

63 The Risk-Neutral Measure and Girsanov's Theorem

Consider a stock that pays dividend at the compounded yield δ. The true price process that is observed in the world is given by

$$\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t) \tag{63.1}$$

where $dZ(t)$ is the unexpected portion of the stock return and $Z(t)$ is a martingale under the true probability distribution.

In risk-neutral pricing, we need a risk-neutral version of the above process. The random part of the process will involve a Brownian motion $\tilde{Z}$ that is martingale under a transformed probability distribution, let's call it Q. This transformed probability distribution is referred to as the **risk-neutral measure.**

To find $\tilde{Z}$ and the associated Q, we let $\tilde{Z}(t) = Z(t) + \frac{\alpha - r}{\sigma}t$. A result, known as **Girsanov's theorem**, asserts the existence of a unique risk-neutral measure Q under which $\tilde{Z}(t)$ is a standard Brownian motion and $\tilde{Z}$ is martingale under Q. Differentiating $\tilde{Z}$ and rearranging yields

$$dZ(t) = d\tilde{Z}(t) - \frac{\alpha - r}{\sigma}dt.$$

Put this back in equation (63.1) we obtain

$$\frac{dS(t)}{S(t)} = (r - \delta)dt + \sigma d\tilde{Z}(t).$$

We refer to this equation as the **risk-neutral price process.** Notice that the volatility is the same for the true price process and the risk-neutral price process.

Example 63.1

The true price process of a stock that pays dividends at the continuously compounded yields δ is

$$\frac{dS(t)}{S(t)} = 0.09dt + \sigma dZ(t).$$

The corresponding risk-neutral price process is

$$\frac{dS(t)}{S(t)} = 0.05dt + \sigma d\tilde{Z}(t).$$

The continuously compounded risk-free interest rate is 0.08. Find δ and α.

Solution.
We know that $r - \delta = 0.05$ or $0.08 - \delta = 0.05$. Solving for δ, we find $\delta = 0.03$. On the other hand, we know that $\alpha - \delta = 0.09$ or $\alpha - 0.03 = 0.09$. Thus, $\alpha = 0.12$ ■

Example 63.2
The risk-neutral price process of a dividend paying stock is given by

$$\frac{dS(t)}{S(t)} = 0.05dt + 0.12d\tilde{Z}(t).$$

It is known that the risk-premium of the stock is 0.14. Find the drift factor of the true price process model.

Solution.
We have $\alpha - r - (\alpha - \delta) = -0.05$ and $\alpha - r = 0.14$. Thus, the drift factor is $\alpha - \delta = 0.14 + 0.05 = 0.19$ ■

Example 63.3
The risk-neutral price process for a nondividend-paying stock is

$$\frac{dS(t)}{S(t)} = 0.07dt + 0.30d\tilde{Z}(t).$$

The expected rate of return on the stock is 0.13. Find the Sharpe ratio of the stock.

Solution.
We have $r - \delta = r - 0 = 0.07$ so that $r = 0.07$. The price volatility of the stock is $\sigma = 0.30$. Thus, the Sharpe ratio of the stock is

$$\frac{\alpha - r}{\sigma} = \frac{0.13 - 0.07}{0.30} = 0.20 \blacksquare$$

Practice Problems

Problem 63.1
Consider a stock that pays dividends. The volatility of the stock price is 0.30. The Sharpe ratio of the stock is 0.12. Find the true price model.

Problem 63.2
The risk-neutral price process of a dividend paying stock is given by

$$\frac{dS(t)}{S(t)} = 0.05dt + 0.12d\tilde{Z}(t).$$

It is known that the drift factor in the true price process is 0.09. Find the risk-premium of the stock.

Problem 63.3
The risk-neutral price model of a dividend paying stock is given by

$$\frac{dS(t)}{S(t)} = 0.05dt + 0.12d\tilde{Z}(t).$$

The continuously compounded yield on the stock is 0.03. Find the continuously compounded risk-free interest rate.

Problem 63.4
The true price process of a stock that pays dividends at the continuously compounded yields δ is

$$\frac{dS(t)}{S(t)} = 0.13dt + \sigma dZ(t).$$

The corresponding risk-neutral price process is

$$\frac{dS(t)}{S(t)} = 0.08dt + \sigma d\tilde{Z}(t).$$

The expected rate of return on the stock is 0.15. Find δ and r.

Problem 63.5
The risk-neutral price process of dividend paying stock is

$$\frac{dS(t)}{S(t)} = 0.04dt + 0.12d\tilde{Z}(t).$$

The continuously compounded yield is 0.03. It is known that the expected rate of return on the stock is twice the risk-free interest rate. Find α.

Problem 63.6 ‡
Assume the Black-Scholes framework.
You are given:
(i) $S(t)$ is the price of a stock at time t.
(ii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 1%.
(iii) The stock-price process is given by

$$\frac{dS(t)}{S(t)} = 0.05dt + 0.25dZ(t)$$

where $\{Z(t)\}$ is a standard Brownian motion under the true probability measure.
(iv) Under the risk-neutral probability measure, the mean of $Z(0.5)$ is -0.03.
Calculate the continuously compounded risk-free interest rate.

64 Single Variate Itô's Lemma

In this section we discuss Itô's Lemma which is essential in the derivation of Black-Scholes Equation.

Lemma 64.1 *(Itô's)*
Assume that the process S satisfies the following stochastic differential equation

$$dS(t) = \mu(S,t)dt + \sigma(S,t)dZ(t)$$

where Z is the standard Brownian motion. Then for any twice continuously differentiable function $f(S,t)$, the change in f is given by

$$df(S,t) = \left(\frac{\partial f}{\partial t} + \mu(S,t)\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2(S,t)\frac{\partial^2 f}{\partial S^2}\right)dt + \sigma(S,t)\frac{\partial f}{\partial S}dZ(t).$$

Proof.
Using Taylor expansion we find

$$\begin{aligned}
df(S,t) =& f(S+dS,t+dt) - f(S,t)\\
=&\frac{\partial f}{\partial S}dS + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2\\
&+\frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial S\partial t}dSdt + \text{terms in}(dt)^{\frac{3}{2}}\text{and higher}
\end{aligned}$$

Now, using the multiplication rules we have:

$$(dt)^\alpha = 0,\ \alpha > 1.$$

$$dt \times dZ = 0$$

$$(dZ)^2 = dt$$

$$(dS)^2 = \sigma^2(S,t)dt$$

$$dS \times dt = 0.$$

Hence,

$$\begin{aligned}
df(S,t) =&\frac{\partial f}{\partial S}(\mu(S,t)dt + \sigma(S,t)dZ(t)) + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2(S,t)dt\\
=&\left(\frac{\partial f}{\partial t} + \mu(S,t)\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2(S,t)\frac{\partial^2 f}{\partial S^2}\right)dt + \sigma(S,t)\frac{\partial f}{\partial S}dZ(t) \blacksquare
\end{aligned}$$

Example 64.1
Suppose that $S(t)$ follows a geometric Brownian motion:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dZ(t).$$

Find $d[\sin S(t)]$.

Solution.
Let $f(S,t) = \sin S(t)$. Then

$$\begin{aligned}\frac{\partial f}{\partial t} =&0\\ \frac{\partial f}{\partial S} =&\cos S(t)\\ \frac{\partial^2 f}{\partial S^2} =&-\sin S(t).\end{aligned}$$

Using Itô's lemma we can write

$$\begin{aligned}d[\sin S(t)] =&\frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS(t))^2 + \frac{\partial f}{\partial S}dS(t)\\ =&-\frac{1}{2}\sin S(t)(dS)^2 + \cos S(t)dS(t)\\ =&-\frac{1}{2}\sin S(t)\sigma^2 S^2(t)dt + \cos S(t)[\alpha S(t)dt + \sigma S(t)dZ(t)]\\ =&\left(\alpha S(t)\cos S(t) - \frac{1}{2}\sigma^2 S^2(t)\sin S(t)\right)dt + \sigma S(t)\cos S(t)dZ(t) \blacksquare\end{aligned}$$

Example 64.2
Suppose that $S(t)$ follows a geometric Brownian motion:

$$dS(t) = (\alpha - \delta)S(t)dt + \sigma S(t)dZ(t).$$

Let f be a twice differentiable function of S and t. Find $df(S,t)$.

Solution.
We have $\mu(S,t) = (\alpha-\delta)S(t)$ and $\sigma(S,t) = \sigma S(t)$. Hence,

$$df(S,t) = \left(\frac{\partial f}{\partial t} + (\alpha-\delta)S(t)\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 f}{\partial S^2}\right)dt + \sigma S(t)\frac{\partial f}{\partial S}dZ(t) \blacksquare$$

Example 64.3
Suppose that $S(t)$ satisfies the geometric Brownian motion

$$dS = (\alpha - \delta)S(t)dt + \sigma S(t)dZ(t).$$

Show that

$$d[\ln S(t)] = (\alpha - \delta - 0.5\sigma^2)dt + \sigma dZ(t).$$

Solution.
Let $f(S,t) = \ln S(t)$. Then

$$\begin{aligned}\frac{\partial f}{\partial t} =&0\\ \frac{\partial f}{\partial S} =&\frac{1}{S}\\ \frac{\partial^2 f}{\partial S^2} =& -\frac{1}{S^2}\end{aligned}$$

Now, the result follows from Itô lemma. Using stochastic integration we find

$$S(t) = S(0)e^{(\alpha-\delta-0.5\sigma^2)t+\sigma Z(t)} \blacksquare$$

Example 64.4 ‡
The price of a stock is a function of time. You are given:
• The expression for a lognormal stock price is $S(t) = S(0)e^{(\alpha-\delta-0.5\sigma^2)t+\sigma Z(t)}$.
• The stock price is a function of the Brownian process Z.
• Itô's lemma is used to characterize the behavior of the stock as a function of $Z(t)$.
Find an expression for dS.

Solution.
We let $f(Z,t) = S(t)$. Thus,

$$\begin{aligned}\frac{\partial f}{\partial t} =&\frac{\partial S(t)}{\partial t} = (\alpha - \delta - 0.5\sigma^2)S(t)\\ \frac{\partial f}{\partial Z} =&\sigma S(t)\\ \frac{\partial^2 f}{\partial Z^2} =&\sigma^2 Z(t).\end{aligned}$$

Using Itô's lemma we can write

$$\begin{aligned}
dS(t) =& \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial Z^2}(dZ(t))^2 + \frac{\partial f}{\partial Z}dZ(t) \\
=&(\alpha - \delta - 0.5\sigma^2)S(t)dt + \frac{1}{2}\sigma^2 S(t)dt + \sigma S(t)dZ(t) \\
=&(\alpha - \delta)S(t)dt + \sigma S(t)dZ(t) \blacksquare
\end{aligned}$$

Practice Problems

Problem 64.1
Assume that $S(t)$ follows an arithmetic Brownian motion: $dS(t) = \alpha dt + \sigma dZ(t)$. Use Itô's Lemma to find $d(2S^2(t))$.

Problem 64.2
Assume that $S(t)$ follows a mean-reverting process: $dS(t) = \lambda(\alpha - S(t))dt + \sigma dZ(t)$. Use Itô's Lemma to find $d(2S^2(t))$.

Problem 64.3
Assume that $S(t)$ follows a geometric Brownian process: $\frac{dS(t)}{S(t)} = \alpha dt + \sigma dZ(t)$. Use Itô's Lemma to find $d(2S^2(t))$.

Problem 64.4
Assume that $S(t)$ follows an arithmetic Brownian motion: $dS(t) = \alpha dt + \sigma dZ(t)$. Use Itô's Lemma to find $d(35S^5(t) + 2t)$.

Problem 64.5 ‡
You are given the following information:
(i) $S(t)$ is the value of one British pound in U.S. dollars at time t.
(ii) $dS(t) = 0.1S(t)dt + 0.4S(t)dZ(t)$.
(iii) The continuously compounded risk-free interest rate in the U.S. is $r = 0.08$.
(iv) The continuously compounded risk-free interest rate in Great Britain is $r^* = 0.10$.
(v) $G(t) = S(t)e^{(r-r^*)(T-t)}$ is the forward price in U.S. dollars per British pound, and T is the maturity time of the currency forward contract.
Based on Itô's Lemma, find an expression for $dG(t)$.

Problem 64.6 ‡
$X(t)$ is an Ornstein-Uhlenbeck process defined by

$$dX(t) = 2[4 - X(t)]dt + 8dZ(t),$$

where $Z(t)$ is a standard Brownian motion.
Let

$$Y(t) = \frac{1}{X(t)}.$$

You are given that

$$dY(t) = \alpha(Y(t))dt + \beta(Y(t))dZ(t)$$

for some functions $\alpha(y)$ and $\beta(y)$.
Determine $\alpha(0.5)$.

Problem 64.7 ‡
Let $\{Z(t)\}$ be a standard Brownian motion. You are given:
(i) $U(t) = 2Z(t) - 2$
(ii) $V(t) = [Z(t)]^2 - t$
(iii)

$$W(t) = t^2 Z(t) - 2\int_0^t sZ(s)ds.$$

Which of the processes defined above has / have zero drift?

Problem 64.8 ‡
The stochastic process $\{R(t)\}$ is given by

$$R(t) = R(0)e^{-t} + 0.05(1 - e^{-t}) + 0.1\int_0^t e^{s-t}\sqrt{R(s)}dZ(s).$$

where $\{Z(t)\}$ is a standard Brownian motion. Define $X(t) = [R(t)]^2$.
Find $dX(t)$.

Problem 64.9 ‡
The price of a stock is governed by the stochastic differential equation:

$$\frac{dS(t)}{S(t)} = 0.03dt + 0.2dZ(t)$$

where $\{Z(t)\}$ is a standard Brownian motion. Consider the geometric average

$$G = [S(1)S(2)S(3)]^{\frac{1}{3}}.$$

Find the variance of $\ln[G]$.

Problem 64.10 ‡
Let $x(t)$ be the dollar/euro exchange rate at time t. That is, at time t, €1 = $\$x(t)$.

Let the constant r be the dollar-denominated continuously compounded risk-free interest rate. Let the constant $r_{€}$ be the euro-denominated continuously compounded risk-free interest rate.
You are given

$$\frac{dx(t)}{x(t)} = (r - r_{€})dt + \sigma dZ(t),$$

where $\{Z(t)\}$ is a standard Brownian motion and σ is a constant.
Let $y(t)$ be the euro/dollar exchange rate at time t. Thus, $y(t) = \frac{1}{x(t)}$. Show that

$$\frac{dy(t)}{y(t)} = (r_{€} - r + \sigma^2)dt - \sigma dZ(t).$$

65 Valuing a Claim on S^a

In this section, we want to compute the price of a claim whose payoff depends on the stock price raised to some power.
Suppose that a stock with an expected instantaneous return of α, dividend yield of δ, and instantaneous velocity σ follows a geometric Brownian motion given by

$$dS(t) = (\alpha - \delta)S(t)dt + \sigma S(t)dZ(t).$$

Consider a claim with payoff $S(T)^a$ at time T. Let's examine the process followed by S^a. Let $f(S,t) = S(t)^a$. Then

$$\begin{aligned}\frac{\partial f}{\partial t} =&0\\ \frac{\partial f}{\partial S} =&aS(t)^{a-1}\\ \frac{\partial^2 f}{\partial S^2} =&a(a-1)S(t)^{a-2}.\end{aligned}$$

Using Itô's lemma we can write

$$\begin{aligned}d[S(t)^a] =&\frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS(t))^2 + \frac{\partial f}{\partial S}dS(t)\\ =&\frac{1}{2}a(a-1)S(t)^{a-2}(dS(t))^2 + aS(t)^{a-1}dS(t)\\ =&\frac{1}{2}a(a-1)S(t)^{a-2}[(\alpha-\delta)S(t)dt + \sigma S(t)dZ(t)]^2 + aS(t)^{a-1}[(\alpha-\delta)S(t)dt + \sigma S(t)dZ(t)]\\ =&\frac{1}{2}a(a-1)S(t)^{a-2}\sigma^2 dt + aS(t)^{a-1}[(\alpha-\delta)S(t)dt + \sigma S(t)dZ(t)]\\ =&[aS(t)^a(\alpha-\delta) + \frac{1}{2}a(a-1)S(t)^a\sigma^2]dt + aS(t)^a\sigma dZ(t)\end{aligned}$$

Hence,

$$\frac{d[S(t)^a]}{S(t)^a} = [a(\alpha-\delta) + \frac{1}{2}a(a-1)\sigma^2]dt + a\sigma dZ(t).$$

Thus, S^a follows a geometric Brownian motion with drift factor $a(\alpha-\delta) + \frac{1}{2}a(a-1)\sigma^2$ and risk $a\sigma dZ(t)$. Now we expect the claim to be perfectly correlated with the stock price S which means that the stock and the claim have equal Sharpe ratio. But the Sharpe ratio of the stock is $(\alpha - r)/\sigma$. Since the volatility of the claim is $a\sigma$ we expect the claim risk-premium to

be $a(\alpha - r)$. Hence, the expected return of the claim is $a(\alpha - r) + r$.
The following result gives us the forward and prepaid forward prices for the claim.

Theorem 65.1
The prepaid forward price of a claim paying $S(T)^a$ at time T is

$$F_{0,T}^{p}[S(T)^a] = e^{-rT}S(0)^a e^{[a(r-\delta)+0.5a(a-1)\sigma^2]T}.$$

The forward price for this claim is

$$F_{0,T}[S(T)^a] = S(0)^a e^{[a(r-\delta)+0.5a(a-1)\sigma^2]T}.$$

Proof.
The risk-neutral price process of the claim is given by

$$\frac{d[S(t)^a]}{S(t)^a} = [a(r-\delta) + \frac{1}{2}a(a-1)\sigma^2]dt + a\sigma d\tilde{Z}(t).$$

The expected value of the claim at time T under the risk-neutral measure, which we denote by E^* is

$$E[S(T)^a] = S(0)^a e^{[a(r-\delta)+0.5a(a-1)\sigma^2]T}.$$

By Example 15.2, the expected price under the risk-neutral measure is just the forward price. That is, the forward price of the claim is

$$F_{0,T}[S(T)^a] = E[S(T)^a] = S(0)^a e^{[a(r-\delta)+0.5a(a-1)\sigma^2]T}.$$

Discounting this expression at the risk-free rate gives us the prepaid forward price

$$F_{0,T}^{p}[S(T)^a] = e^{-rT}S(0)^a e^{[a(r-\delta)+0.5a(a-1)\sigma^2]T} \blacksquare$$

Example 65.1
Examine the cases of $a = 0, 1, 2$.

Solution.
• If $a = 0$ the claim does not depend on the stock price so it is a bond. With $a = 0$ the prepaid forward price is

$$F_{0,T}^{p}[S(T)^0] = e^{-rT}S(0)^0 e^{[0(r-\delta)+0.50(0-1)\sigma^2]T} = e^{-rT}$$

which is the price of a bond that pays \$1 at time T.
• If $a = 1$ then the prepaid forward price is

$$F^p_{0,T}[S(T)] = e^{-rT}S(0)e^{[(r-\delta)+0.5(1-1)\sigma^2]T} = S(0)e^{-\delta T}$$

which is the prepaid forward price on a stock.
• If $a = 2$ then the prepaid forward price is

$$F^p_{0,T}[S(T)^2] = e^{-rT}S(0)^2e^{[2(r-\delta)+0.5(2)(2-1)\sigma^2]T} = S(0)^2e^{(r-2\delta+\sigma^2)T} \blacksquare$$

Example 65.2
Assume the Black-Scholes framework. For $t \geq 0$, let $S(t)$ be the time$-t$ price of a nondividend-paying stock. You are given:
(i) $S(0) = 0.5$
(ii) The stock price process is

$$\frac{dS(t)}{S(t)} = 0.05dt + 0.2dZ(t)$$

where $Z(t)$ is a standard Brownian motion.
(iii) $E[S(1)^a] = 1.4$, where a is a negative constant.
(iv) The continuously compounded risk-free interest rate is 3%.
Determine a.

Solution.
Since

$$S(T) = S(0)e^{(\alpha-\delta-0.5\sigma^2)T+\sigma Z(T)}$$

we can write

$$S(T)^a = S(0)^ae^{a(\alpha-\delta-0.5\sigma^2)T+a\sigma Z(T)}$$

Thus,

$$E[S(T)^a] = S(0)^ae^{[a(\alpha-\delta)+0.5a(a-1)\sigma^2]T}.$$

From (iii), we have

$$1.4 = 0.5^ae^{a(0.05-0)+0.5a(a-1)0.2^2}.$$

Taking the natural logarithm of both sides to obtain

$$\ln 1.4 = a\ln 0.5 + 0.05a + 0.5a(a-1)(0.04).$$

This leads to the quadratic equation

$$0.02a^2 + (\ln(0.5) + 0.03)a - \ln 1.4 = 0.$$

Solving this equation for a we find $a = -0.49985$ or $a = 33.65721$. Since $a < 0$ we discard the positive value $\blacksquare$

Practice Problems

Problem 65.1
Assume that the Black-Scholes framework holds. Let $S(t)$ be the price of a nondividend-paying stock at time $t \geq 0$. The stock's volatility is 18%, and the expected return on the stock is 8%. Find an expression for the instantaneous change $d[S(t)^a]$, where $a = 0.5$.

Problem 65.2
Assume that the Black-Scholes framework holds. Let $S(t)$ be the price of a stock at time $t \geq 0$. The stock's volatility is 20%, the expected return on the stock is 9%, and the continuous compounded yield is 2%. Find an expression for the instantaneous change $d[S(t)^a]$, where $a = 0.5$.

Problem 65.3
Assume that the Black-Scholes framework holds. Let $S(t)$ be the price of a stock at time $t \geq 0$. The stock's volatility is 20%, the expected return on the stock is 9%, and the continuous compounded yield is 2%. Suppose that $S(1) = 60$. Find $E[d[S(1)^{0.5}]]$.

Problem 65.4 ‡
Assume the Black-Scholes framework. For $t \geq 0$, let $S(t)$ be the time$-t$ price of a nondividend-paying stock. You are given:
(i) $S(0) = 0.5$
(ii) The stock price process is

$$\frac{dS(t)}{S(t)} = 0.05dt + 0.2dZ(t)$$

where $Z(t)$ is a standard Brownian motion.
(iii) $E[S(1)^a] = 1.4$, where a is a negative constant.
(iv) The continuously compounded risk-free interest rate is 3%.
Consider a contingent claim that pays $S(1)^a$ at time 1. Calculate the time-0 price of the contingent claim.

Problem 65.5 ‡
Assume that the Black-Scholes framework holds. Let $S(t)$ be the price of a nondividend-paying stock at time $t \geq 0$. The stock's volatility is 20%, and the continuously compounded risk-free interest rate is 4%.

You are interested in contingent claims with payoff being the stock price raised to some power. For $0 \leq t < T$, consider the equation

$$F_{t,T}^{P}[S(T)^x] = S(t)^x$$

where the left-hand side is the prepaid forward price at time t of a contingent claim that pays $S(T)^x$ at time T. A solution for the equation is $x = 1$.
Determine another x that solves the equation.

Problem 65.6 ‡
Assume the Black-Scholes framework. Consider a derivative security of a stock.
You are given:
(i) The continuously compounded risk-free interest rate is 0.04.
(ii) The volatility of the stock is σ.
(iii) The stock does not pay dividends.
(iv) The derivative security also does not pay dividends.
(v) $S(t)$ denotes the time-t price of the stock.
(iv) The time-t price of the derivative security is $[S(t)]^{-\frac{k}{\sigma^2}}$, where k is a positive constant.
Find k.

Problem 65.7 ‡
Assume the Black-Scholes framework.
Let $S(t)$ be the time-t price of a stock that pays dividends continuously at a rate proportional to its price.
You are given:
(i)

$$\frac{dS(t)}{S(t)} = \mu dt + 0.4d\tilde{Z}(t),$$

where $\{\tilde{Z}(t)\}$ is a standard Brownian motion under the risk-neutral probability measure;
(ii) for $0 \leq t \leq T$, the time-t forward price for a forward contract that delivers the square of the stock price at time T is

$$F_{t,T}(S^2) = S^2(t)e^{0.18(T-t)}.$$

Calculate μ.

Problem 65.8 ‡
Assume the Black-Scholes framework. For a stock that pays dividends continuously at a rate proportional to its price, you are given:
(i) The current stock price is 5.
(ii) The stock's volatility is 0.2.
(iii) The continuously compounded expected rate of stock-price appreciation is 5%.
Consider a 2-year arithmetic average strike option. The strike price is

$$A(2) = \frac{1}{2}[S(1) + S(2)].$$

Calculate $\mathrm{Var}[A(2)]$.

The Black-Scholes Partial Differential Equation

In the derivation of the option pricing, Black and Scholes assumed that the stock price follows a geometric Brownian motion and used Itô's lemma to describe the behavior of the option price. Their analysis yields a partial differential equation, which the correct option pricing formula must satisfy. In this chapter, we derive and examine this partial differential equation.

66 Differential Equations for Riskless Assets

In this section we will describe the pricing of a bond or a stock under certainty, i.e., the owner of the asset receives the risk-free return. It turns out that the pricing formula is the solution of a boundary value differential equation. That is, the differential equation describing the change in the price together with a boundary condition determine the price of either the bond or the stock at any point in time.

Let $S(t)$ the value of a risk-free zero-coupon bond that pays \$1 at time T. Then over a small time interval h, the price of the bond changes by $rhS(t)$. That is,

$$S(t+h) - S(t) = rhS(t)$$

or

$$\frac{S(t+h) - S(t)}{h} = rS(t).$$

Letting $h \to 0$ we obtain the differential equation

$$\frac{dS(t)}{dt} = rS(t).$$

Example 66.1

Solve the differential equation

$$\frac{dS(t)}{dt} = rS(t).$$

subject to the condition $S(T) = \$1$.

Solution.

Separating the variables we find

$$\frac{dS(t)}{S(t)} = rdt.$$

Integrate both sides from t to T to obtain

$$\int_t^T \frac{dS}{S} = r\int_t^T dt.$$

Thus,

$$\ln\left(S(T)/S(t)\right) = r(T-t)$$

or

$$S(t) = S(T)e^{-r(T-t)}.$$

Using the **boundary condition**, $S(T) = 1$ we find that the price of the bond is given by the formula

$$S(t) = e^{-r(T-t)}.$$

This solution confirms what we already know: The price of a risk free zero coupon bond at time t is the discount factor back from the time when the bond matures to time t ■

Now, let $S(t)$ be the value of a stock that pays continuous dividend at the rate of δ. Keep in mind that we are pricing the stock under certainty so that the random nature of future stock prices is ignored and the stock receives only the risk-free rate. Then for a small time period h, the current price of the stock is equal to the dividends paid plus the future stock price, discounted back for interest:

$$S(t) = [\delta h S(t) + S(t+h)](1+rh)^{-1}.$$

This can be rewritten in the form

$$S(t+h) - S(t) + \delta h S(t) = rhS(t).$$

This equation says that the stock return is the change in the stock price plus dividend paid. The previous equation can be rewritten in the form

$$\frac{S(t+h) - S(t)}{h} = (r-\delta)S(t).$$

Letting $h \to 0$ we find

$$\frac{dS(t)}{dt} = (r-\delta)S(t).$$

Using the method of separation of variables, the general solution to this equation is

$$S(t) = Ae^{-(r-\delta)(T-t)}.$$

If S_0 is the stock price at time $t = 0$ then we have the **initial boundary condition** $S(0) = S_0$. In this case, $A = S_0 e^{(r-\delta)T}$. Hence,

$$S(t) = S_0 e^{(r-\delta)t}.$$

The price of a risk free stock at time t is the initial price accumulated at a rate of $r-\delta$. The price of a risk free stock at time t is its forward price.
Now, if the stock pays continuous dividends in the amount $D(t)$, then we have the equation

$$S(t) = [S(t+h) + hD(t+h)](1+rh)^{-1}$$

which can be rewritten as

$$S(t+h) - S(t) + hD(t+h) = rhS(t)$$

or

$$\frac{S(t+h)-S(t)}{h} + D(t+h) = rS(t).$$

Letting $h \to 0$ we find the differential equation

$$\frac{dS(t)}{dt} + D(t) = rS(t).$$

If we inpose the terminal boundary condition $S(T) = \tilde{S}$, then by solving the intial value problem by the method of integrating factor[1] we find the solution

$$S(t) = \int_t^T D(s)e^{-r(s-t)}ds + \tilde{S}e^{-r(T-t)}.$$

Example 66.2
A stock pays continuous dividends in the amount of $D(t) = \$8$. The continuously compounded risk-free interest rate is 0.10. Suppose that $S(3) = \$40$. Determine the price of the stock two years from now. Assume that the future random nature of the stock price is ignored.

Solution.
We have

$$\begin{aligned}
S(2) = &\int_2^3 8e^{-0.10(s-2)}ds + 40e^{-0.10(3-2)} \\
=&8e^{0.20}\int_2^3 e^{-0.10s}ds + 40e^{-0.10} \\
=&8e^{0.20}\left[-10e^{-0.10s}\right]_2^3 + 40e^{-0.10} = \$43.81 \blacksquare
\end{aligned}$$

[1]See Practice Problems 66.1-66.3.

Practice Problems

Problem 66.1
Consider the differential equation

$$\frac{dS(t)}{dt} - rS(t) = -D(t).$$

Show that this equation is equivalent to

$$\frac{d}{dt}\left[e^{-rt}S(t)\right] = -e^{-rt}D(t).$$

Problem 66.2
Find the general solution of the differential equation

$$\frac{dS(t)}{dt} - rS(t) = -D(t).$$

Problem 66.3
Find the solution to the boundary value problem

$$\frac{dS(t)}{dt} - rS(t) = -D(t), 0 \leq t \leq T$$

with $S(T) = \tilde{S}$.

Problem 66.4
The price $S(t)$ of a bond that matures in $T \geq 4$ years satisfies the differential equation

$$\frac{dS(t)}{dt} = 0.05S(t)$$

subject to the boundary condition $S(T) = \$100$. Find T if $S(2) = \$70.50$.

Problem 66.5
The price $S(t)$ of a bond that matures in 3 years satisfies the differential equation

$$\frac{dS(t)}{dt} = rS(t)$$

subject to the boundary condition $S(3) = \$100$. Find r if $S(1) = \$81.87$.

Problem 66.6
Find the differential equation satisfied by the expression

$$S(t) == \int_t^T \delta S(s)e^{-0.10(s-t)}ds + \tilde{S}e^{-0.10(T-t)}.$$

67 Derivation of the Black-Scholes PDE

There are several assumptions involved in the derivation of the Black-Scholes equation. It is important to understand these properly so as to see the limitations of the theory. These assumptions are summarised below :
1. The efficient market hypothesis is assumed to be satisfied. In other words, the markets are assumed to be liquid, have price-continuity, be fair and provide all players with equal access to available information. This implies that zero transaction costs are assumed in the Black-Scholes analysis.
2. It is assumed that the underlying asset is perfectly divisible and that short selling with full use of proceeds is possible.
3. Constant risk-free interest rates are assumed. In other words, we assume that there exists a risk-free security which returns \$1 at time T when $e^{-r(T-t)}$ is invested at time t.
4. As we will see later, the Black-Scholes analysis requires continuous trading. This is, of course, not possible in practice as the more frequently one trades, the larger the transaction costs.
5. The prinicple of no arbitrage is assumed to hold.
6. The price of the underlying asset is assumed to follow a geometric Brownian process of the form $dS(t) = \alpha S(t)dt + \sigma S(t)dZ(t)$ where $Z(t)$ is the risk term or the stochastic term.
It should be obvious that none of these principles can be perfectly satisfied. Transaction costs exist in all markets, all securities come in discrete units, short selling with full use of proceeds is very rare, interest rates vary with time and we will later see that there is evidence that the price of most stocks do not precisely follow a geometric Brownian process.

Let $V = V[S,t]$ be the value of a call or a put option written on an underlying asset with value $S(t)$ at time t. Then according to Itô's lemma, V changes over the infinitesimal time interval dt according to

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \alpha S \frac{\partial V}{\partial S}\right) dt + \sigma S \frac{\partial V}{\partial S} dZ(t).$$

Let us now assume that we have one option with value V and Δ shares of the underlying asset where Δ as yet undetermined, with $\Delta > 0$ for shares held long and $\Delta < 0$ for shares held short. The value of this portfolio at any time t is

$$\pi = V[S,t] + \Delta S.$$

Over the time dt, the gain in the value of the portfolio is

$$d\pi = dV + \Delta dS.$$

or

$$d\pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \alpha S \frac{\partial V}{\partial S}\right) dt + \sigma S \frac{\partial V}{\partial S} dZ(t) + \Delta(\alpha S dt + \sigma S dZ(t)).$$

If we choose $\Delta = -\frac{\partial V}{\partial S}$ then the stochastic terms cancel so that the gain is deterministic which means the gain cannot be more or less than the gain in the value of the portfolio were it invested at the risk-free interest rate r. That is, we must have

$$d\pi = r\pi dt = r\left(V - \frac{\partial V}{\partial S} S\right) dt.$$

Now, equating the two expressions of $d\pi$ we find

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.$$

This is the famous Black-Scholes equation for the value of an option.
For a dividend paying asset the corresponding Black-Scholes formula is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta) S \frac{\partial V}{\partial S} - rV = 0.$$

The Black-Scholes equation applies to any derivative asset. What distinguishes the different assets are the **boundary conditions**. For a zero-coupon bond that matures at time T and pays \$1 at maturity, the boundary condition is that it must be worth \$1 at time T. For a prepaid forward contract on a share of stock, the boundary condition is that the prepaid forward contract is worth a share at maturity. For a European call option the boundary condition is $\max\{0, S(T) - K\}$ whereas for a European put the boundary condition is $\max\{0, K - S(T)\}$. A derivation of the solution of the Black-Scholes for European options was discussed in Section 50.

Binary Options

A **binary option** is a type of option where the payoff is either some fixed amount of some asset or nothing at all. The two main types of binary options are the **cash-or-nothing** binary option and the **asset-or-nothing** binary option.[1]

Example 67.1
A cash-or-nothing binary option pays some fixed amount of cash if the option expires in-the-money. Suppose you buy a cash-or-nothing option on a stock with strike price \$100 and payoff of \$350. What is the payoff of the option if the stock price at expiration is \$200? What if the stock price is \$90?

Solution.
If the stock price at expiration is \$200 then the payoff of the option is \$350. If its stock is trading below \$100, nothing is received ■

Example 67.2
A cash-or-nothing call option pays out one unit of cash if the spot is above the strike at maturity. Its Black-Sholes value now is given by

$$V[S(t),t] = e^{-r(T-t)}N(d_2)$$

where

$$d_2 = \frac{\ln(S(t)/K) + (r - \delta - 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Show that V satisfies the Black-Scholes partial differential equation.

Solution.
For this problem, we refer the reader to Section 29 and Section 30 where expressions for $\frac{\partial N(d_1)}{\partial t}$, $\frac{\partial N(d_2)}{\partial t}$, $\frac{\partial N(d_1)}{\partial S}$, and $\frac{\partial N(d_2)}{\partial S}$ were established. We have

$$\begin{aligned}
V_t =& rV + e^{-r(T-t)}\left(\frac{1}{\sqrt{2\pi}}\frac{S}{K}e^{-\frac{d_1^2}{2}}e^{(r-\delta)(T-t)}\right)\left(\frac{\ln(S/K)}{2\sigma(T-t)^{\frac{3}{2}}} - \frac{r-\delta-0.5\sigma^2}{\sigma\sqrt{T-t}} + \frac{r-\delta-0.5\sigma^2}{2\sigma\sqrt{T-t}}\right)\\
V_S =& e^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\cdot\frac{1}{K}\cdot\frac{e^{(r-\delta)(T-t)}}{\sigma\sqrt{T-t}}\\
V_{SS} =& -d_1e^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\cdot\frac{1}{K}\cdot\frac{e^{(r-\delta)(T-t)}}{S\sigma^2(T-t)}
\end{aligned}$$

Thus,

$$V_t + \frac{1}{2}\sigma^2S^2V_{SS} + (r-\delta)SV_S - rV = 0 \blacksquare$$

[1]These options are also called **all-or-nothing** options or **digital** options.

Example 67.3
Suppose that $V = e^{rt} \ln S$ is a solution to Black-Scholes equation. Given that $r = 0.08$ and $\sigma = 0.30$, find δ.

Solution.
We have

$$\begin{aligned} V =& e^{rt} \ln S \\ V_t =& re^{rt} \ln S \\ V_S =& e^{rt} S^{-1} \\ V_{SS} =& -e^{rt} S^{-2} \end{aligned}$$

Substituting into the differential equation we find

$$re^{rt} \ln S + \frac{1}{2}(0.30)^2 S^2(-e^{rt}S^{-2}) + (0.08 - \delta)S(e^{rt}S^{-1}) - re^{rt} \ln S = 0.$$

Reducing this expression we find

$$-0.5(0.3)^2 e^{rt} + (0.08 - \delta)e^{rt} = 0$$

or

$$-0.5(0.30)^2 + 0.08 - \delta = 0.$$

Solving this equation we find $\delta = 3.5\%$ ∎

Practice Problems

Problem 67.1
The price of a zero-coupon bond that pays \$1 at time T is given by $V(t,T) = e^{-r(T-t)}$. Show that V satisfies the Black-Scholes differential equation with boundary condition $V(T,T) = \$1$.

Problem 67.2
The price of a prepaid forward contract on a share of stock is given by $V[S(t),t] = S(t)e^{-\delta(T-t)}$. Show that V satisfies the Black-Scholes differential equation with boundary condition $V[S(T),T] = S(T)$.

Problem 67.3
The underlying asset pays no dividends. Show that the value function $V[S(t),t] = e^{rt}S^{1-\frac{2r}{\sigma^2}}$ satisfies the Black-Scholes partial differential equation.

Problem 67.4
Suppose you were interested in buying binary call options for common shares of ABC company with a strike price of \$50 per share and a specified binary payoff of \$500. What would you receive if the stock is trading above \$50 when the expiration date is reached? What if the stock is trading below \$50 per share at the expiration date.

Problem 67.5
A cash-or-nothing put option pays out one unit of cash if the spot is below the strike at maturity. Its Black-Sholes value now is given by

$$V[S(t),t] = e^{-r(T-t)}N(-d_2) = e^{-r(T-t)}(1 - N(d_2))$$

where

$$d_2 = \frac{\ln(S(t)/K) + (r - \delta - 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Show that V satisfies the Black-Scholes partial differential equation.

Problem 67.6
An asset-or-nothing call option pays out one unit of asset if the spot is above the strike at maturity. Its Black-Sholes value now is given by

$$V[S(t),t] = Se^{-\delta(T-t)}N(d_1)$$

where

$$d_1 = \frac{\ln(S(t)/K) + (r - \delta + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Show that V satisfies the Black-Scholes partial differential equation.

Problem 67.7

An asset-or-nothing put option pays out one unit of asset if the spot is below the strike at maturity. Its Black-Sholes value now is given by

$$V[S(t), t] = Se^{-\delta(T-t)}N(-d_1) = Se^{-\delta(T-t)}(1 - N(d_1))$$

where

$$d_1 = \frac{\ln(S(t)/K) + (r - \delta + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Show that V satisfies the Black-Scholes partial differential equation.

Problem 67.8

When a derivative claim makes a payout $D(t)$ at time t, the Black-Scholes equation takes the form

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + (\alpha - \delta)SV_S + D(t) - rV = 0.$$

You are given the following: $r = 0.05, \delta = 0.02, D(t) = 0.04V(t), V(t) = e^{0.01t}\ln S$. Determine the value of σ.

Problem 67.9 ‡

Assume the Black-Scholes framework. Consider a stock and a derivative security on the stock.

You are given:

(i) The continuously compounded risk-free interest rate, r, is 5.5%.

(ii) The time-t price of the stock is $S(t)$.

(iii) The time-t price of the derivative security is $e^{rt}\ln[S(t)]$.

(iv) The stock's volatility is 30%.

(v) The stock pays dividends continuously at a rate proportional to its price.

(vi) The derivative security does not pay dividends.

Calculate δ, the dividend yield on the stock.

68 The Black-Scholes PDE and Equilibrium Returns

In this section we derive the Black-Scholes partial differential equation based on the condition that the expected return on the option must be equal the equilibrium expected return on the underlying asset.

Let $V = V[S(t), t]$ denote the value of an option with a stock as underlying asset. Using Itô's lemma we can write

$$dV = \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{\partial V}{\partial S}dS.$$

If the stock price follows a geometric Brownian motion then

$$dS = (\alpha - \delta)Sdt + \sigma SdZ(t).$$

Thus, using the mutliplication rules, we can write

$$dV = \left[\frac{1}{2}\sigma^2 S^2 V_{SS} + (\alpha - \delta)SV_S + V_t\right]dt + SV_S\sigma dZ(t).$$

The term involving dZ is the random element or the unexpected return on the option. In contrast, the term involving dt is deterministic or the expected return on the option.

We define the instantaneous expected return on the option by the expression

$$\alpha_{\text{option}} = \frac{0.5\sigma^2 S^2 V_{SS} + (\alpha - \delta)SV_S + V_t}{V}.$$

The unexpected return on a stock is given by $\sigma SdZ(t)$. Likewise, we define the unexpected return on an option by $\sigma_{\text{option}}VdZ(t)$. But from the equation above, the unexpected return is $SV_S\sigma dZ(t)$. Hence, $\sigma_{\text{option}}VdZ(t) = SV_S\sigma dZ(t)$ which implies

$$\sigma_{\text{option}} = \frac{SV_S}{V}\sigma.$$

Since elasticity is defined as the percentage change of option price divided by the percentage change in stock price we find the option's elasticity to be

$$\Omega = \frac{\frac{\partial V}{V}}{\frac{\partial S}{S}} = \frac{\frac{\partial V}{\partial S}}{\frac{V}{S}} = \frac{SV_S}{V}.$$

Therefore,

$$\sigma_{\text{option}} = \Omega\sigma.$$

Since the option value and stock value are driven by the same standard Brownian motion, they must have the same Sharpe ratios. That is,

$$\frac{\alpha - r}{\sigma} = \frac{\alpha_{\text{option}} - r}{\sigma_{\text{option}}}$$

and from this we find the risk-premium on the option

$$\alpha_{\text{option}} - r = \Omega(\alpha - r).$$

The quantity $\Omega(\alpha-r)+r$ is referred to as the **equilibrium expected return on the stock**. It follows that the expected return on the option is equal to the equilibrium expected return on the stock. From this fact we have

$$\begin{aligned}
\alpha_{\text{option}} - r =& \Omega(\alpha - r)\\
\frac{1}{2}\sigma^2 S^2 V_{SS} + (\alpha - \delta)SV_S + V_t - r =& \frac{SV_S}{V}(\alpha - r)\\
\frac{1}{2}\sigma^2 S^2 V_{SS} + (\alpha - \delta)SV_S + V_t - r - \frac{SV_S}{V}(\alpha - r) =& 0\\
V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + (r - \delta)SV_S - rV =& 0
\end{aligned}$$

which is the Black-Scholes equation.

Example 68.1
The following information are given regarding an option on a stock: $r = 0.08, \delta = 0.02, \sigma = 0.30, S(t) = 50, V_t = -3.730, V_S = 0.743$, and $V_{SS} = 0.020$. Find the value of the option.

Solution.
Using the Black-Scholes formula, we find

$$0.5(0.30)^2(50)^2(0.020) + (0.08 - 0.02)(50)(0.743) - 3.730 = 0.08V.$$

Solving this equation for V we find $V = 9.3625$ ■

Example 68.2
The following information are given regarding an option on a stock: $r = 0.08, \delta = 0.02, \sigma = 0.30, S(t) = 50, V_t = -3.730, V_S = 0.743$, and $V_{SS} = 0.020$. Find the instantaneous expected return on the option if the stock risk-premium is 0.13.

Solution.
We have

$$\begin{aligned}\alpha_{\text{option}} =&\frac{0.5\sigma^2S^2V_{SS}+(\alpha-\delta)SV_S+V_t}{V}\\ =&\frac{0.5(0.30)^2(50)^2(0.020)+0.19(50)(0.743)-3.730}{9.3625}\\ =&0.5958 \blacksquare\end{aligned}$$

Example 68.3
Consider a European put option on a nondividend-paying stock with exercise date $T, T>0$. Let $S(t)$ be the price of one share of the stock at time $t, t\geq 0$. For $0\leq t\leq T$, let $P(s,t)$ be the price of one unit of the put option at time t, if the stock price is s at that time. You are given:
(i) $\frac{dS(t)}{S(t)}=0.11dt+\sigma dZ(t)$, where σ is a positive constant and $\{Z(t)\}$ is a Brownian motion.
(ii) $\frac{dP(S(t),t)}{P(S(t),t)}=\gamma(S(t),t)dt+\sigma_P(S(t),t)dZ(t),\ 0\leq t\leq T.$
(iii) $P(S(0),0)=7$.
(iv) At time $t=0$, the cost of shares required to delta-hedge one unit of the put option is 18.25.
(v) The continuously compounded risk-free interest rate is 8%.
Determine $\gamma(S(0),0)$.

Solution.
We have

$$\gamma_{\text{option}}-r=\frac{SV_S}{V}(\alpha-r)$$

which, for this problem, translates to

$$\gamma(S(t),t)-0.08=\frac{S(t)\Delta}{P(S(t),t)}(0.11-0.08).$$

We are given that

$$\frac{S(0)\Delta}{P(S(0),0)}=\frac{-18.25}{7}=-2.61.$$

Hence,

$$\gamma(S(0),0)=-2.61(0.11-0.08)+0.08=0.0017 \blacksquare$$

Remark 68.1
The Black-Scholes equation in the previous section was derived by assuming no arbitrage; this involved hedging the option. The equilibrium and no-arbitrage prices are the same. The equilibrium pricing approach shows that the Black-Scholes equation does not depend on the ability to hedge the option.

Practice Problems

Problem 68.1
The following information are given regarding an option on a stock: $r = 0.08, \sigma = 0.30, S(t) = 50, V_t = -3.730, V_S = 0.743, V_{SS} = 0.020$, and $V = 9.3625$. Find the continuously compounded yield δ.

Problem 68.2
The following information are given regarding an option on a stock: $\delta = 0.02, \sigma = 0.30, S(t) = 50, V_t = -3.730, V_S = 0.743, V_{SS} = 0.020$, and $V = 9.3625$. Find the continuously compounded risk-free rate r.

Problem 68.3
The following information are given regarding an option on a stock: $r = 0.08, \delta = 0.02, \sigma = 0.30, S(t) = 50, V_t = -3.730, V_S = 0.743$, and $V_{SS} = 0.020$. Find the option's elasticity.

Problem 68.4
The following information are given regarding an option on a stock: $r = 0.08, \delta = 0.02, \sigma = 0.30, S(t) = 50, V_t = -3.730, V_S = 0.743$, and $V_{SS} = 0.020$. Find the option's risk-premium if the stock's risk-premium is 0.13.

Problem 68.5
The following information are given regarding an option on a stock: $r = 0.08, \delta = 0.02, \sigma = 0.30, S(t) = 5, V_t = -0.405, V_S = 0.624, V_{SS} = 0.245$. Find the option's elasticity.

Problem 68.6 ‡
Consider a European call option on a nondividend-paying stock with exercise date $T, T > 0$. Let $S(t)$ be the price of one share of the stock at time $t, t \geq 0$. For $0 \leq t \leq T$, let $C(s,t)$ be the price of one unit of the call option at time t, if the stock price is s at that time. You are given:
(i) $\frac{dS(t)}{S(t)} = 0.1dt + \sigma dZ(t)$, where σ is a positive constant and $\{Z(t)\}$ is a Brownian motion.
(ii) $\frac{dC(S(t),t)}{C(S(t),t)} = \gamma(S(t),t)dt + \sigma_C(S(t),t)dZ(t),\ 0 \leq t \leq T$.
(iii) $C(S(0),0) = 6$.
(iv) At time $t = 0$, the cost of shares required to delta-hedge one unit of the call option is 9.
(v) The continuously compounded risk-free interest rate is 4%.
Determine $\gamma(S(0),0)$.

69 The Black-Scholes Equation and the Risk Neutral Pricing

Since α, the expected return on the stock, does not appear in the Black-Scholes equation, the actual expected return on a stock is irrelevant for pricing an option on the stock. The pricing is only dependent on the risk-free rate and so it was observed that the Black-Scholes pricing model is consistent with any possible world where there is no arbitrage. Thus, it makes sense when valuing an option to select the world with the easiest option valuation. The easiet valuation environment is the risk-neutral environment.
As disucssed in Section 63, the true stock price process that is observed in the world follows a Brownian motion given by

$$\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t).$$

The risk-neutral price process is given by

$$\frac{dS(t)}{S(t)} = (r - \delta)dt + \sigma d\tilde{Z}(t)$$

where $\tilde{Z}$ is a standard Brownian process (the risk-neutral process). Notice that the volatility is the same for the true price process and the risk-neutral price process.
Since,

$$dV = (V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + (\alpha - \delta)SV_S)dt + S\sigma V_S dZ(t),$$

the actual expected change per unit of time in the option price is given by

$$\frac{E(dV)}{dt} = V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + (\alpha - \delta)SV_S.$$

Likewise, we define the expected change per unit of time in the option value with respect to the risk-neutral distribution to be

$$\frac{E^*(dV)}{dt} = V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + (r - \delta)SV_S$$

where E^* is the expectation with respect to the risk-neutral probability distribution. Hence, the Black-Scholes equation can be rewritten as

$$\frac{E^*(dV)}{dt} = rV. \tag{69.1}$$

In other words, in the risk-neutral environment, the expected increase in the value of the option is at the risk free rate.

Example 69.1
Given the following information about a European option: $r = 0.08, \delta = 0.02, \sigma = 0.3, V_t = -4.567, V_S = 0.123, V_{SS} = 0.231, S = 25$. Find the expected change of the option value with respect to the risk-neutral distribution.

Solution.
Using the Black-Scholes equation we have

$$rV = -4.567 + \frac{1}{2}(0.3)^2(25)^2(0.231) + (0.08 - 0.02)(25)(0.123) = 2.11.$$

Thus,

$$\frac{E^*(dV)}{dt} = rV = 2.11 \blacksquare$$

In their paper, Black and Scholes established that the solution to equation (69.1) is just the expected value of the derivative payoff under the risk-neutral probability distribution discounted at the risk-free rate. For example, in the case of a European call option on the stock with boundary condition $C(T) = \max\{S(T) - K\}$, the price of the call is given by

$$C = e^{-r(T-t)} \int_K^\infty [S(T) - K] f^*(S(T)) dS(T)$$

where $f^*(S(T))$ equals the risk-neutral probability density function for the stock price at time T, given the observed stock price is $S(t)$ at time $t < T$ and the integral is the expected value of the the payoff on the call in the risk neutral environment.

Example 69.2
Find the premium of a European put in the risk-neutral environment that satisfies equation (69.1).

Solution.
Since the payoff of the put is $K - S(T)$ for $S(T) < K$, the expected payoff in the risk-neutral environment is:

$$\int_0^K [K - S(T)] f^*(S(T)) dS(T).$$

Discounting back to time t at the risk-free rate, the put premium is:

$$P = e^{-r(T-t)} \int_0^K [K - S(T)] f^*(S(T)) dS(T) \blacksquare$$

Remark 69.1
The above result shows that the discounted risk-neutral expectations are prices of derivatives. This is also true for risk-neutral probabilities. See pp. 692-3 of [1].

Practice Problems

Problem 69.1
Given the following information about a European option: $r = 0.08, \delta = 0.02, \sigma = 0.3, V_t = -0.405, V_S = 0.624, V_{SS} = 0.245, S = 5$. Find the expected change per unit time of the option value with respect to the risk-neutral distribution.

Problem 69.2
Given the following information about a European option: $\alpha = 0.05, \delta = 0.01, \sigma = 0.15, V_t = -0.105, V_S = 0.624, V_{SS} = 0.245, S = 5$. Find the expected change per unit time of the option value with respect to the actual probability distribution.

Problem 69.3
The expected change per unit time of a European option with respect to the risk-neutral distribution is found to be 2.11. Assuming a risk-free interest rate of 8%, find the value of the option.

Problem 69.4
Given the following information about a European option: $r = 0.08, \delta = 0.02, \sigma = 0.3, V_S = 0.123, V_{SS} = 0.231, S = 25$. Find the value of V_t if the expected change of the option value with respect to the risk-neutral distribution is 2.11.

Problem 69.5
Find the risk-free interest rate r if the expected change of an option value with respect to the risk-neutral distribution is 1.72 and with option value of 18.1006.

Problem 69.6
Suppose that the instantaneous expected return on an option is 0.3578 and the option value is 9.3625. Find the actual expected change in the option price.

Binary Options

Binary options, also known as digital options, were first introduced in Section 67. In this chapter, we discuss a family of binary options known as **all-or-nothing options**. They include two types of options: cash-or-nothing and asset-or-nothing.

70 Cash-or-Nothing Options

A **cash-or-nothing** call option with strike K and expiration T is an option that pays its owner \$b if $S(T) > K$ and zero otherwise. In the Black-Scholes framework, $Pr(S(T) > K) = N(d_2)$ (See Section 49) where

$$d_2 = \frac{\ln(S(t)/K) + (r - \delta - 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

From Remark 69.1, the price of this call is just the discounted risk-neutral probability. That is,

$$\text{CashCall} = be^{-r(T-t)}N(d_2).$$

Example 70.1
Find the price of a cash-or-nothing put.

Solution.
A cash-or-nothing put option with strike K and expiration T is an option that pays its owner \$b if $S(T) < K$ and zero otherwise. In the Black-Scholes framework, $Pr(S(T) < K) = N(-d_2) = 1 - N(d_2)$ (See Section 49). The price of this put is the discounted risk-neutral probability. That is,

$$\text{CashPut} = be^{-r(T-t)}N(-d_2).$$

Remark 70.1
Note that the pricing formula of a cash-or-nothing options satisfy the Black-Scholes equation. See Example 67.2. Also, note that if $b = K$ we obtain the second term of the Black-Scholes formula.

Example 70.2
Find an expression for $\text{CashCall} + \text{CashPut}$.

Solution.
We have

$$\text{CashCall} + \text{CashPut} = be^{-r(T-t)}N(d_2) + be^{-r(T-t)}(1 - N(d_2)) = be^{-r(T-t)} \blacksquare$$

We next examine the delta of cash-or-nothing options. For a cash-or-nothing call option we have

$$\begin{aligned}\Delta_{\text{CashCall}} =& \frac{\partial}{\partial S}(be^{-r(T-t)}N(d_2)) = be^{-r(T-t)}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial S}\\ =& be^{-r(T-t)} \cdot \frac{e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}} \cdot \frac{1}{S\sigma\sqrt{T-t}}.\end{aligned}$$

Example 70.3
An option will pay its owner \$10 three months from now if the stock price at that time 3 is greater than \$40. You are given: $S_0 = 40, \sigma = 30\%, r = 8\%$, and $\delta = 2\%$. Find the delta of this option.

Solution.

$$\Delta_{\text{CashCall}} = 10e^{-0.08\times 0.25}\frac{e^{-\frac{0.025^2}{2}}}{\sqrt{2\pi}} \cdot \frac{1}{40(0.30)\sqrt{0.25}} = 0.6515 \blacksquare$$

The delta for a cash-or-nothing put is

$$\begin{aligned}\Delta_{\text{CashPut}} =& \frac{\partial}{\partial S}(be^{-r(T-t)}N(-d_2)) = be^{-r(T-t)}\frac{\partial(1-N(d_2)}{\partial S}\\ =& -be^{-r(T-t)} \cdot \frac{e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}} \cdot \frac{1}{S\sigma\sqrt{T-t}}.\end{aligned}$$

Example 70.4
An option will pay its owner \$10 three months from now if the stock price at that time 3 is less than \$40. You are given: $S_0 = 40, \sigma = 30\%, r = 8\%$, and $\delta = 2\%$. Find the delta of this option.

Solution.

$$\Delta_{\text{CashPut}} = -10e^{-0.08\times 0.25}\frac{e^{-\frac{0.025^2}{2}}}{\sqrt{2\pi}} \cdot \frac{1}{40(0.30)\sqrt{0.25}} = -0.6515 \blacksquare$$

Delta-Heding of Cash-or-Nothing Options
Recall the formula for delta in the case of a cash-or-nothing call option:

$$\Delta_{\text{CashCall}} = be^{-r(T-t)} \cdot \frac{e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}} \cdot \frac{1}{S\sigma\sqrt{T-t}}.$$

Since

$$d_2 = \frac{\ln{(S/K)}}{S\sigma\sqrt{T-t}} + \frac{(r-\delta-0.5\sigma^2)\sqrt{T-t}}{S\sigma}$$

we see that as t approaches T, $d_2 \to 0$ for $S_T = K$ and therefore Δ_{CashCall} approaches infinity. For $S_T \neq K$, we have $d_2 \to \pm\infty$ so that Δ_{CashCall}

approaches 0. Thus, if a cash-or-nothing option is getting very close to expiration, and the current stock price is close to K, then a very small movement in the stock price will have a very large effect on the payoff of the option. The delta for an at-the-money cash-or-nothing option gets very big as it approaches expiration. Thus, it is very hard to hedge cash-or-nothing options. That is why cash-or-nothing options are rarely traded.

Example 70.5
An option will pay its owner \$40 three months from now if the stock price at that time 3 is greater than \$40. You are given: $S_0 = 40, \sigma = 30\%, r = 8\%, K = \$40$ and $\delta = 0\%$. Find payoff diagram of this cash-or-nothing call as function of S_T. Compare this diagram with an ordinary call option.

Solution.
The diagrams are shown in Figure 70.1 ■

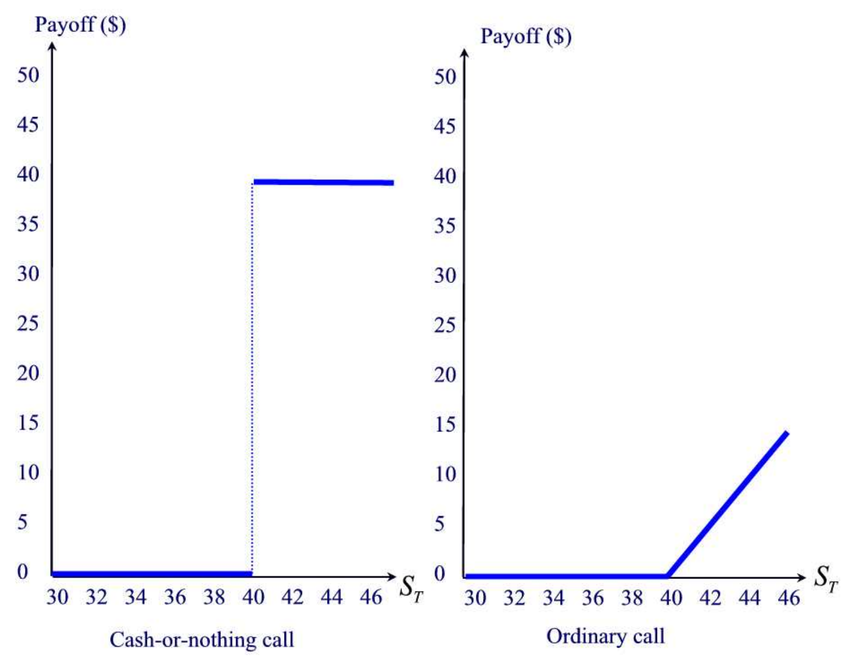


Figure 70.1

Remark 70.2
An ordinary put or call is easier to hedge because the payoff is continuous—there is no discrete jump at the strike price as the option approaches expiration.

Example 70.6
A **collect-on-delivery call** (COD) costs zero initialy, with the payoff at expiration being 0 if $S < K$,and $S - K - P$ if $S \geq K$. The problem in valuing the option is to determine P, the amount the option holder pays if the option is in-the-money at expiration. The premium P is determined once for all when the option is created. Find a formula for P.

Solution.
The payoff on the collect-on-delivery call is that from an ordinary call with with strike price K, minus that from a cash-or-nothing call with with strike K that pays P.
Since we pay nothing initially for the collect-on-delivery call, the initial value of the portfolio must be zero. Thus, the premium on the cash-or-nothing option is equal to the premium of the ordinary European call at time $t = 0$. This implies the equation

$$Pe^{-r(T-t)}N(d_2) = Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2).$$

Solving this equation for P we find

$$P = \frac{Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)}{e^{-r(T-t)}N(d_2)} \blacksquare$$

Practice Problems

Problem 70.1
An option will pay its owner \$10 three years from now if the stock price at that time is greater than \$40. You are given: $S_0 = 40, \sigma = 30\%, r = 8\%$, and $\delta = 2\%$. What is the premium for this option?

Problem 70.2
An option will pay its owner \$10 three years from now if the stock price at that time 3 is less than \$40. You are given: $S_0 = 40, \sigma = 30\%, r = 8\%$, and $\delta = 2\%$. What is the premium for this option?

Problem 70.3
Option A pays \$10 three years from now if the stock price at that time 3 is greater than \$40. Option B pays \$40 three years from now if the stock price at that time 3 is less than \$40. You are given: $S_0 = 40, \sigma = 30\%, r = 8\%$, and $\delta = 0\%$. What is the value of the two options?

Problem 70.4
The premium for a cash-or-nothing call that pays one is 0.5129. The premium for a similar cash-or-nothing put is 0.4673. If the time until expiration is three months, what is r?

Problem 70.5
The premium for a cash-or-nothing call that pays one is 0.5129. The premium for a similar cash-or-nothing put is 0.4673. The continuously-compounded risk-free interest rate is 0.08. If the time until expiration is T years, what is T?

Problem 70.6
An option will pay its owner \$40 three months from now if the stock price at that time 3 is less than \$40. You are given: $S_0 = 40, \sigma = 30\%, r = 8\%, K =$ \$40 and $\delta = 0\%$. Find payoff diagram of this cash-or-nothing put as function of S_T. Compare this diagram with an ordinary put option.

Problem 70.7
A **collect-on-delivery call** (COD) costs zero initialy, with the payoff at expiration being 0 if $S < 90$,and $S - K - P$ if $S \geq 90$. You are also given the following: $r = 0.05, \delta = 0.01, \sigma = 0.4, S = \$70 T - t = 0.5$. Find P.

Problem 70.8
A cash-or-nothing call that will pay \$1 is at-the-money, with one day to expiration. You are given: $\sigma = 20\%, r = 6\%, \delta = 0\%$.
A market maker has written 1000 of these calls. He buys shares of stock in order to delta hedge his position. How much does he pay for these shares of stock?

Problem 70.9 ‡
Assume the Black-Scholes framework. You are given:
(i) $S(t)$ is the price of a nondividend-paying stock at time t.
(ii) $S(0) = 10$
(iii) The stock's volatility is 20%.
(iv) The continuously compounded risk-free interest rate is 2%.
At time $t = 0$, you write a one-year European option that pays 100 if $[S(1)]^2$ is greater than 100 and pays nothing otherwise.
You delta-hedge your commitment.
Calculate the number of shares of the stock for your hedging program at time $t = 0$.

Problem 70.10 ‡
Assume the Black-Scholes framework. For a European put option and a European gap call option on a stock, you are given:
(i) The expiry date for both options is T.
(ii) The put option has a strike price of 40.
(iii) The gap call option has strike price 45 and payment trigger 40.
(iv) The time-0 gamma of the put option is 0.07.
(v) The time-0 gamma of the gap call option is 0.08.
Consider a European cash-or-nothing call option that pays 1000 at time T if the stock price at that time is higher than 40.
Find the time-0 gamma of the cash-or-nothing call option. Hint: Consider a replicating portfolio of the cash-or-nothing option.

71 Asset-or-Nothing Options

An **asset-or-nothing** call option with strike K and expiration T is an option that gives its owner one unit of a share of the stock if $S(T) > K$ and zero otherwise. In the Black-Scholes framework, the value of this option in risk-neutral environment is[1]

$$\text{AssetCall} = e^{-r(T-t)}E[S_t|S_t > K]Pr(S_t > K) = S_0e^{-r(T-t)}e^{(r-\delta)(T-t)}\frac{N(d_1)}{N(d_2)}N(d_2) = S_0e^{-\delta(T-t)}$$

where

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Note that the price given above is the first term of the Black-Scholes formula for a European option.

Example 71.1
An option will give its owner a share of stock three months from now if the stock price at that time is greater than \$40. You are given: $S_0 = \$40, \sigma = 0.30, r = 0.08, \delta = 0$. What is the premium for this option?

Solution.
We have

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln(40/40) + (0.08 - 0 + 0.5(0.3)^2)(0.25)}{0.30\sqrt{0.25}} = 0.208.$$

Thus, $N(d_1) = 0.582386$ and

$$\text{AssetCall} = S_0e^{-\delta(T-t)}N(d_1) = 40e^{-0\times 0.25}(0.582386) = \$23.29 \blacksquare$$

Example 71.2
Find the price of an asset-or-nothing put.

Solution.
An asset-or-nothing put option with strike K and expiration T is an option that gives its owner to receive one unit of a share of stock if $S(T) < K$

[1] See Section 50.

and zero otherwise. In the Black-Scholes framework, the time 0 value of this option is

$$\text{AssetPut} = e^{-r(T-t)}E[S_t|S_t < K]Pr(S_t < K) = S_0e^{-r(T-t)}e^{(r-\delta)(T-t)}\frac{N(-d_1)}{N(-d_2)}N(-d_2) = S_0e^{-\delta(T-t)}N(-d$$

which is the second term in the Black-Scholes formula for a European put ■

Example 71.3
An option will gives its owner to receive a share of stock three years from now if the stock price at that time is less than $40. You are given $S_0 = 40, \sigma = 0.03, r = 0.08, \delta = 0$. What is the premium for this option?

Solution.
We have

$$d_1 = \frac{\ln{(S/K)} + (r - \delta + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln{(40/40)} + (0.08 - 0 + 0.5(0.3)^2)(0.25)}{0.30\sqrt{0.25}} = 0.208.$$

Thus, $N(d_1) = 0.582386$ and

$$\text{AssetPut} = S_0e^{-\delta(T-t)}N(-d_1) = 40e^{-0\times 0.25}(1 - 0.582386) = \$16.70 \blacksquare$$

In monetary value, the owner of a European call gets $S_T - K$, if $S_T > K$. Therefore, a European call is equivalent to the difference between an asset-or-nothing call and a cash-or-nothing call that pays K. That is,

Premium of a European call = Premium of an asset-or-nothing call −
Premium of a cash-or-nothing call that pays K

or

$$\text{BSCall} = Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

which is the Black-Scholes formula for a European call. Likewise, the owner of a European put gets $K - S_T$, if $S_T < K$. Therefore, a European put is equivalent to the difference between a cash-or-nothing put that pays K and an asset-or-nothing put. That is,

Premium of a European put = Premium of a cash-or-nothing put that pays
K − Premium of an asset-or-nothing put

or

$$\text{BSPut} = Ke^{-r(T-t)}N(-d_2) - Se^{-\delta(T-t)}N(-d_1)$$

which is the Black-Scholes formula for a European put.
Next, let's examine the delta of an asset-or-nothing call. We have

$$\begin{aligned}\Delta_{AssetCall} =& \frac{\partial \text{AssetCall}}{\partial S}\\ =& e^{-\delta(T-t)}N(d_1) + Se^{-\delta(T-t)}\frac{\partial N(d_1)}{\partial S}\\ =& e^{-\delta(T-t)}N(d_1) + Se^{-\delta(T-t)}\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S}\\ =& e^{-\delta(T-t)}N(d_1) + Se^{-\delta(T-t)} \cdot \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \cdot \frac{1}{S\sigma\sqrt{T-t}}\\ =& e^{-\delta(T-t)}N(d_1) + e^{-\delta(T-t)} \cdot \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \cdot \frac{1}{\sigma\sqrt{T-t}}\end{aligned}$$

Example 71.4
An option will give its owner a share of stock three months from now if the stock price at that time is greater than \$40. You are given: $S_0 = \$40, \sigma = 0.30, r = 0.08, \delta = 0$. Find the delta of this option.

Solution.
We have

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln(40/40) + (0.08 - 0 + 0.5(0.3)^2)(0.25)}{0.30\sqrt{0.25}} = 0.208.$$

Thus, $N(d_1) = 0.582386$ and

$$\begin{aligned}\Delta_{AssetCall} =& e^{-\delta(T-t)}N(d_1) + e^{-\delta(T-t)} \cdot \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \cdot \frac{1}{\sigma\sqrt{T-t}}\\ =& 0.582386 + \frac{e^{-\frac{0.208^2}{2}}}{\sqrt{2\pi}} \cdot \frac{1}{0.30\sqrt{0.25}}\\ =& 3.185 \blacksquare\end{aligned}$$

Example 71.5
Express the premium of a gap call option with payoff $S_T - K_1$ if $S_T > K_2$ in terms of asset-or-nothing call option and cash-or-nothing call.

Solution.
The owner of a gap call gets $S_T - K_1$, if $S_T > K_2$. Therefore, a gap call is equivalent to the difference between an asset-or-nothing call with $K = K_2$, and a cash-or-nothing call with $K = K_2$ that pays K_1.
Therefore, as discussed previously, the premium of a gap call is

$$\text{AssetCall} - K_1\text{CashCall} = Se^{-\delta(T-t)}N(d_1) - K_1e^{-r(T-t)}N(d_2)$$

where d_1 and d_2 are calculated using $K = K_2$ ■

Remark 71.1
For the same reason as with cash-or-nothing options, the delta for an at-the-money asset-or-nothing option gets very big as it approaches expiration. Thus, it is very hard to hedge asset-or-nothing options. That is why asset-or-nothing options are rarely traded.

Practice Problems

Problem 71.1
An option will give its owner a share of stock three years from now if the stock price at that time is greater than \$100. You are given: $S_0 = \$70, \sigma = 0.25, r = 0.06, \delta = 0.01$. What is the premium for this option?

Problem 71.2
An option will give its owner a share of stock three years from now if the stock price at that time is less than \$100. You are given: $S_0 = \$70, \sigma = 0.25, r = 0.06, \delta = 0.01$. What is the premium for this option?

Problem 71.3
Find an expression for $\text{AssetCall} + \text{AssetPut}$.

Problem 71.4
An option will give its owner a share of stock three years from now if the stock price at that time is greater than \$100. You are given: $S_0 = \$70, \sigma = 0.25, r = 0.06, \delta = 0.01$. What is the delta of this option?

Problem 71.5
Find the delta of an asset-or-nothing put.

Problem 71.6
An option will give its owner a share of stock three years from now if the stock price at that time is less than \$100. You are given: $S_0 = \$70, \sigma = 0.25, r = 0.06, \delta = 0.01$. What is the delta of this option?

Problem 71.7
Express the premium of a gap put option with payoff $K_1 - S_T$ if $S_T < K_2$ in terms of asset-or-nothing put option and cash-or-nothing put.

Problem 71.8
The premium for a cash-or-nothing call is 0.5129. The premium for a similar asset-or-nothing call is 23.30. If $K = 40$, what is the premium for a similar European Call.

Problem 71.9
The premium for a European call is 2.7848. The premium for a similar cash-or-nothing call is 0.5129. The premium for a similar asset-or-nothing call is 23.30. Determine the strike price K.

Problem 71.10
The premium for a asset-or-nothing call is 47.85. The premium for a similar asset-or-nothing put is 40.15. The stock pays no dividends. What is the current stock price?

Problem 71.11 ‡
Your company has just written one million units of a one-year European asset-or-nothing put option on an equity index fund.
The equity index fund is currently trading at 1000. It pays dividends continuously at a rate proportional to its price; the dividend yield is 2%. It has a volatility of 20%.
The option's payoff will be made only if the equity index fund is down by more than 40% at the end of one year. The continuously compounded risk-free interest rate is 2.5%
Using the Black-Scholes model, determine the price of the asset-or-nothing put options.

72 Supershares

In this section, we discuss a type of binary options known as supershares. In an article published in 1976, Nils Hakansson proposed a financial intermediary that would hold an underlying portfolio and issue claims called **supershares** against this portfolio to investors. A supershare is a security, which on its expiration date entitles its owner to a given dollar value proportion of the assets of the underlying portfolio, provided the value of those assets on that date lies between a lower value K_L and an upper value K_U. Otherwise, the supershare expires worthless. That is, the payoff of a supershare is:

$$\text{Payoff} = \begin{cases} 0 & S_T < K_L \\ \frac{S_T}{K_L} & K_L \leq S_T \leq K_U \\ 0 & S_T > K_U \end{cases}$$

Consider a portfolio that consists of $\frac{1}{K_1}$ units of long asset-or-nothing call options with strike K_1 and $\frac{1}{K_1}$ units of short asset-or-nothing call options with strike price K_2. Then the payoff at expiration of this portfolio is exactly the payoff of the supershare mentioned above with $K_L = K_1$ and $K_U = K_2$ with $K_1 < K_2$. Hence,

$$\text{Premium of the supershare} = \tfrac{1}{K_1}\left(AssetCall(K_1) - AssetCall(K_2)\right)$$

That is,

$$SS = \frac{S}{K_1}e^{-\delta(T-t)}[N(d_1) - N(d_1')]$$

where

$$d_1 = \frac{\ln{(S/K_1)} + (r - \delta + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_1' = \frac{\ln{(S/K_2)} + (r - \delta + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Example 72.1
Find the premium of a supershare for the following data: $S = 100, r = 0.1, \delta = 0.05, \sigma = 0.2, K_1 = 100, K_2 = 105$, and $T - t = 0.5$ years.

Solution.
We have

$$d_1 = \frac{\ln{(S/K_1)} + (r - \delta + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln{(100/100)} + (0.1 - 0.05 + 0.5(0.2)^2)(0.5)}{0.2\sqrt{0.5}} = 0.24$$

and

$$d_1' = \frac{\ln{(S/K_2)} + (r - \delta + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln{(100/105)} + (0.1 - 0.05 + 0.5(0.2)^2)(0.5)}{0.2\sqrt{0.5}} = -0.097511.$$

Thus, $N(0.247487) = 0.597734$ and $N(-0.097511) = 0.461160$. The premium of the supershare is

$$SS = \frac{S}{K_1}e^{-\delta(T-t)}[N(d_1) - N(d_1')] = \frac{100}{100}e^{-0.05\times 0.5}[0.597734 - 0.461160] = \$0.1332 \blacksquare$$

Practice Problems

Problem 72.1
Consider a supershare based on a portfolio of nondividend paying stocks with a lower strike of 350 and an upper strike of 450. The value of the portfolio on November 1, 2008 is 400. The risk-free rate is 4.5% and the volatility is 18%. Using this data, calculate the values of d_1 and $N(d_1)$ for $T-t=0.25$.

Problem 72.2
Consider a supershare based on a portfolio of nondividend paying stocks with a lower strike of 350 and an upper strike of 450. The value of the portfolio on November 1, 2008 is 400. The risk-free rate is 4.5% and the volatility is 18%. Using this data, calculate the values of d_1' and $N(d_1')$ for $T-t=0.25$.

Problem 72.3
Consider a supershare based on a portfolio of nondividend paying stocks with a lower strike of 350 and an upper strike of 450. The value of the portfolio on November 1, 2008 is 400. The risk-free rate is 4.5% and the volatility is 18%. Using this data, calculate the price of the supershare option on February 1, 2009.

Problem 72.4
Given the following information regarding a supershare: $S=100, K_1=95, K_2=110, r=0.1, \delta=0.05, \sigma=0.2$, and $T-t=0.5$. Calculate d_1 and $N(d_1)$.

Problem 72.5
Given the following information regarding a supershare: $S=100, K_1=95, K_2=110, r=0.1, \delta=0.05, \sigma=0.2$, and $T-t=0.5$. Calculate d_1' and $N(d_1')$.

Problem 72.6
Given the following information regarding a supershare: $S=100, K_1=95, K_2=110, r=0.1, \delta=0.05, \sigma=0.2$, and $T-t=0.5$. Find the premium of this supershare.

Problem 72.7
Using the previous problem and the results of Example 72.1, what can you conclude about the value of supershares and the value K_2-K_1?

Interest Rates Models

In this chapter we examine pricing models for derivatives with underlying assets either bonds or interest rates. That is, pricing models for bond options and interest rate options. As with derivatives on stocks, prices of interest rates or bonds derivatives are characterized by a partial differential equation that is essentially the same as the Black-Scholes equation.

73 Bond Pricing Model with Arbitrage Opportunity

In this section, we examine a bond pricing model that gives rise to arbitrage opportunities. We examine the hedging of one bond with other bonds. The type of hedging that we consider in this section is referred to as **duration-hedging.**

By analogy to modeling stocks, we will assume that the time-t bond price $P(r,t,T)$ follows an Itô process

$$\frac{dP}{P}=\alpha(r,t)dt-q(r,t)dZ(t). \tag{73.1}$$

Also, we will assume that the short-term interest rate follows the Itô process

$$dr=a(r)dt+\sigma(r)dZ(t). \tag{73.2}$$

We first examine a bond pricing model based on the assumption that the yield curve is flat.[1] We will show that this model gives rise to arbitrage opportunities.

Under the flat yield curve assumption, the price of zero-coupon bonds is given by

$$P(r,t,T)=e^{-r(T-t)}.$$

The purchase of a T_2-year zero-coupon bond ($T_2<T_1$) can be delta-hedged by purchasing N T_1-year zero-coupon bonds and lending $W=-P(r,t,T_2)-N\times P(r,t,T_1)$ at the short-term interest rate r. The number W can be positive or negative. If $W>0$ then we lend. If $W<0$ then we borrow. Also, N can be positive or negative. If $N<0$ (i.e., $t<T_2<T_1$) then we sell rather than buy the T_2-year bond.

Now, let $I(t)$ be the value of the delta-hedged position at time t. Then we have

$$I(t)=N\times P(r,t,T_1)+P(r,t,T_2)+W=0.$$

Since W is invested in short-term (zero-duration) bonds, at the next instant we must have

$$dW(t)=rW(t)dt.$$

[1] In finance, the **yield curve** is the relation between the interest rate (or cost of borrowing) and the time to maturity of the debt for a given borrower. A **flat yield curve** is a yield curve where the interest rate is the same for all maturities.

Also, using Itô's lemma we have

$$\begin{aligned}
dI(t) =& NdP(r,t,T_1) + dP(r,t,T_2) + dW \\
=& N\left(P_r(r,t,T_1)dr + \frac{1}{2}P_{rr}(r,t,T_1)(dr)^2 + P_t(r,t,T_1)dt\right) \\
&+ \left(P_r(r,t,T_2)dr + \frac{1}{2}P_{rr}(r,t,T_2)(dr)^2 + P_t(r,t,T_2)dt\right) + dW(t) \\
=& N\left(-(T_1-t)P(r,t,T_1)dr + \frac{1}{2}(T_1-t)^2\sigma^2 P(r,t,T_1)dt + rP(r,t,T_1)dt\right) \\
&+ \left(-(T_2-t)P(r,t,T_2)dr + \frac{1}{2}(T_2-t)^2\sigma^2 P(r,t,T_2)dt + rP(r,t,T_2)dt\right) + rW(t)dt
\end{aligned}$$

We select N in such a way to eliminate the effect of interest rate change, dr, on the value of the portfolio. Thus, we set

$$N = -\frac{(T_2-t)P(r,t,T_2)}{(T_1-t)P(r,t,T_1)},\ t \neq T_1, t < T_2.$$

Note that the ratio $\frac{T_2-t}{T_1-t}$ is the ratio of the duration of the bond maturing at T_2 to the duration of the bond maturing at T_1. With this choice of N, we have

$$\begin{aligned}
dI(t) =& N\left(\frac{1}{2}(T_1-t)^2\sigma^2 P(r,t,T_1)dt + rP(r,t,T_1)dt\right) \\
&+ \left(\frac{1}{2}(T_2-t)^2\sigma^2 P(r,t,T_2)dt + rP(r,t,T_2)dt\right) + rW(t)dt \\
=& [NP(r,t,T_1) + P(r,t,T_2) + W(t)]rdt + \frac{1}{2}\sigma^2\left(N(T_1-t)^2P(r,t,T_1) + (T_2-t)^2P(r,t,T_2)\right)dt \\
=& \frac{1}{2}\sigma^2\left(-\frac{(T_2-t)P(r,t,T_2)}{(T_1-t)P(r,t,T_1)} \times (T_1-t)^2P(r,t,T_1)dt + (T_2-t)^2P(r,t,T_2)dt\right) \\
=& \frac{1}{2}\sigma^2(T_2-T_1)(T_2-t)P(r,t,T_2)dt \neq 0.
\end{aligned}$$

Thus, this gives rise to an arbitrage since the delta-hedged portfolio is risk-free and has zero investment.

With the above pricing model, we encounter two difficulties with pricing bonds:

- A casual specified model may give rise to arbitrage opportunities.
- In general, hedging a bond portfolio based on duration does not result in a perfect hedge.

Example 73.1
Suppose the yield curve is flat at 8%. Consider 3−year and 6−year zero-coupon bonds. You buy one 3−year bond and sell an appropriate quantity N of the 6−year bond to duration-hedge the position. Determine N.

Solution.
We are given that $t = 0, T_1 = 6, T_2 = 3$, and $r = 0.08$. Thus, $P(0.08, 0, 3) = e^{-0.08(3-0)} = \$0.78663$ and $P(0.08, 0, 6) = e^{-0.08(6-0)} = \$0.61878$. The number of 6−year bonds that must be sold to duration-hedge the position is

$$N = -\frac{(T_2 - t)P(r, t, T_2)}{(T_1 - t)P(r, t, T_1)} = -\frac{(3 - 0) \times 0.78663}{(6 - 0) \times 0.61878} = -0.63562 \blacksquare$$

Example 73.2
Suppose the yield curve is flat at 8%. Consider 3−year and 6−year zero-coupon bonds. You buy one 3−year bond and sell an appropriate quantity N of the 6−year bond to duration-hedge the position. What is the total cost of the duration-hedge strategy? How much will you owe in one day?

Solution.
The total cost of the duration-hedge strategy is

$$W(0) = -P(r, t, T_2) - N \times P(r, t, T_1) = -0.78663 + 0.63562 \times 0.61878 = -\$0.39332.$$

Since $W < 0$, we have to borrow \$0.39332 at the short-term rate of 8% to finance the position. After one day, we owe the lender $0.39332e^{0.08/365} = \$0.3934$ $\blacksquare$

Example 73.3
Suppose the yield curve is flat at 8%. Consider 3−year and 6−year zero-coupon bonds. You buy one 3−year bond and sell an appropriate quantity N of the 6−year bond to duration-hedge the position. Suppose that the yield curve can move up to 8.25% or down to 7.75% over the course of one day. Do you make or lose money on the hedge?

Solution.
The hedged position has an initial value of \$0. After one day, the short-term rate increases to 8.25% and the new value of the position is

$$e^{-0.0825\left(3-\frac{1}{365}\right)} - 0.63562e^{-0.0825\left(6-\frac{1}{365}\right)} - 0.39332e^{\frac{0.08}{365}} = -\$0.00002255.$$

The hedged position has an initial value of \$0. After one day, the short-term rate decreases to 7.75% and the new value of the position is

$$e^{-0.0775\left(3-\frac{1}{365}\right)} - 0.63562e^{-0.0775\left(6-\frac{1}{365}\right)} - 0.39332e^{\frac{0.08}{365}} = -\$0.00002819.$$

Thus, there is a lose of about \$0.00003 ■

Practice Problems

Problem 73.1
Suppose the yield curve is flat at 8%. Consider 2−year and 7−year zero-coupon bonds. You buy one 2−year bond and sell an appropriate quantity N of the 7−year bond to duration-hedge the position. Determine the time-0 prices of the two bonds.

Problem 73.2
Suppose the yield curve is flat at 8%. Consider 2−year and 7−year zero-coupon bonds. You buy one 2−year bond and sell an appropriate quantity N of the 7−year bond to duration-hedge the position. Determine the quantity of the 7−year bonds to be purchased to duration-hedge the position.

Problem 73.3
Suppose the yield curve is flat at 8%. Consider 2−year and 7−year zero-coupon bonds. You buy one 2−year bond and sell an appropriate quantity N of the 7−year bond to duration-hedge the position. Find the total cost of financing this position.

Problem 73.4
Suppose the yield curve is flat at 8%. Consider 2−year and 7−year zero-coupon bonds. You buy one 2−year bond and sell an appropriate quantity N of the 7−year bond to duration-hedge the position. Suppose that the yield curve can move up to 8.5% or down to 7.5% over the course of one day. Do you make or lose money on the hedge?

Problem 73.5
Suppose the yield curve is flat at 6%. Consider 4−year 5%-coupon bond and an 8−year 7%-coupon bond. All coupons are annual. You buy one 4−year bond and purchase an appropriate quantity N of the 8−year bond to duration-hedge the position. Determine the time-0 prices of the two bonds.

Problem 73.6
Suppose the yield curve is flat at 6%. Consider 4−year 5%-coupon bond and an 8−year 7%-coupon bond. All coupons are annual. You buy one 4−year bond and purchase an appropriate quantity N of the 8−year bond to duration-hedge the position. Determine the (modified) durations of each bond. Hint: The modified duration is given by $D = \frac{\frac{\partial P}{\partial t}}{P}$.

Problem 73.7

Suppose the yield curve is flat at 6%. Consider $4-$year 5%-coupon bond and an $8-$year 7%-coupon bond. All coupons are annual. You buy one $4-$year bond and purchase an appropriate quantity N of the $8-$year bond to duration-hedge the position. Determine the number of $8-$year bonds to be purchased for this position.

Problem 73.8

Suppose the yield curve is flat at 6%. Consider $4-$year 5%-coupon bond and an $8-$year 7%-coupon bond. All coupons are annual. You buy one $4-$year bond and purchase an appropriate quantity N of the $8-$year bond to duration-hedge the position. What is the total cost of the duration-hedge strategy? How much will you owe in one day?

Problem 73.9

Suppose the yield curve is flat at 6%. Consider $4-$year 5%-coupon bond and an $8-$year 7%-coupon bond. All coupons are annual. You buy one $4-$year bond and purchase an appropriate quantity N of the $8-$year bond to duration-hedge the position. Suppose that the yield curve can move up to 6.25% over the course of one day. Do you make or lose money on the hedge?

Problem 73.10

Suppose the yield curve is flat at 6%. Consider $4-$year 5%-coupon bond and an $8-$year 7%-coupon bond. All coupons are annual. You buy one $4-$year bond and purchase an appropriate quantity N of the $8-$year bond to duration-hedge the position. Suppose that the yield curve can move down to 5.75% over the course of one day. Do you make or lose money on the hedge?

74 A Black-Scholes Analogue for Pricing Zero-Coupon Bonds

In this section, we continue studying the bond-hedging problem with a general pricing model $P(r,t,T)$ rather than the specific pricing model of the previous section.
At the next instant, the change in the bond's price is

$$\begin{aligned} dP(r,t,T) =& P_r(r,t,T)dr + \frac{1}{2}P_{rr}(r,t,T)(dr)^2 + P_t(r,t,T)dt \\ =& \left[a(r)P_r(r,t,T) + \frac{1}{2}P_{rr}(r,t,T)\sigma(r)^2 + P_t(r,t,T)\right]dt + P_r(r,t,T)\sigma(r)dZ(t) \end{aligned}$$

Define

$$\begin{aligned} \alpha(r,t,T) =& \frac{1}{P(r,t,T)}\left[a(r)P_r(r,t,T) + \frac{1}{2}P_{rr}(r,t,T)\sigma(r)^2 + P_t(r,t,T)\right] \\ q(r,t,T) =& -\frac{1}{P(r,t,T)}P_r(r,t,T)\sigma(r) \end{aligned}$$

In this case, we can describe the change in the price of the bond by the equation

$$\frac{dP(r,t,T)}{P(r,t,T)} = \alpha(r,t,T)dt - q(r,t,T)dZ(t).$$

Thus, the instantaneous rate of return on a bond maturing at time T has a mean $\alpha(r,t,T)$ and standard deviation $q(r,t,T)$. That is, the expected return on the bond over the next instant is $\alpha(r,t,T)$.[1] Note that $P_r(r,t,T) < 0$ so that $q(r,t,T) > 0$.[2]
Now we consider again the delta-hedged portofolio that consists of buying a T_2-year zero-coupon bond, buying N T_1-year zero-coupon bonds, and financing the strategy by lending $W = -P(r,t,T_2) - N \times P(r,t,T_1)$. Let $I(t)$ denote the value of the portfolio at time t. Then

$$I(t) = N \times P(r,t,T_1) + P(r,t,T_2) + W = 0$$

[1]We will assume that $\alpha(r,t,T) > r$ since a bond is expected to earn more than the risk-free rate due to the fact that owning a bond is riskier than lending at the risk-free rate.

[2]Bond prices decrease when interest rates increase because the fixed interest and principal payments stated in the bond will become less attractive to investors.

and

$$\begin{aligned}dI(t) =&N[\alpha(r,t,T_1)dt - q(r,t,T_1)dZ(t)]P(r,t,T_1)\\&+[\alpha(r,t,T_2)dt - q(r,t,T_2)dZ(t)]P(r,t,T_2) + rW(t)dt\end{aligned}$$

For the short-term interest rate to be the only source of uncertainty, that is, for the bond price to be driven only by the short-term interest rate, we choose N such that the dZ terms in the previous equation are eliminated. This occurs when

$$N = -\frac{P(r,t,T_2)q(r,t,T_2)}{P(r,t,T_1)q(r,t,T_1)} = -\frac{P_r(r,t,T_2)}{P_r(r,t,T_1)}.$$

In this case, we have

$$\begin{aligned}dI(t) =&[N\alpha(r,t,T_1)P(r,t,T_1) + \alpha(r,t,T_2)P(r,t,T_2) + rW]dt\\=&[-\alpha(r,t,T_1)\times\frac{P(r,t,T_2)q(r,t,T_2)}{q(r,t,T_1)} + \alpha(r,t,T_2)P(r,t,T_2) - rP(r,t,T_2)\\&+r\frac{P(r,t,T_2)q(r,t,T_2)}{q(r,t,T_1)}]dt\\=&\{q(r,t,T_2)[r-\alpha(r,t,T_1)] + q(r,t,T_1)[\alpha(r,t,T_2)-r]\}\times\frac{P(r,t,T_2)}{q(r,t,T_1)}dt\end{aligned}$$

Since the cost of the portfolio is zero, and its return in not random, to preclude arbitrage we must have $dI(t) = 0$. Thus, we obtain

$$\frac{\alpha(r,t,T_1) - r}{q(r,t,T_1)} = \frac{\alpha(r,t,T_2) - r}{q(r,t,T_2)}.$$

This equation says that the two bonds have the same Sharpe ratio. This is consistent with what we proved about prices of two assets driven by the same random term $dZ(t)$. It follows that all zero-coupon bonds have the same Sharpe ratio regardless of maturity. Denoting the Sharpe ratio for a zero-coupon bond by $\phi(r,t)$ we have

$$\phi(r,t) = \frac{\alpha(r,t,T) - r}{q(r,t,T)}.$$

Substituting the expressions for $\alpha(r,t,T)$ and $q(r,t,T)$ into the Sharpe ratio formula for a zero-coupon bond that matures at time T, we have

$$\begin{aligned}
\phi(r,t) =& \frac{\alpha(r,t,T)-r}{q(r,t,T)}\\
\phi(r,t) =& \frac{\frac{1}{P(r,t,T)}[a(r)P_r(r,t,T)+\frac{1}{2}\sigma(r)^2P_{rr}(r,t,T)+P_t(r,t,T)]-r}{-\frac{1}{P(r,t,T}\sigma(r)P_r(r,t,T)}\\
-\phi(r,t)\sigma(r)P_r(r,t,T) =& a(r)P_r(r,t,T)+\frac{1}{2}\sigma(r)^2P_{rr}(r,t,T)+P_t(r,t,T)-rP(r,t,T)\\
rP(r,t,T) =& \frac{1}{2}\sigma(r)^2P_{rr}(r,t,T)+[a(r)+\sigma(r)\phi(r,t)]P_r(r,t,T)+P_t(r,t,T)
\end{aligned} \tag{74.1}$$

When the short-term interest rate is the only source of uncertainty, equation (74.1) must be satisfied by any zero-coupon bond. Equation (74.1) is analogous to the Black-Scholes equation for Stocks (See Section 67).
As with stocks, we define the Greeks of a bond as follows:

$$\begin{aligned}
\Delta =& \frac{\partial P}{\partial r}\\
\Gamma =& \frac{\partial^2 P}{\partial r^2}\\
\Theta =& \frac{\partial P}{\partial t}
\end{aligned}$$

Example 74.1
Find the Greeks of the bond price model $P(r,t,T)=e^{-r(T-t)}$.

Solution.
We have

$$\begin{aligned}
\Delta =& \frac{\partial P}{\partial r} = -(T-t)e^{-r(T-t)}\\
\Gamma =& \frac{\partial^2 P}{\partial r^2} = (T-t)^2e^{-r(T-t)}\\
\Theta =& \frac{\partial P}{\partial t} = re^{-r(T-t)} \blacksquare
\end{aligned}$$

It follows that the $\Delta\Gamma\Theta$ approximation of the change in price of a zero-coupon bond is given by

$$dP = P_rdr + \frac{1}{2}P_{rr}(dr)^2 + P_tdt = \Delta dr + \frac{1}{2}\Gamma(dr)^2 + \Theta dt.$$

Example 74.2 ‡
For $t \leq T$, let $P(r,t,T)$ be the price at time t of a zero-coupon bond that pays \$1 at time T, if the short-rate at time t is r.
You are given:
(i) $P(r,t,T) = A(t,T)exp[-B(t,T)r]$ for some functions $A(t,T)$ and $B(t,T)$.
(ii) $B(0,3) = 2$.
Based on $P(0.05,0,3)$, you use the delta-gamma approximation to estimate $P(0.03,0,3)$, and denote the value as $P_{Est}(0.03,0,3)$. Find $\frac{P_{Est}(0.03,0,3}{P(0.05,0,3)}$.

Solution.
By the delta-gamma-theta approximation we have

$$P(r(t+dt),t+dt,T)-P(r(t),t,T) \approx [r(t+dt)-r(t)]P_r+\frac{1}{2}[r(t+dt)-r(t)]^2P_{rr}+P_tdt.$$

The question is asking us to use only the delta-gamma approximation so that we will neglect the term P_tdt. Also, we let $r(t+dt) = 0.05, r(t) = 0.03$, and $t + dt \approx t$. Moreover, we have

$$\begin{aligned}
P(r,t,T) =& A(t,T)e^{-B(t,T)r(t)} \\
P_r(r(t),t,T) =& -B(t,T)P(r(t),t,T) \\
P_{rr}(r(t),t,T) =& [B(t,T)]^2P(r(t),t,T)
\end{aligned}$$

Thus,

$$\begin{aligned}
P(0.05,0,3) - P_{Est}(0.03,0,3) =& -(0.05-0.03)B(0,3)P_{Est}(0.03,0,3) \\
&+\frac{1}{2}(0.05-0.03)^2[B(0,3)]^2P_{Est}(0.03,0,3)
\end{aligned}$$

which implies

$$P_{Est}(0.03,0,3)[1-0.02\times 2+0.0002\times 2^2] = P(0.05,0,3).$$

Thus,

$$\frac{P_{Est}(0.03,0,3)}{P(0.05,0,3)} = \frac{1}{1-0.02\times 2+0.0002\times 4} = 1.0408 \blacksquare$$

The **risk-premium** of a zero-coupon bond that matures at time T is the expected return minus the risk-free return which is equal to the Sharpe ratio times the bond's volatility

$$\text{Risk-premium of bond} = \alpha(r,t,T) - r = \phi(r,t)q(r,t,T).$$

Now consider an asset that has a percentage price increase of dr where

$$dr = a(r)dt + \sigma(r)dZ(t).$$

Since the percentage change in price of a zero-coupon bond and the percentage price increase of the asset are driven by the same standard Brownian motion, they must have the same Sharpe ratio:

$$-\frac{a(r) - r}{\sigma(r)} = \frac{\alpha(r,t,T) - r}{q(r,t,T)} = \phi(r,t).$$

where the asset Sharpe ratio is defined by[3]

$$-\frac{a(r) - r}{\sigma(r)}.$$

In this case, the risk-premium of the asset is defined by

$$a(r) - r = -\sigma(r)\phi(r,t).$$

Example 74.3
An asset has a percentage price increse of dr where

$$dr = 0.25(0.10 - r)dt + 0.01dZ.$$

Find the Risk-premium of the asset and the Sharpe ratio.

Solution.
We have $a(r) = 0.25(0.10 - r)$ and $\sigma(r) = 0.01$. Thus, the risk-premium of the asset is $a(r) - r = 0.025 - 1.25r$. The Sharpe ratio is

$$-\frac{a(r) - r}{\sigma(r)} = -\frac{0.025 - 1.25r}{0.01} = \frac{1.25r - 0.025}{0.01} = 125r - 2.5 \blacksquare$$

Example 74.4
The change in a zero-bond price is $-\$0.008651$. Given the following information: $P_r = -1.70126, P_{rr} = 4.85536, dt = \frac{1}{365}$, and $d_r = 0.0052342$. Estimate Δ.

[3] A bond decreases in value as the short-term rate increases whereas the asset value increases as the short-term rate increases. Since the risk-premium of the bond is positive, the risk premium of the asset must be negative. Otherwise, it would be possible to combine the asset with the zero-coupon bond to create a risk-free portfolio that earned more than the risk-free of return.

Solution.
We have

$$\begin{aligned} dP =& P_r dr + \frac{1}{2}P_{rr}(dr)^2 + P_t dt \\ -0.008651 =& -1.70126(0.0052342) + \frac{1}{2}(4.85536)(0.0052342)^2 + P_t\left(\frac{1}{365}\right). \end{aligned}$$

Thus,

$$\Theta = 365\left(-0.008651 + 1.70126(0.0052342) - \frac{1}{2}(4.85536)(0.0052342)^2\right) \approx 0.06834 \blacksquare$$

Practice Problems

Problem 74.1
The price of a zero-coupon bond that matures at time T is given by the expression $P(r,t,T) = A(t,T)e^{-B(t,T)r}$ where $A(t,T)$ and $B(t,T)$ are differentiable functions of t. Find Δ, Γ and Θ.

Problem 74.2
An asset has a percentage price increse of dr where

$$dr = 0.34(0.09 - r)dt + 0.26dZ.$$

Find the Risk-premium of the asset and the Sharpe ratio.

Problem 74.3
The following information are given: $P_r = -1.70126, P_{rr} = 4.85536, P_t = 0.0685, dt = \frac{1}{365}$, and $dr = 0.0052342$. Estimate the change in the price P using the $\Delta\Gamma\Theta$ approximation.

Problem 74.4
Show that $q(r,t,T) = -\sigma(r)\frac{\partial}{\partial r}\ln[P(r,t,T)]$.

Problem 74.5
The price of a zero-coupon bond that matures to \$1 at time T follows an Itô process

$$\frac{dP(r,t,T)}{P(r,t,T)} = \alpha(r,t,T)dt - q(r,t,T)dZ(t).$$

Find $q(r,t,T)$ if $P(r,t,T) = A(t,T)e^{-B(t,T)r}$.

Problem 74.6
Find $\alpha(r,t,T)$ in the previous exercise if the the Sharpe ratio of the zero-coupon bond is a constant ϕ.

75 Zero-Coupon Bond Pricing: Risk-Neutral Process

We next consider the risk-neutral process for the short-term interest rate. For that purpose, we first notice the following

$$\begin{aligned}\frac{dP(r,t,T)}{P(r,t,T)} =&\alpha(r,t,T)dt - q(r,t,T)dZ(t)\\ =&rdt - q(r,t,T)dZ(t) + [\alpha(r,t,T) - r]dt\\ =&rdt - q(r,t,T)\left[dZ(t) - \frac{\alpha(r,t,T)-r}{q(r,t,T)}dt\right]\\ =&rdt - q(r,t,T)[dZ(t) - \phi(r,t)dt]\end{aligned}$$

Using Girsanov's theorem, the transformation

$$d\tilde{Z}(t) = dZ(t) - \phi(r,t)dt$$

transforms the standard Brownian process, $Z(t)$, to a new standard Brownian process that is martingale under the risk-neutral probability measure. In this case, the process for the bond price can be written as

$$\frac{dP(r,t,T)}{P(r,t,T)} = rdt - q(r,t,T)d\tilde{Z}(t).$$

This shows that the expected return on the bond is r. It follows that the Sharpe ratio of a zero-coupon bond under the risk-neutral distribution is 0. The Itô process for r using this new Brownian motion is

$$\begin{aligned}dr =&a(r)dt + \sigma(r)dZ(t)\\ =&a(r)dt + \sigma(r)[d\tilde{Z}(t) + \phi(r,t)]\\ =&[a(r) + \sigma(r)\phi(r,t)]dt + \sigma(r)d\tilde{Z}(t).\end{aligned}$$

It has been proven that the solution to equation (74.1) subject to the boundary condition $P(r,T,T) = \$1$ is given by

$$P(r(t),t,T) = E^*(e^{-R(t,T)})$$

where E^* is the expectation based on the risk-neutral distribution and $R(t,T)$ is the cumulative interest rate given by

$$R(t,T) = \int_t^T r(s)ds.$$

Now by Jensen's inequality[1], $E^*(e^{-R(t,T)}) \neq e^{-E^*(R(t,T))}$. That is, it is not correct to say that the price of a zero-coupon bond is found by discounting at the expected interest rate.
In summary, an approach to modeling zero-coupon bond prices is exactly the same procedure used to price options of stocks:
• We begin with a model that describes the interest rate and then use equation (74.1) a partial differential equation that describes the bond price.
• Next, using the PDE together with the boundary conditions we can determine the price of the bond.

Example 75.1
The realistic process for the short-term interest rate is given by

$$dr = 0.3(0.12 - r)dt + 0.03dZ$$

and the risk-neutral process is given by

$$dr = 0.3(0.125 - r)dt + 0.03d\tilde{Z}.$$

Determine the Sharpe ratio $\phi(r,t)$.

Solution.
We have that

$$0.3(0.125 - r) = 0.3(0.12 - r) + 0.03\phi(r,t).$$

Solving this equation we find

$$\phi(r,t) = 0.05 \blacksquare$$

Delta-Gamma Approximations for Bonds
Using Itô lemma, the change in the price of a zero-coupon bond under the risk-neutral probability measure is given by

$$\begin{aligned}
dP =& P_r(r,t,T)dr + \frac{1}{2}P_{rr}(r,t,T)(dr)^2 + P_t(r,t,T)dt \\
=& P_r(r,t,T)\left([a(r) + \sigma(r)\phi(r,t)]dt + \sigma(r)d\tilde{Z}\right) \\
&+ \frac{1}{2}P_{rr}(r,t,T)\left([a(r) + \sigma(r)\phi(r,t)]dt + \sigma(r)d\tilde{Z}\right)^2 + P_t(r,t,T)dt \\
=& \left(\frac{1}{2}\sigma(r)^2P_{rr}(r,t,T) + [a(r) + \sigma(r)\phi(r,t)]P_r(r,t,T) + P_t(r,t,T)\right)dt + \sigma(r)P_r d\tilde{Z}
\end{aligned}$$

[1] See Section 84.

The expected change of price per unit time is

$$\frac{E^*(dP)}{dt} = \frac{1}{2}\sigma(r)^2 P_{rr}(r,t,T) + [a(r) + \sigma(r)\phi(r,t)]P_r(r,t,T) + P_t(r,t,T).$$

This and equation (74.1) imply

$$\frac{E^*(dP)}{dt} = rP.$$

This says that under the risk-neutral probability distribution measure, a bond has an expected return that is equal to the risk-free rate of return. *The approximation is exact if the interest rate moves one standard deviation.*

Example 75.2
Find the differential of $U(t) = \sigma \int_0^t e^{a(s-t)}\sqrt{r(s)}dZ(s)$.

Solution.
We have

$$\begin{aligned}
dU(t) =& d\left[\sigma e^{-at}\int_0^t e^{as}\sqrt{r(s)}dZ(s)\right]\\
=& d[\sigma e^{-at}]\int_0^t e^{as}\sqrt{r(s)}dZ(s) + \sigma e^{-at} d\left[\int_0^t e^{as}\sqrt{r(s)}dZ(s)\right]\\
=& -a\sigma e^{-at}dt\int_0^t e^{as}\sqrt{r(s)}dZ(s) + \sigma e^{-at}e^{at}\sqrt{r(t)}dZ(t)\\
=& -a\sigma e^{-at}dt\int_0^t e^{as}\sqrt{r(s)}dZ(s) + \sigma\sqrt{r(t)}dZ(t) \blacksquare
\end{aligned}$$

Example 75.3
Suppose that the short-term interest rate is given by

$$r(t) = r(0)e^{-at} + b(1 - e^{-at}) + \sigma\int_0^t e^{a(s-t)}\sqrt{r(s)}dZ(s).$$

Find an expression for $dr(t)$.

Solution.
We have

$$\begin{aligned}
dr(t) =&[-r(0)+b]ae^{-at}dt - a\sigma e^{-at}dt\int_0^t e^{as}\sqrt{r(s)}dZ(s) + \sigma\sqrt{r(t)}dZ(t)\\
=&-a\left[r(0)e^{-at} - be^{-at} + \sigma e^{-at}\int_0^t e^{as}\sqrt{r(s)}dZ(s)\right]dt + \sigma\sqrt{r(t)}dZ(t)\\
=&-a\left[r(0)e^{-at} + b(1-e^{-at}) + \sigma e^{-at}\int_0^t e^{as}\sqrt{r(s)}dZ(s)\right]dt + abdt + \sigma\sqrt{r(t)}dZ(t)\\
=&-ar(t)dt + abdt + \sigma\sqrt{r(t)}dZ(t)\\
=&a(b-r(t))dt + \sigma\sqrt{r(t)}dZ(t) \blacksquare
\end{aligned}$$

Practice Problems

Problem 75.1
The realistic process for the short-term interest rate is given by

$$dr = 0.3(0.12 - r)dt + 0.03d\tilde{Z}.$$

For $t \leq T$, let $P(r, t, T)$ be the price at time t of a zero-coupon bond that pays \$1 at time T, if the short-rate at time t is r. The Sharpe ratio of this bond is 0.05. Find the risk-neutral process of the short-term interest rate.

Problem 75.2
The risk-neutral process of a zero-coupon bond is given by

$$\frac{dP(0.08, 0, 10)}{P(0.08, 0, 10)} = 0.08dt - 0.095d\tilde{Z}.$$

Find the Sharpe ratio $\phi^*(0.08, 0)$ of a zero-coupon bond under the risk-neutral distribution.

Problem 75.3
Suppose that the Sharpe ratio of a zero-coupon bond is $\phi(r, t) = 0.05$. Let $\tilde{Z}(t)$ be the transformation defined in this section. Find $\tilde{Z}(4)$ if $Z(4) = -1.4766$.

Problem 75.4
Suppose that $r(t) = (br(0) - \sigma)e^{-at}$ with $r(0) = \frac{\sigma}{b-1}$, $b > 1$. Find the stochastic differential equation satisfied by r.

Problem 75.5
Suppose that $r(t) = (br(0) - \sigma)e^{-at}$ with $r(0) = \frac{\sigma}{b-1}$, $b > 1$. Find an expression for $R(t, T)$.

Problem 75.6
The risk-neutral process for a zero-coupon bond is given by

$$\frac{dP(r, t, T)}{P(r, t, T)} = 0.08dt - 0.095d\tilde{Z}.$$

Determine $\frac{E^*(dP)}{P}$.

Problem 75.7
The risk-neutral process for a zero-coupon bond is given by

$$\frac{dP(r,t,T)}{P(r,t,T)} = rdt - 0.0453d\tilde{Z}.$$

Determine r if $\frac{E^*(dP)}{P} = 0.07dt$.

76 The Rendleman-Bartter Short-Term Model

In this and the next two sections, we discuss several bond pricing models based on equation (74.1), in which all bond prices are driven by the short-term interest rate r with

$$dr = a(r)dt + \sigma(r)dZ.$$

In this section, we look at the Rendleman-Bartter model. The **Rendleman-Bartter model** in finance is a short rate model describing the evolution of interest rates. It is a type of "one factor model" as it describes interest rate movements as driven by only one source of market risk.
We first start by looking at short-term interest rate model that follows the arithmetic Brownian motion

$$dr = adt + \sigma dZ.$$

This says that the short-rate is normally distributed with mean $r(0)+at$ and variance $\sigma^2 t$. There are several problems with this model, namely:
• The short-rate can assume negative values which is not realistic in practice.
• The process is not mean-reverting because the drift term a is constant. For example, if $a > 0$, the short-rate will drift up over time forever. In practice, short-rate exhibits mean-reversion[1].
• The volatility σ of the short-rate is constant regardless of the rate. In practice, the short-rate is more volatile if rates are high.
To overcome the above mentioned obstacles, the Rendleman-Bartter model replaces the arithmetic motion by a standard geometric Brownian motion, that is, the instantaneous interest rate follows a geometric Brownian motion:

$$dr = ardt + \sigma rdZ$$

where the drift parameter, a, represents a constant expected instantaneous rate of change in the interest rate, while the standard deviation parameter, σ, determines the volatility of the interest rate.
The process for r is of the same type as that assumed for a stock price in

[1]When rates are high, the economy tends to slow down and borrowers require less funds. As a result rates decline. When rates are low, there tends to be a high demand for funds on the part of the borrowers and rates tend to rise

Section 48. That is, the natural logarithm of the rate follows an arithmetic Brownian motion[2]:

$$\ln\left(\frac{r(t)}{r(0)}\right) = (a - 0.5\sigma^2)t + \sigma Z(t)$$

and solving for $r(t)$ we find

$$r(t) = r(0)e^{(a-0.5\sigma^2)t+\sigma Z(t)}.$$

This shows that the rate can never be negative. Also, the volatility increases with the short-term rate. This follows from the fact that the variance of the short-term rate over a small increment of time is

$$\mathrm{Var}(r(t+dt)|r(t)) = r^2\sigma^2 dt.$$

Unlike stock prices, the only disadvantage that was observed with this model is that it does not incorporate mean-reversion.

Example 76.1
The Rendleman-Bartter one-factor interest rate model with the short-term rate is given by the process

$$dr = 0.001rdt + 0.01rdZ(t).$$

Suppose that the relevant Sharpe ratio is $\phi(r) = 0.88$. Find the process satisfied by r under the risk-neutral probability measure.

Solution.
We have

$$dr = [a(r)+\phi(r)\sigma(r)]dt+\sigma(r)d\tilde{Z}(t) = (0.001r+0.88\times 0.01r)dt+0.01d\tilde{Z}(t) \blacksquare$$

Example 76.2
The Rendleman-Bartter one-factor interest rate model with the short-term rate is given by the process

$$dr = 0.001rdt + 0.01rdZ(t).$$

Determine $\mathrm{Var}(r(t+h)|r(t))$.

Solution.
We have $\mathrm{Var}(r(t+h)|r(t)) = r^2\sigma^2 h = 0.01^2 r^2 h$ ■

[2]See Problems 76.3 - 76.4

Practice Problems

Problem 76.1
Consider the Rendleman-Bartter model $dr(t) = ar(t)dt + \sigma r(t)dZ(t)$. Find an expression of $r(t)$ in integral form.

Problem 76.2
Consider the Rendleman-Bartter model $dr(t) = ar(t)dt + \sigma r(t)dZ(t)$. Show that

$$d[\ln r(t)] = (a - 0.5\sigma^2)dt + \sigma dZ(t).$$

Hint: Apply Itô's lemma to the function $f(r,t) = \ln[r(t)]$.

Problem 76.3
Consider the Rendleman-Bartter model $dr(t) = ar(t)dt + \sigma r(t)dZ(t)$. Show that

$$\ln[r(t)] = \ln[r(0)] + (a - 0.5\sigma^2)t + \sigma Z(t).$$

Problem 76.4
Consider the Rendleman-Bartter model $dr(t) = ar(t)dt + \sigma r(t)dZ(t)$. Show that

$$r(t) = r(0)e^{(a-0.5\sigma^2)t+\sigma Z(t)}.$$

Problem 76.5
Consider the Rendleman-Bartter model $dr(t) = ar(t)dt + \sigma r(t)dZ(t)$. Show that for $t > s$ we have

$$r(t) = r(s)e^{(a-0.5\sigma^2)(t-s)+\sigma(Z(t)-Z(s))}.$$

Problem 76.6
Consider the Rendleman-Bartter model $dr(t) = ar(t)dt + \sigma r(t)dZ(t)$. Determine a formula for $E[r(t)|r(s)],\ t > s$. See Section 47.

Problem 76.7
Consider the Rendleman-Bartter model $dr(t) = ar(t)dt + \sigma r(t)dZ(t)$. Determine a formula for $\text{Var}[r(t)|r(s)],\ t > s$. See Section 47.

77 The Vasicek Short-Term Model

The Vasicek model for the short-term interest rate is given by the process

$$dr = a(b - r)dt + \sigma dZ(t).$$

Thus, the interest rate follows an Ornstein-Uhlenbeck process (See equation (59.2)). The term $a(b-r)dt$ induces mean-reversion. The parameter b is the level to which the short term revert. If $r > b$, the short-term rate is expected to decrease toward b. If $r < b$, the short-term rate is expected to increase toward b. The parameter a is a positive number that reflects the speed of the reversion to b. The higher a is, the faster the reversion is.
The Vasicek model is mean-reverting, but it exhibits the two problems of the arithmetic model discussed at the beginning of Section 76, namely:
- Interest rates in this model can become negative.
- The volatility of the short-rate is the same whether the rate is high or low.

When solving for the zero-coupon bond price, it is assumed that the Sharpe ratio for interest rate risk is some constant ϕ. In this case, we have the following differential equation

$$\frac{1}{2}\sigma^2 P_{rr} + [a(b - r) + \sigma\phi]P_r + P_t - rP = 0.$$

For $a \neq 0$, the solution to this differential equation subject to the condition $P(r, T, T) = \$1$ is given by

$$P(r, t, T) = A(t, T)e^{-B(t,T)r(r)}$$

where

$$\begin{aligned}
A(t, T) =& e^{\bar{r}(B(t,T)+t-T) - \frac{B^2(t,T)\sigma^2}{4a}} \\
B(t, T) =& \frac{1 - e^{-a(T-t)}}{a} \\
\bar{r} =& b + \frac{\sigma\phi}{a} - \frac{0.5\sigma^2}{a^2}
\end{aligned}$$

where $\bar{r}$ is the yield to maturity on an infinitely lived bond, that is the value the yield will approach as T goes to infinity.

For $a = 0$, we have

$$\begin{aligned} B(t,T) =& T - t \\ A(t,T) =& e^{0.5\sigma(T-t)^2\phi + \frac{\sigma^2(T-t)^3}{6}} \\ \overline{r} =& \text{undefined} \end{aligned}$$

Example 77.1
Show that $A(0, T-t) = A(t,T)$ and $B(0, T-t) = B(t,T)$.

Solution.
For $a \neq 0$, we have

$$B(0, T-t) = \frac{1 - e^{-a(T-t-0)}}{a} = \frac{1 - e^{-a(T-t)}}{a} = B(t,T)$$

and

$$A(0, T-t) = e^{\overline{r}B(0,T-t)+0-(T-t)) - \frac{B^2(0,T-t)\sigma^2}{4a}} = e^{\overline{r}(B(t,T)+t-T) - \frac{B^2(t,T)\sigma^2}{4a}} = A(t,T).$$

Likewise, we can establish the results for the case $a = 0$ ■

Example 77.2
A Vasicek model for a short-rate is given by

$$dr = 0.2(0.1 - r)dt + 0.02dZ(t).$$

The Sharpe ratio for interest rate risk is $\phi = 0$. Determine the yield to maturity on an infinitely lived bond.

Solution.
We are given: $a = 0.2, b = 0.1, \sigma = 0.02$, and $\phi = 0$. The yield to maturity on an infinitely lived bond is

$$\overline{r} = b + \frac{\sigma\phi}{a} - \frac{0.5\sigma^2}{a^2} = 0.1 + 0 \times \frac{0.02}{0.2} - \frac{0.5 \times 0.02^2}{0.2^2} = 0.095 \blacksquare$$

Example 77.3
A Vasicek model for a short-rate is given by

$$dr = 0.2(0.1 - r)dt + 0.02dZ(t).$$

The Sharpe ratio for interest rate risk is $\phi = 0$. Determine the value of $B(0, 10)$.

Solution.
We use the formula $B(t,T) = \frac{1-e^{-a(T-t)}}{a}$. We are given: $a = 0.2, b = 0.1, \sigma = 0.02, t = 0, T = 10$ and $\phi = 0$. Thus,

$$B(1,2) = \frac{1 - e^{-0.2(10-0)}}{0.2} = 4.323 \blacksquare$$

Now, the price $P(r,t,T)$ of a zero-coupon bond in the Vasicek model follows an Itô process

$$\frac{dP(r,t,T)}{P(r,t,T)} = \alpha(r,t,T)dt - q(r,t,T)dZ.$$

Example 77.4
Show that $q(r,t,T) = \sigma B(t,T)$.

Solution.
We have

$$\begin{aligned} q(r,t,T) =& -\sigma\frac{P_r(r,t,T)}{P(r,t,T)} = -\sigma\frac{\partial P(r,t,T)}{\partial r} \\ =& -\sigma \times -B(t,T) = \sigma B(t,T) \blacksquare \end{aligned}$$

Example 77.5
Show that $\alpha(r,t,T) = r + B(t,T)\sigma\phi$.

Solution.
Since the Sharpe ratio for interest rate risk in the Vasicek model is given by a constant ϕ, we can write

$$\phi = \frac{\alpha(r,t,T) - r}{q(r,t,T)} = \frac{\alpha(r,t,T) - r}{\sigma B(t,T)}.$$

From this, it follows that

$$\alpha(r,t,T) = r + B(t,T)\sigma\phi \blacksquare$$

Example 77.6 ‡
For a Vasicek bond model you are given the following information:
• $a = 0.15$
• $b = 0.1$
• $r = 0.05$
• $\sigma = 0.05$
Calculate the expected change in the interest rate, expressed as an annual rate.

Solution.
The given Vasicek model is described by the process

$$dr = 0.15(0.10 - r)dt + 0.05dZ(t).$$

The expected change in the interest rate is

$$E(dr) = 0.15(0.10 - r)dt = (0.015 - 0.15r)dt.$$

Since $r = 0.05$ we obtain

$$E(dr) = (0.015 - 0.15 \times 0.05)dt = 0.0075dt.$$

The expected change in the interest rate, expressed as an annual rate, is then

$$\frac{E(dr)}{dt} = 0.0075 \blacksquare$$

Practice Problems

Problem 77.1
A Vasicek model for a short-rate is given by

$$dr = 0.34(0.09 - r)dt + 0.26dZ(t).$$

The Sharpe ratio for interest rate risk is $\phi = 0.88$. Determine the yield to maturity on an infinitely lived bond.

Problem 77.2
A Vasicek model for a short-rate is given by

$$dr = 0.34(0.09 - r)dt + 0.26dZ(t).$$

The Sharpe ratio for interest rate risk is $\phi = 0.88$. Determine the value of $B(2, 3)$.

Problem 77.3
A Vasicek model for a short-rate is given by

$$dr = 0.2(0.1 - r)dt + 0.02dZ(t).$$

The Sharpe ratio for interest rate risk is $\phi = 0$. Determine the value of $A(0, 10)$.

Problem 77.4
A Vasicek model for a short-rate is given by

$$dr = 0.2(0.1 - r)dt + 0.02dZ(t).$$

The Sharpe ratio for interest rate risk is $\phi = 0$. Determine the price of a zero-coupon bond with par value of \$100 and that matures in 10 years if the risk-free annual interest rate is 8%.

Problem 77.5 ‡
You are using the Vasicek one-factor interest-rate model with the short-rate process calibrated as

$$dr(t) = 0.6[b - r(t)]dt + \sigma dZ(t).$$

For $t \leq T$, let $P(r,t,T)$ be the price at time t of a zero-coupon bond that pays \$1 at time T, if the short-rate at time t is r. The price of each zero-coupon bond in the Vasicek model follows an Itô process,

$$\frac{dP(r,t,T)}{P(r,t,T)} = \alpha(r,t,T)dt - q(r,t,T)dZ(t), \quad t \leq T.$$

You are given that $\alpha(0.04, 0, 2) = 0.04139761$. Find $\alpha(0.05, 1, 4)$.

Problem 77.6 ‡
You are given:
(i) The true stochastic process of the short-rate is given by

$$dr(t) = [0.09 - 0.5r(t)]dt + 0.3dZ(t),$$

where $\{Z(t)\}$ is a standard Brownian motion under the true probability measure.
(ii) The risk-neutral process of the short-rate is given by

$$dr(t) = [0.15 - 0.5r(t)]dt + \sigma(r(t))d\tilde{Z}(t),$$

where $\{\tilde{Z}(t)\}$is a standard Brownian motion under the risk-neutral probability measure.
(iii) $g(r,t)$ denotes the price of an interest-rate derivative at time t, if the short-rate at that time is r. The interest-rate derivative does not pay any dividend or interest.
(iv) $g(r(t),t)$ satisfies

$$dg(r(t),t) = \mu(r(t), g(r(t),t))dt - 0.4g(r(t),t)dZ(t).$$

Determine $\mu(r,g)$.

Problem 77.7 ‡
You are given:
(i) The true stochastic process of the short-rate is given by

$$dr(t) = [0.008 - 0.1r(t)]dt + 0.05dZ(t),$$

where $\{Z(t)\}$ is a standard Brownian motion under the true probability measure.
(ii) The risk-neutral process of the short-rate is given by

$$dr(t) = [0.013 - 0.1r(t)]dt + 0.05d\tilde{Z}(t),$$

where $\{\tilde{Z}(t)\}$is a standard Brownian motion under the risk-neutral probability measure.
(iii) For $t \leq T$, let $P(r,t,T)$ be the price at time t of a zero-coupon bond that pays \$1 at time T, if the short-rate at time t is r. The price of each zero-coupon bond follows an Itô process,

$$\frac{dP(r,t,T)}{P(r,t,T)} = \alpha(r,t,T)dt - q(r,t,T)dZ(t), \quad t \leq T.$$

Determine $\alpha(0.04, 2, 5)$.

Problem 77.8 ‡
Let $P(r,t,T)$ denote the price at time t of \$1 to be paid with certainty at time T, $t \leq T$, if the short rate at time t is equal to r. For a Vasicek model you are given:

$$\begin{aligned} P(0.04, 0, 2) =& 0.9445 \\ P(0.05, 1, 3) =& 0.9321 \\ P(r^*, 2, 4) =& 0.8960 \end{aligned}$$

Calculate r^*.

78 The Cox-Ingersoll-Ross Short-Term Model

The Cox-Ingersoll-Ross (CIR) bond pricing model assumes that the risk natural process for the short interest rate is given by

$$dr = a(b - r)dt + \sigma\sqrt{r}dZ.$$

Note that the variance of the short-term rate over a small increment of time is

$$\text{Var}(r(t + dt)|r(t)) = \text{Var}(dr) = \sigma^2 r dt.$$

This says that the standard deviation of the short-term rate is proportional to the square root of r. Thus, as r increases the volatility increases. Like the Vasicek model, the drift term $a(b-r)$ induces mean reversion. Also, it should be noted that the interest rate in this model can never become negative. For example, if $r = 0$ in the CIR model, the drift factor $a(b-r)$ becomes $ab > 0$, while the volatility factor is $\sigma\sqrt{0} = 0$, so the interest rate will increase.
The Sharpe ratio in the CIR model takes the form

$$\phi(r, t) = \overline{\phi}\frac{\sqrt{r}}{\sigma}$$

and the bond price satisfies the differential equation

$$\frac{1}{2}\sigma^2 r\frac{\partial^2 P}{\partial r^2} + [a(b - r) + r\overline{\phi}]\frac{\partial P}{\partial r} + \frac{\partial P}{\partial t} - rP = 0.$$

The solution to this equation that satisfies the condition $P(r, T, T) = \$1$ is given by

$$P(r, t, T) = A(t, T)e^{-B(t,T)r(t)}$$

where

$$\begin{aligned}
A(t, T) &= \left[\frac{2\gamma e^{(a+\overline{\phi}+\gamma)(T-t)/2}}{(a + \overline{\phi} + \gamma)(e^{\gamma(T-t)} - 1) + 2\gamma}\right]^{\frac{2ab}{\sigma^2}} \\
B(t, T) &= \frac{2(e^{\gamma(T-t)} - 1)}{(a + \overline{\phi} + \gamma)(e^{\gamma(T-t)} - 1) + 2\gamma} \\
\gamma &= \sqrt{(a + \overline{\phi})^2 + 2\sigma^2}
\end{aligned}$$

As in the Vasicek model, the terms $A(t,T)$ and $B(t,T)$ are independent of the short-term rate. The yield to maturity of a long lived bond in the CIR model is given by

$$\overline{r} = \lim_{T\to\infty} \frac{-\ln[P(r,t,T)]}{T-t} = \frac{2ab}{a+\overline{\phi}+\gamma}.$$

From the process of the bond price

$$\frac{dP(r,t,T)}{P(r,t,T)} = \alpha(r,t,T)dt - q(r,t,T)dZ(t)$$

we find

$$q(r,t,T) = -\sigma(r)\frac{P_r(r,t,T)}{P(r,t,T)} = -\frac{1}{P(r,t,T)}[-B(t,T)]P(r,t,T)\sigma\sqrt{r} = B(t,T)\sigma\sqrt{r}$$

and

$$\begin{aligned}
\phi(r,t) =& \frac{\alpha(r,t,T)-r}{q(r,t,T)}\\
\overline{\phi}\frac{\sqrt{r}}{\sigma} =& \frac{\alpha(r,t,T)-r}{B(t,T)\sigma\sqrt{r}}\\
\alpha(r,t,T) =& r + r\overline{\phi}B(t,T)
\end{aligned}$$

Example 78.1
A CIR short-rate model is described by the process $dr = 0.18(0.1-r)dt + 0.17\sqrt{r}dZ$. Given that the Sharpe ratio is zero. Find $\overline{\phi}$.

Solution.
We have $0 = \phi(r,t) = \overline{\phi}\frac{\sqrt{r}}{\sigma}$. This implies that $\overline{\phi} = 0$ ■

Example 78.2
A CIR short-rate model is described by the process $dr = 0.18(0.1-r)dt + 0.17\sqrt{r}dZ$. Given that the Sharpe ratio is zero. Determine the value of γ.

Solution.
We have $a = 0.18, b = 0.10, \sigma = 0.17$, and $\overline{\phi} = 0$. Thus,

$$\gamma = \sqrt{(a+\overline{\phi})^2 + 2\sigma^2} = \sqrt{0.18^2 + 0 + 2(0.17)^2} = 0.30033 \blacksquare$$

Example 78.3
A CIR short-rate model is described by the process $dr = 0.18(0.1 - r)dt + 0.17\sqrt{r}dZ$. Given that the Sharpe ratio is zero. Determine the value of $B(0,5)$.

Solution.
We have $a = 0.18, b = 0.10, \sigma = 0.17, \overline{\phi} = 0, \gamma = 0.30033, t = 0$, and $T = 5$. Substituting these values into the formula

$$B(t,T) = \frac{2(e^{\gamma(T-t)} - 1)}{(a + \overline{\phi} + \gamma)(e^{\gamma(T-t)} - 1) + 2\gamma}$$

we find $B(0,5) = 3.06520$ ■

Example 78.4
A CIR short-rate model is described by the process $dr = 0.18(0.1 - r)dt + 0.17\sqrt{r}dZ$. Given that the Sharpe ratio is zero. Determine the value of $A(0,5)$.

Solution.
We have $a = 0.18, b = 0.10, \sigma = 0.17, \overline{\phi} = 0, \gamma = 0.30033, t = 0$, and $T = 5$. Substituting these values into the formula

$$A(t,T) = \left[\frac{2\gamma e^{(a+\overline{\phi}+\gamma)(T-t)/2}}{(a + \overline{\phi} + \gamma)(e^{\gamma(T-t)} - 1) + 2\gamma}\right]^{\frac{2ab}{\sigma^2}}$$

we find $A(0,5) = 0.84882$ ■

Example 78.5
A CIR short-rate model is described by the process $dr = 0.18(0.1 - r)dt + 0.17\sqrt{r}dZ$. Given that the Sharpe ratio is zero. Determine the price of a zero-coupon bond that pays \$ in five years if the risk-free interest rate is assumed to be 8%. What is the yield on the bond?

Solution.
We have

$$P(0.08, 0, 5) = A(0,5)e^{-B(0,5)(0.08)} = 0.84882e^{-3.06520 \times 0.08} = \$0.66424.$$

The yield on the bond is

$$\text{yield} = -\frac{\ln\left[P(0.08, 0, 5)0\right.}{5} = 0.08182 \blacksquare$$

Example 78.6
What is the yield on a long lived zero coupon bond under the CIR short-rate model described by the process $dr = 0.18(0.1 - r)dt + 0.17\sqrt{r}dZ$ with zero Sharpe ratio?

Solution.
The yield is

$$\bar{r} = \frac{2ab}{a + \bar{\phi} + \gamma} = \frac{2(0.18)(0.10)}{0.18 + 0 + 0.30033} = 0.075 \blacksquare$$

Vasicek Model Versus CIR Model
We next compare the various features of Vasicek and CIR models:
• In the Vasicek model, interest rates can be negative. In the CIR model, negative interest rates are impossible. As one's time horizon T increases, the likelihood of interest rates becoming negative in the Vasicek model greatly increases as well.
• In the Vasicek model, the volatility of the short-term interest rate is constant. In the CIR model, the volatility of the short-term interest rate increases as the short-term interest rate increases.
• The short-term interest rate in both models exhibit mean reversion.
• In both the Vasicek and the CIR model, the delta and gamma Greeks for a zero-coupon bond are based on the change in the short-term interest rate.
• With a relatively high volatility (lower panel of Figure 78.1), the CIR yields tend to be higher than the Vasicek yields. This occur because the Vasicek yields can be negative.
• With a relatively low volatility (upper panel of Figure 78.1), the mean-reversion effect outweighs the volatility effect, and the Vasicek yields tend to be a bit higher than the CIR yields. Also, both models produce upward sloping yield curves.

An Exogenously Prescribed Time Zero Yield Curve
An **exogenous prescription** of the time-zero yield curve in a financial model would mean that you know−empirically, or from some source external to the model−the data regarding the yields to maturity for many different time horizons, where currently, $t = 0$. Then you would be able to put that data into the model and get consistent results as well as the ability to predict yields to maturity for other time horizons. A model based on m parameters can only be used consistently with exogenously prescribed data consisting of

$m-1$ or fewer data points. To elborate, suppose your model is described by the two-parameter equation $x+y=2$. If we are just given x, we can solve for y consistently with the model. If we are given both x and y, however, we will not always be able to do so. For instance, if $x=3$ and $y=4$, there is no way for $x+y$ to equal 5. If you have even more externally prescribed data, the likelihood of the model working decreases even further.
The Vasicek and CIR have each exactly four inputs: a, b, σ, and r. A yield curve can consist of many more than four yields which means that the two models don't have enough inputs to fully describe an existing yield curve. Another way to say this is that the time zero yield curve cannot be exagenously prescribed using either the Vasicek or CIR model.

Example 78.7
For which of these exogenously prescribed data sets regarding yields to maturity for various time horizons can the Vasicek and CIR models be used? More than one answer may be correct. Each data has the form $[t, T, r]$.
Set A: $[0, 1, 0.22], [0, 2, 0.26], [0, 3, 0.28]$
Set B: $[0, 0.5, 0.04], [0, 5, 0.05]$
Set C: $[0, 1, 0.12], [0, 2, 0.09], [0, 3, 0.18], [0, 6, 0.21], [0, 7, 0.03]$
Set D: $[0, 1, 0.11], [0, 2, 31], [0, 7, 0.34], [0, 8, 0.54]$.

Solution.
The time-zero yield curve for an interest rate model can only be interpreted consistently with the model (in most cases) when the number of data points in the time-zero yield curve is less than the number of parameters in the model. The Vasicek and CIR models each have 4 parameters: a, b, σ, and r. So only data sets with 3 points or fewer can have the models consistently applied to them. Thus, only Sets A and B, with 3 and 2 data points respectively, can have the models applied to them ■

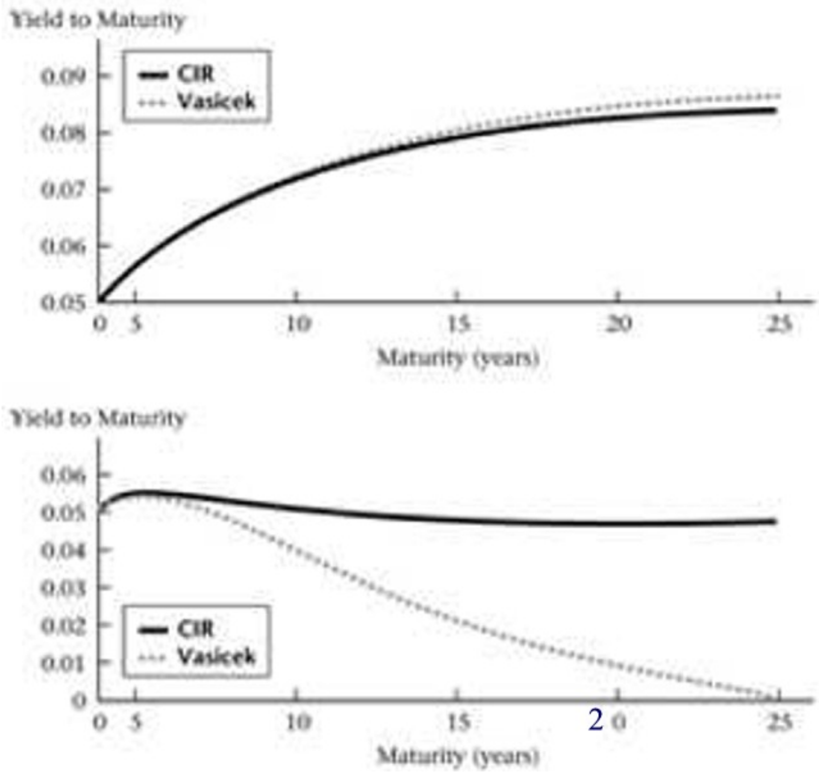

Yield to Maturity
0.09
0.08
0.07
0.06
0.05
CIR
Vasicek
0 5 10 15 20 25
Maturity (years)
Yield to Maturity
0.06
0.05
0.04
0.03
0.02
0.01
0
CIR
Vasicek
0 5 10 15 20 25
Maturity (years)

Figure 78.1

Practice Problems

Problem 78.1
State two shortcomings of the Vasicek model that the CIR does not have.

Problem 78.2
Which of the following statements are true? More than one answer may be correct.
(a) The Vasicek model incorporates mean reversion.
(b) The CIR model incorporates mean reversion.
(c) In the Vasicek model, interest rates can be negative.
(d) In the CIR model, interest rates can be negative.
(e) In the Vasicek model, volatility is a function of the interest rate.
(f) In the CIR model, volatility is a function of the interest rate.

Problem 78.3
Assume the CIR model holds. A particular interest rate follows this Brownian motion:

$$dr = 0.22(0.06 - r)dt + 0.443\sqrt{r}dZ.$$

At some particular time $t, r = 0.11$. Then, r suddenly becomes 0.02. What is the resulting change in the volatility?

Problem 78.4
Assume that the CIR model holds. When a particular interest rate is 0, the Brownian motion it follows is $dr = 0.55dt$.
You know that $a+b = 1.7$ and $a > 1$. What is the drift factor of the Brownian motion in this model when $r = 0.05$?

Problem 78.5
Find the risk-neutral process that corresponds to the CIR model

$$dr = a(b - r)dt + \sigma\sqrt{r}dZ.$$

Problem 78.6
The realistic process of the CIR model is given by

$$dr = a(0.08 - r)dt + 0.04\sqrt{r}dZ$$

and the risk-neutral process is given by

$$dr = 0.2(0.096 - r)dt + 0.04\sqrt{r}dZ.$$

Determine the values of a and $\overline{\phi}$.

Problem 78.7
The CIR short-rate process is given by

$$dr = 0.0192(0.08 - r)dt + 0.04\sqrt{r}dZ$$

Determine the value of $\phi(0.07, t)$.

Problem 78.8
A CIR short-rate model is described by the process $dr = 0.18(0.1 - r)dt + 0.17\sqrt{r}dZ$. Given that the Sharpe ratio is zero. Find the delta and the gamma of a zero-coupon bond that pays \$1 in five years under this model if the risk-free interest rate is assumed to be 0.08.

Problem 78.9
A CIR short-rate model is described by the process $dr = 0.18(0.1 - r)dt + 0.17\sqrt{r}dZ$. Given that the Sharpe ratio is zero. Find the theta of a zero-coupon bond that pays \$1 in five years under this model if the risk-free interest rate is assumed to be 0.08.

Problem 78.10 ‡
The Cox-Ingersoll-Ross (CIR) interest-rate model has the short-rate process:

$$dr(t) = a[b - r(t)]dt + \sigma\sqrt{r(t)}dZ(t),$$

where $\{Z(t)\}$ is a standard Brownian motion.
For $t \leq T$, let $P(r, t, T)$ be the price at time t of a zero-coupon bond that pays \$1 at time T, if the short-rate at time t is r. The price of each zero-coupon bond in the Vasicek model follows an Itô process,

$$\frac{dP(r,t,T)}{P(r,t,T)} = \alpha(r,t,T)dt - q(r,t,T)dZ(t), \quad t \leq T.$$

You are given that $\alpha(0.05, 7, 9) = 0.06$. Find $\alpha(0.04, 11, 13)$.

Problem 78.11 ‡

The short-rate process $\{r(t)\}$ in a Cox-Ingersoll-Ross model follows

$$dr(t) = [0.011 - 0.1r(t)]dt + 0.08\sqrt{r(t)}dZ(t),$$

where $\{Z(t)\}$ is a standard Brownian motion under the true probability measure.

For $t \leq T$, let $P(r,t,T)$ denote the price at time t of a zero-coupon bond that pays \$1 at time T, if the short-rate at time t is r.

You are given:

(i) The Sharpe ratio takes the form $\phi(r,t) = c\sqrt{r}$.

(ii) $\lim_{T\to\infty} \frac{\ln[P(r,0,T)]}{T} = -0.1$ for each $r > 0$.

Find the constant c.

79 The Black Formula for Pricing Options on Bonds

In this section, we discuss the Black model, a variant of the Black-Scholes pricing model, for pricing zero-coupon bond options.
Consider a $s-$year zero-coupon bond. When this bond is purchased at time T then it will pay \$1 at time $T+s$. Consider a forward contract signed at time $t \leq T$ that calls for time$-T$ delivery of the bond maturing at time $T+s$. Let $P_t(T, T+s)$ be the forward price of this contract. That is, $P_t(T, T+s)$ is the time$-t$ price agreed upon to be paid at time T. Let $P(T, T+s) = P_T(T, T+s)$ be the (spot) price of the bond at time T. For $t < T$, we let $F_{t,T}[P(T, T+s)] = P_t(T, T+s)$ be the forward price at time t for an agreement to buy a bond at time T that pays \$ at tim $T+s$. At time t, if you want \$1 at time $T+s$, there are two ways to achieve this:
1. You may buy a bond immediately maturing for \$1 at time $T+s$. This would cost $P(t, T+s)$.
2. You can enter into a forward agreement to buy a bond at time T maturing for \$1 at time $T+s$. At time T, you would pay $F_{t,T}[P(T, T+s)]$.
Therefore, discounting $F_{t,T}[P(T, T+s)]$ to time t, we must have

$$F_{t,T}[P(T, T+s)]P(t, T) = P(t, T+s)$$

or

$$F_{t,T}[P(T, T+s)] = \frac{P(t, T+s)}{P(t, T)}$$

where $P(t, T)$ is the time-t price of a bond that pays \$1 at time T.

Example 79.1
The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8495. The time-0 price of a 3-year zero-coupon bond with par value \$1 is \$0.7722. At current time $t = 0$, what is the forward price for a 1-year bond in year 2?

Solution.
We are asked to find $P_0(T, T+s) = F_{0,T}[P(T, T+s)] = \frac{P(0,T+s)}{P(0,T)}$ with $T = 2$ and $s = 1$. That is,

$$\frac{P(0,3)}{P(0,2)} = \frac{0.7722}{0.8495} = \$0.90901 \blacksquare$$

Now, consider a European call option with strike K, expiration date T, and underlying asset the forward price of a $s-$year zero-coupon bond. The payoff of this option at time T is

$$\text{Call option payoff} = \max\{0, P(T, T+s) - K\}.$$

Under the assumption that the bond forward price is lognormally distributed with constant volatility σ, the Black formula for such a European call option is given by

$$C = P(0,T)[FN(d_1) - KN(d_2)],$$

where

$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(F/K) - (\sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

and where $F = F_{0,T}[P(T, T+s)]$.
The price of an otherwise equivalent European put option on the bond is

$$P = P(0,T)[KN(-d_2) - FN(-d_1)].$$

Since $P(0,T)F = P(0,T+s)$, the Black formula simply uses the time-0 price of a zero-coupon bond that matures at time $T+s$ as the underlying asset.

Example 79.2
The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8495. The time-0 price of a 3-year zero-coupon bond with par value \$1 is \$0.7722. What is the price of a European call option that expires in 2 years, giving you the right to pay \$0.90 to buy a bond expiring in 1 year? The 1-year forward price volatility is 0.105.

Solution.
We have

$$F = \frac{P(0,3)}{P(0,2)} = 0.90901$$

$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln(0.90901/0.90) + 0.5(0.105)^2(2)}{0.105\sqrt{2}} = 0.14133$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.14133 - 0.105\sqrt{2} = -0.00716$$

$$N(d_1) = 0.556195,\ N(d_2) = 0.497144.$$

Thus,

$$C = 0.8495[0.90901 \times 0.556195 - 0.90 \times 0.497144] = \$0.0494 \blacksquare$$

Example 79.3

The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8495. The time-0 price of a 3-year zero-coupon bond with par value \$1 is \$0.7722. What is the price of a European put option that expires in 2 years, giving you the right of selling a bond expiring in 1 year for the price of \$0.90? The 1-year forward price volatility is 0.105.

Solution.

By the Black model, the price is

$$\begin{aligned} P =& P(0,T)[KN(-d_2) - FN(-d_1)] \\ =& 0.8495[0.90(1 - 0.497144) - 0.90901(1 - 0.556195)] = \$0.04175 \blacksquare \end{aligned}$$

Practice Problems

Problem 79.1
The time-0 price of a 1-year zero-coupon bond with par value \$1 is \$0.9259. The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8495. At current time $t = 0$, what is the forward price of a 1-year bond in year 1?

Problem 79.2
The time-0 price of a 1-year zero-coupon bond with par value \$1 is \$0.9259. The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8495. What is the price of a European call option that expires in 1 year, giving you the right to pay \$0.9009 to buy a bond expiring in 1 year? The 1-year forward price volatility is 0.10.

Problem 79.3
The time-0 price of a 1-year zero-coupon bond with par value \$1 is \$0.9259. The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8495. What is the price of a European put option that expires in 1 year, giving you the right of selling a bond expiring in 1 year for the price of \$0.9009? The 1-year forward price volatility is 0.10.

Problem 79.4
The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8853. The time-0 price of a 5-year zero-coupon bond with par value \$1 is \$0.6657. A European call option that expires in 2 years enables you to purchase a 3-year bond at expiration at a purchase price of \$0.7799. The forward price of the bond is lognormally distributed and has a volatility of 0.33. Find the value of F in the Black formula for the price of this call option.

Problem 79.5
The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8853. The time-0 price of a 5-year zero-coupon bond with par value \$1 is \$0.6657. A European call option that expires in 2 years enables you to purchase a 3-year bond at expiration at a purchase price of \$0.7799. The forward price of the bond is lognormally distributed and has a volatility of 0.33. Find the value of d_1 in the Black formula for the price of this call option.

Problem 79.6
The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8853.

The time-0 price of a 5-year zero-coupon bond with par value \$1 is \$0.6657. A European call option that expires in 2 years enables you to purchase a 3-year bond at expiration at a purchase price of \$0.7799. The forward price of the bond is lognormally distributed and has a volatility of 0.33. Find the value of d_2 in the Black formula for the price of this call option.

Problem 79.7
The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8853. The time-0 price of a 5-year zero-coupon bond with par value \$1 is \$0.6657. A European call option that expires in 2 years enables you to purchase a 3-year bond at expiration at a purchase price of \$0.7799. The forward price of the bond is lognormally distributed and has a volatility of 0.33. Assume the Black framework, find the price of this call option.

Problem 79.8
The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8853. The time-0 price of a 5-year zero-coupon bond with par value \$1 is \$0.6657. A European put option that expires in 2 years enables you to sell a 3-year bond at expiration for the price of \$0.7799. The forward price of the bond is lognormally distributed and has a volatility of 0.33. Find the price of this put option.

80 The Black Formula for Pricing FRA, Caplets, and Caps

In this section we can extend the Black formula to price options on interest rates. More specifically, we will use the Black model to price forward rate agreement, interest rate caplets and caps.

Forward Rate Agreement
A **forward rate agreement** (FRA) is an agreement to pay or receive the difference between the prevailing rate at a certain time and a fixed rate R, times some notional amount. For example, suppose that ABC Inc. enters into an FRA with MBF in which Company ABC will receive a fixed rate of 4% for one year on a principal of \$1 million in three years. In return, Company MBF will receive the one-year LIBOR rate, determined in three years' time, on the principal amount. The agreement will be settled in cash in three years.
If, after three years' time, the LIBOR is at 5.5%, the settlement to the agreement will require that ABC Inc. pay MBF. This is because the LIBOR is higher than the fixed rate. Mathematically, \$1 million at 5% generates \$50,000 of interest for ABC Inc. while \$1 million at 5.5% generates \$55,000 in interest for MBF. Ignoring present values, the net difference between the two amounts is \$5,000, which is paid to Company MBF.
Now, if after three years' time, the LIBOR is at 4.5%, the settlement to the agreement will require that MBF pay ABC Inc. This is because the LIBOR is lower than the fixed rate. The amount \$5,000 is paid to Company ABC Inc.
Consider a borrower who wishes to have a loan of \$1 at time T that matures at time $T+s$. To hedge against interest rate fluctuation, the borrower enters into a **forward rate agreement (FRA)** that allows the borrowers to lock-in an interest rate, the forward interest rate. Thus, at maturity time $T+s$ of the FRA, the FRA payoff is given by

$$\text{Payoff to FRA} = R_T(T, T+s) - R_0(T, T+s)$$

where $R_T(T, T+s)$ is the spot $s-$period from time T to time $T+s$ (ot the time$-T$ forward rate from time T to time $T+s$) and $R_0(T, T+s)$ is the time-0 forward interest rate from time T to time $T+s$. If $R_T(T, T+s) > R_0(T, T+s)$, the borrower pays the lender the difference. If $R_T(T, T+s) < R_0(T, T+s)$,

the lender pays the borrower the difference.
But[1] we know that

$$R_t(T, T+s) = \frac{P(t,T)}{P(t,T+s)} - 1$$

so that

$$R_0(T, T+s) = \frac{P(0,T)}{P(0,T+s)} - 1$$

and

$$R_T(T, T+s) = \frac{P(T,T)}{P(T,T+s)} - 1 = \frac{1}{P(T,T+s)} - 1$$

where $P(T,T) = \$1$ (since a bond paying \$1 at expiration will have a value of \$1 right when it expires) so that the payoff of the FRA at time $T+s$ takes the form

$$\text{Payoff to FRA} = \frac{1}{P(T,T+s)} - \frac{P(0,T)}{P(0,T+s)}.$$

We should point out here that $R_0(T, T+s)$ is nonannualized: If you invest \$1 at time T, you will have $[1 + R_0(T, T+s)]$ after s time periods elapse.

Example 80.1
The time-0 price of a 6-year zero-coupon bond that pays \$1 at expiration is \$0.6464. The time-0 price of a 8-year zero-coupon bond that pays \$1 at expiration is \$0.5461. What is the time-0 nonannualized forward interest rate from time $t = 6$ years to time $t = 8$ years?

Solution.
We use the formula of $R_0(T, T+s)$ with $T = 6, s = 2, P(0,6) = 0.6464$, and $P(0,8) = 0.5461$. Thus,

$$R_0(T, T+s) = \frac{P(0,T)}{P(0,T+s)} - 1 = \frac{0.6462}{0.5461} - 1 = 0.18367 \blacksquare$$

Example 80.2
GS has entered into a forward rate agreement which expires in 4 years on a loan to be made in two years and which expires in four years. A zero-coupon bond paying \$1 in 4 years has a price of \$0.5523 today. A zero-coupon bond

[1] See Page 218 of [1].

paying \$1 in 2 years has a price of \$0.8868 today. Now assume that two years have passed and a zero-coupon bond paying \$1 in yet another 2 years has a price of \$0.7769. What will be GS' payoff to the forward rate agreement at the end of 4 years?

Solution.
The payoff of the FRA is given by the formula

$$\text{Payoff to FRA} = \frac{1}{P(T,T+s)} - \frac{P(0,T)}{P(0,T+s)}$$

with $T = s = 2, P(0,2) = 0.8868, P(0,4) = 0.5523$, and $P(2,4) = 0.7769$. Thus,

$$\text{Payoff to FRA} = \frac{1}{0.7769} - \frac{0.8868}{0.5523} = -\$0.3184821582,$$

which means that GS will lose money in this financial transaction ■

Pricing Interest Rate Caplets
Consider a loan of \$1 to be made at time T for a period of length s. One way to hedge against rate flactuation is by buying a caplet. By a **caplet** we mean a European call option with underlying asset a forward rate at the caplet maturity. For a caplet with strike rate K_R and maturity $T+s$, the payoff at maturity is given by

$$\text{Payoff of caplet} = \max\{0, R_T - K_R\} \tag{80.1}$$

where $R_T = R_T(T, T+s)$ is the lending rate between time T and time $T+s$. The caplet permits the borrower to pay the time$-T$ market interest rate if it is below K_R, but receive a payment for the difference in rates if the rate is above K_R.

Example 80.3
GS has bought a caplet which expires in 4 years. The caplet's strike rate of 0.2334. A zero-coupon bond paying \$1 in 4 years has a price of \$0.5523 today. A zero-coupon bond paying \$1 in 2 years has a price of \$0.8868 today. Now assume that two years have passed and a zero-coupon bond paying \$1 in yet another 2 years has a price of \$0.7769. What will be GS' payoff to the caplet at time $t = 4$?

Solution.
We have: $T = s = 2, P(0,2) = 0.8868, P(0,4) = 0.5523, P(2,4) = 0.7769$ and $K_R = 0.2334$. Thus, using the formula

$$R_T(T, T+s) = \frac{1}{P(T, T+s)} - 1$$

we find

$$R_T(2,4) = \frac{1}{0.7769} - 1 = 0.2871669456.$$

The payoff of the caplet at time $T+s$ is given by

$$\text{Payoff of caplet} = \max\{0, R_T - K_R\}.$$

Hence, the final answer is

$$\text{Payoff of caplet} = \max\{0, 0.2871669456 - 0.2334\} = \$0.0537669456 \blacksquare$$

The caplet can be settled at time T. In this case, the value of the caplet at time T is the discounted value of (80.1):

$$\frac{1}{1+R_T}\max\{0, R_T - K_R\}.$$

Note that $\frac{1}{1+R_T}$ is the time$-T$ price of a bond paying \$1 at time $T+s$. The previous expression can be written as

$$\begin{aligned}
\frac{1}{1+R_T}\max\{0, R_T - K_R\} =& \max\{0, \frac{R_T - K_R}{1+R_T}\} \\
=& (1+K_R)\max\{0, \frac{R_T - K_R}{(1+R_T)(1+K_R)}\} \\
=& (1+K_R)\max\left\{0, \frac{1}{1+K_R} - \frac{1}{1+R_T}\right\}
\end{aligned}$$

But the expression

$$\max\left\{0, \frac{1}{1+K_R} - \frac{1}{1+R_T}\right\}$$

is the payoff of a European put option on a bond with strike $\frac{1}{1+K_R}$. Thus, a caplet is equivalent to $(1+K_R)$ European call options on a zero-coupon bond at time T paying \$1 at time $T+s$ and with strike price $\frac{1}{1+K_R}$ and maturity T. The price of the caplet is then found by finding the price of the put option which is found by the Black formula discussed in the previous section.

Example 80.4
The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8495. The time-0 price of a 3-year zero-coupon bond with par value \$1 is \$0.7722. The 1-year forward price volatility is 0.105. What is the price of an interest rate caplet with strike rate 11% (effective annual rate) on a 1-year loan 2 years from now.

Solution.
We are given $T = 2, s = 1$ and $K_R = 0.11$. We first find the price of a European put with strike rate $\frac{1}{1+K_R} = \frac{1}{1.11} = 0.9009$ and maturity in 2 years. We use the Black formula. We have

$$F = \frac{P(0,3)}{P(0,2)} = \frac{0.7722}{0.8495} = 0.90901$$

$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln(0.90901/0.9009) + 0.5(0.105)^2(2)}{0.105\sqrt{2}} = 0.1346$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.1346 - 0.105\sqrt{2} = -0.0139$$

$$N(d_1) = 0.553536, \ N(d_2) = 0.494455.$$

Thus, the price of the put option is

$$P = P(0,T)[KN(-d_2) - FN(-d_1)] = 0.8495[0.9009(1-0.494455) - 0.90901(1-0.553536)] = \$0.04214.$$

The price of the caplet is

$$(1 + K_r)P = 1.11 \times 0.04214 = 0.0468 \blacksquare$$

Pricing Interest Rate Caps
An **interest rate cap** with strike price of K_R that makes payments at time t_i is a series of caplets with common strike price of K_R such that at time t_i the owner of the cap receives a payment in the amount

$$\max\{0, R_{t_i}(t_i, t_{i+1}) - K_R\}$$

whenever $R_{t_i}(t_i, t_{i+1}) > K_R$. The value of the cap is the sum of the values of the caplets that make up the cap.

Example 80.5
Suppose you own an interest rate cap with a strike price of 0.0345. The cap makes payments at times $t = 2, t = 3$, and $t = 4$. The forward rates for these time periods are as follows: $R_2(2,3) = 0.04343, R_3(3,4) = 0.09661, R_4(4,5) = 0.01342$, and $R_5(5,6) = 0.04361$. Find the cap payment for each value of the given t.

Solution.
We have

$$\begin{aligned}
\text{Cap payment at time t=2} &= \max\{0, 0.04343 - 0.0345\} = 0.00893\\
\text{Cap payment at time t=3} &= \max\{0, 0.09661 - 0.0345\} = 0.06211\\
\text{Cap payment at time t=4} &= \max\{0, 0.01342 - 0.0345\} = 0 \ \blacksquare
\end{aligned}$$

Practice Problems

Problem 80.1
Calculate the forward interest rate for a forward rate agreement for a 1-year loan initiated at year 3, the beginning of the fourth year.

Problem 80.2
GS has bought a caplet which expires in 4 years and has a strike rate of 0.2334. A zero-coupon bond paying \$1 in 4 years has a price of \$0.5523 today. A zero-coupon bond paying \$1 in 2 years has a price of \$0.8868 today. Now assume that two years have passed and a zero-coupon bond paying \$1 in yet another 2 years has a price of \$0.7769. What will be GS' payoff to the caplet at time $t = 2$?

Problem 80.3
A caplet with a certain strike rate has a payoff of 0.6556 at time 10 years and a payoff of 0.3434 in 3 years. The caplet is based on the forward rate $R_3(3, 10)$. What is the strike price for this caplet?

Problem 80.4
The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8573. The time-0 price of a 3-year zero-coupon bond with par value \$1 is \$0.7722. The 1-year forward price volatility is 0.105. You want to borrow \$1 for a period 1 year two years from now. To hedge against rate fluctuation you buy a caplet with strike rate 10.75%. This caplet is equivalent to a certain number N of European put option with maturity of 2 years and strike price K. Determine N and K.

Problem 80.5
The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8573. The time-0 price of a 3-year zero-coupon bond with par value \$1 is \$0.7722. The 1-year forward price volatility is 0.105. You want to borrow \$1 for a period 1 year two years from now. To hedge against rate fluctuation you buy a caplet with strike rate 10.75%. This caplet is equivalent to a certain number N of European put option with maturity of 2 years and strike price K. The price of the put is found using the Black formula. Determine the value of F.

Problem 80.6

The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8573. The time-0 price of a 3-year zero-coupon bond with par value \$1 is \$0.7722. The 1-year forward price volatility is 0.105. You want to borrow \$1 for a period 1 year two years from now. To hedge against rate fluctuation you buy a caplet with strike rate 10.75%. This caplet is equivalent to a certain number N of European put option with maturity of 2 years and strike price K. The price of the put is found using the Black formula. Find $d_1, d_2, N(d_1)$, and $N(d_2)$.

Problem 80.7

The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8573. The time-0 price of a 3-year zero-coupon bond with par value \$1 is \$0.7722. The 1-year forward price volatility is 0.105. You want to borrow \$1 for a period 1 year two years from now. To hedge against rate fluctuation you buy a caplet with strike rate 10.75%. This caplet is equivalent to a certain number N of European put option with maturity of 2 years and strike price K. The price P of the put is found using the Black formula. Find P.

Problem 80.8

The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8573. The time-0 price of a 3-year zero-coupon bond with par value \$1 is \$0.7722. The 1-year forward price volatility is 0.105. You want to borrow \$1 for a period 1 year two years from now. To hedge against rate fluctuation you buy a caplet with strike rate 10.75%. Find the price of the caplet.

Problem 80.9

The time-0 price of a 1-year zero-coupon bond with par value \$1 is \$0.9259. The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8495. The 1-year forward price volatility is 0.10. You want to borrow \$1 for a period 1 year one year from now. To hedge against rate fluctuation you buy a caplet with strike rate 11.5%(effectice annual). Find the price of this caplet.

Problem 80.10

The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8495. The time-0 price of a 3-year zero-coupon bond with par value \$1 is \$0.7722. The 1-year forward price volatility is 0.105. You want to borrow \$1 for a period 1 year two years from now. To hedge against rate fluctuation you buy a caplet with strike rate 11.5%(effectice annual). Find the price of this caplet.

Problem 80.11
The time-0 price of a 3-year zero-coupon bond with par value \$1 is \$0.7722. The time-0 price of a 4-year zero-coupon bond with par value \$1 is \$0.7020. The 1-year forward price volatility is 0.11. You want to borrow \$1 for a period 1 year three years from now. To hedge against rate fluctuation you buy a caplet with strike rate 11.5%(effectice annual). Find the price of this caplet.

Problem 80.12
What is the price of a 3-year interest rate cap with strike rate 11.5% (effective annual)?

Binomial Models for Interest Rates

In this chapter we examine binomial interest rate models, in particular the Black-Derman-Toy model.

81 Binomial Model for Pricing of a Zero-Coupon Bond

In this section we discuss pricing a zero-coupon bond based on interest rates. The model is based on a simple binomial tree that models interest rates fluctuations.
The **binomial interest rate tree** is patterned after the binomial option pricing model, this model assumes that interest rates follow a binomial process in which in each period the rate is either higher or lower.
Assume a 1-period, risk-free spot rate r follows a process in which in each period the rate is either equal to a proportion u times its beginning-of-the-period rate or a proportion d times its beginning-of-the-period rate with $u > d$. After one period, there would be two possible one-period spot rate: $r_u = ur_0$ and $r_d = dr_0$. Assuming u and d are constants over different periods, then after two periods there would be three possible rates: $r_{uu} = u^2r_0, r_{ud} = udr_0$, and $r_{dd} = d^2r_0$. This creates a two-period binomial interest rate tree.

Example 81.1
Construct a three-year interest rate binomial tree assuming that the current 1-year spot rate is 10% and that each year rate moves up 1% or down 0.5%, each with risk-neutral probability 0.5

Solution.
The upward parameter is $u = 1.1$ and the downward parameter is $d = 0.95$. The tree is shown in Figure 81.1 ■

Now, the pricing of a zero-coupon bond that pays \$1 at maturity can be determined in much the same way we determined option prices in a binomial stock-price tree. We illustrate the process in the next example.

Example 81.2
Use Figure 81.1, to find the price of 1-year, 2-year and 3-year zero-coupon default-free[1] bonds that pay \$1 at maturity.

Solution.
The price of a 1-year bond is

$$P(0,1) = e^{-0.10} = \$0.9048.$$

[1]A bond that has no default risk.

The two-year bond is priced by working backward along the tree. In the second period, the price of the bond is \$1. One year from today, the bond will have the price e^{-r_u} with probability $p = 0.5$ or e^{-r_d} with probability $1 - p = 0.5$. Thus, the price of the bond is

$$P(0,2) = e^{-0.10}[0.5e^{-0.11} + 0.5e^{-0.095}] = \$0.81671.$$

Likewise, a three-year bond is priced as

$$\begin{aligned}P(0,3) =&e^{-r_0}[pe^{-r_u}(pe^{-r_{uu}} + (1-p)e^{-r_{ud}})\\&+(1-p)e^{-r_d}(pe^{-r_{du}} + (1-p)e^{-r_{dd}})]\\=&e^{-0.10}[0.5e^{-0.11}[0.5e^{-0.121} + 0.5e^{-0.1045}] + 0.5e^{-0.095}[0.5e^{-0.1045} + 0.5e^{-0.09025}]]\\=&\$0.7353 \blacksquare\end{aligned}$$

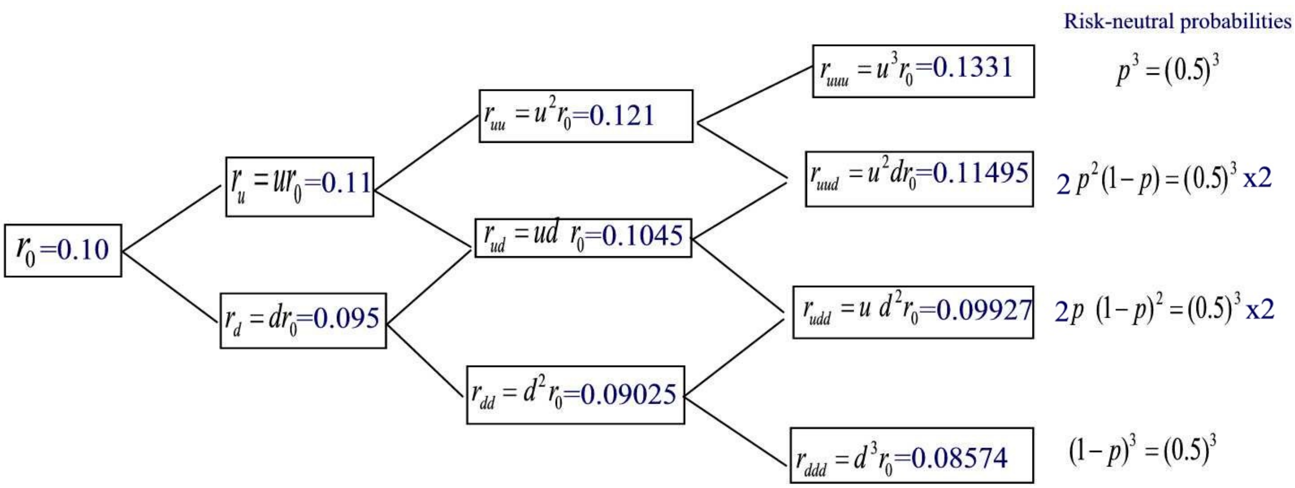


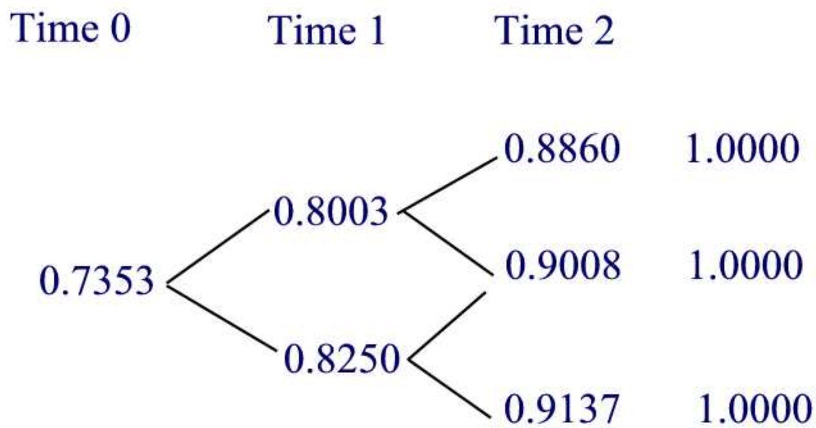
Figure 81.1

Using the interest rate binomial tree one can construct a binomial tree of the prices of a bond. We illustrate this point in the next example.

Example 81.3
Use Figure 81.1, to construct the tree of prices of a 3-year zero-coupon bond that pays \$1 at maturity.

Solution.
The time-2 price of the bond at the node uu is $e^{-0.121} = \$0.8860$. The time-2

price of the bond at the node ud is $e^{-0.1045} = \$0.9008$.The time-1 price of the bond at the node u is $e^{-0.11}(0.8860 \times 0.5 + 0.9008 \times 0.5) = \$0.8003$. We obtain the following tree of the bond prices.



Yields in the Binomial Model
The yield of an $n-$year zero-coupon bond is given by the formula

$$\text{yield} = -\frac{\ln P(0,n)}{n}.$$

Example 81.4
Find the yields of the 1-year, 2-year, and 3-year zero-coupon bond based on the interest rate tree of Example 81.1.

Solution.
The yield of the 1-year bond is

$$\text{yield} = -\frac{\ln P(0,1)}{1} = -\frac{\ln(0.9048)}{1} = 10.004\%$$

The yield of the 2-year bond is

$$\text{yield} = -\frac{\ln P(0,2)}{2} = -\frac{\ln(0.81671)}{2} = 10.12\%$$

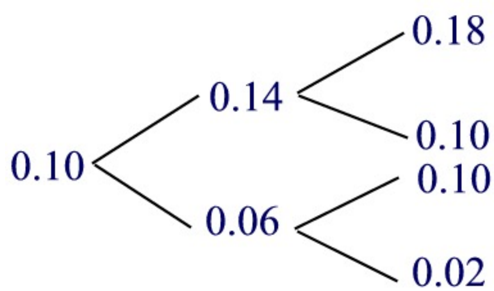
The yield of the 3-year bond is

$$\text{yield} = -\frac{\ln P(0,3)}{3} = -\frac{\ln(0.7353)}{3} = 10.25\% \blacksquare$$

Remark 81.1
Note that in the previous example, the yields increase with maturity. This is not the case when the up and down move are symmetric with probability 0.5. In this case, the yields decrease with maturity. See Problem 81.6. Moreover, these yields are lower than current interest rate, which means that the interest rate binomial model underprice the bonds.

Binomial Interest Rate Model in Bond Option Pricing
The binomial tree used for pricing a zero-coupon bond can be used to price an option with underlying asset a zero-coupon bond. The pricing process is exactly the same as pricing option for stock, just that the price of the underlying asset is bond instead of stock. The only thing to be careful about is that the calculation here assumes there is a non-recombining tree (the binomial tree constructed above are examples of recombining trees).
We illustrate the process of pricing a bond option using an interest rate binomial tree in the next example.

Example 81.5
Given the following interest rate binomial tree.



Consider a 2-year European put option on a 1-year zero-coupon bond with strike price \$0.88. What is the price of this option?

Solution.
The price of the bond at the end of year 2 is $e^{-0.18} = \$0.8353$ at the node uu; $e^{-0.10} = \$0.9048$ at the nodes ud and du; $e^{-0.02} = \$0.9802$. The put option pays off only if the price of the bond is less than \$0.88. This occurs only at the node uu. The put payoff in this case is $0.88 - 0.8353 = \$0.0447$. Using the interest rates along the tree, and accounting for the 0.25 risk-neutral probability of raching that one node, we obtain the put option price of

$$0.0447 \times (0.5)^2 e^{-0.14-0.10} = \$0.0088 \blacksquare$$

Practice Problems

Problem 81.1
Construct a two-year interest rate binomial tree assuming that the current 1-year spot rate is 10% and that each year rate moves up or down 4%, each with risk-neutral probability 0.5.

Problem 81.2
Use Problem 81.1, to find the price of 1-year zero-coupon default-free bonds that pay \$1 at maturity.

Problem 81.3
Use Problem 81.1, to find the price of 2-year zero-coupon default-free bonds that pay \$1 at maturity.

Problem 81.4
Use Problem 81.1, to find the price of 3-year zero-coupon default-free bonds that pay \$1 at maturity.

Problem 81.5
Use Problem 81.1, to construct the tree of prices of a 3-year zero-coupon bond that pays \$1 at maturity.

Problem 81.6
(a) Construct a two-period interest rate tree assuming that the current 1-year rate is 10% and that each year the 1-year rate moves up or down 2%, with probability 0.5.
(b) Find the prices of 1-, 2-, and 3-year zero coupon bonds that pays \$1 at maturity using the above tree.
(c) Find the yields of the bonds in (b).

Problem 81.7
Construct a two-year interest rate binomial tree assuming that the current 1-year spot rate is 12% and that each year rate moves up or down 4%, each with risk-neutral probability 0.5.

Problem 81.8
Using the tree of the previous exercise, consider a 2-year European put option on a 1-year zero-coupon bond with strike price \$0.85. What is the price of this option?

Problem 81.9 ‡
A discrete-time model is used to model both the price of a nondividend-paying stock and the short-term (risk-free) interest rate. Each period is one year.
At time 0, the stock price is $S_0 = 100$ and the effective annual interest rate is $r_0 = 5\%$.
At time 1, there are only two states of the world, denoted by u and d. The stock prices are $S_u = 110$ and $S_d = 95$. The effective annual interest rates are $r_u = 6\%$ and $r_d = 4\%$.
Let $C(K)$ be the price of a 2-year K-strike European call option on the stock.
Let $P(K)$ be the price of a 2-year K-strike European put option on the stock.
Determine $P(108) - C(108)$.

82 The Basics of the Black-Derman-Toy Model

The interest rate models that we have examined so far are arbitrage-free only in a world consistent with their assumptions. In the real world, however, they will generate apparent arbitrage opportunities, in the sense that observed prices will not match the theoretical prices obtained through these models. So the important question is whether a particular interest rate model fits given real-world data. Recall that the Vasicek and CIR models often do not match data exogenous to the models, because the models are based on four parameters while real-world data sets may have more than four members. The Black-Derman-Toy (BDT) model attempts to incorporate an element of calibration[1] to allow applications to real-world data.

The basic idea of the Black-Dermon-Toy model is to construct a recombining binomial tree of short-term interest rates[2], with a flexible enough structure to match the real-world data. The short-term rates are assumed to be log-normally distributed. It is also assumed that the risk-neutral probability of an up move in the interest rate is 0.50.

We next describe the BDT tree: For each period in the tree there are two parameters, R_{ih} and σ_i, where the index i represents the number of time periods that have elapsed since time 0, and h is the length of each time period. The parameter R_{ih} is the rate **level parameter,** and σ_i is the volatility parameter. Both parameters are used to match the tree with observed data.

At each node of the tree the ratio of the up move to the down move is $e^{2\sigma\sqrt{h}}$. For example, at the node r_0, if $r_d = R_h$ then $r_u = R_h e^{2\sigma_1\sqrt{h}}$. Hence, in a single time period, the ratio between adjacent nodes is the same. A general form of the BDT tree is shown in Figure 82.1.

Example 82.1

You are given the following values of a Black-Derman-Toy binomial tree: $r_{dd} = 0.0925$ and $r_{ud} = 0.1366$. Find r_{uu}.

Solution.

We have

$$\frac{r_{uu}}{r_{du}} = \frac{r_{du}}{r_{dd}}$$

or

$$\frac{r_{uu}}{0.1366} = \frac{0.1366}{0.0925}.$$

[1]**Calibration** is the matching of a model to fit observed data.

[2]All interest rates in the BDT are annual effective, not continously compounded rates.

Solving this equation we find $r_{uu} = 0.2017$ ■

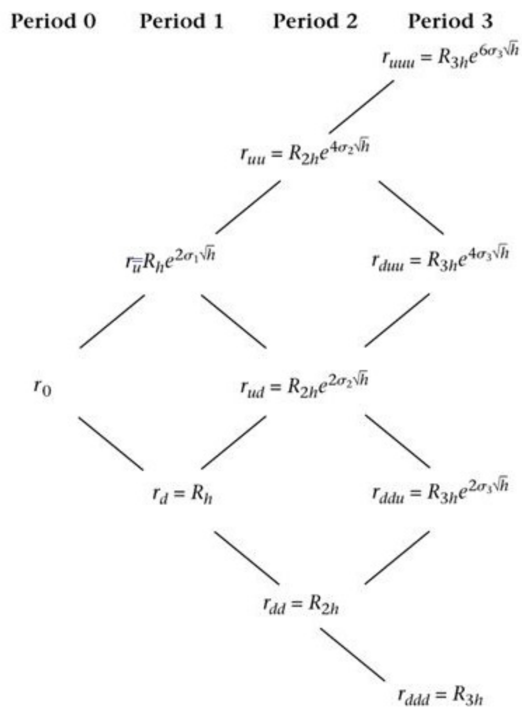


Figure 82.1

We next show through a series of examples how the Black-Derman-Toy model matches the observed data.

Example 82.2
The table below lists market information about bonds that we would like to match using the BDT model. All rates are effective annual rates.

Maturity (years)	Yield to Maturity	Bond Price ($)	Volatility in Year 1
1	10%	0.90901	NA
2	11%	0.8116	10%
3	12%	0.7118	15%
4	12.5%	0.6243	14%

The BDT tree, shown in Figure 82.2, depicts 1-year effective annual rates based on the above table .

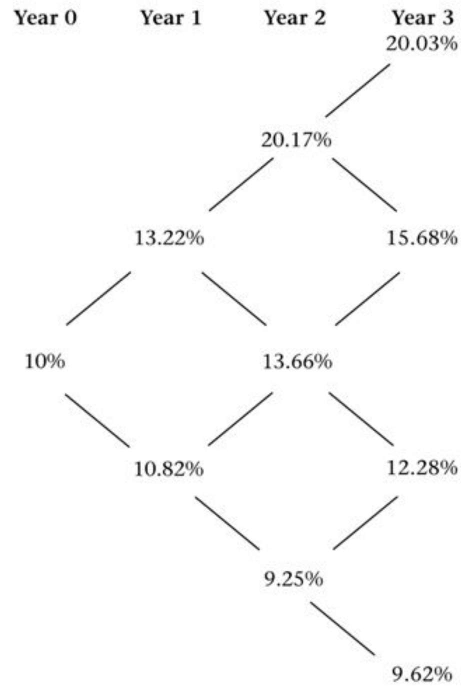


Figure 82.2

Construct the tree of 1-year bond prices implied by the tree in Figure 82.2.

Solution.
The price at each node is $\frac{1}{1+r}$, where r is at that node. Thus, we obtain the tree of prices shown in Figure 82.3 ■

Example 82.3
Verify that the 2- and 3-year zero-coupon bonds priced using BDT match the observed one listed in the table of Example 82.2.

Solution.
The time 0 of a 2-year zero-coupon bond is the discounted expected price at time 0 given by

$$0.9091(0.5 \times 0.8832 + 0.5 \times 0.9023) = \$0.8116.$$

The time 0 of a 3-year zero-coupon bond is the discounted expected price at time 0 given by

$$\begin{aligned}&0.9091[0.5 \times 0.8832(0.5 \times 0.8321 + 0.5 \times 0.8798)\\&\quad+0.5 \times 0.9023(0.5 \times 0.8798 + 0.5 \times 0.9153)] = \$0.7118\end{aligned}$$

The time 0 of a 4-year zero-coupon bond can be shown to be \$0.6243. See Problem 82.5 ■

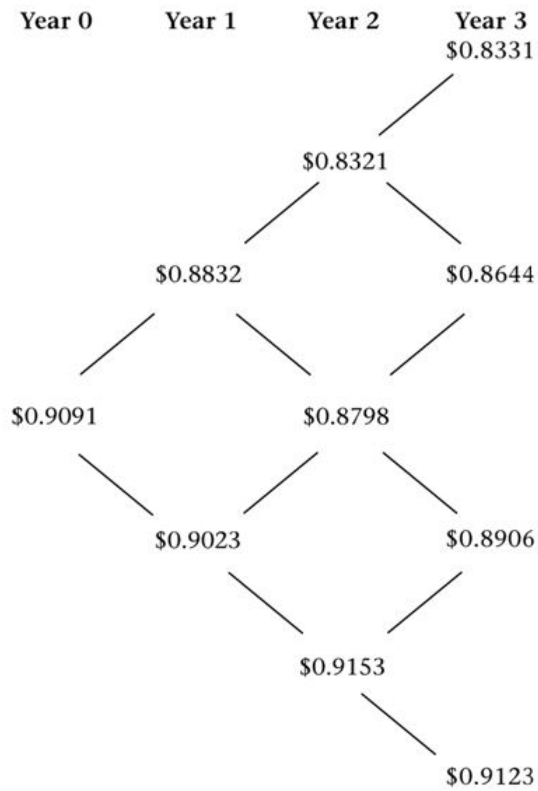


Figure 82.3

Now, let the time$-t$ price of a zero-coupon bond maturing at time T when the time$-t$ short term rate is $r(t)$ be denoted by $P[t,T,r(t)]$. Then the annualized yield of the bond is given by

$$y[t,T,r(t)] = P[t,T,r(t)]^{-\frac{1}{T-t}} - 1.$$

Example 82.4
Verify that the annual yields found through the BDT model match the observed ones in the table of Example 82.2.

Solution.
The time-0 price of a 1-year zero-coupon bond found by the BDT model is \$0.9091. Thus, the time-0 yield of this bond is

$$0.9091^{-\frac{1}{1-0}} - 1 = 10\%.$$

Likewise, the time-0 of a 2-year zero-coupon bond is

$$0.8116^{-\frac{1}{2-0}} - 1 = 11\%.$$

The time-0 of a 3-year zero-coupon bond is

$$0.7118^{-\frac{1}{3-0}} - 1 = 12\%.$$

The time-0 of a 4-year zero-coupon bond is

$$0.6243^{-\frac{1}{4-0}} - 1 = 12.5\% \blacksquare$$

We define the annualized yield volatility by

$$\text{Yield volatility} = \frac{0.5}{\sqrt{h}} \times \ln\left[\frac{y(h, T, r_u)}{y(h, T, r_d)}\right].$$

Example 82.5
Verify that the volatilities implied by the BDT tree match the observed volatilities.

Solution.
We will need to compute the implied bond yields in year 1 and then compute the volatility. For a 2-year bond (1-year bond in year 1) will be worth either \$0.8832 with yield $0.8832^{-\frac{1}{2-1}} - 1$ or \$0.9023 with yield $0.9023^{-\frac{1}{2-1}} - 1$. Thus, the yield volatility in year 1 is

$$0.5 \times \ln\left(\frac{0.8832^{-1} - 1}{0.9023^{-1} - 1}\right) = 10\%.$$

For a 3-year zero-coupon bond (2-year bond in year 1) the price of the bond is either

$$0.8832(0.5 \times 0.8321 + 0.5 \times 0.8798) = \$0.7560$$

with yield $0.7560^{-\frac{1}{3-1}} - 1$ or price

$$0.9023(0.5 \times 0.8798 + 0.5 \times 0.9153) = \$0.8099$$

with yield $0.8099^{-\frac{1}{3-1}} - 1$. Thus, the yield volatility in year 1 is

$$0.5 \times \ln\left(\frac{0.7560^{-\frac{1}{2}} - 1}{0.8099^{-\frac{1}{2}} - 1}\right) = 15\%.$$

Both yield volatilities match the one listed in the table of Example 82.2. We find the year 1 volatility yield of a 4-year zero-coupon bond in Problem 82.6 $\blacksquare$

Practice Problems

Problem 82.1
In a BDT tree, you are given the following: $r_{uu} = 0.2017$ and $r_{dd} = 0.0925$. Find r_{ud}.

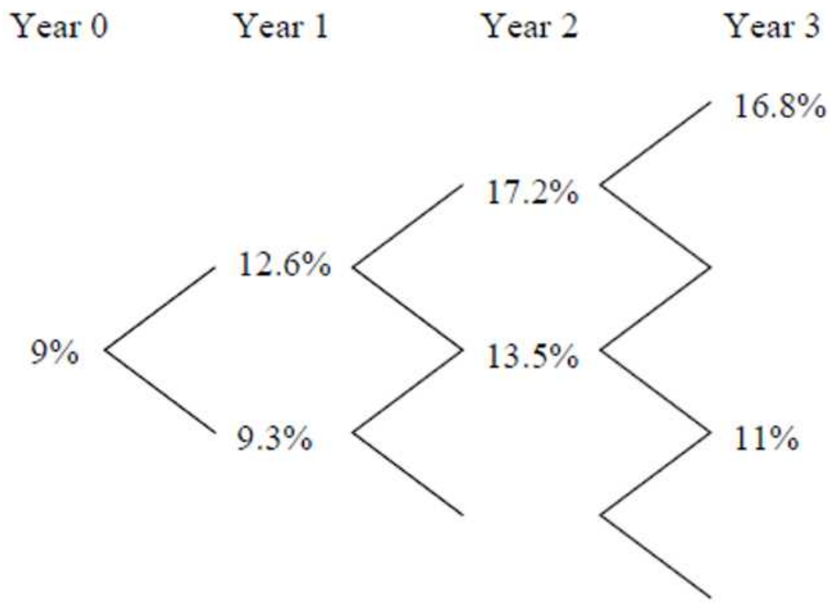
Problem 82.2
If we are given some number of nodes for a particular time period while the others are left blank, it may be possible to fill in the rest of the nodes, as the ratio between every two adjacent nodes in a single time period is the same. From this one can show that for a time i we have

$$r_{u^n d^{i-n}} = r_{d^i} e^{2n\sigma_i\sqrt{h}}.$$

where n is the number of times the interest rate has gone up and $i-n$ is the number of times the interest rate has gone down.
In a BDT model, you are given that $r_{ddddd} = 0.0586$, and $\sigma_5 = 0.21$. Each time period in the binomial tree is 2 years. Find r_{uuudd}.

Problem 82.3
In a BDT model, you are given that $r_{uudddd} = 0.1185$, and $\sigma_6 = 0.43$. Each time period in the binomial tree is 1 year. Find r_{uuuuuu}.

Problem 82.4
You are given the following incomplete Black-Derman-Toy interest rate tree model for the effective annual interest rates:

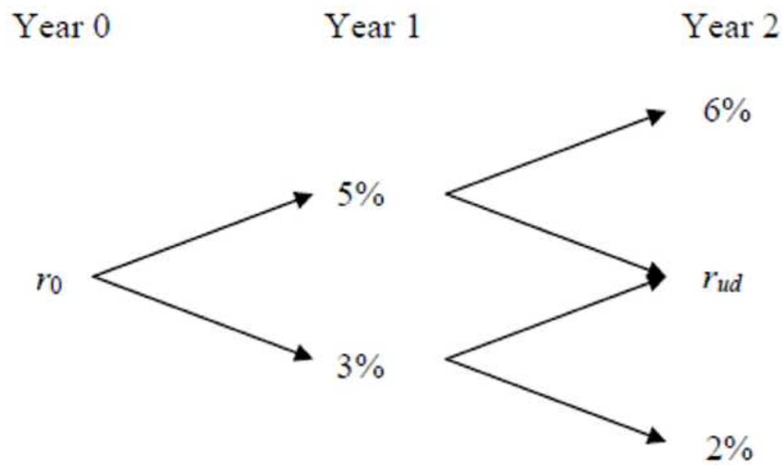


Fill in the three missing interest rates in the model.

Problem 82.5
Verify that the time-0 price of a 4-year zero-coupon bond implied by the BDT model (Example 82.2) is \$0.6243.

Problem 82.6
Verify that the annual yield volatility in year 1 of the 4-year zero-coupon bond implied by the BDT model is 14%.

Problem 82.7
The following is a Black-Derman-Toy binomial tree for effective annual interest rates.



Determine the value of r_{ud}.

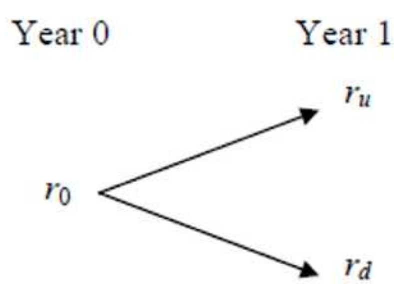
Problem 82.8
Using the BDT tree of the previous problem, Construct the tree of bond prices implied by the BDT model.

Problem 82.9 ‡
Using the tree of Problem 82.7, compute the volatility yield in year 1 of a 3-year zero-coupon bond.

Problem 82.10 ‡
You are given the following market data for zero-coupon bonds with a maturity payoff of $100.

Maturity (years)	Bond Price ($)	Volatility in Year 1
1	94.34	N/A
2	88.50	10%

A 2-period Black-Derman=Toy interest tree is calibrated using the data from above:



(a) Calculate r_0 and σ_1, the volatility in year 1 of the 2-year zero-coupon bond.
(b) Calculate r_d, the effective annual rate in year 1 in the "down" state.

Problem 82.11 ‡
You are to use a Black-Derman-Toy model to determine $F_{0,2}[P(2,3)]$, the forward price for time-2 delivery of a zero-coupon bond that pays 1 at time 3. In the Black-Derman-Toy model, each period is one year. The following effective annual interest rates are given:

$$\begin{aligned} r_d =& 30\% \\ r_u =& 60\% \\ r_{dd} =& 20\% \\ r_{uu} =& 80\% \end{aligned}$$

Determine $1000 \times F_{0,2}[P(2,3)]$.

83 Black-Derman-Toy Pricing of Caplets and Caps

In this section, we use the Black-Derman-Toy tree to price caplets and caps. We first discuss the pricing of a caplet.
Although a caplet makes a payment at the end of each year, it is valued at the beginning of the year. Thus, the value of a $T-$year caplet with strike rate K_R at time $T-1$ is the discounted value of the year$-T$ payoff. It is given by the formula

$$\text{Notional} \times \frac{\max\{0, R_{T-1} - K_R\}}{1 + R_{T-1}}$$

where $R_{T-1} = r_x$ is a rate in year $T-1$ and x is some product of u's and d's (up and down movements in the interest rates).

Example 83.1
The BDT interest rate tree is given in Figure 83.1.

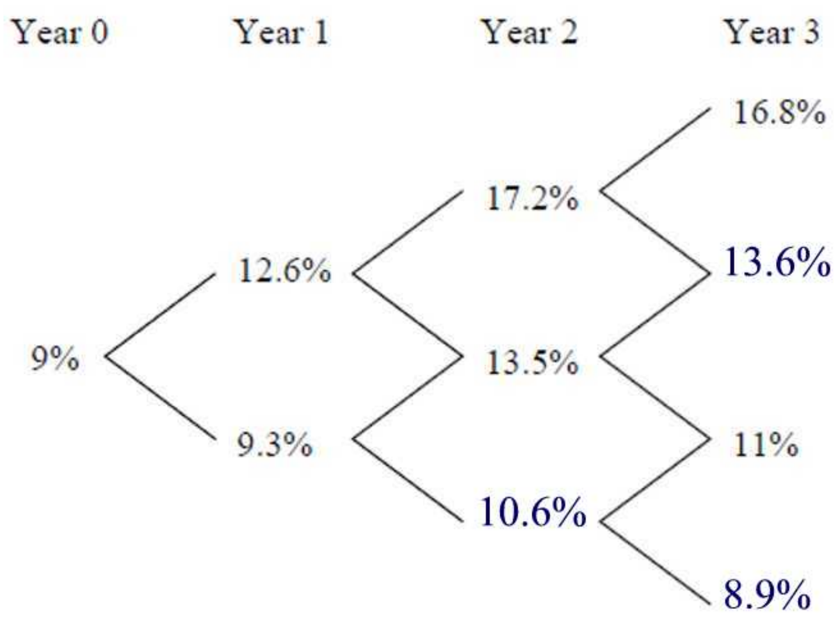


Figure 83.1

Consider a year-4 caplet with strike rate of 10.5%. What are the year-3 caplet values?

Solution.
A year-4 caplet at year 3 has the following four values:

$$\begin{aligned}
\text{Notional} \times \frac{\max\{0, r_{uuu} - K_R\}}{1 + r_{uuu}} =& 100 \times \frac{0.168 - 0.105}{1.168} = 5.394 \\
\text{Notional} \times \frac{\max\{0, r_{uud} - K_R\}}{1 + r_{uud}} =& 100 \times \frac{0.136 - 0.105}{1.136} = 2.729 \\
\text{Notional} \times \frac{\max\{0, r_{udd} - K_R\}}{1 + r_{udd}} =& 100 \times \frac{0.111 - 0.105}{1.11} = 0.450 \\
\text{Notional} \times \frac{\max\{0, r_{ddd} - K_R\}}{1 + r_{ddd}} =& 0 \blacksquare
\end{aligned}$$

Now, remember that the risk-neutral probability on an up movement in the short-term interest rate is always 0.5 in the BDT model. Thus, if P_x is the expected value of a caplet at some node in a BDT binomial tree, where x is some product of u's and d's (up and down movements in the interest rates), then we can obtain P_x using the following formula:

$$P_x = (1 + r_x)^{-1}[0.5P_{xu} + 0.5P_{xd}].$$

Example 83.2
Using the information of the previous exercise, find the time−2 value of the caplet.

Solution.
At time−2 there are three nodes. The values of the caplet at these nodes are

$$\begin{aligned}
(1 + 0.172)^{-1}[0.5 \times 5.394 + 0.5 \times 2.729] =& 3.4654 \\
(1 + 0.135)^{-1}[0.5 \times 2.729 + 0.5 \times 0.450] =& 1.4004 \\
(1 + 0.106)^{-1}[0.5 \times 0.450 + 0.5 \times 0] =& 0.2034 \blacksquare
\end{aligned}$$

Example 83.3
Find the time−1 and time−0 values of the caplet of Example 83.1

Solution.
At year 1, there are two nodes. The values of the caplet at these nodes are

$$\begin{aligned}
(1 + 0.126)^{-1}[0.5 \times 3.4654 + 0.5 \times 1.4004] =& 2.1607 \\
(1 + 0.093)^{-1}[0.5 \times 1.4004 + 0.5 \times 0.2034] =& 0.7337.
\end{aligned}$$

The time−0 value of the caplet is

$$(1 + 0.09)^{-1}[0.5 \times 2.1607 + 0.5 \times 0.7337] = \$1.3277 \blacksquare$$

Next, we discuss the pricing of a cap. Recall that an interest rate cap pays the difference between the realized interest rate in a period and the interest cap rate, if the difference is positive. Let us illustrate how a cap works. Suppose you borrow \$1,000,000 for 5 years at a floating rate which is determined once a year. The initial rate is 6%, and the cap is 7%. The floating rate for years 2-5 are 5.5%, 8%, 9.5%, 6.5%. The amount of interest without the cap and with the cap are shown in the table below.

End of Year	Interest payment without a cap(\$)	Interest payment with a cap(\$)
1	60,000	60,000
2	55,000	55,000
3	80,000	70,000
4	95,000	70,000
5	65,000	65,000

Example 83.4
Find the price of a 4-year cap with strike rate 10.5% using the BDT model given in Figure 83.1 for the notional amount of \$100.

Solution.
The value of the cap is the sum of the values of the year-3 cap payments, year-2 cap payments, and year-1 cap payments. The cap payment in year-3 has the following three values:

$$\begin{aligned}
\text{Notional} \times \frac{\max\{0, r_{uuu} - K_R\}}{1 + r_{uuu}} =&100 \times \frac{0.168 - 0.105}{1.168} = 4.5377 \\
\text{Notional} \times \frac{\max\{0, r_{uud} - K_R\}}{1 + r_{uud}} =&100 \times \frac{0.136 - 0.105}{1.136} = 2.7289 \\
\text{Notional} \times \frac{\max\{0, r_{udd} - K_R\}}{1 + r_{udd}} =&100 \times \frac{0.11 - 0.105}{1.11} = 0.4505 \\
\text{Notional} \times \frac{\max\{0, r_{ddd} - K_R\}}{1 + r_{ddd}} =&0
\end{aligned}$$

The node at the bottom is 0 because 8.9% is smaller than 10.5%.
The year-3 cap payment in year-2 has the following three possible values:

$$\begin{aligned}(1+0.172)^{-1}[0.5\times 4.5377+0.5\times 2.7289] =&3.1000\\(1+0.135)^{-1}[0.5\times 2.7289+0.5\times 0.4505] =&1.4006\\(1+0.106)^{-1}[0.5\times 0.4505+0.5\times 0] =&0.2037.\end{aligned}$$

The year-3 cap payment in year-1 has the two possible values

$$\begin{aligned}(1+0.126)^{-1}[0.5\times 3.1000+0.5\times 1.4006] =&1.9985\\(1+0.093)^{-1}[0.5\times 1.4006+0.5\times 0.2037] =&0.7339\end{aligned}$$

Thus, the year-3 cap payment value is

$$(1+0.09)^{-1}[0.5\times 1.9985+0.5\times 0.7339]=1.2534.$$

We next find the value of the year-2 cap payment. The cap payment in year 2 has the following three possible values

$$\begin{aligned}\text{Notional}\times\frac{\max\{0,r_{uu}-K_R\}}{1+r_{uu}} =&100\times\frac{0.172-0.105}{1.172}=5.7167\\\text{Notional}\times\frac{\max\{0,r_{ud}-K_R\}}{1+r_{ud}} =&100\times\frac{0.135-0.105}{1.135}=2.6432\\\text{Notional}\times\frac{\max\{0,r_{dd}-K_R\}}{1+r_{dd}} =&100\times\frac{0.106-0.105}{1.106}=0.0904\end{aligned}$$

The year-2 cap payment in year-1 has the two possible values

$$\begin{aligned}(1+0.126)^{-1}[0.5\times 5.7167+0.5\times 2.6432] =&3.7122\\(1+0.093)^{-1}[0.5\times 2.6432+0.5\times 0.0904] =&1.2505\end{aligned}$$

The year-2 cap payment has the value

$$(1+0.09)^{-1}[0.5\times 3.7122+0.5\times 1.2505]=2.2765.$$

The cap payment in year-1 has the following two possible values

$$\begin{aligned}\text{Notional}\times\frac{\max\{0,r_u-K_R\}}{1+r_u} =&100\times\frac{0.126-0.105}{1.126}=1.8650\\\text{Notional}\times\frac{\max\{0,r_d-K_R\}}{1+r_d} =&0\end{aligned}$$

The year-1 cap payment has the value

$$(1 + 0.09)^{-1}[0.5 \times 1.8650 + 0.5 \times 0] = 0.8555$$

Finally, the 4-year cap has the value

$$1.2534 + 2.2765 + 0.8555 = \$4.3854 \blacksquare$$

Practice Problems

Problem 83.1
In a particular Black-Derman-Toy binomial tree, you are given $r_{uuu} = 0.1, r_{uuuu} = 0.11$, and $r_{uuud} = 0.09$. A particular caplet has a strike rate of 0.08 and a notional amount of \$1. What is the expected value of a caplet at node uuu of the BDT binomial tree? That is, find P_{uuu}.

Problem 83.2
The price of a caplet today is \$3. The caplet has a strike rate of 0.06 and a notional amount of \$100. The one-year short-term interest rate today is 0.05. In one year, the one-year short-term interest rate will be either 0.1 or Q. Find Q using a one-period BDT binomial tree.

Problem 83.3
You are given the following short-term interest rates in a Black-Derman-Toy binomial tree: $r_0 = 0.04, r_d = 0.02, r_u = 0.11, r_{dd} = 0.037, r_{du} = 0.074, r_{uu} = 0.148$. A 2-year caplet (which actually pays at time $t = 3$ years) has a strike rate of 0.07. The caplet has a notional amount of \$100. What is the current value of this caplet?

Problem 83.4
You are given the following short-term interest rates in a Black-Derman-Toy binomial tree: $r_0 = 0.05, r_d = 0.04, r_u = 0.09, r_{dd} = 0.06, r_{uu} = 0.18, r_{ddd} = 0.04, r_{uud} = 0.16$. A 3-year caplet (which actually pays at time $t = 4$ years) has a strike rate of 0.07. The caplet has a notional amount of \$100. What is the current value of this caplet?

Problem 83.5
Find the price of a 4-year cap with strike rate 12% using the BDT model given in Figure 83.1 for the notional amount of \$100.

Supplement

This supplement includes a discussion of Jensen's inequality and its applications. It also includes a discussion of utility theory and its relation to risk-neutral pricing of a stock.

84 Jensen's Inequality

In this section, we discuss Jensen's inequality as applied to probability theory. We first start with the following definition: A function f is said to be **convex** on the open interval (a, b) if and only if for any u, v in (a, b) and any $0 \le \lambda \le 1$, we have

$$f(\lambda u + (1 - \lambda)v) \le \lambda f(u) + (1 - \lambda)f(v).$$

Graphically, this is saying that for any point on the line segment connecting u and v, the value of f at that point is smaller than the value of the corresponding point on the line line segement connecting $(u, f(u))$ and $(v, f(v))$. Thus, the arc from $f(u)$ to $f(v)$ is below the line segment connecting $f(u)$ and $f(v)$. See Figure 84.1.

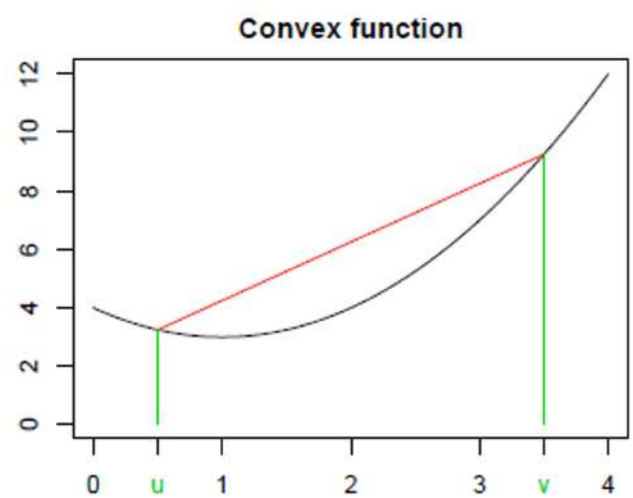


Figure 84.1

It is easy to see that a differentiable function f is convex if and only if the slope of the tangent line at a point is less than or equal to the slope of any secant line connecting the point to another point on the graph of f.
The following result gives a characterization of convex functions.

Proposition 84.1
A twice differentiable function f is convex in (a, b) if and only if $f''(x) \ge 0$ in (a, b).

Proof.
Suppose that f is convex in (a,b). Fix $a<v<b$ and let $a<u<b$. Then

$$f'(u) \leq \frac{f(v)-f(u)}{v-u}$$

and

$$f'(v) \geq \frac{f(u)-f(v)}{u-v}.$$

Thus,

$$\frac{f'(u)-f'(v)}{u-v} \geq \frac{f(v)-f(u)}{v-u} - \frac{f(u)-f(v)}{u-v} = 0.$$

Hence,

$$f'(v) = \lim_{u \to v} \frac{f'(u)-f'(v)}{u-v} \geq 0.$$

Now, suppose that $f''(x) \geq 0$ for all x in (a,b). This means that f' is increasing in (a,b). Let $a<u<b$. Define the function $g(v) = f(v) - f'(u)(v-u)$. Then $g'(v) = f'(v) - f'(u) \geq 0$ if $v \geq u$ and $g'(v) \leq 0$ if $v \leq u$. Thus, g is increasing to the right of u and decreasing to the left of u. Hence, if $v<u$ we have

$$0 \leq g(u) - g(v) = f(u) - f(v) + f'(u)(v-u)$$

which can be written as

$$f'(u) \leq \frac{f(v)-f(u)}{v-u}.$$

For $v>u$ we have

$$0 \leq g(v) - g(u) = f(v) - f(u) - f'(u)(v-u)$$

which can written as

$$f'(u) \leq \frac{f(v)-f(u)}{v-u}.$$

It follows that f is convex ■

Example 84.1
Show that the exponential function $f(x) = e^x$ is convex on the interval $(-\infty, \infty)$.

Solution.
Since $f''(x) \geq 0$ for all real number x, the exponential function is convex by the previous proposition ∎

Proposition 84.2 *(Jensen)*
If f is a convex function in (a, b) and X is a random variable with values in (a, b) then

$$E[f(X)] \geq f[E(X)].$$

Proof.
The equation of the tangent line to the graph of f at $E(X)$ is given by the equation

$$y = f[E(X)] + f'[E(X)](x - E(X)).$$

Since the function is convex, the graph of $f(x)$ is at or above the tangent line. Therefore,

$$f(x) \geq f[E(X)] + f'[E(X)](x - E(X)).$$

Taking expected values of both sides:

$$E[f(X)] \geq f[E(X)] + f'[E(X)](E(X) - E(X)) = f[E(X)] \ ∎$$

Example 84.2
Let X be a binomial random variable that assumes the two values 1 and 2 with equal probability. Verify that $E[e^X] \leq e^{E(X)}$.

Solution.
We have $E(X) = 0.5 \times 1 + 0.5 \times 2 = 1.5$ so that $e^{E(X)} = e^{1.5} = 4.482$. On the other hand, $E[e^X] = 0.5e + 0.5e^2 = 5.054 > e^{E(X)}$ ∎

Example 84.3
Assume that the price of a stock at expiration is a binomial random variable and is equally likely to be 35 or 45. Verify that Jensen's inequality holds for the payoff on an 40 strike call.

Solution.
Let $f(x) = \max\{0, x - K\}$. Then f is a convex function. We have

$$E(X) = 0.5 \times 35 + 0.5 \times 45 = 40$$

so that

$$f[E(X)] = \max\{0, 40 - 40\} = 0.$$

On the other hand,

$$E[f(X)] = 0.5 \times f(45) + 0.5 \times f(35) = 0.5 \times 5 + 0.5 \times 0 = 2.5.$$

Hence, $E[f(X)] \geq f[E(X)]$ ∎

We say that a function f is **concave** on the interval (a, b) if and only if for any u, v in (a, b) and any $0 \leq \lambda \leq 1$, we have

$$f(\lambda u + (1 - \lambda)v) \geq \lambda f(u) + (1 - \lambda)f(v).$$

Proposition 84.3
f is concave in (a, b) if and only if $-f$ is convex in (a, b).

Proof. Let u, v be in (a, b) and $0 \leq \lambda \leq 1$. Then f is concave if and only if $f(\lambda u + (1 - \lambda)v) \geq \lambda f(u) + (1 - \lambda)f(v)$ which is equivalent to $(-f)f(\lambda u + (1-\lambda)v) \leq \lambda(-f)(u) + (1-\lambda)(-f)(v)$. That is, if and only if $-f$ is convex ∎

Remark 84.1
It follows from Propositions and that f is concave in (a, b) if and only if $f''(x) \leq 0$ in (a, b).

Example 84.4
Show that the function $f(x) = \ln x$ is concave in $(0, \infty)$.

Solution.
This follows from the above proposition since $f''(x) = -\frac{1}{x^2} < 0$ ∎

Practice Problems

Problem 84.1
Show that if f is convex on (a, b) then

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

for all x, y in (a, b).

Problem 84.2
Show that the function $f(x) = -\ln x$ is convex for $x > 0$.

Problem 84.3
Show that if f is a concave function in (a, b) and X is a random variable with values in (a, b) then

$$E[f(X)] \geq f[E(X)].$$

Problem 84.4
Show that for any random variable X, we have $E[X^2] \geq [E(X)]^2$.

Problem 84.5
Assume that the price of a stock at expiration is equally likely to be 50, 100, or 150. Verify that Jensen's inequality holds for the payoff on an 80 strike call.

Problem 84.6
Assume that the price of a stock at expiration is equally likely to be 60, 70, or 110. Verify that Jensen's inequality holds for the payoff on an 80 strike put.

85 Utility Theory and Risk-Neutral Pricing

In economics, the well-being of investors is not measured in dollars, but in utility. **Utility** is a measure of satisfaction. An investor is assumed to have an utility function that translates an amount of wealth to his/her amount of satisfaction. It is clearly an increasing function since an investor is more satisfied with more wealth than less wealth. However, it was observed by economists that the marginal utility of each extra dollar of wealth decreases as ones wealth increases. That is, if the wealth keeps increasing, at one point, an additional dollar of wealth will make the investor less happy than the previous dollars. Therefore, we can say that the utility function increases at a decreasing rate which means in terms of calculus that the second derivative of the utility function is negative. According to the previous section, we can say that a utility function is a concave function. If we denote that utility function by U, by Jensen's inequality we can write

$$E[U(X)] \leq U[E(X)].$$

In other words, the expected utility associated with a set of possible outcomes is less than the utility of getting the expected value of these outcomes for sure.

Example 85.1
Does the function $U(x) = \frac{x}{x+100}, x \geq 0$ represent a utility function?

Solution.
A utility function must have a positive first derivative and a negative second derivative. We have

$$U'(x) = \frac{100}{(x+100)^2} > 0$$

and

$$U''(x) = -\frac{200}{(x+100)^3} < 0.$$

Hence, U can be a utility function ■
Investors are risk-averse which means that an investor will prefer a safer investment to a riskier investment that has the same expected return. The next example illustrates this point.

Example 85.2
Suppose that your utility function is given by the formula $U(x) = \frac{x}{x+100}$ where x is your wealth. You have the option of owning either a stock A with predicted future prices of \$11 with probabiltiy of 50% and \$13 with probabiltiy 50% or a riskier stock B with predicted future prices of \$10 with probability of 80% and \$20 with probability of 20%.
(a) Find the expected value of each stock.
(b) Evaluate the utility of each stock.

Solution.
(a) The expected value of stock A is $0.5\times 11+0.5\times 13=\$12$. The expected value of stock B is $0.8\times 10+0.2\times 20=\$12$.
(b) The utilty value of owning stock A is

$$0.5\times\frac{11}{11+100}+0.5\times\frac{13}{13+100}=0.1071.$$

The utilty value of owning stock B is

$$0.8\times\frac{10}{10+100}+0.2\times\frac{20}{20+100}=0.106.$$

This shows that the utility declines with the riskier investment. In order for the riskier stock to be attractive for an investor, it must have higher expected return than the safer stock ■
In a fair game where gains and losses have the same expected return, a risk-averse investor will avoid such a bet. We illustrate this point in the example below.

Example 85.3
Given that an investor's with current wealth of 35 has a utility function given by $U(x)=1-e^{-\frac{x}{100}}$, where x denotes the total wealth. The investor consider making a bet depending on the outcome of flipping a coin. The investor will receive 5 is the coin comes up heads, oherwise he will lose 5.
(a) What is the investor's utility if he/she does not make the bet?
(b) What is the investor's expected utility from making the bet?

Solution.
(a) The investor's utility for not making the bet is $1-e^{-\frac{35}{100}}=0.295$.
(b) The investor's utility for making the bet is

$$0.5(1-e^{-\frac{40}{100}})+0.5(1-e^{-\frac{30}{100}})=0.294 \blacksquare$$

Now recall that the Black-Scholes pricing model of a stock assumes a risk-neutral environment. We will next discuss why risk-neutral pricing works using utility theory.
We will assume a world with only two assets: A risk-free bond and a risky stock.
Suppose that the economy in one period (for simplicity of the argument, we assume a period of one year) will have exactly two states: A high state with probabiltiy p and a low state with probabiltiy $1-p$..
Let U_H be value today of \$1 received in the future in the high state of the economy. Let U_L be value today of \$1 received in future in the low state of the economy. Since utility function is convex, the marginal utility of a dollar is more when there is less wealth. Thus, $U_H < U_L$.

Example 85.4 *(Valuing the risk-free bond)*
Assume that the high and low states are equally likely, and $U_H = 0.90$ and $U_L = 0.98$. What would one pay for a zero coupon bond today that pays \$1 in one year? What is the risk-free rate of return?

Solution.
The bond will pay \$1 in each state. The risk-free bond price is

$$0.5 \times 0.90 + 0.5 \times 0.98 = \$0.94.$$

The risk-free rate of return is

$$r = \frac{1}{0.94} - 1 = 6.4\% \blacksquare$$

In general[1], the price of a zero-coupon bond is given by

$$pU_H + (1-p)U_L$$

and the risk-free rate of return on a zero-coupon bond is given by

$$r = \frac{1}{pU_H + (1-p)U_L} - 1.$$

Thus, the price of a zero-coupon bond is just $\frac{1}{1+r}$.

[1]See page 371 of [1]

Now, let C_H be the price of the stock in the high state and C_L the price in the low state. The price of the risky stock is given by

$$\text{Price of stock} = pU_HC_H + (1-p)U_LC_L.$$

The expected future value on the stock is

$$pC_H + (1-p)C_L.$$

Therefore, the expected return on the stock is given by

$$\alpha = \frac{pC_H + (1-p)C_L}{pU_HC_H + (1-p)U_LC_L} - 1.$$

Example 85.5 *(Valuing bond)*
Assume that the high and low states are equally likely, and $U_H = 0.90$ and $U_L = 0.98$. The cash flow of the non-dividend paying stock is \$100 in the low state and \$200 in the high state.
(a) What is the price of the stock?
(b) What is expected future value of the stock?
(c) What is the expected return on the stock?

Solution.
(a) The price of the stock is

$$\text{Price of stock} = pU_HC_H+(1-p)U_LC_L = 0.5\times 0.90\times 200+0.5\times 0.98\times 100 = 139.$$

(b) The expected future value on the stock is

$$pC_H + (1-p)C_L = 0.5 \times 200 + 0.5 \times 100 = 150.$$

(c) The expected return on the stock is given by

$$\alpha = \frac{pC_H + (1-p)C_L}{pU_HC_H + (1-p)U_LC_L} - 1 = \frac{150}{139} - 1 = 7.9\% \blacksquare$$

The current price of the asset can be rewritten as follows

$$\begin{aligned}
pU_HC_H + (1-p)U_LC_L =&[pU_HC_H + (1-p)U_LC_L] \times \frac{1+r}{1+r}\\
=&[p(1+r)U_HC_H + (1-p)(1+r)U_LC_L] \times \frac{1}{1+r}\\
=&[p_uC_H + (1-p_u)C_L] \times \frac{1}{1+r}
\end{aligned}$$

where the high-state risk-neutral probability is

$$p_u = \frac{pU_H}{pU_H + (1-p)U_L}$$

and the low-state risk-neutral-probability is

$$1 - p_u = \frac{(1-p)U_L}{pU_H + (1-p)U_L}.$$

Thus, by taking the expected future value of the stock using the risk-neutral probabilities and discount back at the risk-free rate, we get the correct price for the stock.

Example 85.6
Assume that the high and low states are equally likely, and $U_H = .90$ and $U_L = .98$. In the high state of the economy a non-dividend paying stock will be worth 200, while in the low state it will be worth 100. Use the risk-neutral approach to find the price of the stock.

Solution.
The risk-neutral probability in the high state is

$$p_u = \frac{pU_H}{pU_H + (1-p)U_L} = \frac{0.5 \times 0.90}{0.5 \times 0.90 + 0.5 \times 0.98} = 47.87\%$$

and the risk-neutral probability in the low state is

$$1 - p_u = 1 - 0.4787 = 52.13\%.$$

On the other hand,

$$\frac{1}{1+r} = pU_H + (1-p)U_L = 0.5 \times 0.90 + 0.5 \times 0.98 = 0.94.$$

Thus, the price of the stock is

$$(1+r)^{-1}[p_uC_H + (1-p_u)C_L] = 0.94[0.4787 \times 200 + 0.5213 \times 100] = \$139$$

which matches the price found using real probabilties in Example 85.5 ■

This risk-neutral pricing worked because the same actual probabilities and the same utility weights, U_L and U_H, applied to both the risk-free bond and the risky asset. This will be true for traded assets. Thus, risk-neutral pricing works for traded assets and derivatives on traded assets. Risk-neutral pricing will not work if the asset can not be either traded, hedged, or have its cashflows duplicated by a traded assets.[2]

[2] See page 374 of [1].

Practice Problems

Problem 85.1
Verify that the function $U(x) = 1 - e^{-\alpha x}, \alpha > 0$, is a valid utility function?

Problem 85.2
You are an art collector with wealth consisting of \$40,000. There is a painting that you would like to consider buying. The painting is for sale for \$10,000. There is an annual probability of 10% of total loss of the painting, and no partial loss of the painting is possible. What will be the maximum premium that you have to pay to insure the painting using the utility function $U(x) = 1 - e^{-0.0001x}$?

For the remaining problems, you are given the following information:
- In one year the economy will be in one of two states, a high state or a low state.
- The value today of \$1 received one year from now in the high state is $U_H = 0.87$.
- The value today of \$1 received one year from now in the low state is $U_L = 0.98$.
- The probability of the high state is 0.52.
- The probability of the low state is 0.48.
- XYZ stock pays no dividends.
- The value of XYZ stock one year from now in the high state is $C_H = \$180$.
- The value of XYZ stock one year from now in the low state is $C_L = \$30$.

Problem 85.3
What is the price of a risk-free zero-coupon bond?

Problem 85.4
What is the risk-free rate?

Problem 85.5
What is p_u, the risk-neutral probability of being in the high state?

Problem 85.6
What is the price of XYZ Stock?

Problem 85.7
What is expected return on the stock?

Problem 85.8
What is the price of 1-year European call option on XYZ stock with strike \$130 under real probabilities?

Problem 85.9
What is the price of 1-year European call option on XYZ stock with strike \$130 under risk-neutral probabilities?

Answer Key

Section 1

1.1 We have the following payoff tables.

		Payoff at Time T	
Transaction	Payoff at Time 0	$S_T \le K$	$S_T > K$
Short sale of stock	S	$-S_T$	$-S_T$
Buy a call	$-C$	0	$S_T - K$
Total	$-C + S$	$-S_T$	$-K$

		Payoff at Time T	
Transaction	Payoff at Time 0	$S_T \le K$	$S_T > K$
Buy a put	$-P$	$K - S_T$	0
Sell a bond	$PV_{0,T}(K)$	$-K$	$-K$
Total	$-P + PV_{0,T}(K)$	$-S_T$	$-K$

Both positions guarantee a payoff of $\max\{-K, -S_T\}$. By the no-arbitrage principle they must have same payoff at time $t = 0$. Thus,

$$S - C = -P + PV_{0,T}(K)$$

1.2 We have the following payoff tables.

		Payoff at Time T	
Transaction	Payoff at Time 0	$S_T \le K$	$S_T > K$
Buy a stock	$-S$	S_T	S_T
Sell a call	C	0	$K - S_T$
Total	$-S + C$	S_T	K

		Payoff at Time T	
Transaction	Payoff at Time 0	$S_T \leq K$	$S_T > K$
Buy a bond	$-PV_{0,T}(K)$	K	K
Sell a put	P	$S_T - K$	0
Total	$P - PV_{0,T}(K)$	S_T	K

Both positions guarantee a payoff of $\min\{K, S_T\}$. By the no-arbitrage principle they must have same payoff at time $t = 0$. Thus,

$$-S + C = P - PV_{0,T}(K).$$

1.3 We have the following payoff tables.

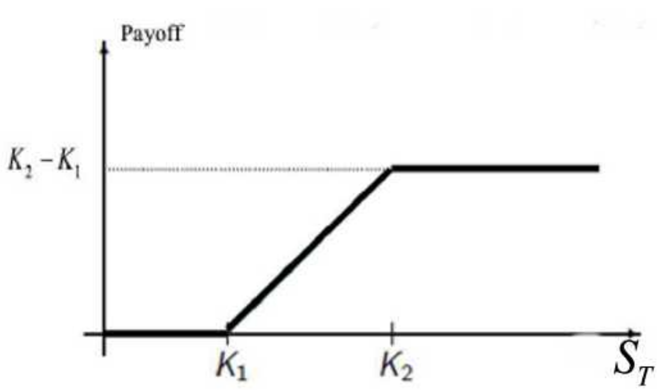
		Payoff at Time T	
Transaction	Payoff at Time 0	$S_T \leq K$	$S_T > K$
Short a stock	S	$-S_T$	$-S_T$
Sell a put	P	$S_T - K$	0
Total	$S + P$	$-K$	$-S_T$

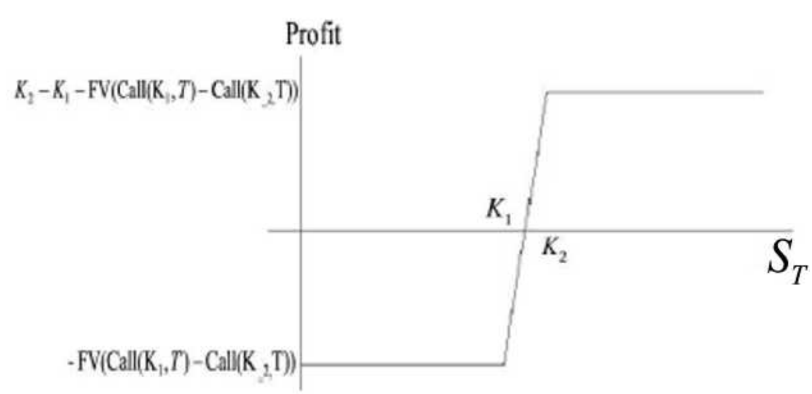
		Payoff at Time T	
Transaction	Payoff at Time 0	$S_T \leq K$	$S_T > K$
Sell a bond	$PV_{0,T}(K)$	$-K$	$-K$
Sell a call	C	0	$K - S_T$
Total	$C + PV_{0,T}(K)$	$-K$	$-S_T$

Both positions guarantee a payoff of $\min\{-K, -S_T\}$. By the no-arbitrage principle they must have same payoff at time $t = 0$. Thus,

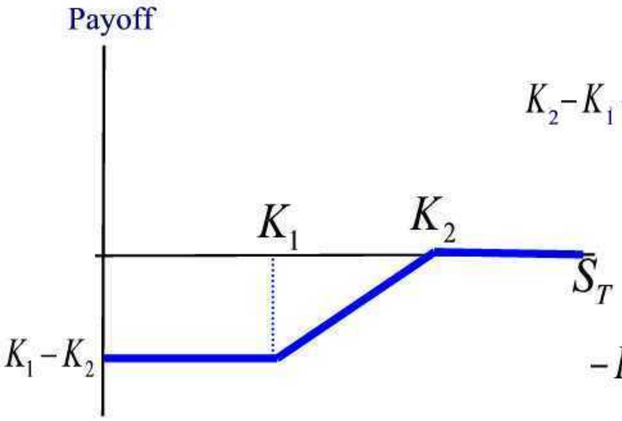
$$S + P = C + PV_{0,T}(K).$$

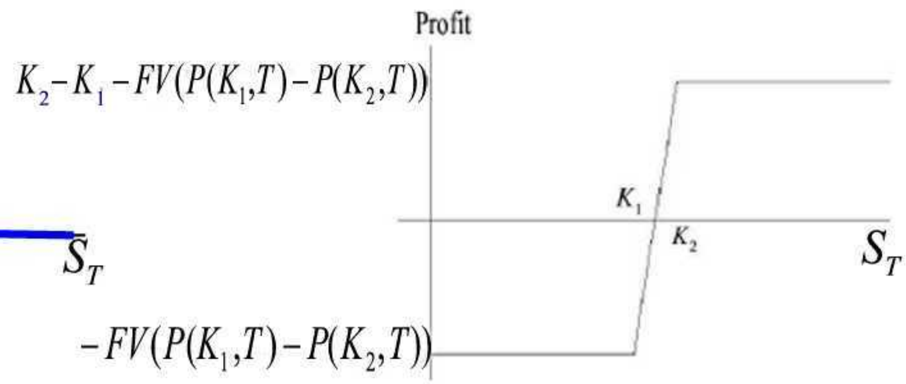
1.4 (a)





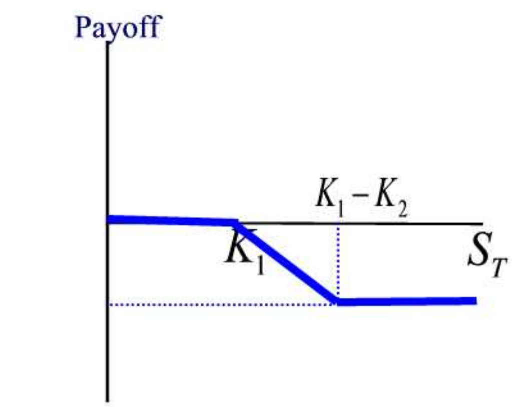
(b) The profit diagram of bull put spread coincides with the profit diagram of the bull call spread.

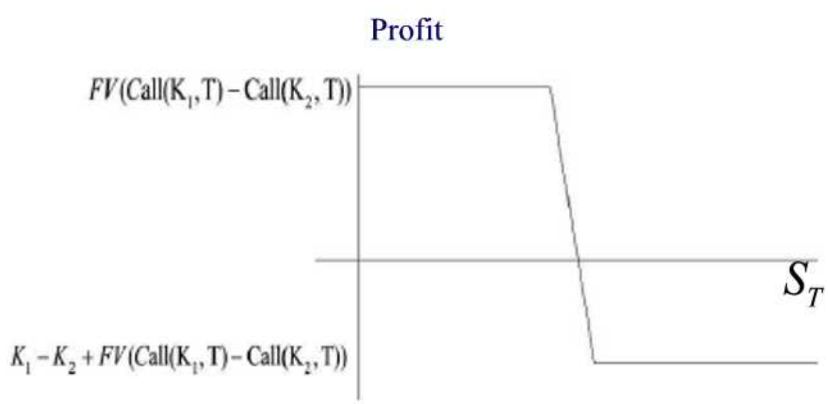




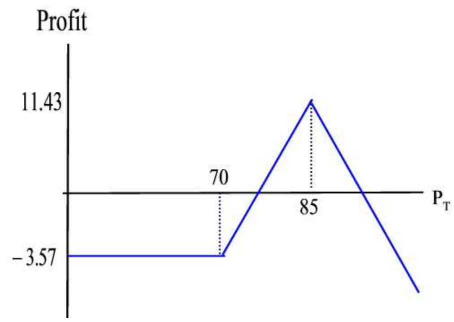
1.5

$$\begin{cases} FV(C(K_1,T)-C(K_2,T)) & \text{if } S_T \le K_1 \\ K_1 - S_T + FV(C(K_1,T)-C(K_2,T)) & \text{if } K_1 < S_T \le< K_2 \\ K_1 - K_2 + FV(C(K_1,T)-C(K_2,T)) & \text{if } K_2 \le S_T \end{cases}$$





1.6

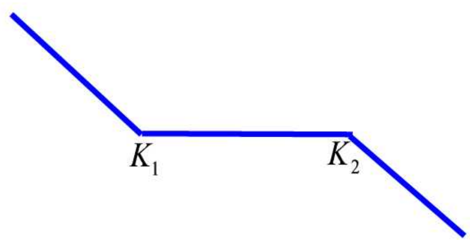


1.7 The profit function is

$$\max\{0, K_1 - S_T\} - \max\{0, S_T - K_2\} + FV(C(K_2,T) - P(K_1,T))$$

which is

$$\begin{cases} K_1 - S_T + FV(C(K_2,T) - P(K_1,T)) & \text{for } S_T \leq K_1 \\ FV(C(K_2,T) - Put(K_1,T)) & \text{for } K_1 < S_T < K_2 \\ K_2 - S_T + FV(C(K_2,T) - P(K_1,T)) & \text{for } K_2 \leq \ S_T \end{cases}$$



1.8

$$S_T + \max\{0, K_1 - S_T\} - \max\{0, S_T - K_2\} + FV(C(K_2,T) - P(K_1,T) - S_0)$$

which is

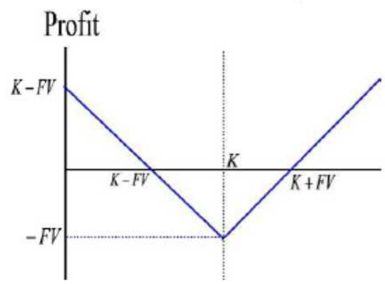
$$\begin{cases} K_1 + FV(C(K_2,T) - P(K_1,T) - S_0) & \text{for } S_T \leq K_1 \\ S_T + FV(C(K_2,T) - P(K_1,T) - S_0) & \text{for } K_1 < S_T < K_2 \\ K_2 + FV(C(K_2,T) - P(K_1,T) - S_0) & \text{for } K_2 \leq S_T \end{cases}$$

1.9 The payoff of a straddle is the sum of the payoff of a purchased call and a purchased put. That is,

$$\max\{0, K - S_T\} + \max\{0, S_T - K\} = |S_T - K|.$$

The profit function is given by

$$|S_T - K| - FV(C(K,T) + P(K,T)).$$

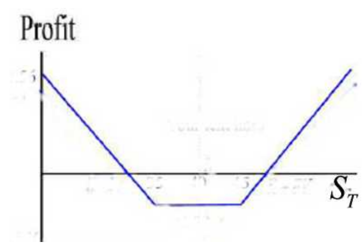


1.10 The profit of a strangle is

$$\max\{0, K_1 - S_T\} + \max\{0, S_T - K_2\} - FV(P(K_1,T) + C(K_2,T))$$

or

$$\begin{cases} K_1 - S_T - FV(P(K_1,T) + C(K_2,T)) & \text{if } S_T \leq K_1 \\ -FV(P(K_1,T) + C(K_2,T)) & \text{if } K_1 < S_T < K_2 \\ S_T - K_2 - FV(P(K_1,T) + C(K_2,T)) & \text{if } K_2 \leq S_T. \end{cases}$$



1.11 The initial cost of the position is

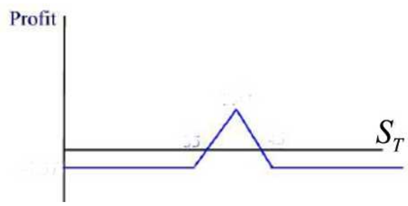
$$C(K_1,T) - C(K_2,T) - P(K_2,T) + P(K_3,T).$$

Letting FV denote the future value of the net premium, the profit is given by

$$\max\{0, S_T - K_1\} + \max\{0, K_3 - S_T\} - \max\{0, S_T - K_2\} - \max\{0, K_2 - S_T\} - FV$$

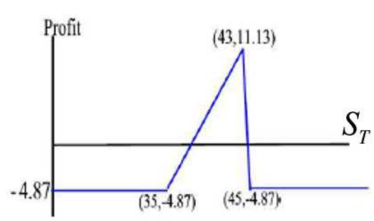
or

$$\begin{cases} K_3 - K_2 - FV & \text{if } S_T \leq K_1 \\ S_T - K_1 - K_2 + K_3 - FV & \text{if } K_1 < S_T < K_2 \\ -S_T - K_1 + K_2 + K_3 - FV & \text{if } K_2 \leq S_T < K_3 \\ K_2 - K_1 - FV & \text{if } K_3 \leq S_T \end{cases}$$



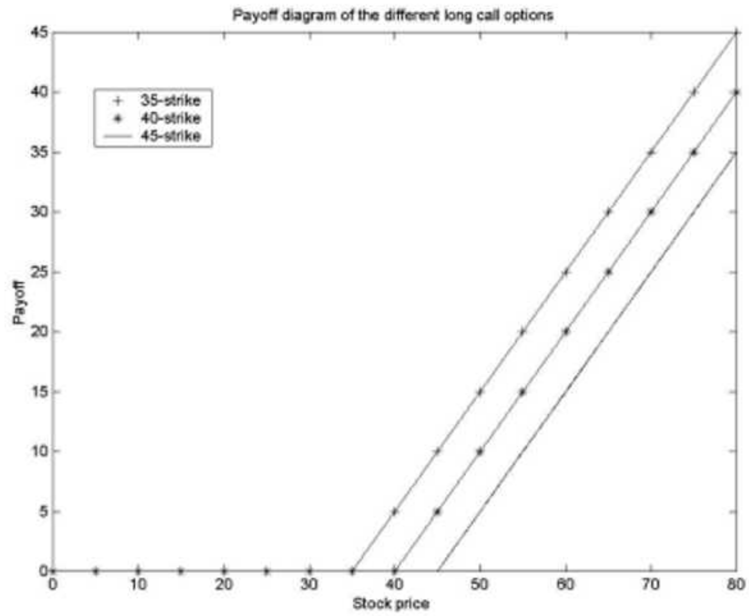
1.12 The profit is

$$\begin{cases} -4.87 & \text{if } S_T \leq 35 \\ 2S_T - 74.87 & \text{if } 35 < S_T < 43 \\ -8S_T + 355.13 & \text{if } 43 \leq S_T < 45 \\ -4.87 & \text{if } 45 \leq S_T \end{cases}$$

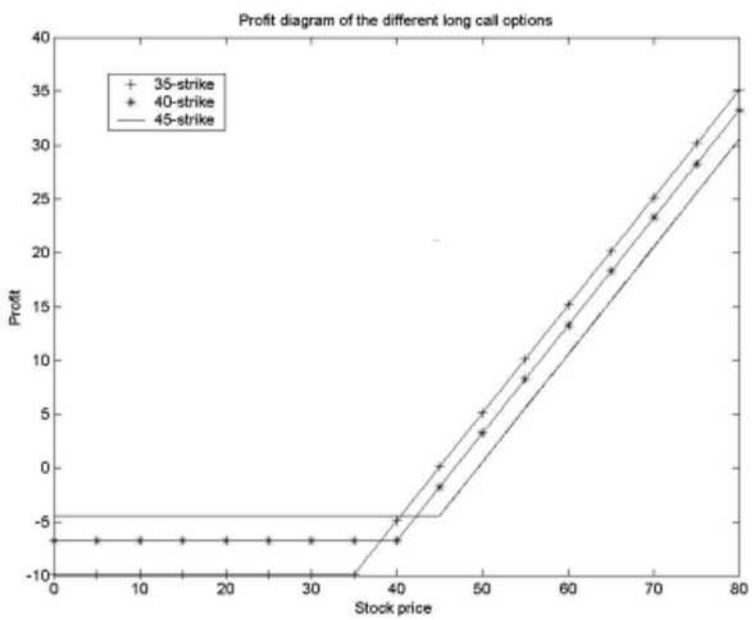


1.13

(a)



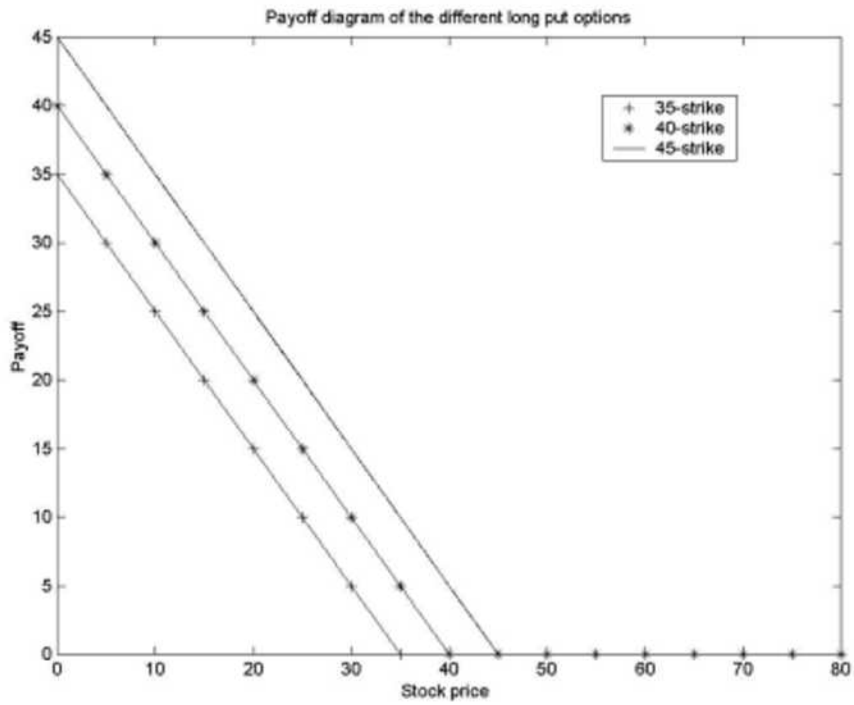
(b)



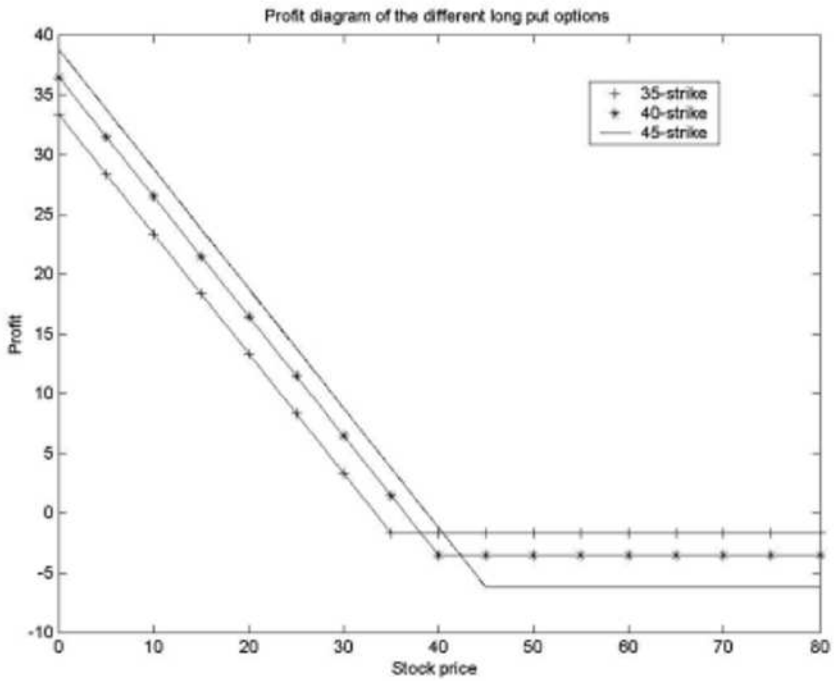
1.14 Suppose that $S_T > K$. Then the put option will not be exercised. However, the long call will be exercised and the trader will pay K and own the stock. If $S_T \leq K$, the call option will not be exercised but the put option will enable the trader to pay K and buy the stock from the buyer of the put.

Hence, in either case, the trader pays K and owns the stock. This is exactly a long forward on the stock with forward price K.

1.15 (a)



(b)



1.16

Derivative Position	Maximum Loss	Maximum Gain	Position with Respect to Underlying Asset	Strategy
Long Forward	−Forward Price	Unlimited	Long(buy)	Guaranteed price
Short Forward	Unlimited	Forward Price	Short(sell)	Guaranteed price
Long Call	−FV(premium)	Unlimited	Long(buy)	Insures against high price
Short Call	unlimited	FV(premium)	Short(sell)	Sells insurance against high price
Long Put	−FV(premium)	Strike price - FV(premium)	Short(sell)	Insures against low price
Short Put	FV(premium) - Strike Price	FV(premium)	Long(buy)	Sells insurance against low price ■

1.17 13.202%

1.18 $4.30

1.19 Portfolio (I): Collar
Portfolio (II): Straddle
Portfolio (III): Strangle
Portfolio (IV): Bull spread

Section 2

2.1 (C)

2.2 (A)

2.3
• The forward contract has a zero premium, while the synthetic forward requires that we pay the net option premium.
• With the forward contract, we pay the forward price, while with the synthetic forward we pay the strike price.

2.4 The cost of buying the asset is $-S_0$ and the cost from selling the call option is $C(K,T)$. The cost from selling the put option is $P(K,T)$ and the cost of buying the zero-coupon bond is $-PV_{0,T}(K)$. From Equation (2.1) with $PV_{0,T}(F_{0,T}) = S_0$ we can write

$$-S_0 + C(K,T) = -PV_{0,T}(K) + P(K,T)$$

which establishes the equivalence of the two positions.

2.5 The cost of the short-selling the asset is S_0 and that of selling the put option is $P(K,T)$. The cost of selling the call option is $C(K,T)$ and that of taking out a loan is $PV_{0,T}(K)$. From Equation (2.1) we can write

$$S_0 + P(K,T) = C(K,T) + PV_{0,T}(K)$$

which establishes the equivalence of the two positions.

2.6 8%

2.7 \$77

2.8 3.9%

2.9 (a) The cost of buying the index is $-S_0$ and that of buying the put option is $-P(K,T)$. The cost of buying the call option is $-C(K,T)$ and that of buying the zero-coupon bond is $-PV_{0,T}(K)$. From put-call parity equation we can write

$$-S_0 - P(K,T) = -C(K,T) - PV_{0,T}(K)$$

which establishes the equivalence of the two positions.
(b) The cost of the short index is S_0 and that of buying the call option is $-C(K,T)$. The cost of buying the put option is $-P(K,T)$ and the taking out the loan is $PV_{0,T}(K)$ (selling a zero-coupon bond). From the put-call parity equation we can write

$$S_0 - C(K,T) = -P(K,T) + PV_{0,T}(K)$$

which establishes the equivalence of the two positions.

2.10 \$71.69

2.11 \$480

2.12 \$1020

2.13 2%

2.14 The cost from buying a call and selling a put is $C - P = S_0 - Ke^{-rT} = 1000 - 1025(1.05)^{-1} = 23.81$. That is, the call is more expensive than the put. Since the call is the long position, we must spend \$23.81 to buy a call and sell a put.

Section 3

3.1 (B)

3.2 \$5.84

3.3 \$100

3.4 \$1

3.5 \$46.15

3.6 \$41.75

3.7 2%

3.8 \$46.16

3.9 2%

3.10 1.08285

3.11 \$5.52

3.12 \$1.47

3.13 \$92.60

3.14 \$0.73

3.15 \$4.45

3.16 \$79.58

3.17 \$0.2999

3.18 \$2.51

Section 4

4.1 (D)

4.2 (E)

4.3 (B) and (D)

4.4 6.96%

4.5 $91.13

4.6 0.4%

4.7 Purchase a put, sell a call, and borrow the present value of the strike price of the options

4.8 Buy a put with strike price K and expiration time T, buy a share of stock and borrow the present value of the strike and dividends

4.9 Buy a call with strike price K and expiration time T, sell a share of stock and lend the present value of the strike and dividends

4.10 $2.395

4.11 0.005

Section 5

5.1 \$0.7003

5.2 \$0.10

5.3 £0.1627

5.4 6%

5.5 (a) \$1.177 (b) \$1.17

5.6 9.97%

5.7 \$0.04376

5.8 \$0.0023/Rupee

5.9 \$0.00025

5.10 Since the put-call parity is violated, there is an arbitrage opportunity

Section 6

6.1 \$1145.12

6.2 \$1153.21

6.3 \$943.41

6.4 \$0.76994

6.5 (a) \$85.302 (b) \$251.12

6.6 \$4.17

6.7 17.98%

6.8 \$194.85

6.9 \$955.83

6.10 \$178.17

Section 7

7.1 \$61.45

7.2 \$3

7.3 2%

7.4 \$7700.56

7.5 2.77 years

7.6 \$7.91

7.7 \$11.76

7.8 \$11

7.9 \$47.89

7.10 \$7.91

Section 8

8.1 If you view the share of Intel stock as the underlying asset, the option is a call. The option gives you the privilege to buy the stock by paying \$35. If you view \$35 as the underlying asset then the option is a put option. This option gives you the right to sell \$35 for the price of one share of the stock.

8.2 €0.0374

8.3 The strike is €1.0870 and the premium is €0.0407

8.4 €0.0175

8.5 \$0.1072

8.6 €0.055

8.7 \$0.1321

8.8 €0.0348

8.9 \$0.2123

8.10 €0.0073

8.11 \$0.0564

8.12 ¥42.7733

Section 9

9.1 (D)

9.2 (E)

9.3 \$3.66

9.4 \$2.01

9.5 \$0.6597

9.6 \$6.3998

9.7 (I) True (II) False (III) False

9.8 Suppose that $C_{\text{Amer}}(K,T) < C_{\text{Eur}}(K,T)$. We will show that this creates an arbitrage opportunity. Consider the position of selling the European call and buying the American call. The net cash flow $C_{\text{Eur}}(K,T) - C_{\text{Amer}}(K,T)$ would be invested at the risk-free rate r.
If the owner of the European call chooses to exercise the option at the expiration date T. Exercise the Americal option and deliver the stock. The amount due is

$$(C_{\text{Eur}}(K,T) - C_{\text{Amer}}(K,T))e^{rT} > 0.$$

If the European option is not exercised, the American option can be allowed to expire and the amount due is

$$(C_{\text{Eur}}(K,T) - C_{\text{Amer}}(K,T))e^{rT} > 0.$$

In either case, an arbitrage opportunity occurs.

9.9 Suppose that $P_{\text{Eur}}(K,T) > PV_{0,T}(K)$. Sell the put and buy a zero-coupon bond with par-value K for the price $PV_{0,T}(K)$. The payoff table of this position is given next.

		Exercise or Expiration	
Transaction	Time 0	$S_t \leq K$	$S_t > K$
Sell European put	$P_{\text{Eur}}(K,T)$	$S_t - K$	0
Buy Bond	$-PV_{0,T}(K)$	K	K
Total payoff	$P_{\text{Eur}}(K,T) - PV_{0,T}(K) > 0$	S_t	K

Thus, the assumption $P_{\text{Eur}}(K,T) > PV_{0,T}(K)$ leads to an arbitrage

9.10 Suppose that $C_{\text{Eur}}(K,T) < S_0 - K$. Consider the position of buying the call option, short selling the stock and lending K. The payoff table of this position is given next.

		Expiration at Time T	
Transaction	Time 0	$S_T \leq K$	$S_T > K$
Buy call	$-C_{\text{Eur}}(K,T)$	0	$S_T - K$
Sell stock	S_0	$-S_T$	$-S_T$
Lend K	$-K$	Ke^rT	Ke^rT
Total payoff	$S_0 - C_{\text{Eur}}(K,T) - K > 0$	$Ke^{rT} - S_T \geq 0$	$Ke^{rT} - K > 0$

Thus, the assumption $C_{\text{Eur}}(K,T) < S_0 - K$ leads to an arbitrage.

9.11 Suppose that $C_{\text{Eur}}(K,T) < S_0 - PV_{0,T}(K)$. Consider the position of buying the call option, selling the stock and lending $PV_{0,T}(K)$. The payoff table of this position is given next.

		Expiration at Time T	
Transaction	Time 0	$S_T \leq K$	$S_T > K$
Buy call	$-C_{\text{Eur}}(K,T)$	0	$S_T - K$
Sell stock	S_0	$-S_T$	$-S_T$
Lend $PV_{0,T}(K)$	$-PV_{0,T}(K)$	K	K
Total payoff	$S_0 - C_{\text{Eur}}(K,T) - PV_{0,T}(K) > 0$	$K - S_t \geq 0$	0

Thus, the assumption $C_{\text{Eur}}(K,T) < S_0 - PV_{0,T}(K)$ leads to an arbitrage.

9.12 We first find $PV_{0,T}(K) = Ke^{-rT} = 100e^{-0.10\times 0.5} = \$95.12$. We know that the price of a call option increases as the stock price increases so that either (I) or (II) contains the graph of $C_{\text{Amer}} = C_{\text{Eur}}$. Now we have the following bounds

$$S(0) \geq C_{\text{Amer}} = C_{\text{Eur}} \geq \max\{0, S(0) - 95.12\}.$$

The left portion of the inequality above describes the upper boundary (II) and the right portion describes the lower boundary. Thus, the shaded-region (II) contains the graph of $C_{\text{Amer}} = C_{\text{Eur}}$.
For the put options we have

$$100 \geq P_{\text{Amer}} \geq P_{\text{Eur}} \geq \max\{0, 95.12 - S(0)\}.$$

The region bounded below by $\max\{0, 95.12 - S(0)\}$ and bounded above by $\pi = 100$ is not given by (III) or (IV). But we have a tighter bound on the European put given by

$$\max\{0, 95.12 - S(0)\} \leq P_{\text{Eur}} \leq 95.12.$$

Thus, the shaded area (IV) contains the graph of P_{Eur}. We also have a tighter bound for P_{Amer} given by

$$100 \geq P_{\text{Amer}} \geq \max\{0, 100 - S(0)\}.$$

Thus, the shaded area (III) contains the graph of P_{Amer}.

Section 10

10.1 (D)

10.2 (E) and (F)

10.3 (A), (B), (D), and (F)

10.4 No

10.5 No

10.6 Possible

10.7 Possible

10.8 \$4

10.9 All options MAY be optimal (but not necessarily so)

10.10
An American call option can be exercised early if the exercise price is higher than the option value. By the generalized put-call parity

$$C_{\text{Amer}} \geq C_{\text{Eur}} \geq F^P_{t,T}(S_t) - F^P_{t,T}(Q_t) = S_t - Q_t.$$

Since the exercise price $S_t - Q_t$ can not be higher than the call option value C_{Amer}, it is never optimal to exercise early.

10.11 An American put option can be exercised early if the exercise price is higher than the option value. By the generalized put-call parity

$$P_{\text{Amer}} \geq P_{\text{Eur}} \geq F^P_{t,T}(Q_t) - F^P_{t,T}(S_t) = Q_t - S_t.$$

Since the exercise price $Q_t - S_t)$ can not be higher than the put option value P_{Amer}, it is never optimal to exercise early.

10.12 It is not optimal to exercise any of these special put options immediately.

Section 11

11.1 (A) True (B) True for nondividend-paying stock but false for dividend-paying stock (C) False (D) True

11.2 (A) It is always the case that, of two call options with the same time to expiration, the option with the lower strike price is worth more− as the difference between the current price of the stock and the strike price of the option would be more positive for any current price of the stock. Hence, $C_{\text{Amer}}(43,1) \geq C_{\text{Amer}}(52,1)$. Thus, $C_{\text{Amer}}(43,2) \geq C_{\text{Amer}}(43,1) \geq C_{\text{Amer}}(52,1)$.
(B) is true because $C_{\text{Amer}}(43,2) \geq C_{\text{Amer}}(43,1)$.
(C) is true because $C_{\text{Amer}}(43,2) \geq C_{\text{Eur}}(43,2)$.
(D) is false; indeed, $P_{\text{Amer}}(56,4) \geq P_{\text{Amer}}(56,3)$.
(E) is false, because it may be the case that the written company is bankrupt, in which case the put options will be worth the present value of the strike price; with the same strike price and longer time to expiration, the longer-term European put option will have a smaller present value factor by which the strike price is multiplied.

11.3 \$9.61

11.4 \$60.45

11.5 (E)

11.6 (D)

11.7 We buy the put with T years to expiration and sell the put with t years to expiration. The payoff of the longer-lived put at time T is $\max\{0, K_T - S_T\}$. The payoff of the shorter-lived put at time T is $-\max\{0, K_t - S_t\}$ accumulated from t to T.
• Suppose $S_T < K_T$. Then the payoff of the longer-lived put is $K_T - S_T$. At time t, the written put will expire. Suppose $S_t < K_t$. Then the payoff of the shorter-lived put is $S_t - K_t$. Suppose that we keep the stock we receive and borrow to finance the strike price, holding this position until the second option expires at time T. At time t, sell the stock and collect S_T. Thus, the time-T value of this time-t payoff is $S_T - K_te^{r(T-t)} = S_T - K_T$. Hence, the

total payoff of the position is 0.
• Suppose $S_T < K_T$. Then the payoff of the longer-lived put is $K_T - S_T$. Suppose $S_t \geq K_t$. Then the payoff of the shorter-lived put is 0 and this accumulates to 0 at time T. Thus, the total payoff of the position is $K_T - S_T$.
• Suppose $S_T \geq K_T$. Then the payoff of the longer-lived put is 0. Suppose $S_t < K_t$. Then the payoff of the shorter-lived put is $S_t - K_t$. Suppose that we keep the stock we receive and borrow to finance the strike price, holding this position until the second option expires at time T. At time t, sell the stock and collect S_T. Thus, the time-T value of this time-t payoff is $S_T - K_t e^{r(T-t)} = S_T - K_T$. Hence, the total payoff of the position is $S_T - K_T$.
• Suppose $S_T \geq K_T$. Then the payoff of the longer-lived put is 0. Suppose $S_t \geq K_t$. Then the payoff of the shorter-lived call is 0 and this accumulates to 0 at time T. Thus, the total payoff of the position is 0.
It follows that in order to avoid arbitrage, the longer-lived put can't sell for less than the shorter-lived put.

11.8 See the previous exercise and Example 11.3. The accumulated value of the arbitrage strategy at the end of nine months is \$0.80.

11.9 We know that the call option with 6-month expiration must be worth more than the call option with 3-month expiration. Thus, buying the 6-month option and selling the 3-month option result in a net cash flow is \$2.

11.10 (a) First, notice that $207.63 = 198e^{0.19\times 0.25}$. We have $C(0.5) = 23.50 > C(0.75)$. We buy one call option with strike price \$207.63 and time to expiration nine months from now. We sell one call option with strike price of \$198 and time to expiration six months from now. The payoff table is given next.

		Payoff at Time 9 months			
		$S_{0.75} \leq 207.63$		$S_{0.75} > 207.63$	
		Payoff at Time 6 months			
Transaction	Time 0	$S_{0.5} \leq 198$	$S_{0.5} > 198$	$S_{0.5} \leq 198$	$S_{0.5} > 198$
Sell $C0.5)$	$C(0.5)$	0	$207.63 - S_{0.75}$	0	$S_{0.75} - 207.63$
Buy $C(0.75)$	$-C0.75)$	0	0	$S_{0.75} - 207.63$	$207.63 - S_{0.75}$
Total Payoff	$C(0.5) - C(0.75)$	0	$207.63 - S_{0.75}$	$S_{0.75} - 207.63$	0

(b) \$4.36

Section 12

12.1 (A) and (C)

12.2 (B), (C), and (D)

12.3 We will establish the result using the usual strategy of a no-arbitrage argument. Let's assume that the inequality above does not hold, that is, $P(K_1) > P(K_2)$. We want to set up a strategy that pays us money today. We can do this by selling the low-strike put option and buying the high-strike put option (this is put bear spread). We then need to check if this strategy ever has a negative payoff in the future. Consider the following payoff table:

		Expiration or Exercise		
Transaction	Time 0	$S_t < K_1$	$K_1 \leq S_t \leq K_2$	$S_t > K_2$
Sell $P(K_1)$	$P(K_1)$	$S_t - K_1$	0	0
Buy $P(K_2)$	$-P(K_2)$	$K_2 - S_t$	$K_2 - S_t$	0
Total	$P(K_1) - P(K_2)$	$K_2 - K_1$	$K_2 - S_t$	0

Every entry in the row labeled "Total" is nonnegative. Thus, by selling the low-strike put and buying the high-strike put we are guaranteed not to lose money at time T. This is an arbitrage. Hence, to prevent arbitrage, we must have $P(K_2) \geq P(K_1)$. If the options are Americans then we have to take in consideration the possibility of early exercise of the written put. If that happens at time $t < T$, we can simply exercise the purchased option, earning the payoffs in the table with T being replaced by t. If it is not optimal to exercise the purchased option, we can sell it, and even get a higher payoff than exercising the purchased option.

12.4 We will use the strategy of a no-arbitrage argument. Assume $P(K_2) - P(K_1) - (K_2 - K_1) > 0$. We want to set up a strategy that pays us money today. We can do this by selling the high-strike put option, buying the low-strike put option (this is put bull spread), and lending the amount $K_2 - K_1$. We then need to check if this strategy ever has a negative payoff in the future. Consider the following payoff table:

		Expiration or Exercise		
Transaction	Time 0	$S_t < K_1$	$K_1 \leq S_t \leq K_2$	$S_t > K_2$
Sell $P(K_2)$	$P(K_2)$	$S_t - K_2$	$S_t - K_2$	0
Buy $P(K_1)$	$-P(K_1)$	$K_1 - S_t$	0	0
Lend $K_2 - K_1$	$K_1 - K_2$	$e^{rt}(K_2 - K_1)$	$e^{rt}(K_2 - K_1)$	$e^{rt}(K_2 - K_1)$
Total	$P(K_2) - P(K_1)$ $-(K_2 - K_1)$	$e^{rt}(K_2 - K_1)$ $-(K_2 - K_1)$	$e^{rt}(K_2 - K_1)$ $-(K_2 - S_t)$	$e^{rt}(K_2 - K_1)$

Every entry in the row labeled "Total" is nonnegative. Thus, by selling the high-strike put, buying the law-strike put and lending $K_2 - K_1$ we are guaranteed not to lose money at time T. This is an arbitrage. Hence, to prevent arbitrage, we must have

$$P(K_2) - P(K_1) \leq K_2 - K_1.$$

In the case of American options, if the written put is exercised, we can duplicate the payoffs in the table by throwing our option away (if $K_1 \leq S_t \leq K_2$) or exercising it (if $S_t < K_1$). Since it never makes sense to discard an unexpired option, and since exercise may not be optimal, we can do at least as well as the payoff in the table if the options are American.

12.5 Replace $K_2 - K_1$ by $PV_{0,T}(K_2 - K_1)$ is the previous problem.

12.6 (a) Property (12.3) (b) sell the 50-strike put option and buy the 55-strike put option (a put bear spread)(c)

		Expiration or Exercise		
Transaction	Time 0	$S_t < 50$	$50 \leq S_t \leq 55$	$S_t > 55$
Sell $P(50)$	7	$S_t - 50$	0	0
Buy $P(55)$	-6	$55 - S_t$	$55 - S_t$	0
Total	$+1$	5	$55 - S_t$	0

Note that we initially receive money, and that at expiration the profit is non-negative. We have found arbitrage opportunities.

12.7 (a) Property (12.2) (b) sell the 50-strike call option and buy the 55-strike call option (a call bear spread) (c)

		Expiration or Exercise		
Transaction	Time 0	$S_t < 50$	$50 \leq S_t \leq 55$	$S_t > 55$
Sell $C(50)$	16	0	$50 - S_t$	$50 - S_t$
Buy $C(55)$	-10	0	0	$S_t - 55$
Total	$+6$	0	$50 - S_t \geq -5$	-5

Note that we initially receive more money than our biggest possible exposure in the future.

12.8 (D)

12.9 (a) We have

		Expiration or Exercise		
Transaction	Time 0	$S_1 < 70$	$70 \leq S_1 \leq 75$	$S_1 > 75$
Sell $P(75)$	13.50	$S_1 - 75$	$S_1 - 75$	0
Buy $P(70)$	-8.75	$70 - S_1$	0	0
Lend 4.75	-4.75	5.09	5.09	5.09
Total		0.09	$S_1 - 69.91$	5.09

(b) \$0.09 (c) \$3.09.

12.10 (I) and (III)

Section 13

13.1 0.759

13.2 \$45.64

13.3 \$23.5

13.4 \$108

13.5 We will show the required inequality by using the usual strategy of a no-arbitrage argument. Let's assume that the inequality above does not hold, that is, $P(\lambda K_1 + (1-\lambda)K_2) > \lambda P(K_1) + (1-\lambda)P(K_2)$ or $P(K_3) > \lambda P(K_1) + (1-\lambda)P(K_2)$ where $K_3 = \lambda K_1 + (1-\lambda)K_2$. Note that $K_3 = \lambda(K_1 - K_2) + K_2 < K_2$ and $K_1 = (1-\lambda)K_1 + \lambda K_1 < (1-\lambda)K_2 + \lambda K_1 = K_3$. We want to set up a strategy that pays us money today. We can do this by selling one put option with strike price K_3, buying λ put options with strike price K_1, and buying $(1-\lambda)$ put options with strike K_2. The payoff table of this position is given next.

		Expiration or exercise			
Transaction	Time 0	$S_t < K_1$	$K_1 \leq S_t \leq K_3$	$K_3 \leq S_t \leq K_2$	$S_t > K$
Sell 1 K_3- strike put	$P(K_3)$	$S_t - K_3$	$S_t - K_3$	0	0
Buy λ K_1-strike puts	$-\lambda P(K_1)$	$\lambda(K_1 - S_t)$	0	0	0
Buy $(1-\lambda)$ K_2-strike puts	$-(1-\lambda)P(K_2)$	$(1-\lambda)(K_2 - S_t)$	$(1-\lambda)(K_2 - S_t)$	$(1-\lambda)(K_2 - S_t)$	0
Total	$P(K_3) - \lambda P(K_1)$ $-(1-\lambda)P(K_2)$	0	$\lambda(S_t - K_1)$	$(1-\lambda)(K_2 - S_t)$	0

Note that

$$\begin{aligned}
S_t - K_3 + (1-\lambda)(K_2 - S_t) =& \lambda S_t - K_3 + (1-\lambda)K_2 \\
=& \lambda S_t - \lambda K_1 \\
=& \lambda(S_t - K_1)
\end{aligned}$$

and

$$S_t - K_3 + \lambda(K_1 - S_t) + (1-\lambda)(K_2 - S_t) = S_t - \lambda K_1 - (1-\lambda)K_2 + \lambda(K_1 - S_t) + (1-\lambda)(K_2 - S_t) = 0.$$

The entries in the row "Total" are all nonnegative. In order to avoid arbitrage, the initial cost must be non-positive. That is

$$P(\lambda K_1 + (1-\lambda)K_2) \le \lambda P(K_1) + (1-\lambda)P(K_2).$$

13.6 Let $\lambda = \frac{K_3-K_2}{K_3-K_1}$. We have $K_1 < K_2$ so that $K_3 - K_2 < K_3 - K_1$ so that $0 < \lambda < 1$. Also, we note that

$$K_2 = \frac{K_3-K_2}{K_3-K_1}K_1 + \frac{K_2-K_1}{K_3-K_1}K_3 = \lambda K_1 + (1-\lambda)K_3.$$

Using convexity we can write

$$P(K_2) \le \frac{K_3-K_2}{K_3-K_1}P(K_1) + \frac{K_2-K_1}{K_3-K_1}P(K_3).$$

13.7 From the previous exercise we found

$$P(K_2) \le \frac{K_3-K_2}{K_3-K_1}P(K_1) + \frac{K_2-K_1}{K_3-K_1}P(K_3)$$

which is equivalent to

$$(K_3-K_1)P(K_2) \le (K_3-K_2)P(K_1) + (K_2-K_1)P(K_3)$$

or

$$(K_3-K_1)P(K_2)-(K_3-K_2)P(K_2)-(K_2-K_1)P(K_3) \le (K_3-K_2)P(K_1)-(K_3-K_2)P(K_2).$$

Hence,

$$(K_2-K_1)[P(K_2)-P(K_3)] \le (K_3-K_2)[P(K_1)-P(K_2)]$$

and the result follows by dividing through by the product $-(K_2-K_1)(K_3-K_2)$.

13.8 (a) We are given $K_1 = 80, K_2 = 100$, and $K_3 = 105$. Since

$$\frac{C(K_1)-C(K_2)}{K_2-K_1} = 0.65 < \frac{C(K_3)-C(K_2)}{K_3-K_2} = 0.8$$

the convexity property is violated.
(b) We find

$$\lambda = \frac{K_3-K_2}{K_3-K_1} = \frac{105-100}{105-80} = 0.2.$$

Thus, a call butterfly spread is constructed by selling one 100-strike call, buying 0.2 units of 80-strike calls and 0.8 units of 105-strike calls. To buy and sell round lots, we multiply all the option trades by 5.
(c) The payoff table is given next.

		Expiration or exercise			
Transaction	Time 0	$S_t < 80$	$80 \leq S_t \leq 100$	$100 \leq S_t \leq 105$	$S_t > 105$
Sell 5 100−strike calls	45	0	0	$500 - 5S_t$	$500 - 5S_t$
Buy one 80−strike call	−22	0	$S_t - 80$	$S_t - 80$	$S_t - 80$
Buy 4 105−strike call	−20	0	0	0	$4S_t - 420$
Total	+3	0	$S_t - 80$	$420 - 4S_t$	0

Note that we initially receive money and have non-negative future payoffs. Therefore, we have found an arbitrage possibility, independent of the prevailing interest rate.

13.9 (a) We are given $K_1 = 80, K_2 = 100$, and $K_3 = 105$. Since

$$\frac{P(K_2) - P(K_1)}{K_2 - K_1} = 0.85 > \frac{P(K_3) - P(K_2)}{K_3 - K_2} = 0.76$$

the convexity property for puts is violated.
(b) We find

$$\lambda = \frac{K_3 - K_2}{K_3 - K_1} = \frac{105 - 100}{105 - 80} = 0.2.$$

Thus, a put butterfly spread is constructed by selling one 100-strike put, buying 0.2 units of 80-strike puts and 0.8 units of 105-strike puts. To buy and sell round lots, we multiply all the option trades by 5.
(c) The payoff table is given next.

		Expiration or exercise			
Transaction	Time 0	$S_t < 80$	$80 \leq S_t \leq 100$	$100 \leq S_t \leq 105$	$S_t > 105$
Sell 5 100−strike puts	105	$5S_t - 500$	$5S_t - 500$	0	0
Buy one 80−strike put	−4	$80 - S_t$	0	0	0
Buy four 105−strike puts	−99.2	$420 - 4S_t$	$420 - 4S_t$	$420 - 4S_t$	0
Total	1.8	0	$S_t - 80$	$420 - 4S_t$	0

Note that we initially receive money and have non-negative future payoffs. Therefore, we have found an arbitrage possibility, independent of the prevailing interest rate.

13.10

Stock at Expiration	Accumulated Strategy Profits
48	$0.05e^{0.10\times0.5} = \$0.05256$
53	$0.05256 + 3 = 3.05256$
59	$0.05256 + 1 = 1.05256$
63	0.05256

Section 14

14.1 \$11.54

14.2 0.35866

14.3 12%

14.4 \$9.79

14.5 \$541.40

14.6 73.374%

14.7 \$10.21

14.8 \$15.693

14.9 Option is underpriced and an arbitrage opportunity arises by selling a synthetic call option and at the same time buying the actual call option.

14.11 \$187,639.78

Section 15

15.1 \$56

15.2 We will show algebraically that $0 \leq \Delta \leq 1$: If $uS \leq K$ then $C_u = C_d = 0$ and therefore $\Delta = 0$. If $uS > K \geq dS$ then $C_u = uS - K$ and $C_d = 0$. In this case,

$$\Delta = e^{-\delta h}\frac{uS - K}{uS - dS} \leq 1.$$

If $uS > dS > K$ then $C_u = uS - K$ and $C_d = dS - K$. In this case,

$$\Delta = e^{-\delta h}\frac{uS - dS}{S(u - d)} = e^{-\delta h} \leq 1.$$

Similar argument holds in the down state.

15.3 $\Delta = \frac{2}{3}$ and $B = \$18.46$

15.4 \$8.8713

15.5 $\frac{3}{4}$

15.6 $\Delta = \frac{1}{3}$ and $B = \$28.54$

15.7 \$135

15.8 15.42%

15.9 \$1.8673

15.10 0.4286 of a share

15.11 0.5

15.12 (E)

Section 16

16.1 1.026%

16.2 \$100.62

16.3 We have

$$\begin{aligned} r_{t,t+nh} =& \ln \frac{S_{t+nh}}{S_t} = \ln\left(\frac{S_{t+nh}}{S_{t+(n-1)h}} \times \frac{S_{t+(n-1)h}}{S_{t+(n-2)h}} \times \cdots \times \frac{S_{t-1}}{S_t}\right) \\ =& \sum_{i=1}^{n} \ln \frac{S_{t+ih}}{S_{t+(i-1)h}} \\ =& \sum_{i=1}^{n} r_{t+(i-1)h,t+ih} \end{aligned}$$

16.4 $u = 6.5535$ and $d = 0.1791$

16.5 (a) $1/(e^{\sigma\sqrt{h}} + 1)$ (b) If $h_1 > h_2$ then $e^{\sigma\sqrt{h_1}} + 1 > e^{\sigma\sqrt{h_2}} + 1$ so that $\frac{1}{e^{\sigma\sqrt{h_1}}+1} < \frac{1}{e^{\sigma\sqrt{h_2}}+1}$. This shows that p_u is decreasing as a function of h. Finally,

$$\lim_{h\to 0} \frac{1}{e^{\sigma\sqrt{h}} + 1} = \frac{1}{1+1} = \frac{1}{2}.$$

16.6 \$533.9805

16.7 10.54%

16.8 30%

16.9 45.69%

16.10 \$0.1844

Section 17

17.1 \$10.731

17.2 \$117.44

17.3 \$14.79

17.4 \$12.33

17.5 \$29.34

17.6 \$24

17.7 \$7.91

17.8 \$0.1345

17.9 \$107.59

Section 18

18.1 \$23.66

18.2 \$6.65

18.3 \$28.11

18.4 We have

$$\begin{aligned}
P =& e^{-rh}[p_uP_u + (1-p_u)P_d] \\
=& e^{-rh}\left(p_ue^{-rh}[p_uP_{uu} + (1-p_u)P_{ud}] + (1-p_u)e^{-rh}[p_uP_{du} + (1-p_u)P_{dd}]\right) \\
=& e^{-2rh}(p_u^2P_{uu} + p_u(1-p_u)P_{ud} + p_u(1-p_u)P_{du} + (1-p_u)^2P_{dd}) \\
=& e^{-2rh}[p_u^2P_{uu} + 2p_u(1-p_u)P_{ud} + (1-p_u)^2P_{dd}]
\end{aligned}$$

18.5 \$54.77

18.6 \$25.59

18.7 \$14.37

18.8 \$3.98

18.9 \$0.1375

18.10 \$46.15

Section 19

19.1 \$30

19.2 \$131.89

19.3 \$70

19.4 \$9.44

19.5 \$17.26

19.6 \$6.57

19.7 \$7.13

19.8 \$11.12

19.9 \$2.06

19.10 \$39.73

19.11 \$10.76

19.12 \$14.16

19.13 \$115

Section 20

20.1 \$0.775

20.2 \$2.91

20.3 42%

20.4 \$0.05

20.5 \$0.1416

20.6 \$0.01036

20.7 48.56%

20.8 FC 1.26

20.9 \$3.0824

20.10 \$0.2256

Section 21

21.1 \$24.76

21.2 0.5511

21.3 $u = 1.2, d = 0.9$

21.4 \$6.43

21.5 \$6.43

21.6 \$380.81

21.7 61.79%

21.8 1

21.9 \$7.44

21.10 0.088

Section 22

22.1 By exercising, the option holder
• Surrenders the underlying asset and loses all related future dividends.
• Receives the strike price and therefore captures the interest from the time of exercising to the time of expiration.
• Loses the insurance/flexibility implicit in the put. The option holder is not protected anymore when the underlying asset has a value greater than the strike price at expiration. It follows that for a put, interest encourages early exercising while dividends and insurance weigh against early exercise.

22.2 The answer is smaller than $80

22.3 Yes

22.4 $364.80

22.5 $38.18

22.6 (F)

22.7 (A) and (B)

22.8 (A) and (B)

22.9 12.1548 years

Section 23

23.1 \$13353.25

23.2 0.4654 years

23.3 15.77%

23.4 1.82%

23.5 −8.31%

23.6 (a) 42.82% (b) 14.32%

23.7 19.785%

23.8 51.576%

23.9 56.71%

Section 24

24.1 Expected value is 0 and variance is 1

24.2 n

24.3 We have $Z_{n+1} - Z_n = \sum_{i=1}^{n+1} Y_i - \sum_{i=1}^{n} Y_i = Y_{n+1}$.

24.4 $S_{n+1} = S_n + Y_n$.

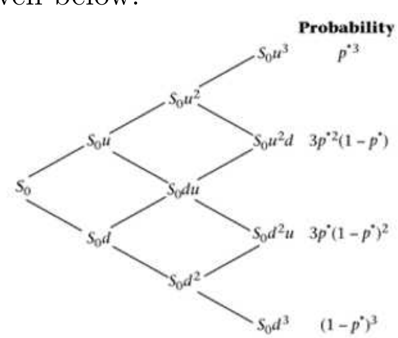
24.5 2.32

24.6 562.5 years

24.7 \$149.53

24.8 6

24.9 The answer is given below.



where $p^* = p_u$.

24.10 17.28%

Section 25

25.1 454.86

25.2 30%

25.3 13.40%

25.4 \$5.03

25.5 50.44%

25.6 \$5.18

25.7 (a) 55.94% (b) 15%

25.8 0.4979

25.9 \$13.27

25.10 \$5.88

Section 26

26.1 0.3085

26.2 0.2468

26.3 0.2074

26.4 0.82636

26.5 0.02424

26.6 7

26.7 $\frac{1}{12}$

26.8 \$118.38

26.9 104

Section 27

27.1 \$4.0513

27.2 \$2.39

27.3 0.574

27.4 \$9.01

27.5 We have

$$\begin{aligned}
d_1 =& \frac{\ln\left(\frac{Se^{-\delta T}}{Ke^{-rT}}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} \\
=& \frac{\ln Se^{-\delta T} - \ln Ke^{-rT} + 0.5\sigma^2 T}{\sigma\sqrt{T}} \\
=& \frac{\ln S + \ln e^{-\delta T} - \ln K - \ln e^{-rT} + 0.5\sigma^2 T}{\sigma\sqrt{T}} \\
=& \frac{\ln S - \delta T - \ln K + rT + 0.5\sigma^2 T}{\sigma\sqrt{T}} \\
=& \frac{\ln S/K + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}
\end{aligned}$$

27.6 \$12.334

27.7 \$20.9176

27.8 \$1.926

27.9 \$0.3064.

Section 28

28.1 (B)

28.2 \$2.0141

28.3 \$57.3058

28.4 \$13.7252

28.5 \$5.0341

28.6 (a) \$0.02425 (b) \$0.05751

28.7 €0.0425

28.8 (a) \$31.25 (b) \$40.668

28.9 (B) and (D)

28.10 \$9.105

28.11 \$12.722

28.12 \$46.155

28.13 \$3.62

28.14 \$4.36

28.15 \$19.57

28.16 \$7,624,,538.365

28.17 \$2.662

Section 29

29.1 we have

$$\begin{aligned}
\frac{\partial N(d_2)}{\partial d_2} =&\frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}}\\
=&\frac{1}{\sqrt{2\pi}}e^{-\frac{(d_1-\sigma\sqrt{T-t})^2}{2}}\\
=&\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}e^{d_1\sigma\sqrt{T-t}}e^{-\frac{\sigma^2(T-t)}{2}}\\
=&\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}e^{\ln{(S/K)}+(r-\delta+\frac{1}{2}\sigma^2)(T-t)}e^{-\frac{\sigma^2(T-t)}{2}}\\
=&\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\cdot\frac{S}{K}\cdot e^{(r-\delta)(T-t)} \blacksquare
\end{aligned}$$

29.2

$$\Delta_{\text{Put}} = -e^{-\delta(T-t)}N(-d_1) = -e^{-\delta(T-t)}(1-N(d_1)) = \Delta_{\text{Call}} - e^{-\delta(T-t)}$$

29.3 \$33.8

29.4 (a)).5748 (b) -0.3391

29.5 (B)

29.6 -0.4258

29.7 (D)

29.8 (a) 0.52 for the call and -0.48 for the put (b) 0.38 for the call and -0.62 for the put

29.9 \$79.38

29.10 (B)

29.11 $\Delta = 0.3972, \Gamma = 0.0452, V = 0.10907$

29.12 (D)

Section 30

30.1 We have

$$\begin{aligned}
\theta_{\text{Put}} =& rKe^{-r(T-t)}N(-d_2) - \delta Se^{-\delta(T-t)}N(-d_1) - \frac{Se^{-\delta(T-t)}N'(d_1)\sigma}{2\sqrt{T-t}}\\
=& rKe^{-r(T-t)}(1-N(d_2)) - \delta Se^{-\delta(T-t)}(1-N(d_1)) - \frac{Se^{-\delta(T-t)}N'(d_1)\sigma}{2\sqrt{T-t}}\\
=& rKe^{-r(T-t)} - \delta Se^{-\delta(T-t)} + \left[\delta Se^{-\delta(T-t)}N(d_1) - rKe^{-r(T-t)}N(d_2) - \frac{Se^{-\delta(T-t)}N'(d_1)\sigma}{2\sqrt{(T-t)}}\right]\\
=& \theta_{\text{Call}} + rKe^{-r(T-t)} - \delta Se^{-\delta(T-t)}
\end{aligned}$$

30.2 See the proof of Proposition 30.2.

30.3 (a) \$54.98 (b) \$59.52 (c) \$56.40

30.4 $\theta = -0.0136, \rho = 0.1024, \psi = -0.1232$

30.5 $\theta = -0.0014, \rho = 0.0336$

30.6 $\Delta = -0.384139, \Gamma = 0.0450, V = 0.1080, \theta = -0.005, \rho = -0.0898$

30.7 $\Delta = -0.6028, \Gamma = 0.0454, V = 0.1091, \theta = -0.0025, \rho = -0.1474$

30.8 $\Delta = 0.2187, \Gamma = -0.0004, V = -0.0010, \theta = -0.0025, \rho = 0.0576$

30.9 We have

$$\begin{aligned}
\Delta_{\text{Call}}^{K_1} - \Delta_{\text{Call}}^{K_2} =& N(d_1^{K_1}) - N(d_1^{K_2})\\
=& -(1-N(d_1^{K_1})) + (1-N(d_1^{K_2})\\
=& -N(-d_1^{K_1}) + N(-d_1^{K_2})\\
=& \Delta_{\text{Put}}^{K_1} - \Delta_{\text{Put}}^{K_2}
\end{aligned}$$

30.10 0.018762

Section 31

31.1 As the strike decreases the option becomes more in the money. But we know that the elasticity of a call decreases as the option is more in-the-money.

31.2 (B) and (C)

31.3 -4.389

31.4 -7.3803

31.5 2.4

31.6 3.3973

31.7 0.6210

31.8 -4

31.9 5.7521

31.10 1.3916

31.11 0.9822

Section 32

32.1 (a) 0.644 (b) 0.664

32.2 0.32098

32.3 0.0912

32.4 (a) $-\$22.2025$ (b) \$26.3525

32.5 0.32098

32.6 0.27167

32.7 -0.16

32.8 5.0543

32.9 0.960317

32.10 3435

32.11 5.227

- **32.12** 58.606

32.13 $\Omega_C = 4.66785, \Omega_P = -4.5714$

32.14 0.16

32.15 -13.0263

32.16 0.34

Section 33

33.1 \$3.202

33.2 \$8

33.3 -6%

33.4 (C)

33.5 \$0.295

33.6 A reverse calendar spread is the opposite to a calendar spread so a move in the stock price away from the strike price will increase the investor's profit. A move in the the stock price toward the strike price will result in a maximum loss.

33.7 \$4

33.8 Sell Option A and buy Option C

33.9 Sell option A and buy option C

Section 34

34.1 0.33077

34.2 \$217.10

34.3 0.3682

34.4 0.1438

34.5 No

34.6 (B)

34.7 0.12

Section 35

35.1 (a) −0.28154 (b) \$1028.90 (c) A loss of \$1.82 a share (d) \$0.46

35.2 (a) A saving of \$1022.66 (b) \$0.43 (c) −0.06

35.3 \$1.38

35.4 (a) A saving of \$350 (b) −\$2.07 (c) \$1.05

35.5 $\sigma\sqrt{h} = \frac{0.3}{\sqrt{365}}$.

Day	0	1	2	3	4	5
Stock Price	40	40.642	40.018	39.403	38.797	39.420

35.6 We have the following table

	Day					
	0	1	2	3	4	5
Stock	40.000	40.642	40.018	39.403	38.797	39.420
Call	278.04	315.00	275.57	239.29	206.14	236.76
Δ	0.5824	0.6232	0.5827	0.5408	0.4980	0.5406
Investment	2051.58	2217.66	2056.08	1891.60	1725.95	1894.27
Interest		−0.45	−0.49	−0.45	−0.41	−0.38
Capital Gain		0.43	0.51	0.46	0.42	0.38
Daily Profit		−0.02	0.02	0.01	0.01	0.00

35.7 A net loss of \$206.64

35.8 \$24.12

Section 36

36.1 0.5824

36.2 \$298.2

36.3 \$298.4

36.4 \$176.132

36.5 \$177.10

36.6 0.01587

36.7 0.65

36.8 \$148.56

36.9 \$3.014

36.10 \$86.49

36.11 \$3.6925

36.12 \$85.2235

Section 37

37.1 \$2.3446

37.2 -1.910

37.3 \$7.333

37.4 $-\$1428.17$

37.5 (II) and (III)

37.6 $-\$0.0312$

37.7 -0.4290

37.8 -8.946

Section 38

38.1 Because it is optimal to exercise, the option price is $P = K - S$. Hence, $\Delta = \frac{\partial P}{\partial S} = -1, \Gamma = \frac{\partial^2 P}{\partial S^2} = 0$, and $\theta = \frac{\partial P}{\partial t} = 0$. In this case, equation (38.1) becomes

$$\frac{1}{2}\sigma^2 S_t^2 \times 0 + rS_t \times (-1) + 0 = r(K - S).$$

This leads to $rK = 0$ which is impossible since $r > 0$ and $K > 0$.

38.2 \$2.59525

38.3 \$5.77

38.4 -13.05

38.5 \$30

38.6 0.01

38.7 0.4118

38.8 0.00094624

38.9 -0.0444

38.10 0.0011

38.11 The stock moves up and down by 0.5234

Section 39

39.1 118.14

39.2 13.713

39.3 \$579.83

39.4 \$0.4792

39.5 We first try to gamma-neutralize the portfolio using the 35-strike puts; then we can delta-neutralize the remaining net delta with shares. The gamma of the portfolio to be hedged is just the gamma of the 100 put options. That is, $100 \times 0.25 = 25$. Let n be the number of the 35-strike puts to be sold to bring the gamma of the portfolio to 0. Then, n satisfies the equation $25 + 0.50n = 0$ or $n = -50$. Thus, 50 of the 35-strike put options must be sold. In this case, the net delta of the portfolio is

$$100(-0.05) + 5(1) + 50(0.10) = 5.$$

Hence, the investor must sell 5 shares of stock A to delta-neutralize the portfolio.

39.6 We first try to gamma-neutralize the portfolio using the Call-II calls; then we can delta-neutralize the remaining net delta with shares. The gamma of the portfolio to be hedged is just the gamma of the 1000 units of call-I. That is, $1000 \times (-0.0651) = -65.1$. Let n be the number of the call-II calls needed to bring the gamma of the portfolio to 0. Then, n satisfies the equation $-65.1 + 0.0746n = 0$ or $n = 872.7$. Thus, 872.7 of the call-II cal options must be purchased. In this case, the net delta of the portfolio is

$$1000(-0.5825) + 872.7(0.7773) = 95.8.$$

Hence, the investor must sell 95.8 shares of stock to delta-neutralize the portfolio.

Section 40

40.1 \$17

40.2 \$6

40.3 \$2.53

40.4 (a) \$6.16 (b) \$6.55

40.5 (a) \$7.19 (b) \$7.45

40.6 (a) \$4.20 (b) \$4.02

40.7 (a) By Proposition 40.1 we have $G(T) \leq A(T)$ so that $K - A(T) \leq K - G(T)$. Thus, $\max\{0, K - A(T)\} \leq \max\{0, K - G(T)\}$
(b) By Proposition 40.1 we have $G(T) \leq A(T)$ so that $S_T - A(T) \leq S_T - G(T)$. Thus, $\max\{0, S_T - A(T)\} \leq \max\{0, S_T - G(T)\}$
(c) By Proposition 40.1 we have $G(T) \leq A(T)$ so that $G(T) - S_T \leq A(T) - S_T$. Thus, $\max\{0, G(T) - S_T\} \leq \max\{0, A(T) - S_T\}$

40.8 \$10

40.9 \$0.402

Section 41

41.1 Suppose that the barrier level H is less than the strike price K. Then the up-and-out call is knocked out and has zero payoff. Hence, the up-and-in call is the same as an ordinary call.
If the barrier level H is greater than K, the down-and-out put is knocked out. Hence, the down-and-in put is the same as an ordinary put.

41.2 \$0

41.3 \$15.50

41.4 \$9.36

41.5 \$18

41.6 \$5.09

41.7 \$5.30

41.8 \$10

41.9 \$40

41.10 \$4.5856

Section 42

42.1 \$87.66

42.2 10.13%

42.3 \$1.054

42.4 \$3.688

42.5 \$0.8354

42.6 \$17.84

42.7 \$2.158

42.8 (a) \$2.215 (b) \$3.20

Section 43

43.1 0.4419

43.2 -0.3657

43.3 0.1303

43.4 \$15.69

43.5 \$11

43.6 (a) $0.1540S_t$ (b) 0.1540 shares of the stock (c) \$7.5476

43.7 (a) $0.04716S_t$ (b) must have 0.04716 of the shares today (c) \$4.716

43.8 \$14.5

43.9 \$2.85

Section 44

44.1 When the strike and the trigger are equal, the payoffs of the gap option and the ordinary option are equal so that both have the same price. The statement is true.

44.2 Using Figure 44.2, we see that for $K_2 > K_1$, negative payoffs are possible. Increasing K_2 will create more negative payoffs and hence should reduce the premium. The given statement is true.

44.3 Using Figure 44.2, we see that for a fixed K_2, increasing K_1 will reduce the negative payoffs. Thus, the premium must not be reduced. The given statement is false.

44.4 The premium is positive when the trigger is equal to the strike price. From Figure 44.2, K_2 can be increased until the premium price is almost zero.

44.5 \$3.0868

44.6 \$5.20

44.7 (B)

44.8 \$4.21

44.9 585.88

Section 45

45.1 \$9.3716

45.2 \$1.49595

45.3 \$10.54

45.4 \$6.41

45.5 \$0.674

Section 46

46.1 (a) 0.2389 (b) 0.1423 (c) 0.6188 (d) 88

46.2 (a) 0.7517 (b) 0.8926 (c) 0.0238

46.3 (a) 0.5 (b) 0.9876

46.4 (a) 0.4772 (b) 0.004

46.5 2%

46.6 (a) 0.9452 (b) 0.8186

46.7 0.224

46.8 75

46.9 (a) 0.4721 (b) 0.1389 (c) 0.6664 (d) 0.58

46.10 0.01287

46.11 0.86

46.12 (a) 0.1736 (b) 1071.20

46.13 0.1151

46.14 (a) 0.1056 (b) 362.84

46.15 1:18 PM

46.16 (a) 0.2514 (b) 0.7496 (c) 16.92

46.17 $X_1 + X_2$ is a normal distribution with mean 10 and variance 12.913

Section 47

47.1 (C)

47.2 (A) and (D)

47.3 $E(Y) = 3.08, \text{Var}(Y) = 2.695$

47.4 1.02

47.5 We have for $x > 0$

$$F_{\frac{1}{Y}}(x) = Pr(\frac{1}{Y} \leq x) = Pr(Y \geq \frac{1}{x}) = 1 - F_Y(1/x).$$

Taking the derivative of both sides we find

$$f_{\frac{1}{Y}}(x) = \frac{1}{x^2}f_Y(1/x) = \frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln x+\mu}{\sigma}\right)^2},\ x > 0$$

and 0 otherwise. Thus, $\frac{1}{Y}$ is lognormal with parameters $-\mu$ and σ^2

47.6 0.70

47.7 668.48 hours

47.8 the mean is 67846.43 and the standard deviation is 197,661.50

47.9 The paramters are $\mu = 5.0944$ and $\delta = 0.0549$

47.10 0.9816

47.11 0.8686

47.12 51.265

47.13 Mean = 90.017 and median = 7.389

Section 48

48.1 Relative stock price is \$1.7062. Holding period return is 70.62%

48.2 19.72%

48.3 3%

48.4 3.04%

48.5 0.11

48.6 \$103.28

48.7 \$86.27

48.8 \$97.27

48.9 (a) $100e^{(0.1-\frac{1}{2}\times 0.6^2)\times 2+0.6\sqrt{2}Z}$ (b) \$122.14 (c) \$85.21

48.10 \$90.4932

48.11 \$125.38 (up) and \$88.04 (down)

48.12 \$148.13

48.13 (a) 0.0885 (mean) and 0.1766 (standard deviation) (b) 10.41%

48.14 34.056%

Section 49

49.1 The monthly continuously compounded mean return is 0.00875 and the monthly standard deviation is 0.0866.

49.2 (a) There is 68.27% probability that a number is within one standard deviation from the mean for a standard normal random variable. Thus, there is 68.27% that the monthly continuously compounded return is between $0.00875 - 0.0866$ and $0.00875 + 0.0866$. Hence, $r^L = -0.0788$ and $r^U = 0.09544$.
(b) There is 95.45% probability that a number is within two standard deviations from the mean for a standard normal random variable. Thus, there is 95.45% that the monthly continuously compounded return is between $0.00875 - 2 \times 0.0866$ and $0.00875 + 2 \times 0.0866$. Hence, $r^L = -0.1644$ and $r^U = 0.1819$.

49.3 (a) $S^L = 46.22$ and $S^U = 55.06$ (b) $S^L = 42.42$ and $S^U = 59.97$.

49.4 0.638031

49.5 \$112.61

49.6 Suppose that $P(S_t < S_0) > 50\%$. Since $P(S_t < S_0) = N(-\hat{d}_2)$ we must have $-\hat{d}_2 > 0$ or $\hat{d}_2 < 0$. But

$$\hat{d}_2 = \frac{\ln S_0 - \ln K + (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}} = \frac{\ln S_0 - \ln S_0 + (0.10 - 0 - 0.5(0.4)^2 t}{0.4\sqrt{t}} = 0.05\sqrt{t} > 0.$$

Hence, we must have $P(S_t < S_0) < 50$.

49.7 The probability that the option expires in the money is

$$P(S_T < S_0 e^{rT}) = N(-\hat{d}_2)$$

where

$$\hat{d}_2 = \frac{\ln S_0 - \ln K + (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}} = \frac{[(\alpha - r) - 0.5\sigma^2]\sqrt{T}}{\sigma}.$$

Hence,

$$P(S_T < S_0e^{rT}) = N\left(\frac{[0.5\sigma^2 - (\alpha - r)]\sqrt{T}}{\sigma}\right)$$

Since $\alpha - r > 0.5\sigma^2$, we find $0.5\sigma^2 - (\alpha - r) < 0$. It follows that as T increases the $-\hat{d}_2$ becomes more and more negative and this decreases the probability that the put option expires in the money.

49.8 0.20

49.9 The biannual continuously compounded mean return is 0.044375 and the biannual standard deviation is 0.176777

49.10 (a) $r^L = -0.132402$ and $r^U = 0.221152$ (b) $r^L = -0.309179$ and $r^U = 0.397929$

49.11 (a) $S^L = 65.70$ and $S^U = 93.56$ (b) $S^L = 55.05$ and $S^U = 111.66$.

49.12 0.431562

49.13 0.242

49.14 \$0.3927

Section 50

50.1 $E(S_t|S_t < K) = \frac{S_0 e^{(\alpha-\delta)t} N(-\hat{d}_1)}{P(S_t<K)} = S_0 e^{(\alpha-\delta)t} \frac{N(-\hat{d}_1)}{N(-\hat{d}_2)}$.

50.2 For a European call option with strike K, the price is defined as the expected value of $e^{-rt}\max\{0, S_t - K\}$. That is,

$$P(S, K, \sigma, r, t, \delta) = E[e^{-rt}\max\{0, K - S_t\}]$$

Now we proceed to establish the Black-Scholes formula based on risk-neutral probability:

$$\begin{aligned}
P(S, K, \sigma, r, t, \delta) =& E[e^{-rt}\max\{0, K - S_t\}] \\
=& e^{-rt}\int_0^K (K - S_t) g^*(S_t, S_0) dS_t \\
=& e^{-rt} E^*(K - S_t|S_t < K) P^*(S_t < K) \\
=& e^{-rt} E^*(K|S_t < K) P^*(S_t < K) - e^{-rt} E^*(S_t|S_t < K) P^*(S_t < K) \\
=& Ke^{-rt} N(-d_2) - S_0 e^{-\delta t} N(-d_1)
\end{aligned}$$

which is the celebrated Black-Scholes formula for a European put option.

50.3 We have

$$\begin{aligned}
E(S_T|S_T < 100) =& S_0 e^{(\alpha-\delta)T} \frac{N(-\hat{d}_1)}{N(-\hat{d}_2)} \\
=& 100 \frac{N(-0.5\sigma\sqrt{T})}{N(0.5\sigma\sqrt{T})} \\
=& 100 \frac{1 - N(0.5\sigma\sqrt{T})}{N(0.5\sigma\sqrt{T})} = \frac{100}{N(0.5\sigma\sqrt{T})} - 100
\end{aligned}$$

We conclude from this expression that as T increases, $N(0.5\sigma\sqrt{T})$ increases and therefore $E(S_T|S_T < 50)$ decreases.

50.4 We have

$$PE(S_t|S_t < K) = S_0 e^{(\alpha-\delta)t} N(-\hat{d}_1) = 100N(-0.5\sigma\sqrt{T}).$$

It follows that as T increases $N(-0.5\sigma\sqrt{T})$ decreases and therefore the partial expectation decreases.

50.5 \$130.0176

50.6 36.35

50.7 \$58.83

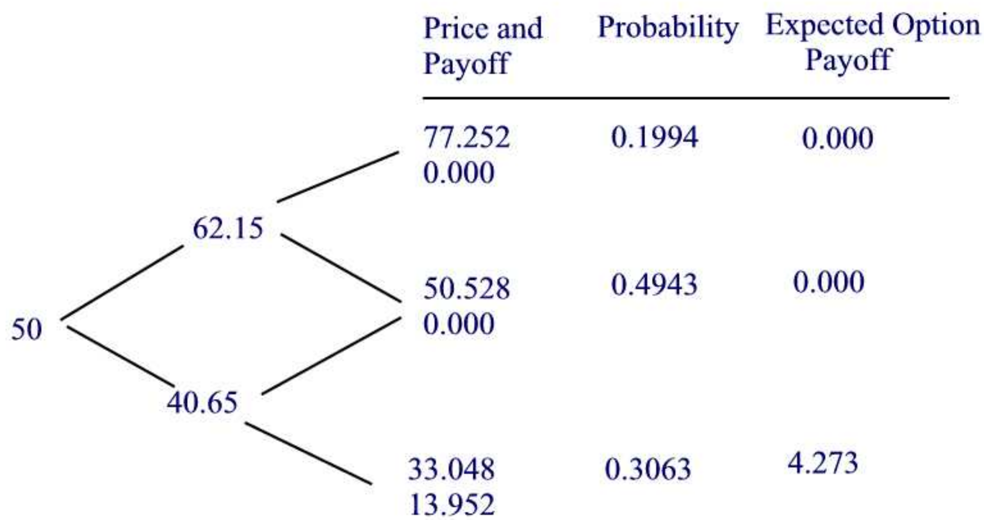
50.8 \$97.51

50.9 (a) \$17.50 (b) \$35

Section 51

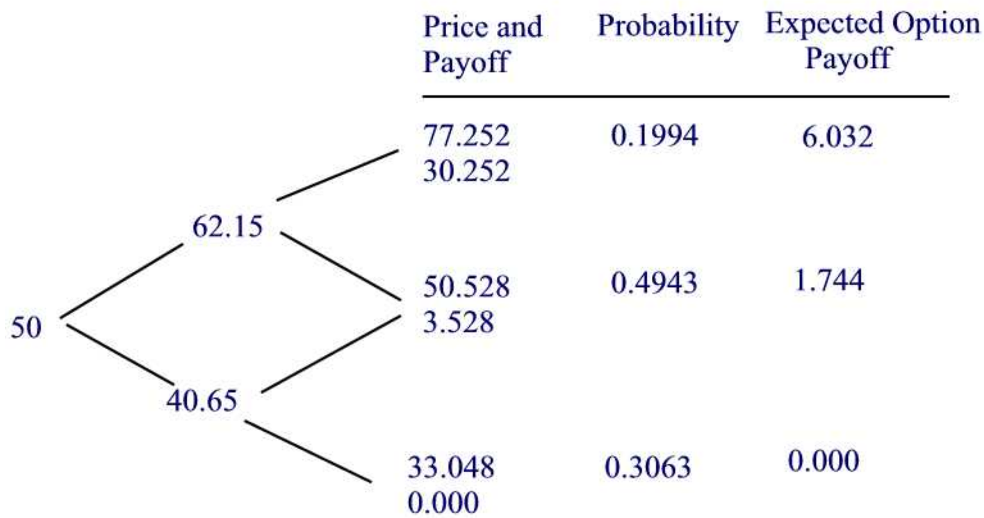
51.1 $P(uu) = 0.1994, P(ud) = 0.4943, P(dd) = 0.3063$

51.2



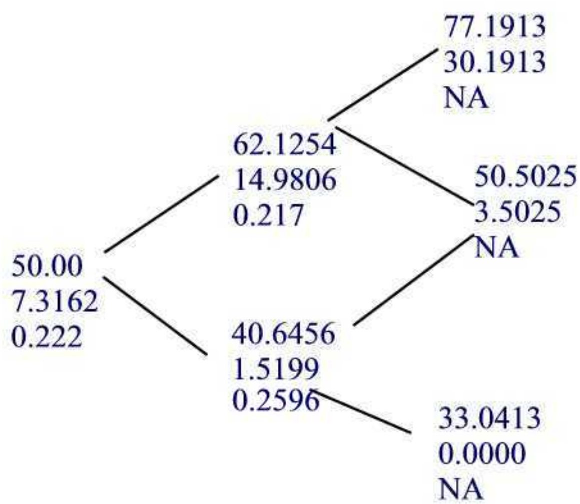
51.3 \$3.662

51.4



51.5 \$7.97

51.6



51.7 \$7.221

Section 52

52.1 (a) The pdf is given by

$$f(x) = \begin{cases} \frac{1}{4} & 3 \leq x \leq 7 \\ 0 & \text{otherwise} \end{cases}$$

(b) 0 (c) 0.5.

52.2 (a) The pdf is given by

$$f(x) = \begin{cases} \frac{1}{10} & 5 \leq x \leq 15 \\ 0 & \text{otherwise} \end{cases}$$

(b) 0.3 (c) $E(X) = 10$ and Var(X) $= 8.33$.

52.3 (a)

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

(b) b.

52.4 -0.137

52.5 -1.282

52.6 -1.28

52.7 $z_1 = -0.8099, z_2 = 1.18, z_3 = -1.95996$

Section 53

53.1 We find

$$\begin{aligned}
S_{\frac{1}{3}} =&100e^{(0.11-0.03-0.5\times 0.3^2)\times\frac{1}{3}+0.3\times\sqrt{\frac{1}{3}}\times(-1.175)} = 82.54\\
S_{\frac{2}{3}} =&82.54e^{(0.11-0.03-0.5\times 0.3^2)\times\frac{1}{3}+0.3\times\sqrt{\frac{1}{3}}\times(1.126)} = 101.49\\
S_1 =&101.49e^{(0.11-0.03-0.5\times 0.3^2)\times\frac{1}{3}+0.3\times\sqrt{\frac{1}{3}}\times(0)} = 102.68
\end{aligned}$$

53.2 We have

$$\begin{aligned}
S_{\frac{1}{4}} =&40e^{(0.08-0.00-0.5\times 0.3^2)\times 0.25+0.3\times\sqrt{0.25}\times(-1.774)} = 30.92\\
S_{\frac{2}{4}} =&30.92e^{(0.08-0.00-0.5\times 0.3^2)\times 0.25+0.3\times\sqrt{0.25}\times(-0.192)} = 30.31\\
S_{\frac{3}{4}} =&30.31e^{(0.08-0.00-0.5\times 0.3^2)\times 0.25+0.3\times\sqrt{0.25}\times(0.516)} = 33.04\\
S_1 =&33.04e^{(0.08-0.00-0.5\times 0.3^2)\times 0.25+0.3\times\sqrt{0.25}\times(0.831)} = 37.76
\end{aligned}$$

53.3 We find

$$\begin{aligned}
S_1^1 =&50e^{(0.15-0.5\times 0.3^2)\times 2+0.3\times\sqrt{2}\times 2.12} = 151.63\\
S_1^2 =&50e^{(0.15-0.5\times 0.3^2)\times 2+0.3\times\sqrt{2}\times(-1.77)} = 29.11\\
S_1^3 =&50e^{(0.15-0.5\times 0.3^2)\times 2+0.3\times\sqrt{2}\times 0.77} = 85.52
\end{aligned}$$

Thus, the mean of the three simulated prices is

$$\frac{151.63+29.11+85.52}{3} = \$88.75.$$

Section 54

54.1 \$3.99

54.2 \$8.72

54.3 21,218 draws

54.4 0.0346

54.5 3986

Section 55

55.1 \$28.36

55.2 We have $Z(1) = N^{-1}(0.9830) = 2.12, Z(2) = N^{-1}(0.0384) = -1.77$, and $Z(3) = N^{-1}(0.7794) = 0.77$. Thus,

$$\begin{aligned} S_{1/3} =& 50e^{(0.15-0.5\times0.3^2)\times\frac{1}{3}+0.3\times\sqrt{\frac{1}{3}}\times2.12} = \$74.76 \\ S_{2/3} =& 74.76e^{(0.15-0.5\times0.3^2)\times\frac{1}{3}+0.3\times\sqrt{\frac{1}{3}}\times(-1.77)} = \$56.98 \\ S_1 =& 56.98e^{(0.15-0.5\times0.3^2)\times\frac{1}{3}+0.3\times\sqrt{\frac{1}{3}}\times0.77} = \$67.43 \end{aligned}$$

Thus, the arithmetic mean of the three simulated prices is

$$\frac{74.76 + 56.98 + 67.43}{3} = \$66.39$$

and the geometric mean is

$$(74.76 \times 56.98 \times 67.43)^{\frac{1}{3}} = \$65.98.$$

55.3 \$16.39

55.4 \$3.61

55.5 \$1.04

55.6 \$0.00

55.7 \$15.98

55.8 \$4.02

55.9 \$1.45

55.10 \$0.00

55.11 (a) \$5.19 (b) \$4.44.

Section 56

56.1 (a) 3.0984 (b) 1.8032 (c) 1.1968

56.2 0.4375

56.3 0.75

56.4 0.875

56.5 (a) 0.764211 (b) \$1.872

56.6 (a) 1.0959 (b) 2.45

Section 57

57.1 \$2.17

57.2 0.78375

57.3

i	1	2	3	4	5	6	7	8
$\hat{U}_i$	0.122	0.44735	0.7157	0.86205	0.0793	0.4736	0.625325	0.825375
Z_i	−1.16505	−0.13236	0.570115	1.089576	−1.4098	−0.06622	0.319497	0.936045

57.4 2.499376

57.5 (a) 0.038, 0.424, 0.697, 0.797 (b) $-1.77438, -0.19167, 0.515792, 0.830953$
(c) \$24.33, \$39.11, \$48.36, \$53.29

57.6 \$30.92, \$30.31, \$33.04, \$37.75

57.7 \$4.75

Section 58

58.1 12

58.2 20

58.3 This is just the continuity of Z at 0.

58.4 Let $W(t) = Z(t+\alpha) - Z(\alpha)$. We have • $W(0) = Z(\alpha) - Z(\alpha) = 0$.
• $W(t+s) - W(t) = Z(t+s+\alpha) - Z(\alpha) - Z(t+\alpha) + Z(\alpha) = Z(t+\alpha+s) - Z(t+\alpha)$ which is normally distributed with mean 0 and variance s.
• $W(t+s_1) - W(t) = Z(t+\alpha+s_1) - Z(t+\alpha)$ and $W(t) - W(t-s_2) = Z(t+\alpha) - Z(t+\alpha-s_2)$ are independent random variables.
• $\lim_{h\to 0} W(t+h) = \lim_{h\to 0} Z(t+h+\alpha) - Z(\alpha) = Z(t+\alpha) - Z(\alpha) = W(t)$. That is, $W(t)$ is continuous.
It follows that $\{Z(t+\alpha) - Z(\alpha)\}_{t\geq 0}$ is a Brownian motion.

58.5 Let $W(t) = Z(1) - Z(1-t)$. We have • $W(0) = Z(1) - Z(1) = 0$.
• $W(t+s) - W(t) = Z(1) - Z(1-t-s) - Z(1) + Z(1-t) = Z(1-t) - Z(1-t-s) = Z((1-t-s)+s) - Z(1-t-s)$ which is normally distributed with mean 0 and variance s.
• $W(t+s_1) - W(t) = Z(1-t) - Z(1-t-s_1)$ and $W(t) - W(t-s_2) = Z(1-t+s_2) - Z(1-t)$ are independent random variables.
• $\lim_{h\to 0} W(t+h) = \lim_{h\to 0} Z(1) - Z(1-t-h) = Z(1) - Z(1-t) = W(t)$. That is, $W(t)$ is continuous.
It follows that $\{Z(1) - Z(1-t)\}_{0\leq t\leq 1}$ is a Brownian motion.

58.6 Let $W(t) = s^{-\frac{1}{2}} Z(st)$. We have • $W(0) = s^{-\frac{1}{2}} Z(0) = 0$.
• $W(t+x) - W(t) = s^{-\frac{1}{2}} Z(s(t+x)) - s^{-\frac{1}{2}} Z(st)$ which is normally distributed with mean 0 and variance sx.
• $W(t+s_1) - W(t) = s^{-\frac{1}{2}} Z(st+ss_1) - s^{-\frac{1}{2}} Z(st)$ and $W(t) - W(t-s_2) = s^{-\frac{1}{2}} Z(st) - s^{-\frac{1}{2}} Z(st-ss_2)$ are independent random variables.
• $\lim_{h\to 0} W(t+h) = \lim_{h\to 0} s^{-\frac{1}{2}} Z(s(t+h)) = s^{-\frac{1}{2}} Z(st) = W(t)$. That is, $W(t)$ is continuous.
It follows that $\{s^{-\frac{1}{2}} Z(st)\}_{t\geq 0}$ is a Brownian motion.

58.7 We have

$$\begin{aligned}\left|\sum_{i=1}^{n}[Z(ih)-Z((i-1)h)]^3\right| =&\left|\sum_{i=1}^{n}\left(Y_{ih}\sqrt{h}\right)^3\right|\\ =&\left|\sum_{i=1}^{n}Y_{ih}^3h^{\frac{3}{2}}\right|\\ \le&\sum_{i=1}^{n}h^{\frac{3}{2}}\\ =&\frac{T^{\frac{3}{2}}}{\sqrt{n}}\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\sum_{i=1}^{n}[Z(ih)-Z((i-1)h)]^3=0.$$

Section 59

59.1 We have $X(t) = \alpha t + \sigma Z(t)$, where $Z(t)$ is a normal distribution with mean 0 and variance t. Hence, $\frac{X(t)-\alpha t}{\sigma}$ is normally distributed with mean 0 and variance t.

59.2 We have $X(t) = \alpha t + \sigma Z(t)$, where $Z(t)$ is a normal distribution with mean 0 and variance t. Thus, $\frac{X(t)-\alpha t}{\sigma\sqrt{t}} = \frac{Z(t)}{\sqrt{t}}$ is normally distributed with mean 0 and variance 1.

59.3 0.1585

59.4 $\alpha = 0, \lambda = 0.40$, and $\sigma = 0.20$.

59.5 $\alpha = 0.12, \lambda = 0.4, \sigma = 0.30$.

59.6 $X(t) = 0.24e^{-0.4t} + 0.12(1 - e^{-0.4t}) + 0.30\int_0^t e^{-0.4(t-s)}dZ(s)$.

59.7 7.1524

59.8 637.474

59.9 \$6.27

59.10 (a) 5.15 (b) 1.29 (c) 0.0014

59.11 0.95254

Section 60

60.1 \$633.34

60.2 \$1.725

60.3 0.414

60.4 0.063

60.5 (a) 3.34815 (b) 1.55 (c) 0.883

60.6 At time t, the fund owns x shares of stock and bonds worth y. Then $W(t) = x(t)S(t)+y(t)$. But $x(t)S(t) = \phi W(t)$ so that $x(t) = \frac{\phi W(t)}{S(t)}$. Likewise, $(1-\phi)W(t) = y(t)$. Since the stock pays no dividends, the value of the fund at the next instant is

$$dW = xdS + yrdt = \frac{\phi W(t)}{S(t)}dS + (1-\phi)W(t)rdt$$

or

$$\frac{dW}{W} = \phi\frac{dS}{S} + (1-\phi)rdt.$$

In other words, the instantaneous return on the fund is a weighted average of the return on the stock and the risk free rate.
Since

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dZ(t),\ t \geq 0$$

we can write

$$\frac{dW}{W} = [\alpha\phi + (1-\alpha)r]dt + \phi\sigma dZ(t).$$

The solution to this equation is

$$W(t) = W(0)e^{(\alpha\phi+(1-\alpha)r-0.5\phi^2\sigma^2)t+\phi\sigma Z(t)}.$$

On the other hand, $S(t)$ is given by

$$S(t) = S(0)e^{(\alpha-0.5\sigma^2)t+\sigma dZ(t)}$$

so that

$$\left[\frac{S(t)}{S(0)}\right]^{\phi} = e^{(\alpha\phi - 0.5\phi\sigma^2)t + \phi\sigma Z(t)}.$$

Hence,

$$\begin{aligned}
W(t) =&W(0)e^{(\alpha\phi+(1-\alpha)r-0.5\phi^2\sigma^2)t+\phi\sigma Z(t)}\\
=&W(0)e^{(\alpha\phi-0.5\phi\sigma^2)t+\phi\sigma Z(t)}e^{[(1-\alpha)r-0.5\phi^2\sigma^2+0.5\phi\sigma^2]t}\\
=&W(0)\left[\frac{S(t)}{S(0)}\right]^{\phi}e^{[(1-\phi)r+0.5\phi\sigma^2(1-\phi)]t}\\
=&W(0)\left[\frac{S(t)}{S(0)}\right]^{\phi}e^{(1-\phi)(r+0.5\phi\sigma^2)t}
\end{aligned}$$

60.7 (i) True (ii) True (iii) True

Section 61

61.1 $(dX)^2 = \sigma^2 dt$ and $DX \times dt = 0$.

61.2 1428dt

61.3 499.52dt

61.4 We have

$$\int_0^T [dZ(t)]^4 = \int_0^T (dZ(t))^2 \times (dZ(t))^2 = \int_0^t dt \times dt = \int_0^T 0 = 0$$

61.5 $X = Z(T) + T$

61.6 Mean = Variance = T

61.7 0.01dt

61.8 (i) True (ii) True (iii) True.

61.9 $V_{2.4}^2(W) < V_{2.4}^2(Y) < V_{2.4}^2(X)$.

Section 62

62.1 4.28%

62.2 1.316

62.3 0.643

62.4 3.152%

62.5 The Sharpe ratio of the first stock is $\phi_1 = \frac{0.07-0.03}{0.12} = 0.333$. For the second stock, we have $\phi_2 = \frac{0.05-0.03}{0.11} = 0.182$. Since $\phi_1 > \phi_2$, an arbitrage opportunity occurs by buying $\frac{1}{0.12S_1}$ shares of stock 1, selling $\frac{1}{0.11S_2}$ of stock 2, and lending $\frac{1}{0.11} - \frac{1}{0.12}$. The arbitrage profit is

$$(\phi_1 - \phi_2)dt = (0.333 - 0.182)dt = 0.151dt.$$

62.6 0.06

62.7 0.1405

62.8 2%

62.9 0.065

62.10 Four shares.

Section 63

63.1 $\frac{dS(t)}{S(t)} = 0.036dt + 0.3dZ(t).$

63.2 0.04

63.3 8%

63.4 $\delta = 2\%, r = 10\%$

63.5 14%

63.6 4.5%

Section 64

64.1 $d(2S^2(t)) = (4\alpha S(t) + 2\sigma^2)dt + 4S(t)\sigma dZ(t)$.

64.2 $d(2S^2(t)) = (4\alpha S(t)\lambda - 4S^2\lambda + 2\sigma^2)dt + 4S(t)\sigma dZ(t)$.

64.3 $d(2S^2(t)) = (4\alpha S^2(t) + 2\sigma^2 S^2(t))dt + 4S^2(t)\sigma dZ(t)$.

64.4 $d(35S^5(t) + 2t) = (175\alpha S^4(t) + 350S^3(t)\sigma^2 + 2)dt + 175S^4(t)\sigma dZ(t)$.

64.5 $dG(t) = G(t)[0.12dt + 0.4dZ(t)]$.

64.6 7

64.7 All three.

64.8 $dX(t) = (0.11\sqrt{X(t)} - 2X(t))dt + 0.2X^{\frac{3}{4}}(t)dZ(t)$.

64.9 0.06222

64.10 Let $f(x(t),t) = \frac{1}{x(t)}$. Then $f_t = 0, f_x = -[x(t)]^{-2}$, and $f_{xx} = 2[x(t)]^{-3}$. Now, by Itô's lemma we have

$$\begin{aligned}
dy(t) =& d[f(x(t),t)] = f_t dt + f_x dx(t) + \frac{1}{2}f_{xx}(dx(t))^2 \\
=& 0 - [x(t)]^{-2}dx(t) + \frac{1}{2}\times 2[x(t)]^{-3}(dx(t))^2 \\
=& -[x(t)]^{-1}\left(\frac{dx(t)}{x(t)}\right) + [x(t)]^{-1}\left(\frac{dx(t)}{x(t)}\right)^2 \\
=& -[x(t)]^{-1}[(r - r_{€})dt + \sigma dZ(t)] + [x(t)]^{-1}[(r - r_{€})dt + \sigma dZ(t)]^2 \\
=& -y(t)[(r - r_{€})dt + \sigma dZ(t)] + y(t)(\sigma^2)dt
\end{aligned}$$

Therefore,

$$\frac{d(y(t)}{y(t)} = (r_{€} - r + \sigma^2)dt - \sigma dZ(t)$$

Section 65

65.1 $d[S(t)^a] = 0.03595\sqrt{S(t)}dt + 0.09\sqrt{S(t)}dZ(t)$.

65.2 $d[S(t)^a] = 0.03\sqrt{S(t)}dt + 0.10\sqrt{S(t)}dZ(t)$.

65.3 0.232dt

65.4 1.37227

65.5 1 or -2

65.6 0.08

65.7 0.01

65.8 1.51038

Section 66

66.1 We have
$$\frac{d}{dt}\left[e^{-rt}S(t)\right] = e^{-rt}\frac{S(t)}{dt} - re^{-rt}S(t).$$
Thus,
$$e^{-rt}\frac{S(t)}{dt} - re^{-rt}S(t) = -e^{-rt}D(t).$$
Multiplying through by e^{rt} we find
$$\frac{dS(t)}{dt} - rS(t) = -D(t).$$

66.2 Intgerating both sides of the equation in the previous problem with respect to t we find
$$e^{-rt}S(t) = -\int e^{-rt}D(t)dt + C$$
or
$$S(t) = Ce^{rt} - e^{rt}\int e^{-rt}D(t)dt.$$

66.3 Integrate the equation
$$\frac{d}{dt}\left[e^{-rt}S(t)\right] = -e^{-rt}D(t)$$
from t to T we find
$$e^{-rT}S(T) - e^{-rt}S(t) = -\int_t^T e^{-rs}D(s)ds$$
and this reduces to
$$S(t) = \int_t^T e^{-r(s-t)}D(s)ds + \tilde{S}e^{-r(T-t)}.$$

66.4 9 years

66.5 10%

66.6 $\frac{dS(t)}{dt} = (0.10 - \delta)S(t).$

Section 67

67.1 We have

$$\begin{aligned} V_t =& re^{-r(T-t)} \\ V_S =& 0 \\ V_{SS} =& 0 \end{aligned}$$

Thus,

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = rV - rV = 0.$$

The boundary condition is: $V(T,T) = \$1$.

67.2 We have

$$\begin{aligned} V_t =& \delta V \\ V_S =& e^{-\delta(T-t)} \\ V_{SS} =& 0 \end{aligned}$$

Thus,

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + (r-\delta)SV_S - rV = \delta V + (r-\delta)V - rV = 0.$$

The boundary condition is $V[S(T),T] = S(T)$.

67.3 We have

$$\begin{aligned} V_t =& rV \\ V_S =& \left(1 - \frac{2r}{\sigma^2}\right) e^{rt} S^{-\frac{2r}{\sigma^2}} \\ V_{SS} =& -\frac{2r}{\sigma^2}\left(1 - \frac{2r}{\sigma^2}\right) e^{rt} S^{-1-\frac{2r}{\sigma^2}} \end{aligned}$$

Thus,

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = rV - r\left(1 - \frac{2r}{\sigma^2}\right) e^{rt} S^{1-\frac{2r}{\sigma^2}} + rS\left(1 - \frac{2r}{\sigma^2}\right) e^{rt} S^{-\frac{2r}{\sigma^2}} - rV = 0.$$

67.4 When the stock is trading for \$50 at expiration, you would receive the \$500 payoff for your option contract. If the stock is trading below \$50 per share at the expiration date, you receive nothing.

67.5 For this problem, we refer the reader to Section 29 and Section 30 where expressions for $\frac{\partial N(d_1)}{\partial t}$, $\frac{\partial N(d_2)}{\partial t}$, $\frac{\partial N(d_1)}{\partial S}$, and $\frac{\partial N(d_2)}{\partial S}$ were established. We have

$$\begin{aligned}
V_t =& rV - e^{-r(T-t)}\left(\frac{1}{\sqrt{2\pi}}\frac{S}{K}e^{-\frac{d_1^2}{2}}e^{(r-\delta)(T-t)}\right)\left(\frac{\ln(S/K)}{2\sigma(T-t)^{\frac{3}{2}}} - \frac{r-\delta-0.5\sigma^2}{\sigma\sqrt{T-t}} + \frac{r-\delta-0.5\sigma^2}{2\sigma\sqrt{T-t}}\right)\\
V_S =& -e^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\cdot\frac{1}{K}\cdot\frac{e^{(r-\delta)(T-t)}}{\sigma\sqrt{T-t}}\\
V_{SS} =& d_1e^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\cdot\frac{1}{K}\cdot\frac{e^{(r-\delta)(T-t)}}{S\sigma^2(T-t)}
\end{aligned}$$

Thus,

$$V_t + \frac{1}{2}\sigma^2S^2V_{SS} + (r-\delta)SV_S - rV = 0.$$

67.6 We have

$$\begin{aligned}
V_t =& \delta V + Se^{-\delta(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\left(\frac{\ln(S/K)}{2\sigma(T-t)^{\frac{3}{2}}} - \frac{r-\delta+0.5\sigma^2}{\sigma\sqrt{T-t}} + \frac{r-\delta+0.5\sigma^2}{2\sigma\sqrt{T-t}}\right)\\
V_S =& e^{-\delta(T-t)}N(d_1) + Se^{-\delta(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\cdot\frac{1}{S\sigma\sqrt{T-t}}\\
V_{SS} =& e^{-\delta(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\cdot\frac{1}{S\sigma\sqrt{T-t}} - d_1e^{-\delta(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\cdot\frac{1}{\sigma^2(T-t)}
\end{aligned}$$

Thus,

$$V_t + \frac{1}{2}\sigma^2S^2V_{SS} + (r-\delta)SV_S - rV = 0.$$

67.7 We have

$$\begin{aligned}
V_t =&\delta V - Se^{-\delta(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\left(\frac{\ln(S/K)}{2\sigma(T-t)^{\frac{3}{2}}} - \frac{r-\delta+0.5\sigma^2}{\sigma\sqrt{T-t}} + \frac{r-\delta+0.5\sigma^2}{2\sigma\sqrt{T-t}}\right)\\
V_S =&e^{-\delta(T-t)}N(-d_1) - Se^{-\delta(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\cdot\frac{1}{S\sigma\sqrt{T-t}}\\
V_{SS} =&-e^{-\delta(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\cdot\frac{1}{S\sigma\sqrt{T-t}} + d_1e^{-\delta(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\cdot\frac{1}{\sigma^2(T-t)}
\end{aligned}$$

Thus,

$$V_t + \frac{1}{2}\sigma^2S^2V_{SS} + (r-\delta)SV_S - rV = 0.$$

67.8 24.5%

67.9 1%

Section 68

68.1 2%

68.2 8%

68.3 3.968

68.4 0.51584

68.5 4.316

68.6 0.13

Section 69

69.1 0.057825

69.2 0.0887

69.3 26.375

69.4 -4.571

69.5 9.5%

69.6 3.35

Section 70

70.1 \$4.205

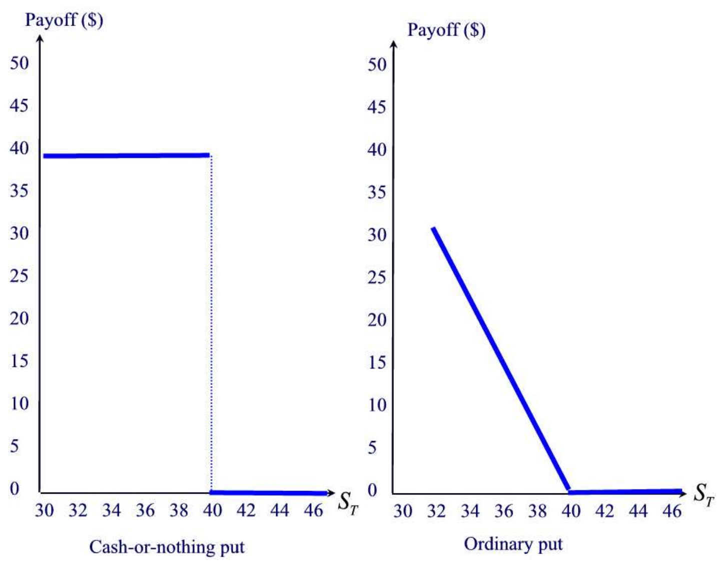
70.2 \$3.662

70.3 \$9.80

70.4 8%

70.5 0.25 years

70.6



70.7 \$15.79

70.8 \$38,100

70.9 19.55 shares

70.10 -2

Section 71

71.1 \$26.98

71.2 \$40.95

71.3 $S_0e^{-\delta(T-t)}$

71.4 1.25

71.5 $e^{-\delta(T-t)}N(-d_1) - e^{-\delta(T-t)} \cdot \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \cdot \frac{1}{\sigma\sqrt{T-t}}$

71.6 -0.28

71.7 $K_1\text{CashPut} - \text{AssetPut} = K_1e^{-r(T-t)}N(-d_2) - Se^{-\delta(T-t)}N(-d_1)$

71.8 \$2.7848

71.9 \$40

71.10 \$88

71.11 \$3,626,735

Section 72

72.1 $d_1 = 1.654$ and $N(1.654) = 0.950936$

72.2 $d_1' = -1.1387$ and $N(-1.1387) = 0.1274$

72.3 \$0.94126

72.4 $d_1 = 0.6102$ and $N(d_1) = 0.729135$

72.5 $d_1' = -0.4265$ and $N(d_1') = 0.334872$

72.6 \$0.4048

72.7 Fixing the parameters S, r, σ, δ, and $T - t$, we conclude that the supershare premium increases in value when the difference $K_2 - K_1$ increases.

Section 73

73.1 \$0.85 and \$0.57

73.2 -0.426326

73.3 Borrow \$0.61

73.4 Lost money

73.5 \$0.95916 and \$1.0503

73.6 3.7167 and 6.4332

73.7 Sell 0.5276 shares of 8-year bonds

73.8 \$0.39573

73.9 A profit is made

73.10 A profit is made

Section 74

74.1 We have

$$\begin{aligned}\Delta =&\frac{\partial P}{\partial r} = B(t,T)A(t,T)e^{-B(t,T)r}\\ \Gamma =&\frac{\partial^2 P}{\partial r^2} = B(t,T)^2A(t,T)e^{-B(t,T)r}\\ \Theta =&\frac{\partial P}{\partial t} = \frac{\partial A(t,T)}{\partial t}e^{-B(t,T)r} - \frac{\partial B(t,T)}{\partial t}rA(t,T)e^{-B(t,T)r}.\end{aligned}$$

74.2 The risk-premium of the asset is $a(r) - r = 0.0306 - 1.34r$ and the Sharpe ratio is

$$-\frac{a(r)-r}{\sigma(r)} = -\frac{0.0306 - 1.34r}{0.26} = \frac{1.34r - 0.0306}{0.26}.$$

74.3 -0.008651

74.4 We have

$$\frac{\partial}{\partial r}\ln\left[P(r,t,T)\right] = \frac{P_r(r,t,T)}{P(r,t,T)}.$$

Thus,

$$-\sigma(r)\frac{\partial}{\partial r}\ln\left[P(r,t,T)\right] = -\sigma(r)\frac{P_r(r,t,T)}{P(r,t,T)} = q(r,t,T).$$

74.5 $\sigma(r)B(t,T)$

74.6 $r + \phi\sigma(r)B(t,T)$.

Section 75

75.1 $dr = (0.0375 - 0.3r)dt + 0.03d\tilde{Z}$

75.2 0

75.3 -1.6766

75.4 $dr = (-abr(0) + a\sigma)e^{-at}dt$

75.5 $R(t,T) = \frac{r(0)}{a}(e^{-at} - e^{-aT})$

75.6 $0.08dt$

75.7 7%

Section 76

76.1 $r(t)=r(0)+\int_0^t ar(s)ds+\int_0^t\sigma(s)dZ(s).$

76.2 Consider the function $f(r,t)=\ln[r(t)]$. Then

$$\begin{aligned}\frac{\partial f}{\partial t}=&0\\ \frac{\partial f}{\partial r}=&\frac{1}{r}\\ \frac{\partial^2 f}{\partial t^2}=&-\frac{1}{r^2}\end{aligned}$$

Applying Itô's lemma to the function $f(r,t)$ we find

$$\begin{aligned}d[\ln[r(t)]]=&\frac{\partial f}{\partial t}dt+\frac{\partial f}{\partial r}dr+\frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dr)^2\\ =&\frac{1}{r}(ardt+\sigma rdZ)-\frac{1}{2}\frac{1}{r^2}(ardt+\sigma rdZ)^2\\ =&adt+\sigma dZ-\frac{1}{2}\sigma^2dt\\ =&(a-0.5\sigma^2)dt+\sigma dZ\end{aligned}$$

76.3 From the previous problem we found that

$$d[\ln r(t)]=(a-0.5\sigma^2)dt+\sigma dZ(t).$$

Integrating from 0 to t we find

$$\begin{aligned}\ln[r(t)]=&\ln[r(0)]+\int_0^t(a-0.5\sigma^2)ds+\int_0^t\sigma dZ(s)\\ =&(a-0.5\sigma^2)t\sigma Z(t)\end{aligned}$$

76.4 Follows from the previous exercise by exponentiation.

76.5 We have

$$\frac{r(t)}{r(s)}=e^{(a-0.5\sigma^2)t\sigma Z(t)}\cdot e^{(a-0.5\sigma^2)s\sigma Z(s)}=e^{(a-0.5\sigma^2)(t-s)+\sigma[Z(t)-Z(s)]}.$$

76.6 $r(s)e^{a(t-s)}$.

76.7 $r^2(s)e^{2a(t-s)}(e^{\sigma^2(t-s)}-1)$.

Section 77

77.1 0.471

77.2 0.848

77.3 0.5777

77.4 \$40.83

77.5 0.05167

77.6 $(r + 0.008)g$

77.7 0.05296

77.8 0.07989

Section 78

78.1 The interest rate in the Vasicek model can become negative a situation that is impossible for the CIR model. The variance in the Vasicek model does not change with the short-rate as opposed to the CIR model.

78.2 The only true statements are (a),(b),(c), and (f).

78.3 -0.0843

78.4 0.48673

78.5 $dr(t) = [a(b - r) + \overline{\phi} r]dt + \sigma\sqrt{r}d\tilde{Z}$.

78.6 $a = 0.24$ and $\overline{\phi} = 0.04$

78.7 0.2646

78.8 $\Delta = -2.03603$ and $\Gamma = 6.2408$

78.9 0.067683

78.10 0.048

78.11 0.2386

Section 79

79.1 0.91749

79.2 \$0.0418

79.3 \$0.0264

79.4 0.75195

79.5 0.15514

79.6 -0.31155

79.7 \$0.11311

79.8 \$0.13786

Section 80

80.1 $R_0(3,4) = \frac{P(0,3)}{P(0,4)} - 1$

80.2 0.04177

80.3 \$0.2535

80.4 $N = 1.1075$ and$K = 0.9029$

80.5 0.9007

80.6 $d_1 = 0.05782, d_2 = -0.09067, N(d_1) = 0.523054, N(d_2) = 0.463877$

80.7 \$0.04671

80.8 0.05173

80.9 0.027652

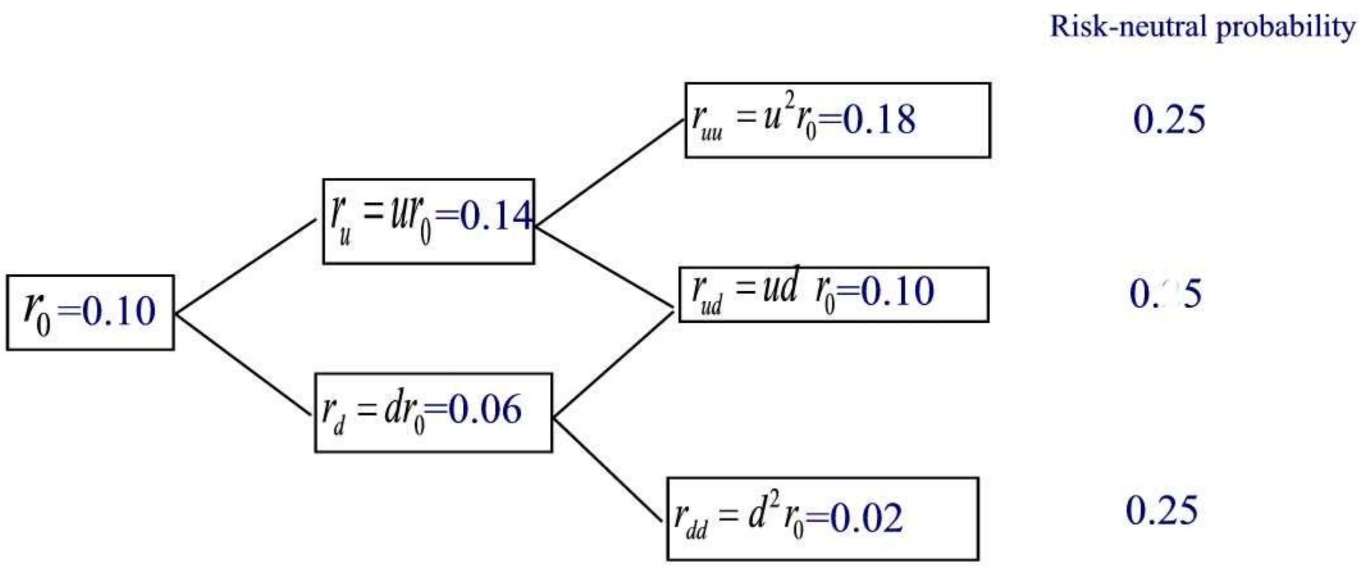
80.10 0.045046

80.11 0.0538545

80.12 \$0.1266

Section 81

81.1

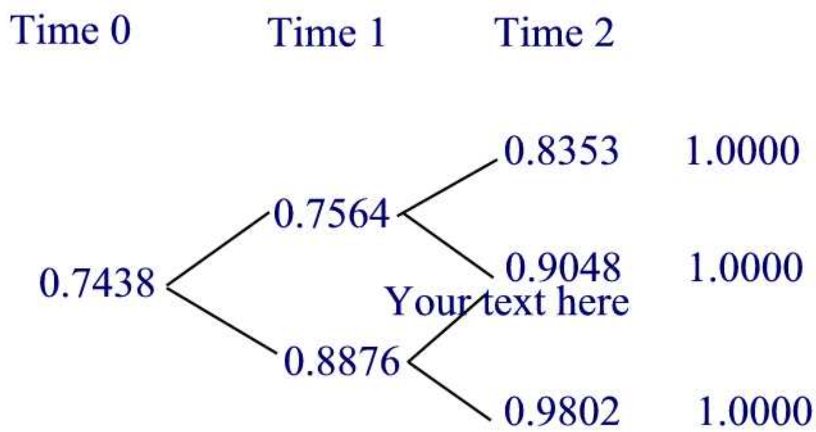


81.2 \$0.9048

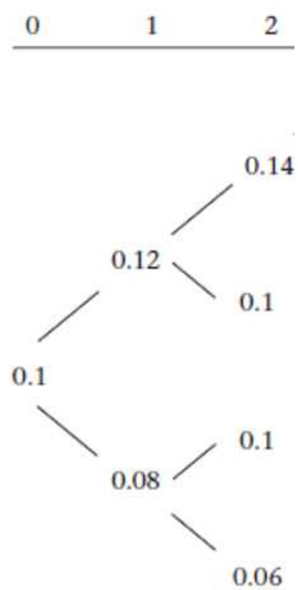
81.3 \$0.8194

81.4 \$0.7438

81.5



81.6 (a) The tree is given below



(b) (b) and (c) The price of 1-year bond is

$$P(0,1) = e^{-0.10} = \$0.9048$$

and the yield is

$$-\frac{\ln(0.9048)}{1} = 0.10.$$

The price of 2-year bond is

$$P(0,2) = e^{-0.10}(0.5e^{-0.12} + 0.5e^{-0.08}) = \$0.8189$$

and the yield is

$$-\frac{\ln{(0.8189)}}{2} = 0.0999.$$

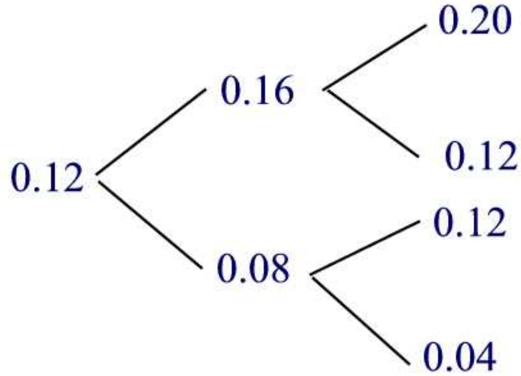
The price of 3-year bond is

$$P(0,3) = e^{-0.10} = \$0.7416$$

and the yield is

$$-\frac{\ln{(0.7416)}}{3} = 0.0997$$

81.7



81.8 \$0.0059

81.9 -2.3429

Section 82

82.1 0.1366

82.2 0.3482

82.3 3.696

82.4 $r_{dd} = 0.106, r_{uud} = 0.136, r_{ddd} = 0.089$

82.5 The time-0 of the 4-year zero-coupon bonddiscounted expected price at time 0 is given by

$$
\begin{aligned}
&0.9091 \times \{0.5 \times 0.8832[0.5 \times 0.8321(0.5 \times 0.8331 + 0.5 \times 0.8644) + 0.5 \times 0.8798(0.5 \times 0.8644 + 0.5 \times \\
&\quad + 0.5 \times 0.9023[0.5 \times 0.8798(0.5 \times 0.8644 + 0.5 \times 0.8906) + 0.5 \times 0.9153(0.5 \times 0.8906 + 0.5 \times 0.91 \\
&\quad = \$0.6243
\end{aligned}
$$

82.6 For a 4-year zero-coupon bond (3-year bond in year 1) the price of the bond is either

$$0.8832[0.5\times0.8321(0.5\times0.8331+0.5\times0.8644)+0.5\times0.8798(0.5\times0.8644+0.5\times0.8906)] = \$0.6532$$

with yield $0.6532^{-\frac{1}{4-1}} - 1$ or price

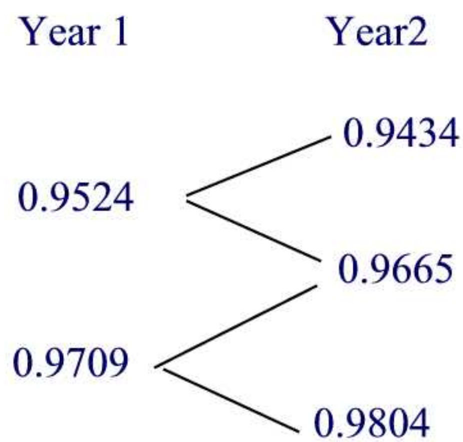
$$0.9023[0.5\times0.8798(0.5\times0.8644+0.5\times0.8906)+0.5\times0.9153(0.5\times0.8906+0.5\times0.9123)] = \$0.7205$$

with yield $0.7205^{-\frac{1}{4-1}} - 1$. Thus, the yield volatility in year 1 is

$$0.5 \times \ln\left(\frac{0.6532^{-\frac{1}{3}} - 1}{0.7205^{-\frac{1}{3}} - 1}\right) = 14\%$$

82.7 0.034641

82.8



82.9 0.264

82.10 $r_0 = 0.06$ and $\sigma_1 = 10\%$ (b) 5.94%

82.11 711.55

Section 83

83.1 0.016455

83.2 0.089

83.3 1.64

83.4 4.33

83.5 \$3.9075

Section 84

84.1 The result follows from the definition by taking $\lambda = \frac{1}{2}$.

84.2 We have $f''(x) = \frac{1}{x^2} > 0$ for all $x > 0$. Thus, by Proposition 84.1, f is convex for $x > 0$.

84.3 If f is concave in (a, b) then $-f$ is convex so that by Jensen's inequality we have $E[-f(X)] \leq -f[E(x)]$. Multiplying both sides by -1 we find $E[f(X)] \geq f[E(X)]$.

84.4 Let $f(x) = x^2$. Then $f''(x) = 2 > 0$ so that f is convex. Applying Jensen's inequality, we find $E[X^2] \geq [E(X)]^2$.

84.5 The payoff of a call

$$f(x) = \max\{0, x - K\}$$

is a convex function. We have

$$E(X) = \frac{1}{3} \times 50 + \frac{1}{3} \times 100 + \frac{1}{3} \times 150 = 100.$$

Thus, $f[E(X)] = \max\{0, 100 - 80\} = 20$. On the other hand,

$$E[f(X)] = \frac{1}{3} \times f(50) + \frac{1}{3} \times f(100) + \frac{1}{3} \times f(150) = \frac{1}{3} \times 20 + \frac{1}{3} \times 70 = 30.$$

Hence, $E[f(X)] \geq f[E(X)]$.

84.6 The payoff of a put

$$f(x) = \max\{0, K - x\}$$

is a convex function. We have

$$E(X) = \frac{1}{3} \times 60 + \frac{1}{3} \times 70 + \frac{1}{3} \times 110 = 80.$$

Thus, $f[E(X)] = \max\{0, 80 - 80\} = 0$. On the other hand,

$$E[f(X)] = \frac{1}{3} \times f(60) + \frac{1}{3} \times f(70) + \frac{1}{3} \times f(110) = \frac{1}{3} \times 20 + \frac{1}{3} \times 10 = 10.$$

Hence, $E[f(X)] \geq f[E(X)]$.

Section 85

85.1 A utility function must have a positive first derivative and a negative second derivative. We have

$$U'(x) = \alpha e^{-\alpha x} > 0$$

and

$$U''(x) = -\alpha^2 e^{-\alpha x} < 0.$$

Hence, U is a valid utility function.

85.2 1602.98

85.3 0.9228

85.4 8.366%

85.5 49.025%

85.6 95.544

85.7 13.037%

85.8 22.62

85.9 22.62

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