Each question is worth 10 points. Show your work and provide explanation of your answers.

1. Find an example of a stochastic process \((X_n)\) in discrete time in each of the following cases:

   (a) \((X_n)\) is a martingale and Markovian.

   (b) \((X_n)\) is Markovian but not a martingale.

   (c) \((X_n)\) is neither Markovian nor a martingale.

   (d) \((X_n)\) is a martingale but not Markovian.

2. Let \(S_n\) be the simple random walk taking independent steps \(X\) of \(\pm 1\), with \(\mathbb{P}(X = 1) = p > \frac{1}{2}\), and starting at \(S_0 = 0\).

   Let \(\mathcal{F}_n = \sigma(S_0, S_1, \ldots, S_n)\) and \(Y_n = \left(\frac{1-p}{p}\right)^{S_n}\). Show that \((Y_n)\) is a \((\mathcal{F}_n)\)-martingale.

3. Let \(S_n\) be the simple random walk taking independent steps \(X\) of \(\pm 1\), with \(\mathbb{P}(X = 1) = p > \frac{1}{2}\), and starting at \(S_0 = 0\). Let \(\mathcal{F}_n = \sigma(S_0, S_1, \ldots, S_n)\).

   Compute the Doob decomposition of the \((\mathcal{F}_n)\)-submartingale \((S_n)\).

4. Consider a martingale \((X_n)\) with respect to the filtration \((\mathcal{F}_n)\). Let \((\mathcal{G}_n)\) be the natural filtration of \((X_n)\), that is \(\mathcal{G}_n = \sigma(X_0, X_1, \ldots, X_n), n \geq 0\).

   Show that \((X_n)\) is a \((\mathcal{G}_n)\)-martingale. More generally, show that if \((X_n)\) is adapted to a filtration \((\mathcal{G}'_n)\) such that \(\mathcal{G}'_n \subseteq \mathcal{F}_n\) for all \(n\), then \((X_n)\) is a \((\mathcal{G}'_n)\)-martingale. Hence the martingale property is stable with respect to “shrinking” filtrations.

5. Let \((X_n, \mathcal{F}_n)\) be a square-integrable martingale with \(X_0 = 0\) a.s..

   Let \(D_j = X_j - X_{j-1}\), for \(j \geq 1\), be its differences. Show that

   \[\mathbb{E}(X_n^2) = \sum_{j=1}^{n} \mathbb{E}(D_j^2).\]