
MA(1): $X_t = Z_t + \theta_1 Z_{t-1}$ where $Z_t \sim WN(0, \sigma^2_Z)$, $|\theta_1| < 1$.

Calculated last time:

$$\gamma_X(k) = Cov(X_t, X_{t+k}) = E(Z_t + \theta Z_{t-1})(Z_{t+k} + \theta Z_{t+k-1}) = \begin{cases} (1 + \theta^2_1)\sigma^2_Z & \text{if } k = 0, \\ \theta_1 \sigma^2_Z & \text{if } k = 1, -1, \\ 0 & \text{if } |k| > 1. \end{cases}$$

$\rho_X(1) = \theta_1/(1 + \theta^2_1)$, $\rho_X(k) = 0, \forall k : k > 1$.

Note: since $|\theta_1| < 1$ it follows that $\rho_X(1) < .5$.

PICTURES OF $\rho_X(k)$:

Further analysis of MA(1):

3.3.2.1 Shift Operator $B$ (defined on pp. 29-30 of [BD]); Invertibility ($\S$2.3 of [BD] defines invertibility for ARMA(1,1))

Note on why $|\theta_1| < 1$: It has to do with the property of invertibility which roughly says that the current observation does not depend overwhelmingly on observations in the remote past.

Def. The TS $X_t$ is invertible if the shocks $Z_t$ can be expressed via values of $X_t$ as convergent series $Z_t = X_t + \pi_1 X_{t-1} + \pi_2 X_{t-2} + \ldots$.

Note that if the TS is invertible, one can write $X_t = Z_t - \pi_1 X_{t-1} - \ldots$. Since the series converges, the coefficients $\pi_k \rightarrow 0$ and thus, contribution of the first terms only (the most recent values) is important.

To study invertibility of MA(1) model, we introduce the shift operator $B$:

Def. For any TS $X$ define the shift operator $B$:

$$BX_t = X_{t-1}, B^k X_t = X_{t-k}$$
Write MA(1) as: 

\[ X_t = (1 + \theta_1 B) Z_t \equiv \theta(B) Z_t. \]

Then,

\[ Z_t = \frac{1}{1 - (-\theta_1 B)} X_t = (1 + (-\theta_1) B + (-\theta_1)^2 B^2 + \ldots) X_t \]

\[ = X_t - \theta_1 X_{t-1} + \ldots + (-1)^k \theta_1^k X_{t-k} + \ldots \]

However, the last series converges only if \(|\theta_1| < 1\) (geometric series).

**Note on model fitting, when one is given an ACF \(\rho_X(k)\):** TS is MA(1) when \(\rho_X(1) \neq 0\), but the values of ACF for other lags is zero. Note that when \(|\theta_1| < 1\), then the value of the model coefficient \(\theta_1\) can be obtained uniquely from the value of \(\rho_X(1)\):

the equation \(\theta_1^2 \rho_X(1) - \theta_1 + \rho_X(1) = 0\) has two roots; but their product is equal to 1 (Vietta’s Theorem). Thus, only one satisfies the condition \(|\theta_1| < 1\).

Since usually we have an estimate of \(\rho_X(1)\) and not the real value, the value of \(\theta_1\), obtained as before, will be a preliminary estimate.

### 3.3.3 Moving average of order \(q\) MA(\(q\))

\[ X_t = Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q} \text{ where } Z_t \sim WN(0, \sigma_Z^2). \]

**Stationarity:** MA models are always stationary with **ACF**:

\[ \rho_X(k) = \frac{\theta_k + \theta_1 \theta_{k+1} + \ldots + \theta_{q-k} \theta_q}{1 + \theta_1^2 + \ldots + \theta_q^2}, \quad k = 1, 2, \ldots, q \]

\[ \rho_X(k) = 0, \quad k > q. \]

Note: the ACF is cut off beyond the order \(q\) of the model.

**Derivation of the ACF** – shows that autocovariance function depends on time-lag only, i.e. stationarity of the model:
Invertibility – Restrictions on coefficients: As in the MA(1) case above, write

\[ X_t = (1 + \theta_1 B + \ldots + \theta_q B^q) Z_t \equiv \theta(B) Z_t \text{ with } \theta(z) = 1 + \theta_1 z + \ldots + \theta_q z^q. \]

Invertibility means writing \( X_t = \theta(B) Z_t \) in form

\[ Z_t = \frac{1}{\theta(B)} X_t = (1 + \pi_1 B + \pi_2 B^2 + \ldots) X_t \equiv X_t + \pi_1 X_{t-1} + \pi_2 X_{t-2} + \ldots \]

This is possible only if the function \( \theta(z)^{-1} \) has a representation as a convergent series. This means that \( \theta(z) \neq 0 \) for \( |z| \leq 1 \) or that the roots of the polynomial \( \theta(z) \) fall outside the unit circle.

3.3.3.1 Example: Moving average of order two MA(2).

\[ X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} \]

Restrictions on coefficients \( \theta_1, \theta_2 \) to assure invertibility: Write

\[ 1 + \theta_1 z + \theta_2 z^2 = (1 - \alpha_1 z)(1 - \alpha_2 z) \]

and check that \( |\alpha_i| < 1 \) (Note that the roots of \( \theta(z) \) are then \( 1/\alpha_i \) and satisfy \( |1/\alpha_i| > 1 \).) Write

\[ \theta_1 = -(\alpha_1 + \alpha_2); \quad \theta_2 = \alpha_1 \alpha_2 \]

After some algebra, one gets the following triangle on \((\theta_1, \theta_2)\) plane:

\[ \begin{align*}
\theta_2 + \theta_1 &> -1 \\
\theta_2 - \theta_1 &> -1 \\
-1 &< \theta_2 < 1
\end{align*} \]

Note that \( \theta_1 \) is allowed to be larger than 1.

Stationarity: MA(2) is stationary and with ACF (special case of formulae in 3.3.3):

\[ \rho_X(k) = \begin{cases} 
\frac{\theta_1 + \theta_1 \theta_2}{1 + \theta_1^2 + \theta_2^2} & \text{if } k = 1, -1, \\
\frac{\theta_2}{1 + \theta_1^2 + \theta_2^2} & \text{if } k = 2, -2, \\
0 & \text{if } |k| > 2.
\end{cases} \]

As in MA(1) case, we accept MA(2) model if ACF = 0 after lag 2. We can obtain preliminary estimates of the parameters \( \theta_1, \theta_2 \) from the estimates of ACF obtained from data.