4.1.
The random variable $X$ is not a Binomial random variable, since the balls are selected without
replacement (after we pick a ball, we do not put it back). For this reason, the probability $p$ of
choosing a red ball changes from trial to trial.

4.2.
If the sampling in Exercise 4.1 is conducted with replacement (we pick a ball, note the color and put
it back), then $X$ is a Binomial random variable with $n = 2$ independent trials, and
$p = P[\text{red ball}] = 3/5$, which remains constant from trial to trial.

4.5.
With $n = 6$ and three different values of $p$, the values of $P(X = k) = \binom{6}{k} p^k q^{6-k}$ are calculated for
$k = 0, 1, 2, 3, 4, 5, \text{ and } 6$. The values of $P(X = k)$ are shown in the table below.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$p = .1$</th>
<th>$p = .5$</th>
<th>$p = .9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.531441</td>
<td>0.03125</td>
<td>0.00032</td>
</tr>
<tr>
<td>1</td>
<td>0.352494</td>
<td>0.15625</td>
<td>0.0064</td>
</tr>
<tr>
<td>2</td>
<td>0.098415</td>
<td>0.3125</td>
<td>0.0512</td>
</tr>
<tr>
<td>3</td>
<td>0.01458</td>
<td>0.3125</td>
<td>0.2048</td>
</tr>
<tr>
<td>4</td>
<td>0.001215</td>
<td>0.15625</td>
<td>0.4096</td>
</tr>
<tr>
<td>5</td>
<td>0.000054</td>
<td>0.03125</td>
<td>0.32768</td>
</tr>
<tr>
<td>6</td>
<td>0.000001</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The probability histogram is given below. Notice that the probability distribution for $p = .1$ and $p =
.9$ are mirror images. When $p$ is small, the distribution is skewed to the left, when $p$ is large, the
distribution is skewed to the right.
4.18. The random variable \( X \) is approximately Binomial, with \( n = 25 \) and probability of success \( p = P[\text{CEO is aware of information superhighway}] = .5 \).

Using Table 1 from the textbook, for \( n = 25 \), we get:

\[ a. \quad P[X = 25] = P[X \leq 25] - P[X < 25] = P[X \leq 24] - P[X < 24] = 1 - 1 = 0 \]

\[ b. \quad P[X \geq 10] = 1 - P[X \leq 9] = 1 - .115 = .885 \]

\[ c. \quad P[X = 10] = P[X \leq 10] - P[X < 10] = P[X \leq 10] - P[X < 9] = .212 - .115 = .097 \]

Note: we can calculate the probabilities using the formula for Binomial probability; however, for parts (b) the calculation will be very long (table is more convenient in this case). See below part (c), using the formula:

\[
P[X = 10] = \binom{25}{10} 0.5^{10} (1-0.5)^{25-10} = \frac{25!}{10!15!} (0.5)^{10} (0.5)^{15} = 0.09
\]

4.20.

\[ a. \] Let \( X \) = number of Japanese who feel their products are superior to American products. Then, \( X \) has a Binomial distribution (the alternative outcomes are ‘feel products are superior’ and ‘don’t feel products are superior’), with \( n = 50 \) and probability of success \( p = P[\text{Japanese feel their products are superior}] = .71 \). That is,

\[
P(X = k) = \binom{50}{k} (.71)^k (.29)^{50-k} \quad \text{for} \quad k = 0, 1, 2, \ldots, 50
\]

\[ b. \] If \( X \) = number of Japanese who feel the US would be the number one economic power in the next century, then \( X \) has a Binomial distribution with \( n = 50 \) and \( p = P[\text{Japanese feel US will be #1 economic power}] = .42 \). That is,

\[
P(X = k) = \binom{50}{k} (.42)^k (.58)^{50-k} \quad \text{for} \quad k = 0, 1, 2, \ldots, 50
\]

\[ c. \] For the random variable \( X \) described in part \( b \),

\[
\mu = np = 50(.42) = 21 \quad \text{and} \quad \sigma = \sqrt{npq} = \sqrt{(50)(.42)(.58)} = 3.490
\]

4.21.

Define \( X \) to be the number of people qualifying for favorable rates. Then \( p = P[\text{qualify}] = .7 \) and \( n = 5 \).

\[ a. \quad P[X = 5] = \binom{5}{5} (.7)^5 (.3)^0 = \left( \frac{5}{5} \right) (0.7)^5 (0.3)^0 = (.7)^5 = .16807
\]

(or, from Table 1: \( P[X = 5] = 1 - P[X \leq 4] = .168 \))

\[ b. \quad P[X \geq 4] = p(4) + p(5) = \binom{5}{4} (.7)^4 (.3)^1 + \binom{5}{5} (.7)^5 (.3)^0 = .36015 + .16807 = .52822
\]

(or, from Table 1: \( P[X \geq 4] = 1 - P[X \leq 3] = .528 \))
4.24.
Using a calculator, we find \( e^{-\mu} = 0.301194 \). Then

\[
p(x) = \frac{\mu^x e^{-\mu}}{x!} = \frac{(1.2)^x e^{-1.2}}{x!}
\]

a. \( p(0) = \frac{(1.2)^0 \cdot 0.301194}{0!} = 0.301194 \)

b. \( p(1) = \frac{(1.2)^1 \cdot 0.301194}{1!} = 0.3614328 \)

c. \( P[x \leq 2] = p(0) + p(1) + \frac{(1.2)^2 \cdot 0.301194}{2!} = 0.879487 \)

d. \( P[x > 1] = 1 - P[x \leq 1] = 1 - p(0) - p(1) = 1 - 0.301194 - 0.3614328 = 0.3373732 \)

4.31.
The number of accidents follows an approximate Poisson distribution with \( \mu = 3.5 \), and

\[
p(x) = \frac{(3.5)^x e^{-3.5}}{x!}
\]

a. \( p(0) = \frac{(3.5)^0 e^{-3.5}}{0!} = 0.030197 \)

b. For this Poisson random variable, \( \mu = 3.5 \) and \( \sigma = \sqrt{\mu} = \sqrt{3.5} = 1.871 \).

The value \( x = 7 \) lies \((7-3.5)/1.871 = 1.871\) standard deviations above the mean. This is not an unlikely occurrence.

c. The value \( x = 9 \) lies \((9-3.5)/1.871 = 2.94\) standard deviations above the mean. This is an unlikely occurrence, if in fact \( \mu = 3.5 \). Perhaps the mean has changed.

5.1.
Denoting by \( A(z_0) \) the area between the mean 0 and the value \( z_0 \), we have:

a. The area between \( Z = 0 \) and \( Z = 1.6 \) is \( A = A(1.6) = 0.4452 \)

b. The area between \( Z = 0 \) and \( Z = 1.83 \) is \( A = A(1.83) = 0.4664 \)

5.4.

a. The area between \( Z = -1.4 \) and \( Z = 1.4 \) is \( A = A(-1.4) + A(1.4) = 2A(1.4) = 2(0.4192) = 0.8384 \)

b. The area between \( Z = -2 \) and \( Z = 2 \) is \( A = A(-2.0) + A(2.0) = 2A(2.0) = 2(0.4772) = 0.9544 \)

c. The area between \( Z = -3 \) and \( Z = 3 \) is \( A = A(-3.0) + A(3.0) = 2A(3.0) = 2(0.4987) = 0.9974 \)

5.10.
We want to find the value \( z_0 \) such that \( P[-z_0 < Z < z_0] = 0.4714 \) (see Figure 5.6 below). That is, \( A(z_0) + A(-z_0) = 0.4714 \Rightarrow 2A(z_0) = 0.4714 \Rightarrow A(z_0) = 0.2357 \)

From the Table, we can see that the value \( z_0 \) is 0.63
5.24.
Define $X$ to be the year-end bonus received by a worker for meeting quality production and profitability targets in 1993 and suppose that $X$ is Normally distributed with mean $\mu = 2800$ and standard deviation $\sigma = 500$.

a. To find $P[X > 3500]$, we first need to standardize $X$, i.e. to calculate
$$z = \frac{x - \mu}{\sigma} = \frac{3500 - 2800}{500} = 1.4.$$ 
Then, $P[X > 3500] = P[Z > 1.4] = 0.5 - 0.4192 = 0.0808$

b. It is necessary to find two values of $X$, say $x_1$ and $x_2$, such that $P[x_1 < X < x_2] = 0.95$
Recall that for the standard normal distribution, 95% of the measurements fall between $z_1 = -1.96$ and $z_2 = 1.96$. That is, the probability statement above will be satisfied if we let
$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{x_1 - 2800}{500} = -1.96 \text{ and } z_2 = \frac{x_2 - \mu}{\sigma} = \frac{x_2 - 2800}{500} = 1.96.$$ 
Solving for $x_1$ and $x_2$, we have:
$$x_1 = 2800 - 1.96(500) = 1820 \text{ and } x_2 = 2800 + 1.96(500) = 3780$$
**Note:** we can also find the values with the help of the Table.
$z_2$ will be the value of $Z$ such that the area between $z_2$ and 0 is $0.95/2 = 0.475$.
$z_1$ will be equal to the negative of $z_2$ (because of the symmetry of the Normal distribution).

5.25.
Define $X =$ cost of natural gas per metric cubic foot (MCF) and suppose that $X$ is normally distributed with $\mu = 6.00$ and $z_1 = \frac{x_1 - \mu}{\sigma} = \frac{x_1 - 2800}{500} = -1.96$ and $\sigma = 1.20$.

a. To find $P[7.60 < X < 8.00]$, calculate
\[ z_1 = \frac{x_1 - \mu}{\sigma} = \frac{x_1 - 6.00}{1.20} = \frac{7.60 - 6.00}{1.20} = 1.33 \text{ and} \]
\[ z_2 = \frac{x_2 - \mu}{\sigma} = \frac{x_2 - 6.00}{1.20} = \frac{8.00 - 6.00}{1.20} = 1.67 \]

Then \( P [7.60 < X < 8.00] = P [1.33 < Z < 1.67] = .4525 - .4082 = .0443 \)

b. The median cost per MCF for natural gas is that value, \( m \), such that \( P [x > m] = P [x < m] = .5 \). For the standard normal random variable, \( Z \), the median value, which has area .50 to its left and to its right, is \( Z = 0 \). The corresponding value for \( X \), which defines the median, is

\[
m - 6.00 = 0 \text{ or } m = 1.20(0) + 6.00 = 6.00
\]

Alternatively, we can just recall that for any symmetric distribution, the median is equal to the mean.

c. The lower and upper quartile of the standard normal distribution are those values, say \( z_1 \) and \( z_2 \), which have area .25 to their left and right, respectively. Using Table 3, we search for a value of \( z_0 \) that gives an area of .5 - .25 = .25 between 0 and itself (\( A(z_0) = .25 \)). This value is \( z_0 = .675 \), so that \( z_1 = - .675 \) and \( z_2 = .675 \).

The corresponding values of \( X \) for this particular normal distribution, with \( \mu = 6.00 \) and \( \sigma = 1.20 \), are found by solving the equations,

\[
z_1 = \frac{x_1 - \mu}{\sigma} = \frac{x_1 - 6.00}{1.20} = -.675 \text{ and } z_2 = \frac{x_2 - \mu}{\sigma} = \frac{x_2 - 6.00}{1.20} = .675
\]

Therefore, \( x_1 = 5.19 \) and \( x_2 = 6.81 \).

5.26.

a. From Table 1 (for Binomial distribution),
\[ P[8 \leq X \leq 10] = P[X \leq 10] - P[X \leq 7] = .902 - .512 = .390 \]

b. Calculate \( \mu = np = 7.5 \) and \( \sigma = \sqrt{25(3)(.7)} = 2.2913 \). The probability of interest is the area under the binomial probability histogram corresponding to the rectangles \( X = 8, 9, \) and 10 in Figure 5.14 below.

To approximate this area, use the “correction for continuity” and find the area under a normal curve with mean \( \mu = 7.5 \) and \( \sigma = 2.2913 \) between \( x_1 = 8 - 0.5 = 7.5 \) and \( x_2 = 10 + 0.5 = 10.5 \). The \( Z \)-values corresponding to the two values of \( X \) are:

\[
z_1 = \frac{7.5 - 7.5}{2.2913} = 0 \text{ and } z_2 = \frac{10.5 - 7.5}{2.2913} = 1.31
\]
The approximated probability is \( P [7.5 < X < 10.5] = P [0 \leq Z \leq 1.31] = .4049 \), which is not too far from the actual probability calculated in part a.

5.64.
For this exercise, it is given that the population of bolt diameter is normally distributed with \( \mu = .498 \) and \( \sigma = .002 \).

The fraction of acceptable bolts will be those which lie in the interval from .496 to .504. All others are unacceptable. The desired fraction of acceptable bolts is calculated, and the fraction of unacceptable bolts is obtained by subtracting from the total probability, which is 1.

The fraction of acceptable bolts is, then

\[
P [.496 \leq X \leq .504] = P \left[ \frac{.496 - .498}{.002} < Z < \frac{.504 - .498}{.002} \right]
\]

\[= P [-1 \leq Z \leq 3] = .3413 + .4987 = .8400\]

and the fraction of unacceptable bolts is 1-.84 = .16

See also figure below.
5.67.  

a. Using the binomial tables and indexing \( n = 25 \) and \( p = .4 \) in Table 1,

\[
P[X \leq 11] = P[X \leq 11] - P[X \leq 7] = .732 - .154 = .578
\]

b. To use the normal approximation to the binomial distribution, calculate

\[
\mu = np = 25(.4) = 10 \quad \text{and} \quad \sigma = \sqrt{npq} = \sqrt{25(.4)(.6)} = 2.449
\]

The desired probability is the area inside the rectangles formed by the histogram for \( x = 8, 9, 10, \) and 11 in the figure below.
Using the correction for continuity to include the entire area under the rectangles, the approximated probability is

\[ P [7.5 < X < 11.5] = P \left[ \frac{7.5 - 10}{2.449} < Z < \frac{11.5 - 10}{2.449} \right] = P [-1.02 < Z < .61] = .3461 + .2291 = .5752 \]

5.71.
The random variable \( X \) = size of the freshman class. That is, the admissions office will send letters of acceptance to (or accept deposits from) a certain number of qualified students. Of these students, a certain number will actually enter the freshman class. Since the experiment results in one of two outcomes (enter or not enter), the random variable \( X \), the number of students entering the freshman class, has a Binomial distribution with

- \( n \) = number of deposits accepted
- \( p = P \) [student enters freshman class] = .8

a. It is necessary to find a value for \( n \) such that \( P [X \leq 120] = .95 \). Note that

\[ \mu = np = .8n \text{ and } \sigma = \sqrt{npq} = \sqrt{16n} \]

Using the normal approximation, we need to find a value of \( n \) such that \( P[X \leq 120.5] = .95 \). The \( Z \)-value corresponding to \( X = 120.5 \) is

\[ z = \frac{x - \mu}{\sigma} = \frac{120.5 - .8n}{\sqrt{16n}} \]

From Table 3, the \( z \)-value corresponding to an area of .05 in the right tail of the normal distribution is 1.645.
Then, \( \frac{120.5 - .8n}{\sqrt{1.6n}} = 1.645 \).

Solving for \( n \) in the above equation, we obtain the following quadratic equation:

\[ .8n + .658 \sqrt{n} - 120.5 = 0 \]

Let \( v = \sqrt{n} \). Then the equation takes the form

\[ av^2 + bv + c = 0 \]

which may be solved using the quadratic formula,

\[ v = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Substituting above, we get:

\[ v = \frac{-0.658 \pm \sqrt{0.433 + 4(96.4)}}{1.6} \]

Since \( v \) (the square root of the number of students \( n \)) must be positive, the desired root is

\[ v = \sqrt{n} = 18.990 \div 1.6 = 11.869 \text{ or } n = (11.869)^2 = 140.86 \]

Thus, 141 deposits should be accepted.

b. Once \( n = 141 \) has been determined, the mean and standard deviation of the distribution are easily calculated as \( \mu = np = 141(.8) = 112.8 \) and \( \sigma = \sqrt{npq} = \sqrt{22.56} = 4.750 \)

Then, the approximation for \( P[X \leq 105] \) is

\[ P[X \leq 104.5] = P[Z \leq \frac{104.5 - 112.8}{4.750}] = P[Z \leq -1.75] = .5 -.4599 = .0401 \]
Additional Problems

(1)  
P [Z > z_0] = 0.05  
We need to find $z_0$ such that the area between 0 and itself is $0.5 - 0.05 = 0.45$.  
From the Table, we get $z_0 = 1.65$

(2)  
P [Z > z_0] = 0.025  
We need to find $z_0$ such that the area between 0 and itself is $0.5 - 0.025 = 0.475$.  
From the Table, we get $z_0 = 1.96$

(3)  
P [Z > z_0] = 0.005  
We need to find $z_0$ such that the area between 0 and itself is $0.5 - 0.005 = 0.495$.  
From the Table, we get $z_0 = 2.58$