Introduction to Wavelets

Ling Zhu

March 2014
Outline

1 Overview

2 Brief Review of Fourier Analysis

3 Wavelets
   - The Haar System
   - Smoother Wavelet Bases
Why Wavelets?

Wavelets: *ondelette*. First developed by French scientists.

Wavelet analysis is a refinement of Fourier analysis, which describes an input signal in terms of its frequency components.

Advantages:

- Spatial adaptivity
- Multi-resolution analysis

Used in:

- Signal processing: denoising
- Image analysis
- Data compression
The Discrete Fourier Transform

Let $f \in L^2[-\pi, \pi]$, where $L^2[a, b] = \{ f : \int_a^b f^2(x) dx < \infty \}$ is the square-integrable function space. Then Fourier analysis states that $f$ can be expressed as an infinite sum of dilated cosine and sine functions:

$$f(x) = \frac{1}{2} a_0 + \sum_{j=1}^{\infty} \left( a_j \cos(jx) + b_j \sin(jx) \right), \quad (1)$$

where the coefficients can be computed using:

$$a_j = \frac{1}{\pi} \langle f, \cos(j\cdot) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(jx) dx, \quad j = 0, 1, \ldots, \quad (2)$$

$$b_j = \frac{1}{\pi} \langle f, \sin(j\cdot) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(jx) dx, \quad j = 1, 2, \ldots, \quad (3)$$
Fourier Basis

Definition

A sequence of function \( \{f_j\} \) is said to be a complete orthonormal system if the \( f_j \)'s are pairwise orthogonal, \( \|f_j\| = 1 \) for each \( j \), and the only function orthogonal to each \( f_j \) is the zero function.

Let

\[
g_j(x) = \pi^{-1/2} \sin(jx) \quad \text{for} \quad j = 1, 2, \ldots
\]

and

\[
h_j(x) = \pi^{-1/2} \cos(jx) \quad \text{for} \quad j = 1, 2, \ldots
\]

with

\[
h_0(x) = 1/\sqrt{2\pi} \quad \text{on} \quad x \in [-\pi, \pi].
\]

Then, the set \( \{h_0, g_j, h_j : j = 1, 2, \ldots\} \) is a complete orthonormal system for \( L^2[-\pi, \pi] \).
A Simple Example

In reality, the summation in (1) can be well-approximated (in the $L^2$ sense) by a finite sum with upper summation limit index $J$:

$$S_J(x) = \frac{1}{2}a_0 + \sum_{j=1}^{J} (a_j \cos(jx) + b_j \sin(jx))$$  \hspace{1cm} (4)

Consider the following example function:

$$f(x) = \begin{cases} 
  x + \pi, & -\pi \leq x \leq -\pi/2 \\
  -\pi/2, & -\pi/2 < x \leq \pi/2 \\
  \pi - x, & \pi/2 < x \leq \pi.
\end{cases}$$  \hspace{1cm} (5)

Figure 1 shows the truncated Fourier series representations (4) for $J = 1, 2, $ and $3$. 
A Simple Example

Figure 1: An example function and its Fourier sum representation
The Haar Function (mother wavelet)

\[ \psi(x) = \begin{cases} 
  1, & 0 \leq x < \frac{1}{2} \\
  -1, & \frac{1}{2} \leq x < 1 \\
  0, & \text{otherwise.} 
\end{cases} \]  

(6)

Figure 2: The Haar function
Haar Wavelets

Each wavelet born of the mother wavelet can be written as:

\[ \psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k), \]

where \( j \) is the dilation index and \( k \) is the translation index.

Figure 3: Haar wavelet examples
Haar Wavelets

**Theorem**

The set \( \{ \psi_{j,k}, j, k \in \mathbb{Z} \} \) constitutes a complete orthonormal system for \( L^2(\mathbb{R}) \).

See reference for the proof.

Basically, from the proof we know that for each level \( j \), one can construct \( f^j \) using Haar functions to approximate the original function. This approximation can be written as the sum of the next coarser approximation \( f^{j-1} \) and a detail function \( g^{j-1} \). As the index \( j \) increases, the approximations become finer. Each detail function \( g^j \) can be written as a linear combination of the \( \psi_{j,k} \) functions.
Multi-resolution Analysis

Define the function space $V_j$, $j \in \mathbb{Z}$ to be:

$$V_j = \left\{ f \in L^2(R) : f \text{ is piecewise constant on } [k2^{-j}, (k+1)2^{-j}), k \in \mathbb{Z} \right\}$$  \hspace{1cm} (8)

Denote $P_j f$ to be the projection of a function $f$ onto the space $V_j$, then

$$P_j f = P_{j-1} f + g_{j-1}$$  \hspace{1cm} (9)

$$= P_{j-1} f + \sum_{k \in \mathbb{Z}} \langle f, \psi_{j-1,k} \rangle \psi_{j-1,k}$$  \hspace{1cm} (10)

$$= \ldots$$

$$= P_{j_0} f + \sum_{\ell=j_0}^{j-1} \sum_{k \in \mathbb{Z}} \langle f, \psi_{\ell,k} \rangle \psi_{\ell,k}.$$  \hspace{1cm} (11)
Haar Scaling Functions

Haar Scaling function (father wavelet): \( \phi(x) = I_{[0,1)}(x) \).

Its dilates and translates: \( \phi_{j,k}(x) = 2^{j/2} \phi(2^j x - k) \).

Then, the set \( \{ \phi_{j,k}, k \in \mathbb{Z} \} \) is an orthonormal basis for \( V_j \).

Hence,

\[
P^j f = \sum_{k} c_{j,k} \phi_{j,k}, \tag{12}\]

where the scaling function coefficients can be computed:

\[
c_{j,k} = \langle f, \phi_{j,k} \rangle = \int_{-\infty}^{\infty} f(x) \phi_{j,k}(x) dx. \tag{13}\]
Decomposition of the Space $V_j$

Define the space for the detail functions:

$$W_j = \text{span}\{\psi_{j,k}, k \in \mathbb{Z}\}. \quad (14)$$

It’s easily seen that $\langle \phi_{j,k}, \psi_{j',k'} \rangle = 0$, when $j \leq j'$. Hence, from (9) we have:

$$V_j = V_{j-1} \oplus W_{j-1} \quad (15)$$
$$= V_{j-2} \oplus W_{j-2} \oplus W_{j-1} \quad (16)$$
$$= \ldots$$
$$= V_{j_0} \oplus \bigoplus_{\ell=j_0}^{j-1} W_{\ell} \quad (17)$$
$$= \bigoplus_{\ell=-\infty}^{j-1} W_{\ell}. \quad (18)$$
Relationship Between Coefficients

For the Haar scaling function coefficients:

\[ c_{j,k} = \frac{(c_{j+1,2k} + c_{j+1,2k+1})}{\sqrt{2}} \]  (19)

For the Haar wavelet coefficients \( d_{j,k} \triangleq \langle f, \psi_{j,k} \rangle \),

\[ d_{j,k} = \frac{(c_{j+1,2k} - c_{j+1,2k+1})}{\sqrt{2}}, \]  (20)

since

\[ \psi_{j,k}(x) = \frac{(\phi_{j+1,2k}(x) - \phi_{j+1,2k+1}(x))}{\sqrt{2}}. \]  (21)
# Coefficients for the Example Function

## Haar scaling function coefficients:

<table>
<thead>
<tr>
<th>k</th>
<th>j=-1</th>
<th>j=0</th>
<th>j=2</th>
</tr>
</thead>
<tbody>
<tr>
<td>-7</td>
<td>0</td>
<td>0</td>
<td>0.0132</td>
</tr>
<tr>
<td>-6</td>
<td>0</td>
<td>0</td>
<td>0.2734</td>
</tr>
<tr>
<td>-5</td>
<td>0</td>
<td>0</td>
<td>0.6269</td>
</tr>
<tr>
<td>-4</td>
<td>0</td>
<td>0.0093</td>
<td>0.9774</td>
</tr>
<tr>
<td>-3</td>
<td>0</td>
<td>0.6366</td>
<td>1.1107</td>
</tr>
<tr>
<td>-2</td>
<td>0.4567</td>
<td>1.4765</td>
<td>1.1107</td>
</tr>
<tr>
<td>-1</td>
<td>2.1548</td>
<td>1.5708</td>
<td>1.1107</td>
</tr>
<tr>
<td>0</td>
<td>2.1548</td>
<td>1.5708</td>
<td>1.1107</td>
</tr>
<tr>
<td>1</td>
<td>0.4567</td>
<td>1.4765</td>
<td>1.1107</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.6366</td>
<td>1.1107</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0.0093</td>
<td>0.9774</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0.6269</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0.2734</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0.0132</td>
</tr>
</tbody>
</table>

## Haar wavelet coefficients:

<table>
<thead>
<tr>
<th>k</th>
<th>j=-1</th>
<th>j=0</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>0</td>
<td>-0.0093</td>
</tr>
<tr>
<td>-3</td>
<td>0</td>
<td>-0.2500</td>
</tr>
<tr>
<td>-2</td>
<td>-0.4435</td>
<td>-0.0943</td>
</tr>
<tr>
<td>-1</td>
<td>-0.0667</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0.0636</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.4496</td>
<td>0.0900</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.2500</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0.0107</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Approximation of the Example Function Using Haar Wavelets

Figure 4: An example function and its piecewise constant approximation
Figure 5: The piecewise constant detail functions between successive approximations for the example function
Daubechies Family of Wavelets:

The wavelets in the Daubechies family form an orthonormal basis for $L^2(\mathbb{R})$, and also have compact support. Members of this family are indexed by an integer $N$, with the smoothness of the functions increases as $N$ increases.
Daubechies Family of Wavelets:

**Figure 6:** Three examples of scaling function/wavelet sets from the Daubechies compactly supported family.
Approximation of the Example Function Using Daubechies Wavelets

Figure 7: An example function and representation using Daubechies wavelets with N=5
Reference:


Thank you!