Homework 4 due on Wednesday May 1st
Questions 1 and 2 are worth 10 points. Questions 3 and 4 are worth 15 points.
Question 5 is worse 15 bonus points.

Show your work and provide explanation of your answers.

1. Let $X_n$ be i.i.d. with $\Pr(X_k = +1) = \Pr(X_k = -1) = 0.5$. Define $M_n = \sum_{k=1}^n X_k/k$. Show that $M_n$ (called the random harmonic series) converges with probability 1.

2. Let $(X_n, \mathcal{F}_n)$ be a martingale such that $(X_{n+1}/X_n) \in L^1$ for all $n$. Show that $X_{n+1}/X_n$ and $X_n/X_{n-1}$ are uncorrelated and that $\mathbb{E}(X_n/X_m) = 1$ for all $n > m$.

3. Let $X_1, X_2, \cdots$ be i.i.d. and set $S_n = X_1 + \cdots + X_n$. Let $\xi$ be such that $\mathbb{E}(e^{\xi X_1}) = 1$. Show that $\Pr(S_k \geq a$ for some $k) \leq e^{-a\xi}$.

4. Let $S_n$ be a simple symmetric random walk starting at $S_0 = a$. Let

$$M_n = \sum_{k=0}^n S_k - \frac{1}{3}S_n^3.$$ 

Let $\tau = \min\{n : S_n = 0$ or $S_n = N\}$. Show that $(M_n, \mathcal{F}_n)$ is a martingale and therefore

$$\mathbb{E}\left\{\sum_{k=0}^\tau S_k\right\} = a + \frac{1}{3}(aN^2 - a^3).$$

5. In the following questions $S$ and $T$ are stopping times for the filtration $(\mathcal{F}_n)_{n\geq0}$. We recall that $\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leq n\} \in \mathcal{F}_n, \forall n \geq 0\}$.

(a) (1 pts) If $T \equiv n$, show that $\mathcal{F}_T = \mathcal{F}_n$.

(b) (2 pts) Show that $S \wedge T = \min(S, T)$ is a stopping time.

(c) (2 pts) Show that $S \vee T = \max(S, T)$ is a stopping time.

(d) (2 pts) Show that $\alpha T$ is a stopping time for $\alpha \geq 1$, $\alpha$ integer.

(e) (2 pts) Show that $T$ is $\mathcal{F}_T$-measurable.

(f) (3 pts) Show that $\{S < T\}$, $\{S \leq T\}$, and $\{S = T\}$ are all in $\mathcal{F}_S \cap \mathcal{F}_T$.

(g) (3 pts) Show that if $S \leq T$ then $\mathcal{F}_S \subset \mathcal{F}_T$. 
