HOMEWORK 1

1. Let $X, Y$ be two independent random variables exponentially distributed over $\mathbb{R}_+$ with parameters $\lambda > 0$ and $\mu > 0$ respectively.

(a) Compute the probability density of the pair $(T = X + Y, Z = X - Y)$.

(b) Deduce the probability density of $T = X + Y$ (hint: treat the particular case $\lambda = \mu$ separately).

(c) Compute the probability density of $U = \min\{X, Y\}$ and identify its distribution.

2. Let $(Y_1, Y_2, \cdots)$ be a sequence of i.i.d. Geometric random variables on $\mathbb{N}^*$ with parameter $0 < p < 1$ (that is $P(Y = n) = q^{n-1}p$ for $n \geq 1$ and $q = 1 - p$).

(a) Compute the generating function of $Y$: $G_Y(z) = E(z^Y)$.

(b) Compute the generating function $G_{X_n}(z) = E(z^{X_n})$ of the sum $X_n = Y_1 + \cdots + Y_n$.

(c) Prove the negative binomial formula

$$
(1 - u)^{-n} = \sum_{j=0}^{\infty} \binom{n + j - 1}{j} u^j
$$

for $|u| < 1$, $n \geq 1$, and where \( \binom{n-1}{0} = 1 \).

(d) Deduce $P(X_n = k)$ for $k \geq n$. Hint: expand $G_{X_n}(z)$ in powers of $z$ by using the negative binomial formula, and identify the coefficient of $z^k$.

(e) Compute the conditional distribution of $(X_1, X_2, \cdots, X_n)$ given $X_{n+1} = M + 1$ for $M \geq n$. That is compute

$$
P(X_1 = k_1, X_2 = k_2, \cdots, X_n = k_n | X_{n+1} = M + 1).
$$

Hint: express everything in terms of the $Y$'s.

(f) Deduce that, given $X_{n+1} = M + 1$, the points $(X_1, X_2, \cdots, X_n)$ are obtained by ordering in increasing order $n$ numbers chosen at random without replacement among $\{1, 2, \cdots, M\}$, and that independently of $0 < p < 1$.

(g) Compute the probability that there are no two consecutive numbers when choosing at random $n$ numbers without replacement among $\{1, 2, \cdots, M\}$. Hint: write

$$
P(\text{no two consecutive}) = P(Y_2 > 1, Y_3 > 1, \cdots, Y_n > 1 | X_{n+1} = M + 1) = a_{M+1},
$$

$$
q^{n-1} = \sum_{m=n+1}^{\infty} P(Y_2 > 1, Y_3 > 1, \cdots, Y_n > 1 | X_{n+1} = m) P(X_{n+1} = m)
$$

and use expansions in powers of $q$ in order to identify the coefficient $a_{M+1}$.

(h) Application to Lotto: choose six out of forty nine, probability of no two consecutive?