Homework 6 due on Wednesday May 15

Questions are worth 10 points.

Show your work and provide explanation of your answers.

1. Suppose that \((X_t)_{0 \leq t \leq 1}\) is a family of independent real-valued random variables. Show that if \((X_t)\) is continuous in probability (that is \(X_{t+h} \rightarrow X_t\) in probability as \(h \rightarrow 0\)), then \(X_t(\omega) = f(t)\) for almost every \(\omega\) and some deterministic continuous function \(f\).

Hint: Show that the cdf of \(X_t\) takes values in \(\{0, 1\}\). Then, \(f(t)\) is the point where it jumps by 1.

2. Let \(X_1, X_2, \cdots\) be a sequence of i.i.d. \(\mathcal{N}(0, 1)\) random variables. Define \(Y_1 = X_1, Y_2 = rX_1 + \sqrt{1 - r^2}X_2\) and in general, \(Y_n = r^{n-1}X_1 + \sqrt{1 - r^2}(r^{n-2}X_2 + r^{n-3}X_3 + \cdots + X_n),\) where \(|r| < 1\).

Show that \((Y_n)\) is a stationary Gaussian sequence and compute its auto covariance function.

3. (a) Let \((W_t)\) be the Wiener process, and \(c > 0\). Show that \(\frac{1}{\sqrt{c}}W_{ct}\) is also a Wiener process.

(b) Let \((W_t, B_t)\) be two independent Wiener processes. Show that \(\frac{1}{\sqrt{2}}(W_t + B_t)\) is a Wiener process.

(c) Show that \(W_1 - W_{1-t}\) is a Wiener process.

4. Let \((W_t)\) be the Wiener process. Compute the mean and auto-covariance functions of the processes \((e^{W_t})\) and \((\int_0^t W_u du)\). Is either one a Gaussian process?

5. Let \((W_t)\) be the Wiener process and \(g(t)\) be a \(C^1\) deterministic function on \([0, T]\). Using the definition of \(\int_0^T g(t)dW_t\) by integration by parts, show that it is a \(\mathcal{N}(0, \int_0^T g(t)^2 dt)\) random variable.

6. Let \((W_t)\) be the Wiener process, For \(\alpha > 0\), define \(Z_t = e^{-\frac{\alpha t}{2}}W_{e^\alpha t}\). Show that \((Z_t)\) is a stationary Gaussian process and compute its mean and autocovariance function.