A Reading of the Theory of Life Contingency Models:
A Preparation for Exam MLC/3L

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Preliminary Draft

To My Daughter
Nadia

Contents

Preface

The objective of this book is to present the basic aspects of the theory of insurance, concentrating on the part of this theory related to life insurance. An understanding of the basic principles underlying this part of the subject will form a solid foundation for further study of the theory in a more general setting.
This is the fourth of a series of books intended to help individuals to pass actuarial exams. The topics in this manuscript parallel the topics tested on Course MLC/3L of the Society of Actuaries exam sequence. The primary objective of the course is to increase students' understanding of the topics covered, and a secondary objective is to prepare students for a career in actuarial science.
The recommended approach for using this book is to read each section, work on the embedded examples, and then try the problems. Answer keys are provided so that you check your numerical answers against the correct ones. Problems taken from previous SOA/CAS exams will be indicated by the symbol ‡.
A calculator, such as the one allowed on the Society of Actuaries examinations, will be useful in solving many of the problems here. Familiarity with this calculator and its capabilities is an essential part of preparation for the examination.
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 If you encounter inconsistencies or errors, I would appreciate hearing from you: mfinan@atu.edu

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Prerequisite Material

Life contingency models are models that deal with the payments (or benefits) to a policyholder that are contingent on the continued survival (or death) of the person. We refer to these payments as **contingent payments.** The theory of insurance can be viewed as the theory of contingent payments. The insurance company makes payments to its insureds contingent upon the occurrence of some event, such as the death of the insured, an auto accident by an insured, and so on. The insured makes premium payments to the insurance company contingent upon being alive, having sufficient funds, and so on.

Two important ingredients in the study of contingent models:

• The first is to represent the contingencies mathematically and this is done by using probability theory. Probabilistic considerations will, therefore, play an important role in the discussion that follows. An overview of probability theory is presented in Chapter 2 of the book.

• The other central consideration in the theory of insurance is the time value of money. Both claims and premium payments occur at various, possibly random, points of time in the future. Since the value of a sum of money depends on the point in time at which the funds are available, a method of comparing the value of sums of money which become available at different points of time is needed. This methodology is provided by the theory of interest. An overview of the theory of interest is presented in Chapter 1.

Brief Review of Interest Theory

A typical part of most insurance contracts is that the insured pays the insurer a fixed premium on a periodic (usually annual or semi-annual) basis. Money has time value, that is, \$1 in hand today is more valuable than \$1 to be received one year. A careful analysis of insurance problems must take this effect into account. The purpose of this chapter is to examine the basic aspects of the theory of interest. Readers interested in further and thorough discussions of the topics can refer to [3].
Compound interest or discount will always be assumed, unless specified otherwise.

1 The Basics of Interest Theory

A dollar received today is worth more than a dollar received tomorrow. This is because a dollar received today can be invested to earn interest. The amount of interest earned depends on the rate of return that can be earned on the investment. The **time value of money** (abbreviated TVM) quantifies the value of a dollar through time.
Compounding is the term used to define computing a future value. **Discounting** is the term used to define computing a present value. We use the **Discount Rate** or **Compound Rate** to determine the present value or future value of a fixed lump sum or a stream of payments.
Under compound interest, the future value of \$1 invested today over t periods is given by the **accumulation function**

$$a(t) = (1+i)^t.$$

Thus, \$1 invested today worths $1+i$ dollars a period later. We call $1+i$ the **accumulation factor**.

The function $A(t) = A(0)a(t)$ which represents the accumulation of an investment of $A(0)$ for t periods is called the **amount function.**

Example 1.1
Suppose that $A(t) = \alpha t^2 + 10\beta$. If X invested at time 0 accumulates to \$500 at time 4, and to \$1,000 at time 10, find the amount of the original investment, X.

Solution.
We have $A(0) = X = 10\beta; A(4) = 500 = 16\alpha + 10\beta$; and $A(10) = 1000 = 100\alpha + 10\beta$. Using the first equation in the second and third we obtain the following system of linear equations

$$\begin{aligned} 16\alpha + X =& 500 \\ 100\alpha + X =& 1000. \end{aligned}$$

Multiply the first equation by 100 and the second equation by 16 and subtract to obtain $1600\alpha + 100X - 1600\alpha - 16X = 50000 - 16000$ or $84X = 34000$. Hence, $X = \frac{34000}{84} = \$404.76$ ■

Now, let n be a positive integer. The n^{th}**period** of time is defined to be the period of time between $t = n - 1$ and $t = n$. More precisely, the period normally will consist of the time interval $n - 1 \le t \le n$. We next introduce the first measure of interest which is developed using the accumulation function. Such a measure is referred to as the **effective rate of interest**:
The effective rate of interest is the amount of money that one unit invested at the beginning of a period will earn during the period, with interest being paid at the end of the period.

Example 1.2
You buy a house for \$100,000. A year later you sell it for \$80,000. What is the effective rate of return on your investment?

Solution.
The effective rate of return is

$$i = \frac{80,000 - 100,000}{100,000} = -20\%$$

which indicates a 20% loss of the original value of the house ■

If i_n is the effective rate of interest for the n^{th} time period then we can write
$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{(1+i)^n - (1+i)^{n-1}}{(1+i)^{n-1}} = i.$$
That is, under compound interest, the effective rate of interest is constant and is equal to the parameter i appearing in the base of the exponential form of $a(t)$.

Example 1.3
It is known that \$600 invested for two years will earn \$264 in interest. Find the accumulated value of \$2,000 invested at the same rate of annual compound interest for three years.

Solution.
We are told that $600(1+i)^2 = 600 + 264 = 864$. Thus, $(1+i)^2 = 1.44$ and solving for i we find $i = 0.2$. Thus, the accumulated value of investing \$2,000 for three years at the rate $i = 20\%$ is $2,000(1+0.2)^3 = \$3,456$ ■

Example 1.4
At a certain rate of compound interest, 1 will increase to 2 in a years, 2 will increase to 3 in b years, and 3 will increase to 15 in c years. If 6 will increase to 10 in n years, find an expression for n in terms of a, b, and c.

Solution.
If the common rate is i, the hypotheses are that

$$\begin{aligned}
1(1+i)^a =& 2 \rightarrow \ln 2 = a \ln(1+i) \\
2(1+i)^b =& 3 \rightarrow \ln \frac{3}{2} = b \ln(1+i) \\
3(1+i)^c =& 15 \rightarrow \ln 5 = c \ln(1+i) \\
6(1+i)^n =& 10 \rightarrow \ln \frac{5}{3} = n \ln(1+i)
\end{aligned}$$

But
$$\ln \frac{5}{3} = \ln 5 - \ln 3 = \ln 5 - (\ln 2 + \ln 1.5).$$
Hence,

$$n \ln(1+i) = c \ln(1+i) - a \ln(1+i) - b \ln(1+i) = (c - a - b) \ln(1+i)$$

and this implies $n = c - a - b$ ■

The accumulation function is used to find future values. In order to find present values of future investments, one uses the **discount function** defined by the ratio $\frac{1}{(1+i)^t}$.

Example 1.5
What is the present value of \$8,000 to be paid at the end of three years if the interest rate is 11% compounded annually?

Solution.
Let FV stand for the future value and PV for the present value. We want to find PV. We have $FV = PV(1+i)^3$ or $PV = FV(1+i)^{-3}$. Substituting into this equation we find $PV = 8000(1.11)^{-3} \approx \$5,849.53$ ■

Parallel to the concept of effective rate of interest, we define the **effective rate of discount** for the n^{th} time period by

$$d_n = \frac{a(n) - a(n-1)}{a(n)} = \frac{(1+i)^n - (1+i)^{n-1}}{(1+i)^n} = \frac{i}{1+i} = d.$$

That is, under compound interest, the effective rate of discount is constant. Now, \$1 a period from now invested at the rate i worths $\nu = \frac{1}{1+i}$ today. We call ν the **discount factor** since it discounts the value of an investment at the end of a period to its value at the beginning of the period.

Example 1.6
What is the difference between the following two situations?
(1) A loan of \$100 is made for one year at an effective rate of interest of 5%.
(2) A loan of \$100 is made for one year at an effective rate of discount of 5%.

Solution.
In both cases the fee for the use of the money is the same which is \$5. That is, the amount of discount is the same as the amount of interest. However, in the first case the interest is paid at the end of the period so the borrower was able to use the full \$100 for the year. He can for example invest this money at a higher rate of interest say 7% and make a profit of \$2 at the end of the transaction. In the second case, the interest is paid at the beginning of the period so the borrower had access to only \$95 for the year. So, if this

amount is invested at 7% like the previous case then the borrower will make a profit of \$1.65. Also, note that the effective rate of interest is taken as a percentage of the balance at the beginning of the year whereas the effective rate of discount is taken as a percentage of the balance at the end of the year ■

Using the definitions of ν and d we have the following relations among, i, d, and ν :
- $i = \frac{d}{1-d}$.
- $d = \frac{i}{1+i}$.
- $d = i\nu$.
- $d = 1 - \nu$.
- $id = i - d$.

Example 1.7
The amount of interest earned for one year when X is invested is \$108. The amount of discount earned when an investment grows to value X at the end of one year is \$100. Find X, i, and d.

Solution.
We have $iX = 108$, $\frac{i}{1+i}X = 100$. Thus, $\frac{108}{1+i} = 100$. Solving for i we find $i = 0.08 = 8\%$. Hence, $X = \frac{108}{0.08} = 1,350$ and $d = \frac{i}{1+i} = \frac{2}{27} \approx 7.41\%$ ■

<u>Nominal Rates of Interest and Discount</u>
When interest is paid (i.e., reinvested) more frequently than once per period, we say it is "payable" ("convertible", "compounded") each fraction of a period, and this fractional period is called the **interest conversion period.**
A **nominal rate of interest** $i^{(m)}$ payable m times per period, where m is a positive integer, represents m times the effective rate of compound interest used for each of the mth of a period. In this case, $\frac{i^{(m)}}{m}$ is the effective rate of interest for each mth of a period. Thus, for a nominal rate of 12% compounded monthly, the effective rate of interest per month is 1% since there are twelve months in a year.
Suppose that 1 is invested at a nominal rate $i^{(m)}$ compounded m times per measurement period. That is, the period is partitioned into m equal fractions of a period. At the end of the first fraction of the period the accumulated value is $1 + \frac{i^{(m)}}{m}$. At the end of the second fraction of the period the accumulated value is $\left(1 + \frac{i^{(m)}}{m}\right)^2$. Continuing, we find that the accumulated value

at the end of the mth fraction of a period, which is the same as the end of one period, is $\left(1+\frac{i^{(m)}}{m}\right)^m$ and at the end of t years the accumulated value is

$$a(t)=\left(1+\frac{i^{(m)}}{m}\right)^{mt}.$$

Example 1.8
Find the accumulated value of $3,000 to be paid at the end of 8 years with a rate of compound interest of 5%
(a) per annum;
(b) convertible quarterly;
(c) convertible monthly.

Solution.
(a) The accumulated value is $3,000\left(1+\frac{0.05}{1}\right)^8 \approx \$4,432.37$.
(b) The accumulated value is $3,000\left(1+\frac{0.05}{4}\right)^{8\times 4} \approx \$4,464.39$.
(c) The accumulated value is $3,000\left(1+\frac{0.05}{12}\right)^{8\times 12} \approx \$4,471.76$ ■

Next we describe the relationship between effective and nominal rates. If i denotes the effective rate of interest per one measurement period equivalent to $i^{(m)}$ then we can write

$$1+i=\left(1+\frac{i^{(m)}}{m}\right)^m$$

since each side represents the accumulated value of a principal of 1 invested for one year. Rearranging we have

$$i=\left(1+\frac{i^{(m)}}{m}\right)^m-1$$

and

$$i^{(m)}=m[(1+i)^{\frac{1}{m}}-1].$$

For any $t\geq 0$ we have

$$(1+i)^t=\left(1+\frac{i^{(m)}}{m}\right)^{mt}.$$

Example 1.9
(a) Find the annual effective interest rate i which is equivalent to a rate of compound interest of 8% convertible quarterly.
(b) Find the compound interest rate $i^{(2)}$ which is equivalent to an annual effective interest rate of 8%.
(c) Find the compound interest rate $i^{(4)}$ which is equivalent to a rate of compound interest of 8% payable semi-annually.

Solution.
(a) We have

$$1+i=\left(1+\frac{0.08}{4}\right)^4 \Rightarrow i=\left(1+\frac{0.08}{4}\right)^4-1\approx 0.08243216$$

(b) We have

$$1+0.08=\left(1+\frac{i^{(2)}}{2}\right)^2 \Rightarrow i^{(2)}=2[(1.08)^{\frac{1}{2}}-1]\approx 0.07846.$$

(c) We have

$$\left(1+\frac{i^{(4)}}{4}\right)^4=\left(1+\frac{i^{(2)}}{2}\right)^2 \Rightarrow i^{(4)}=4[(1.04)^{\frac{1}{2}}-1]\approx 0.0792 \blacksquare$$

In the same way that we defined a nominal rate of interest, we could also define a nominal rate of discount, $d^{(m)}$, as meaning an effective rate of discount of $\frac{d^{(m)}}{m}$ for each of the mth of a period with interest paid at the beginning of a mth of a period.
The accumulation function with the nominal rate of discount $d^{(m)}$ is

$$a(t)=\left(1-\frac{d^{(m)}}{m}\right)^{-mt}, \quad t\geq 0.$$

Example 1.10
Find the present value of \$8,000 to be paid at the end of 5 years using an annual compound interest of 7%
(a) convertible semiannually.
(b) payable in advance and convertible semiannually.

Solution.
(a) The answer is

$$\frac{8,000}{\left(1+\frac{0.07}{2}\right)^{5\times 2}} \approx \$5,671.35$$

(b) The answer is

$$8000\left(1-\frac{0.07}{2}\right)^{5\times 2} \approx \$5,602.26 \blacksquare$$

If d is the effective discount rate equivalent to $d^{(m)}$ then

$$1-d=\left(1-\frac{d^{(m)}}{m}\right)^m$$

since each side of the equation gives the present value of 1 to be paid at the end of the measurement period. Rearranging, we have

$$d=1-\left(1-\frac{d^{(m)}}{m}\right)^m$$

and solving this last equation for $d^{(m)}$ we find

$$d^{(m)}=m[1-(1-d)^{\frac{1}{m}}]=m(1-\nu^{\frac{1}{m}}).$$

Example 1.11
Find the present value of \$1,000 to be paid at the end of six years at 6% per year payable in advance and convertible semiannually.

Solution.
The answer is

$$1,000\left(1-\frac{0.06}{2}\right)^{12} = \$693.84 \blacksquare$$

There is a close relationship between nominal rate of interest and nominal rate of discount. Since $1-d=\frac{1}{1+i}$, we conclude that

$$\left(1+\frac{i^{(m)}}{m}\right)^m = 1+i=(1-d)^{-1}=\left(1-\frac{d^{(n)}}{n}\right)^{-n}. \qquad (1.1)$$

If $m=n$ then the previous formula reduces to

$$\left(1+\frac{i^{(n)}}{n}\right)=\left(1-\frac{d^{(n)}}{n}\right)^{-1}.$$

Example 1.12
Find the nominal rate of discount convertible semiannually which is equivalent to a nominal rate of interest of 12% per year convertible monthly.

Solution.
We have

$$\left(1 - \frac{d^{(2)}}{2}\right)^{-2} = \left(1 + \frac{0.12}{12}\right)^{12}.$$

Solving for $d^{(2)}$ we find $d^{(2)} = 0.11591$ ■

Note that formula (1.1) can be used in general to find equivalent rates of interest or discount, either effective or nominal, converted with any desired frequency.

<u>**Force of Interest**</u>
Effective and nominal rates of interest and discount each measures interest over some interval of time. Effective rates of interest and discount measure interest over one full measurement period, while nominal rates of interest and discount measure interest over $m$ths of a period.
In this section we want to measure interest at any particular moment of time. This measure of interest is called the **force of interest** and is defined by

$$\delta_t = \frac{d}{dt} \ln [a(t)]$$

which under compound interest we have

$$\delta_t = \ln (1 + i) = \delta.$$

Example 1.13
Given the nominal interest rate of 12%, compounded monthly. Find the equivalent force of interest δ.

Solution.
The effective annual interest rate is

$$i = (1 + 0.01)^{12} - 1 \approx 0.1268250.$$

Hence, $\delta = \ln (1 + i) = \ln (1.1268250) \approx 0.119404.$ ■

Example 1.14
A loan of \$3,000 is taken out on June 23, 1997. If the force of interest is 14%, find each of the following
(a) The value of the loan on June 23, 2002.
(b) The value of i.
(c) The value of $i^{(12)}$.

Solution.
(a) $3,000(1+i)^5 = 3,000e^{5\delta} = 3,000e^{0.7} \approx \$6,041.26$.
(b) $i = e^{\delta} - 1 = e^{0.14} - 1 \approx 0.15027$.
(c) We have

$$\left(1 + \frac{i^{(12)}}{12}\right)^{12} = 1 + i = e^{0.14}.$$

Solving for $i^{(12)}$ we find $i^{(12)} \approx 0.14082$ ∎

Since

$$\delta_t = \frac{d}{dt}\ln(a(t))$$

we can find δ_t given $a(t)$. What if we are given δ_t instead, and we wish to derive $a(t)$ from it?
From the definition of δ_t we can write

$$\frac{d}{dr}\ln a(r) = \delta_r.$$

Integrating both sides from 0 to t we obtain

$$\int_0^t \frac{d}{dr}\ln a(r)dr = \int_0^t \delta_r dr.$$

Hence,

$$\ln a(t) = \int_0^t \delta_r dr.$$

From this last equation we find

$$a(t) = e^{\int_0^t \delta_r dr}.$$

Example 1.15
A deposit of \$10 is invested at time 2 years. Using a force of interest of $\delta_t = 0.2 - 0.02t$, find the accumulated value of this payment at the end of 5 years.

Solution.
The accumulated value is

$$A(5) = 10\frac{a(5)}{a(2)} = 10e^{\int_2^5 (0.2-0.02t)dt} = 10e^{\left[0.2t-0.01t^2\right]_2^5} \approx \$14.77 \blacksquare$$

2 Equations of Value and Time Diagrams

Interest problems generally involve four quantities: principal(s), investment period length(s), interest rate(s), accumulated value(s). If any three of these quantities are known, then the fourth quantity can be determined. In this section we introduce equations that involve all four quantities with three quantities are given and the fourth to be found.

In calculations involving interest, the value of an amount of money at any given point in time depends upon the time elapsed since the money was paid in the past or upon time which will elapse in the future before it is paid. This principle is often characterized as the recognition of the **time value of money.** We assume that this principle reflects only the effect of interest and does not include the effect of inflation. Inflation reduces the purchasing power of money over time so investors expect a higher rate of return to compensate for inflation. As pointed out, we will neglect the effect of inflation when applying the above mentioned principle.

As a consequence of the above principle, various amounts of money payable at different points in time cannot be compared until all the amounts are accumulated or discounted to a common date, called the **comparison date,** is established. The equation which accumulates or discounts each payment to the comparison date is called the **equation of value.**

One device which is often helpful in the solution of equations of value is the **time diagram**. A time diagram is a one-dimensional diagram where the only variable is time, shown on a single coordinate axis. We may show above or below the coordinate of a point on the time-axis, values of money intended to be associated with different funds. A time diagram is not a formal part of a solution, but may be very helpful in visualizing the solution. Usually, they are very helpful in the solution of complex probems.

Example 2.1

In return for a payment of \$1,200 at the end of 10 years, a lender agrees to pay \$200 immediately, \$400 at the end of 6 years, and a final amount at the end of 15 years. Find the amount of the final payment at the end of 15 years if the nominal rate of interest is 9% converted semiannually.

Solution.

The comparison date is chosen to be $t = 0$. The time diagram is given in Figure 2.1.

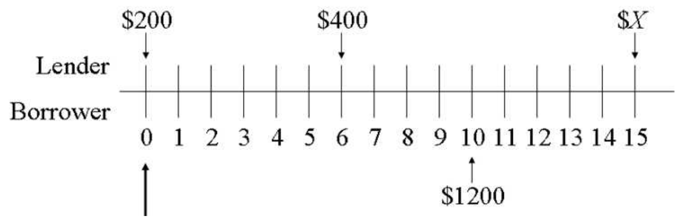


Figure 2.1

The equation of value is

$$200 + 400(1 + 0.045)^{-12} + X(1 + 0.045)^{-30} = 1200(1 + 0.045)^{-20}.$$

Solving this equation for X we find $X \approx \$231.11$ ■

Example 2.2
Investor A deposits 1,000 into an account paying 4% compounded quarterly. At the end of three years, he deposits an additional 1,000. Investor B deposits X into an account with force of interest $\delta_t = \frac{1}{6+t}$. After five years, investors A and B have the same amount of money. Find X.

Solution.
Consider investor A's account first. The initial 1,000 accumulates at 4% compounded quarterly for five years; the accumulated amount of this piece is

$$1,000\left(1 + \frac{0.04}{4}\right)^{4\times 5} = 1000(1.01)^{20}.$$

The second 1,000 accumulates at 4% compounded quarterly for two years, accumulating to

$$1,000\left(1 + \frac{0.04}{4}\right)^{4\times 2} = 1000(1.01)^{8}.$$

The value in investor A's account after five years is

$$A = 1000(1.01)^{20} + 1000(1.01)^{8}.$$

The accumulated amount of investor B's account after five years is given by

$$B = Xe^{\int_0^5 \frac{dt}{6+t}} = Xe^{\ln\left(\frac{11}{6}\right)} = \frac{11}{6}X.$$

The equation of value at time $t = 5$ is

$$\frac{11}{6}X = 1000(1.01)^{20} + 1000(1.01)^{8}.$$

Solving for X we find $X \approx \$1,256.21$ ■

3 Level Annuities

A series of payments made at equal intervals of time is called an **annuity.** An annuity where payments are guaranteed to occur for a fixed period of time is called an **annuity-certain.** In what follows we review the terminology and notation partained to annuity-certain. By a **level-annuity** we mean an annuity with fixed payments.

3.1 Level Annuity-Immediate

An annuity under which payments of 1 are made at the end of each period for n periods is called an **annuity−immediate** or **ordinary annuity.** The cash stream represented by the annuity can be visualized on a time diagram as shown in Figure 3.1 .

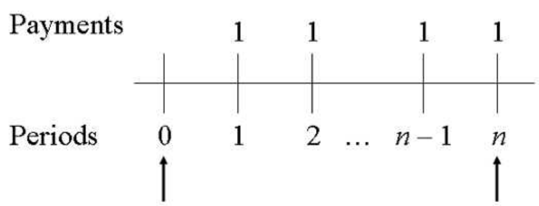


Figure 3.1

The first arrow shows the beginning of the first period, at the end of which the first payment is due under the annuity. The second arrow indicates the last payment date−just after the payment has been made.
The present value of the annuity at time 0 is given by

$$a_{\overline{n}|} = \nu + \nu^2 + \cdots + \nu^n = \frac{1 - \nu^n}{i}.$$

That is, the present value of the annuity is the sum of the present values of each of the n payments.

Example 3.1
Calculate the present value of an annuity−immediate of amount \$100 paid annually for 5 years at the rate of interest of 9%.

Solution.
The answer is $100a_{\overline{5}|} = 100\frac{1-(1.09)^{-5}}{0.09} \approx 388.97$ ■

The accumulated value of an annuity−immediate right after the nth payment is made is given by

$$s_{\overline{n}|} = (1+i)^n a_{\overline{n}|} = \frac{(1+i)^n - 1}{i}.$$

Example 3.2
Calculate the future value of an annuity−immediate of amount \$100 paid annually for 5 years at the rate of interest of 9%.

Solution.
The answer is $100s_{\overline{5}|} = 100 \times \frac{(1.09)^5-1}{0.09} \approx \$598.47$ ■

With $a_{\overline{n}|}$ and $s_{\overline{n}|}$ as defined above we have

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i.$$

A **perpetuity-immediate** is an annuity with infinite number of payments with the first payment occurring at the end of the first period. Since the term of the annuity is infinite, perpetuities do not have accumulated values. The present value of a perpetuity-immediate is given by

$$a_{\overline{\infty}|} = \lim_{n\to\infty} a_{\overline{n}|} = \frac{1}{i}.$$

Example 3.3
Suppose a company issues a stock that pays a dividend at the end of each year of \$10 indefinitely, and the companies cost of capital is 6%. What is the value of the stock at the beginning of the year?

Solution.
The answer is $10 \cdot a_{\overline{\infty}|} = 10 \cdot \frac{1}{0.06} = \$166.67$ ■

3.2 Level Annuity-Due

An **annuity−due** is an annuity for which the payments are made at the beginning of the payment periods. The cash stream represented by the annuity can be visualized on a time diagram as shown in Figure 3.2 .

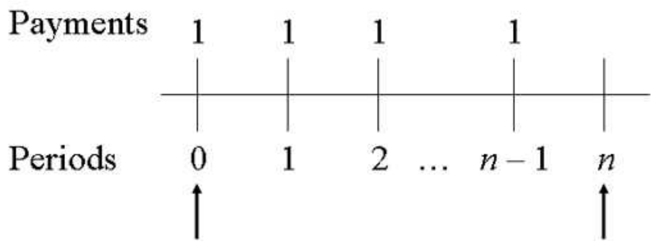


Figure 3.2

The first arrow shows the beginning of the first period at which the first payment is made under the annuity. The second arrow appears n periods after arrow 1, one period after the last payment is made.

The present value of an annuity-due is given by

$$\ddot{a}_{\overline{n}|} = 1 + \nu + \nu^2 + \cdots + \nu^{n-1} = \frac{1-\nu^n}{d}.$$

Example 3.4
Find $\ddot{a}_{\overline{8}|}$ if the effective rate of discount is 10%.

Solution.
Since $d = 0.10$, we have $\nu = 1 - d = 0.9$. Hence, $\ddot{a}_{\overline{8}|} = \frac{1-(0.9)^8}{0.1} = 5.6953279$ ■

Example 3.5
What amount must you invest today at 6% interest rate compounded annually so that you can withdraw \$5,000 at the beginning of each year for the next 5 years?

Solution.
The answer is $5000\ddot{a}_{\overline{5}|} = 5000 \cdot \frac{1-(1.06)^{-5}}{0.06(1.06)^{-1}} = \$22,325.53$ ■

The accumulated value at time n of an annuity-due is given by

$$\ddot{s}_{\overline{n}|} = (1+i)^n \ddot{a}_{\overline{n}|} = \frac{(1+i)^n - 1}{i\nu} = \frac{(1+i)^n - 1}{d}.$$

Example 3.6
What amount will accumulate if we deposit \$5,000 at the beginning of each year for the next 5 years? Assume an interest of 6% compounded annually.

Solution.
The answer is $5000\ddot{s}_{\overline{5}|} = 5000 \cdot \frac{(1.06)^5-1}{0.06(1.06)^{-1}} = \$29,876.59$ ■

With $\ddot{a}_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$ as defined above we have
(i) $\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$.
(ii) $a_{\overline{n}|} = \nu\ddot{a}_{\overline{n}|}$.
(iii) $\ddot{s}_{\overline{n}|} = (1+i)s_{\overline{n}|}$.
(iv) $s_{\overline{n}|} = \nu\ddot{s}_{\overline{n}|}$.
(v) $\frac{1}{\ddot{a}_{\overline{n}|}} = \frac{1}{\ddot{s}_{\overline{n}|}} + d$.

A **perpetuity-due** is an annuity with infinite number of payments with the first payment occurring at the beginning of the first period. Since the term of the annuity is infinite, perpetuities do not have accumulated values. The present value of a perpetuity-due is given by

$$\ddot{a}_{\overline{\infty}|} = \lim_{n\to\infty} \ddot{a}_{\overline{n}|} = \frac{1}{d}.$$

Example 3.7
What would you be willing to pay for an infinite stream of \$37 annual payments (cash inflows) beginning now if the interest rate is 8% per annum?

Solution.
The answer is $37\ddot{a}_{\overline{\infty}|} = \frac{37}{0.08(1.08)^{-1}} = \$499.50$ ■

3.3 Level Continuous Annuity

In this section we consider annuities with a finite term and an infinte frequency of payments . Formulas corresponding to such annuities are useful as approximations corresponding to annuities payable with great frequency such as daily.
Consider an annuity in which a very small payment dt is made at time t and these small payments are payable continuously for n interest conversion periods. Let i denote the periodic interest rate. Then the total amount paid during each period is

$$\int_{k-1}^{k} dt = [t]_{k-1}^{k} = \$1.$$

Let $\overline{a}_{\overline{n}|}$ denote the present value of an annuity payable continuously for n interest conversion periods so that 1 is the total amount paid during each interest conversion period. Then the present value can be found as follows:

$$\overline{a}_{\overline{n}|} = \int_0^n \nu^t dt = \left.\frac{\nu^t}{\ln \nu}\right|_0^n = \frac{1-\nu^n}{\delta}.$$

With $\overline{a}_{\overline{n}|}$ defined above we have

$$\overline{a}_{\overline{n}|} = \frac{i}{\delta}a_{\overline{n}|} = \frac{d}{\delta}\ddot{a}_{\overline{n}|} = \frac{1-e^{-n\delta}}{\delta}.$$

Example 3.8
Starting four years from today, you will receive payment at the rate of \$1,000 per annum, payable continuously, with the payment terminating twelve years from today. Find the present value of this continuous annuity if $\delta = 5\%$.

Solution.
The present value is $PV = 1000\nu^4 \cdot \overline{a}_{\overline{8}|} = 1000e^{-0.20} \cdot \frac{1-e^{-0.40}}{0.05} = \$5,398.38$ ■

Next, let $\overline{s}_{\overline{n}|}$ denote the accumulated value at the end of the term of an annuity payable continuously for n interest conversion periods so that 1 is the total amound paid during each interest conversion period. Then

$$\overline{s}_{\overline{n}|} = (1+i)^n\overline{a}_{\overline{n}|} = \int_0^n (1+i)^{n-t}dt = \frac{(1+i)^n-1}{\delta}.$$

It is easy to see that

$$\overline{s}_{\overline{n}|} = \frac{e^{n\delta}-1}{\delta} = \frac{i}{\delta}s_{\overline{n}|} = \frac{d}{\delta}\ddot{s}_{\overline{n}|}.$$

Example 3.9
Find the force of interest at which the accumulated value of a continuous payment of 1 every year for 8 years will be equal to four times the accumulated value of a continuous payment of 1 every year for four years.

Solution.
We have

$$\begin{aligned}\overline{s}_{\overline{8}|} =& 4\overline{s}_{\overline{4}|}\\ \frac{e^{8\delta}-1}{\delta} =& 4\cdot\frac{e^{4\delta}-1}{\delta}\\ e^{8\delta}-4e^{4\delta}+3 =& 0\\ (e^{4\delta}-3)(e^{4\delta}-1) =& 0\end{aligned}$$

If $e^{4\delta}=3$ then $\delta=\frac{\ln 3}{4}\approx 0.0275=2.75\%$. If $e^{4\delta}=1$ then $\delta=0$, an extraneous solution ■

With $\overline{a}_{\overline{n}|}$ and $\overline{s}_{\overline{n}|}$ as defined above we have

$$\frac{1}{\overline{a}_{\overline{n}|}}=\frac{1}{\overline{s}_{\overline{n}|}}+\delta.$$

The present value of a perpetuity payable continuously with total of 1 per period is given by

$$\overline{a}_{\overline{\infty}|}=\lim_{n\to\infty}\overline{a}_{\overline{n}|}=\frac{1}{\delta}.$$

Example 3.10
A perpetuity paid continuously at a rate of 100 per year has a present value of 800. Calculate the annual effective interest rate used to calculate the present value.

Solution.
The equation of value at time $t=0$ is

$$800=\frac{100}{\delta}=\frac{100}{\ln(1+i)}.$$

Thus,

$$i=e^{\frac{1}{8}}-1=13.3\%\ \blacksquare$$

4 Varying Annuities

In the previous section we considered annuities with level series of payments, that is, payments are all equal in values. In this section we consider annuities with a varying series of payments. Annuities with varying payments will be called **varying annuities**. In what follows, we assume that the payment period and interest conversion period coincide.

4.1 Varying Annuity-Immediate

Any type of annuities can be evaluated by taking the present value or the accumulated value of each payment seperately and adding the results. There are, however, several types of varying annuities for which relatively simple compact expressions are possible. The only general types that we consider vary in either arithmetic progression or geometric progression.

Payments Varying in an Arithmetic Progression

First, let us assume that payments vary in arithmetic progression. The first payment is 1 and then the payments increase by 1 thereafter, continuing for n years as shown in the time diagram of Figure 4.1.

Figure 4.1

The present value at time 0 of such annuity is

$$(Ia)_{\overline{n}|} = \nu + 2\nu^2 + 3\nu^3 + \cdots + n\nu^n = \frac{\ddot{a}_{\overline{n}|} - n\nu^n}{i}.$$

The accumulated value at time n is given by

$$(Is)_{\overline{n}|} = (1+i)^n (Ia)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i} = \frac{s_{\overline{n+1}|} - (n+1)}{i}.$$

Example 4.1

The following payments are to be received: \$500 at the end of the first year, \$520 at the end of the second year, \$540 at the end of the third year and so

on, until the final payment is \$800. Using an annual effective interest rate of 2%
(a) determine the present value of these payments at time 0;
(b) determine the accumulated value of these payments at the time of the last payment.

Solution.
In n years the payment is $500 + 20(n-1)$. So the total number of payments is 16. The given payments can be regarded as the sum of a level annuity immediate of \$480 and an increasing annuity-immediate $\$20, \$40, \cdots, \$320$.
(a) The present value at time $t = 0$ is

$$480a_{\overline{16}|} + 20(Ia)_{\overline{16}|} = 480(13.5777) + 20(109.7065) = \$8,711.43.$$

(b) The accumulated value at time $t = 16$ is

$$480s_{\overline{16}|} + 20(Is)_{\overline{16}|} = 480(18.6393) + 20(150.6035) = \$11,958.93 \blacksquare$$

Next, we consider a decreasing annuity-immediate with first payment n and each payment decreases by 1 for a total of n payments as shown in Figure 4.2.

Figure 4.2

In this case, the present value one year before the first payment (i.e., at time $t = 0$) is given by

$$(Da)_{\overline{n}|} = n\nu + (n-1)\nu^2 + \cdots + \nu^n = \frac{n - a_{\overline{n}|}}{i}.$$

The accumulated value at time n is given by

$$(Ds)_{\overline{n}|} = (1+i)^n(Da)_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|}}{i} = (n+1)a_{\overline{n}|} - (Ia)_{\overline{n}|}.$$

Example 4.2
John receives \$400 at the end of the first year, \$350 at the end of the second year, \$300 at the end of the third year and so on, until the final payment of \$50. Using an annual effective rate of 3.5%, calculate the present value of these payments at time 0.

Solution.
In year n the payment is $400 - 50(n-1)$. Since the final payment is 50 we must have $400 - 50(n-1) = 50$. Solving for n we find $n = 8$. Thus, the present value is

$$50(Da)_{\overline{8|}} = 50 \cdot \frac{8 - a_{\overline{8|}}}{0.035} = \$1,608.63 \blacksquare$$

Example 4.3
Calculate the accumulated value in Example 4.2.

Solution.
The answer is $50(Ds)_{\overline{8|}} = 50 \cdot \frac{8(1.035)^8 - s_{\overline{8|}}}{0.035} = \$2,118.27 \blacksquare$

Besides varying annuities immediate, it is also possible to have varying perpetuity−immediate. Consider the perpetuity with first payment of 1 at the end of the first period and then each successive payment increases by 1. In this case, the present value is given by

$$(Ia)_{\overline{\infty|}} = \lim_{n\to\infty}(Ia)_{\overline{n|}} = \frac{1}{i} + \frac{1}{i^2}.$$

Example 4.4
Find the present value of a perpetuity-immediate whose successive payments are 1, 2, 3, 4,$\cdots$ at an effective rate of 6%.

Solution.
We have $(Ia)_{\overline{\infty|}} = \frac{1}{i} + \frac{1}{i^2} = \frac{1}{0.06} + \frac{1}{0.06^2} = \$294.44 \blacksquare$

Remark 4.1
The notion of perpetuity does not apply for the decreasing case.

Payments Varying in a Geometric Progression
Next, we consider payments varying in a geometric progression. Consider an annuity-immediate with a term of n periods where the interest rate is i per period, and where the first payment is 1 and successive payments increase in geometric progression with common ratio $1 + k$. The present value of this annuity is

$$\nu + \nu^2(1+k) + \nu^3(1+k)^2 + \cdots + \nu^n(1+k)^{n-1} = \frac{1 - \left(\frac{1+k}{1+i}\right)^n}{i - k}$$

provided that $k \neq i$. If $k = i$ then the original sum consists of a sum of n terms of ν which equals to $n\nu$.

Example 4.5
The first of 30 payments of an annuity occurs in exactly one year and is equal to \$500. The payments increase so that each payment is 5% greater than the preceding payment. Find the present value of this annuity with an annual effective rate of interest of 8%.

Solution.
The present value is given by

$$PV = 500 \cdot \frac{1 - \left(\frac{1.05}{1.08}\right)^{30}}{0.08 - 0.05} = \$9,508.28 \blacksquare$$

For an annuity-immediate with a term of n periods where the interest rate is i per period, and where the first payment is 1 and successive payments decrease in geometric progression with common ratio $1-k$. The present value of this annuity is

$$\nu + \nu^2(1-k) + \nu^3(1-k)^2 + \cdots + \nu^n(1-k)^{n-1} = \frac{1 - \left(\frac{1-k}{1+i}\right)^n}{i+k}$$

provided that $k \neq i$. If $k = i$ then the original sum becomes

$$\nu + \nu^2(1-i) + \nu^3(1-i)^2 + \cdots + \nu^n(1-i)^{n-1} = \frac{1}{2i}\left[1 - \left(\frac{1-i}{1+i}\right)^n\right]$$

Finally, we consider a perpetuity with payments that form a geometric progression where $0 < 1+k < 1+i$. The present value for such a perpetuity with the first payment at the end of the first period is

$$\nu + \nu^2(1+k) + \nu^3(1+k)^2 + \cdots = \frac{\nu}{1-(1+k)\nu} = \frac{1}{i-k}.$$

Observe that the value for these perpetuities cannot exist if $1+k \geq 1+i$.

Example 4.6
What is the present value of a stream of annual dividends, which starts at 1 at the end of the first year, and grows at the annual rate of 2%, given that the rate of interest is 6% ?

Solution.
The present value is $\frac{1}{i-k} = \frac{1}{0.06-0.02} = 25 \blacksquare$

4.2 Varying Annuity-Due

In this section, we examine the case of an increasing annuity-due. Consider an annuity with the first payment is 1 at the beginning of year 1 and then the payments increase by 1 thereafter, continuing for n years. A time diagram of this situation is given in Figure 4.3.

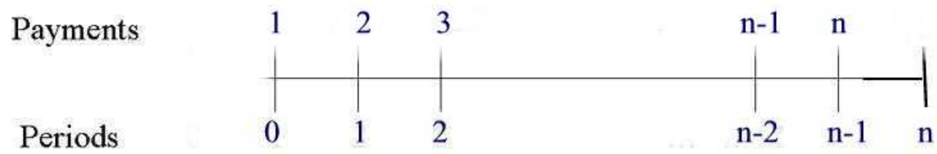


Figure 4.3

The present value for this annuity-due (at time $t = 0$) is

$$(I\ddot{a})_{\overline{n}|} = 1 + 2\nu + 3\nu^2 + \cdots + n\nu^{n-1} = \frac{\ddot{a}_{\overline{n}|} - n\nu^n}{d}$$

and the accumulated value at time n is

$$(I\ddot{s})_{\overline{n}|} = (1+i)^n (I\ddot{a})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d} = \frac{s_{\overline{n+1}|} - (n+1)}{d}.$$

Example 4.7
Determine the present value and future value of payments of \$75 at time 0, \$80 at time 1 year, \$85 at time 2 years, and so on up to \$175 at time 20 years. The annual effective rate is 4%.

Solution.
The present value is $70\ddot{a}_{\overline{21}|} + 5(I\ddot{a})_{\overline{21}|} = \$1,720.05$ and the future value is $(1.04)^{21}(1,720.05) = \$3919.60$ ■

In the case of a decreasing annuity-due where the first payment is n and each successive payment decreases by 1, the present value at time 0 is

$$(D\ddot{a})_{\overline{n}|} = n + (n-1)\nu + \cdots + \nu^{n-1} = \frac{n - a_{\overline{n}|}}{d}$$

and the accumulated value at time n is

$$(D\ddot{s})_{\overline{n}|} = (1+i)^n (D\ddot{a})_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|}}{d}.$$

Example 4.8
Calculate the present value and the accumulated value of a series of payments of \$100 now, \$90 in 1 year, \$80 in 2 years, and so on, down to \$10 at time 9 years using an annual effective interest rate of 3%.

Solution.
The present value is $10(D\ddot{a})_{\overline{10}|} = 10 \cdot \frac{10-8.530203}{0.03/1.03} = \$504.63$ and the accumulated value is $(1.03)^{10}(504.63) = \$678.18$ ■

A counterpart to a varying perpetuity-immediate is a varying perpetuity-due which starts with a payment of 1 at time 0 and each successive payment increases by 1 forever. The present value of such perpetuity is given by

$$(I\ddot{a})_{\overline{\infty}|} = \lim_{n\to\infty}(I\ddot{a})_{\overline{n}|} = \frac{1}{d^2}.$$

Example 4.9
Determine the present value at time 0 of payments of \$10 paid at time 0, \$20 paid at time 1 year, \$30 paid at time 2 years, and so on, assuming an annual effective rate of 5%.

Solution.
The answer is $10(I\ddot{a})_{\overline{\infty}|} = \frac{10}{d^2} = 10\left(\frac{1.05}{0.05}\right)^2 = \$4,410.00$ ■

Next, we consider payments varying in a geometric progression. Consider an annuity-due with a term of n periods where the interest rate is i per period, and where the first payment is 1 at time 0 and successive payments increase in geometric progression with common ratio $1+k$. The present value of this annuity is

$$1 + \nu(1+k) + \nu^2(1+k)^2 + \cdots + \nu^{n-1}(1+k)^{n-1} = (1+i)\frac{1-\left(\frac{1+k}{1+i}\right)^n}{i-k}$$

provided that $k \neq i$. If $k = i$ then the original sum consists of a sum of n terms of 1 which equals to n.

Example 4.10
An annual annuity due pays \$1 at the beginning of the first year. Each subsequent payment is 5% greater than the preceding payment. The last payment is at the beginning of the 10th year. Calculate the present value at:
(a) an annual effective interest rate of 4%;
(b) an annual effective interest rate of 5%.

Solution.
(a) $PV = (1.04)\frac{\left[1-\left(\frac{1.05}{1.04}\right)^{10}\right]}{(0.04-0.05)} = \$10.44.$
(b) Since $i = k$, $PV = n = \$10.00$ ■

For an annuity-due with n payments where the first payment is 1 at time 0 and successive payments decrease in geometric progression with common ratio $1-k$. The present value of this annuity is

$$1 + \nu(1-k) + \nu^2(1-k)^2 + \cdots + \nu^{n-1}(1-k)^{n-1} = (1+i)\frac{1-\left(\frac{1-k}{1+i}\right)^n}{i+k}$$

provided that $k \neq i$. If $k = i$ then the original sum is

$$1 + \nu(1-i) + \nu^2(1-i)^2 + \cdots + \nu^{n-1}(1-i)^{n-1} = \frac{1}{2d}\left[1-\left(\frac{1-i}{1+i}\right)^n\right].$$

Example 4.11 ‡
Matthew makes a series of payments at the beginning of each year for 20 years. The first payment is 100. Each subsequent payment through the tenth year increases by 5% from the previous payment. After the tenth payment, each payment decreases by 5% from the previous payment. Calculate the present value of these payments at the time the first payment is made using an annual effective rate of 7%.

Solution.
The present value at time 0 of the first 10 payments is

$$100\left[\frac{1-\left(\frac{1.05}{1.07}\right)^{10}}{0.07-0.05}\right]\cdot(1.07) = 919.95.$$

The value of the 11th payment is $100(1.05)^9(0.95) = 147.38$. The present value of the last ten payments is

$$147.38\left[\frac{1-\left(\frac{0.95}{1.07}\right)^{10}}{0.07+0.05}\right]\cdot(1.07)(1.07)^{-10} = 464.71.$$

The total present value of the 20 payments is $919.95 + 464.71 = 1384.66$ ■

Finally, the present value of a perpetuity with first payment of 1 at time

0 and successive payments increase in geometric progression with common ration $1+k$ is

$$1+\nu(1+k)+\nu^2(1+k)^2+\cdots=\frac{1}{1-(1+k)\nu}=\frac{1+i}{i-k}.$$

Observe that the value for these perpetuities cannot exist if $1+k\geq 1+i$.

Example 4.12
Perpetuity A has the following sequence of annual payments beginning on January 1, 2005:

$$1,3,5,7,\cdots$$

Perpetuity B is a level perpetuity of 1 per year, also beginning on January 1, 2005.
Perpetuity C has the following sequence of annual payments beginning on January 1, 2005:

$$1,1+r,(1+r)^2,\cdots$$

On January 1, 2005, the present value of Perpetuity A is 25 times as large as the present value of Perpetuity B, and the present value of Perpetuity A is equal to the present value of Perpetuity C. Based on this information, find r.

Solution.
The present value of Perpetuity A is $\frac{1}{d}+\frac{2(1+i)}{i^2}$.
The present value of Perpetuity B is $\frac{1}{d}$.
The present value of Perpetuity C is $\frac{1+i}{i-r}$.
We are told that

$$\frac{1+i}{i}+\frac{2(1+i)}{i^2}=\frac{25(1+i)}{i}.$$

This is equivalent to

$$12i^2+11i-1=0.$$

Solving for i we find $i=\frac{1}{12}$. Also, we are told that

$$\frac{1+i}{i}+\frac{2(1+i)}{i^2}=\frac{1+i}{i-r}$$

or

$$25(12)(1+\frac{1}{12})=\frac{1+\frac{1}{12}}{\frac{1}{12}-r}.$$

Solving for r we find $r=0.08=8\%$ ■

4.3 Continuous Varying Annuities

In this section we look at annuities in which payments are being made continuously at a varying rate.
Consider an annuity for n interest conversion periods in which payments are being made continuously at the rate $f(t)$ at exact moment t and the interest rate is variable with variable force of interest δ_t. Then $f(t)e^{-\int_0^t \delta_r dr}dt$ is the present value of the payment $f(t)dt$ made at exact moment t. Hence, the present value of this $n-$period continuous varying annuity is

$$PV = \int_0^n f(t)e^{-\int_0^t \delta_r dr}dt. \tag{4.1}$$

Example 4.13
Find an expression for the present value of a continuously increasing annuity with a term of n years if the force of interest is δ and if the rate of payment at time t is t^2 per annum.

Solution.
Using integration by parts process, we find

$$\begin{aligned}
\int_0^n t^2 e^{-\delta t}dt =& -\frac{t^2}{\delta}e^{-\delta t}\Big]_0^n + \frac{2}{\delta}\int_0^n te^{-\delta t}dt \\
=& -\frac{n^2}{\delta}e^{-\delta n} - \left[\frac{2t}{\delta^2}e^{-\delta t}\right]_0^n + \frac{2}{\delta^2}\int_0^n e^{-\delta t}dt \\
=& -\frac{n^2}{\delta}e^{-\delta n} - \frac{2n}{\delta^2}e^{-\delta n} - \left[\frac{2}{\delta^3}e^{-\delta t}\right]_0^n \\
=& -\frac{n^2}{\delta}e^{-\delta n} - \frac{2n}{\delta^2}e^{-\delta n} - \frac{2}{\delta^3}e^{-\delta n} + \frac{2}{\delta^3} \\
=& \frac{2}{\delta^3} - e^{-\delta n}\left[\frac{n^2}{\delta} + \frac{2n}{\delta^2} + \frac{2}{\delta^3}\right] \blacksquare
\end{aligned}$$

Under compound interest, i.e., $\delta_t = \ln(1+i)$, formula (4.1) becomes

$$PV = \int_0^n f(t)\nu^t dt.$$

Under compound interest and with $f(t) = t$ (an increasing annuity), the present value is

$$(\overline{I}\overline{a})_{\overline{n|}} = \int_0^n t\nu^t dt = \frac{\overline{a}_{\overline{n|}} - n\nu^n}{\delta}$$

and the accumulated value at time n years is

$$(\bar{I}\bar{s})_{\overline{n}|} = (1+i)^n(\bar{I}\bar{a})_{\overline{n}|} = \frac{\bar{s}_{\overline{n}|} - n}{\delta}.$$

Example 4.14
Sam receives continuous payments at an annual rate of $8t+5$ from time 0 to 10 years. The continuously compounded interest rate is 9%.
(a) Determine the present value at time 0.
(b) Determine the accumulated value at time 10 years.

Solution.
(a) The payment stream can be split into two parts so that the present value is

$$8(\bar{I}\bar{a})_{\overline{10}|} + 5\bar{a}_{\overline{10}|}.$$

Since

$$\begin{aligned}
i =& e^{0.09} - 1 = 9.4174\% \\
\bar{a}_{\overline{10}|} =& \frac{1-(1.094174)^{-10}}{0.09} = 6.59370 \\
(\bar{I}\bar{a})_{\overline{10}|} =& \frac{6.59370 - 10(1.094174)^{-10}}{0.09} = 28.088592
\end{aligned}$$

we obtain

$$8(\bar{I}\bar{a})_{\overline{10}|} + 5\bar{a}_{\overline{10}|} = 8 \times 28.088592 + 5 \times 6.59370 = 257.68.$$

(b) The accumulated value at time 10 years is

$$257.68 \times (1.094174)^{10} = 633.78 \blacksquare$$

For a continuous payable continuously increasing perpetuity (where $f(t) = t$), the present value at time 0 is

$$(\bar{I}\bar{a})_{\overline{\infty}|} = \lim_{n\to\infty} \frac{\bar{a}_{\overline{n}|} - n\nu^n}{\delta} = \lim_{n\to\infty} \frac{\frac{1-(1+i)^{-n}}{\delta} - n(1+i)^{-n}}{\delta} = \frac{1}{\delta^2}.$$

Example 4.15
Determine the present value of a payment stream that pays a rate of $5t$ at time t. The payments start at time 0 and they continue indefinitely. The annual effective interest rate is 7%.

Solution.
The present value is

$$5(\overline{I}\overline{a})_{\overline{\infty}|} = \frac{5}{[\ln(1.07)]^2} = 1,092.25 \blacksquare$$

We conclude this section by considering the case of a continuously decreasing continuously payable stream in which a continuous payment is received from time 0 to time n years. The rate of payment at time t is $f(t) = n - t$, and the force of interest is constant at δ. The present value is

$$\begin{aligned}(\overline{D}\overline{a})_{\overline{n}|} =& n\overline{a}_{\overline{n}|} - (\overline{I}\overline{a})_{\overline{n}|}\\ =& n\frac{1-\nu^n}{\delta} - \frac{\overline{a}_{\overline{n}|} - n\nu^n}{\delta}\\ =& \frac{n - \overline{a}_{\overline{n}|}}{\delta}\end{aligned}$$

and the accumulated value

$$(\overline{D}\overline{s})_{\overline{n}|} = \frac{n(1+i)^n - \overline{s}_{\overline{n}|}}{\delta}.$$

Example 4.16
Otto receives a payment at an annual rate of $10 - t$ from time 0 to time 10 years. The force of interest is 6%. Determine the present value of these payments at time 0.

Solution.
Since

$$i = e^{0.06} - 1 = 6.184\%$$

$$\overline{a}_{\overline{10}|} = \frac{1 - (1.06184)^{-10}}{0.06} = 7.5201$$

the present value is then

$$(\overline{D}\overline{a})_{\overline{10}|} = \frac{10 - 7.5201}{0.06} = 41.33 \blacksquare$$

4.4 Continuously Payable Varying Annuities

In this section, we consider annuities where payments are made at a continuous rate but increase/decrease at discrete times. To elaborate, consider first an increasing annuity with payments made continuously for n conversion periods with a total payments of 1 at the end of the first period, a total payments of 2 in the second period, $\cdots$, a total payments of n in the nth period. The present value of such annuity is

$$I(\overline{a})_{\overline{n}|} = \overline{s}_{\overline{1}|}\nu + 2\overline{s}_{\overline{1}|}\nu^2 + \cdots + n\overline{s}_{\overline{1}|}\nu^n = \overline{s}_{\overline{1}|}(Ia_{\overline{n}|}) = \frac{\ddot{a}_{\overline{n}|} - n\nu^n}{\delta}.$$

The accumulated value of this annuity is

$$I(\overline{s}_{\overline{n}|}) = (1+i)^n I(\overline{a})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{\delta}.$$

Next, we consider a decreasing annuity with payments made continuously for n conversion periods with a total payments of n at the end of the first period, a total payments of $n-1$ for the second period, $\cdots$, a total payments of 1 for the nth period. The present value at time 0 of such annuity is

$$D(\overline{a})_{\overline{n}|} = n\overline{s}_{\overline{1}|}\nu + (n-1)\overline{s}_{\overline{1}|}\nu^2 + \cdots + \overline{s}_{\overline{1}|}\nu^n = \overline{s}_{\overline{1}|}(Da_{\overline{n}|}) = \frac{n - a_{\overline{n}|}}{\delta}.$$

The accumulated value at time n of this annuity is

$$D(\overline{s})_{\overline{n}|} = (1+i)^n D(\overline{a}_{\overline{n}|}) = \frac{n(1+i)^n - s_{\overline{n}|}}{\delta}.$$

5 Annuity Values on Any Date: Deferred Annuity

Evaluating annuities thus far has always been done at the beginning of the term(either on the date of, or one period before the first payment) or at the end of the term (either on the date of, or one period after the last payment). In this section, we shall now consider evaluating the
(1) present value of an annuity more than one period before the first payment date,
(2) accumulated value of an annuity more than one period after the last payment date,
(3) current value of an annuity between the first and last payment dates.
We will assume that the evaluation date is always an integral number of periods from each payment date.

(1) **Present values more than one period before the first payment date**
Consider the question of finding the present value of an annuity$-$immediate with periodic interest rate i and $m+1$ periods before the first payment date. Figure 5.1 shows the time diagram for this case where "?" indicates the present value to be found.

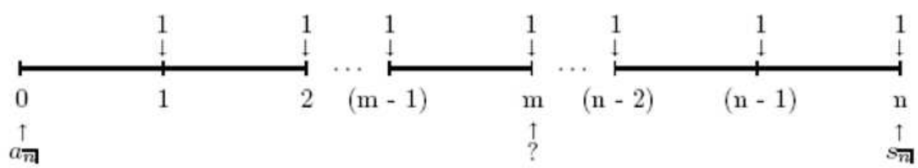


Figure 5.1

The present value of an $n-$period annuity$-$immediate $m+1$ periods before the first payment date (called a **deferred annuity** since payments do not begin until some later period) is the present value at time m discounted for m time periods, that is, $v^m a_{\overline{n}|}$. It is possible to express this answer strictly in terms of annuity values. Indeed,

$$a_{\overline{m+n}|} - a_{\overline{m}|} = \frac{1-\nu^{m+n}}{i} - \frac{1-\nu^m}{i} = \frac{\nu^m - \nu^{m+n}}{i} = \nu^m \frac{1-\nu^n}{i} = \nu^m a_{\overline{n}|}.$$

Such an expression is convenient for calculation, if interest tables are being used.

Example 5.1
Exactly 3 years from now is the first of four \$200 yearly payments for an annuity-immediate, with an effective 8% rate of interest. Find the present value of the annuity.

Solution.
The answer is $200\nu^2 a_{\overline{4}|} = 200(a_{\overline{6}|} - a_{\overline{2}|}) = 200(4.6229 - 1.7833) = \$567.92$ ■

The deferred−annuity introduced above uses annuity−immediate. It is possible to work with a deferred annuity−due. In this case, one can easily see that the present value is given by

$$\nu^m \ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{m+n}|} - \ddot{a}_{\overline{m}|}.$$

Example 5.2
Calculate the present value of an annuity−due paying annual payments of 1200 for 12 years with the first payment two years from now. The annual effective interest rate is 6%.

Solution.
The answer is $1200(1.06)^{-2}\ddot{a}_{\overline{12}|} = 1200(\ddot{a}_{\overline{14}|} - \ddot{a}_{\overline{2}|}) = 1200(9.8527 - 1.9434) \approx 9,491.16$ ■

(2) Accumulated values more than 1 period after the last payment date
Consider the question of finding the accumulated value of an annuity−immediate with periodic interest rate i and m periods after the last payment date. Figure 5.2 shows the time diagram for this case where "?" indicates the sought accumulated value.

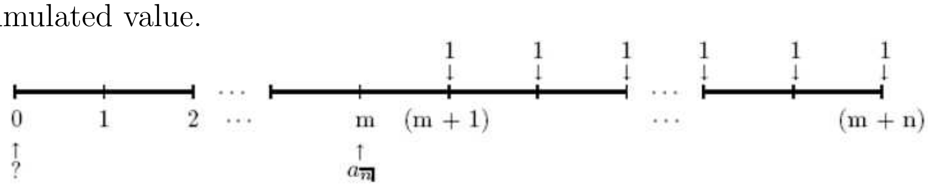


Figure 5.2

The accumulated value of an $n-$period annuity−immediate m periods after the last payment date is the accumulated value at time n accumulated for m

time periods, that is, $(1+i)^m s_{\overline{n}|}$. Notice that

$$\begin{aligned} s_{\overline{m+n}|} - s_{\overline{m}|} =& \frac{(1+i)^{m+n}-1}{i} - \frac{(1+i)^m - 1}{i} \\ =& \frac{(1+i)^{m+n} - (1+i)^m}{i} = (1+i)^m \frac{(1+i)^n - 1}{i} = (1+i)^m s_{\overline{n}|} \end{aligned}$$

Example 5.3
For four years, an annuity pays \$200 at the end of each year with an effective 8% rate of interest. Find the accumulated value of the annuity 3 years after the last payment.

Solution.
The answer is $200(1+0.08)^3 s_{\overline{4}|} = 200(s_{\overline{7}|} - s_{\overline{3}|}) = 200(8.9228 - 3.2464) =$ \$1135.28 ■

It is also possible to work with annuities−due instead of annuities−immediate. The reader should verify that

$$(1+i)^m \ddot{s}_{\overline{n}|} = \ddot{s}_{\overline{m+n}|} - \ddot{s}_{\overline{m}|}.$$

Example 5.4
A monthly annuity−due pays 100 per month for 12 months. Calculate the accumulated value 24 months after the first payment using a nominal rate of 4% compounded monthly.

Solution.
The answer is $100\left(1+\frac{0.04}{12}\right)^{12} \ddot{s}_{\overline{12}|\frac{0.04}{12}} = 1,276.28$ ■

(3) **Current value between the first and last payment date**
Next, we consider the question of finding the present value of an $n-$period annuity−immediate after the payment at the end of mth period where $1 \leq m \leq n$. Figure 5.3 shows the time diagram for this case.

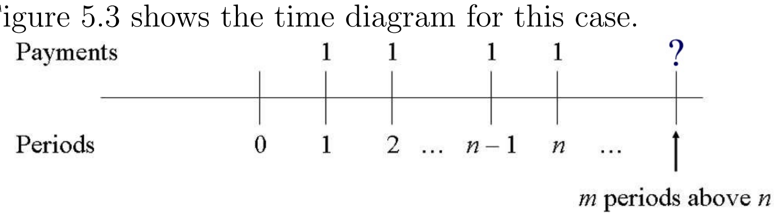


Figure 5.3

The current value of an $n-$period annuity$-$immediate immediately upon the mth payment date is the present value at time 0 accumulated for m time periods which is equal to the accumulated value at time n discounted for $n-m$ time periods, that is,

$$(1+i)^m a_{\overline{n}|} = \nu^{n-m} s_{\overline{n}|}.$$

One has the following formula,

$$(1+i)^m a_{\overline{n}|} = \nu^{n-m} s_{\overline{n}|} = s_{\overline{m}|} + a_{\overline{n-m}|}.$$

Example 5.5
For four years, an annuity pays \$200 at the end of each half$-$year with an 8% rate of interest convertible semiannually. Find the current value of the annuity immediately upon the 5th payment (i.e., middle of year 3).

Solution.
The answer is $200(1.04)^5 a_{\overline{8}|0.04} = 200(s_{\overline{5}|0.04} + a_{\overline{3}|0.04}) = 200(5.4163 + 2.7751) =$ \$1,638.28 ■

For annuity$-$due we have a similar formula for the current value

$$(1+i)^m \ddot{a}_{\overline{n}|} = \nu^{n-m} \ddot{s}_{\overline{n}|} = \ddot{s}_{\overline{m}|} + \ddot{a}_{\overline{n-m}|}.$$

Example 5.6
Calculate the current value at the end of 2 years of a 10 year annuity due of \$100 per year using a discount rate of 6%.

Solution.
We have $(1+i)^{-1} = 1-d = 1-0.06 = 0.94$ and $i = \frac{0.06}{0.94}$. Thus, $100(.94)^{-2}\ddot{a}_{\overline{10}|} =$ \$870.27 ■

Up to this point, we have assumed that the date is an integral number of periods. In the case the date is not an integral number of periods from each payment date, the value of an annuity is found by finding the value on a date which is an integral number of periods from each payment date and then the value on this date is either accumulated or discounted for the fractional period to the actual evaluation date. We illustrate this situation in the next example.

Example 5.7
An annuity−immediate pays \$1000 every six months for three years. Calculate the present value of this annuity two months before the first payment using a nominal interest rate of 12% compounded semiannually.

Solution.
The present value at time $t = 0$ is

$$1000a_{\overline{6}|0.06} = 1000\frac{1-(1.06)^{-6}}{0.06} = \$4917.32.$$

Let j be the interest rate per 2-month. Then $1+j = (1+0.06)^{\frac{1}{3}}$. The present value two months before the first payment is made is

$$4917.32(1.06)^{\frac{2}{3}} = \$5112.10 \blacksquare$$

A Brief Review of Probability Theory

One aspect of insurance is that money is paid by the company only if some event, which may be considered random, occurs within a specific time frame. For example, an automobile insurance policy will experience a claim only if there is an accident involving the insured auto. In this chapter a brief outline of the essential material from the theory of probability is given. Almost all of the material presented here should be familiar to the reader. A more thorough discussion can be found in [2]. Probability concepts that are not usually covered in an introductory probability course will be introduced and discussed in futher details whenever needed.

6 Basic Definitions of Probability

In probability, we consider **experiments** whose results cannot be predicted with certainty. Examples of such experiments include rolling a die, flipping a coin, and choosing a card from a deck of playing cards.
By an **outcome** or **simple event** we mean any result of the experiment. For example, the experiment of rolling a die yields six outcomes, namely, the outcomes 1,2,3,4,5, and 6.
The **sample space** Ω of an experiment is the set of all possible outcomes for the experiment. For example, if you roll a die one time then the experiment is the roll of the die. A sample space for this experiment could be $\Omega = \{1, 2, 3, 4, 5, 6\}$ where each digit represents a face of the die.
An **event** is a subset of the sample space. For example, the event of rolling an odd number with a die consists of three simple events $\{1, 3, 5\}$.

Example 6.1
Consider the random experiment of tossing a coin three times.
(a) Find the sample space of this experiment.
(b) Find the outcomes of the event of obtaining more than one head.

Solution.
We will use T for tail and H for head.
(a) The sample space is composed of eight simple events:

$$\Omega = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}.$$

(b) The event of obtaining more than one head is the set

$$\{THH, HTH, HHT, HHH\}\blacksquare$$

The **complement** of an event E, denoted by E^c, is the set of all possible outcomes not in E. The **union** of two events A and B is the event $A \cup B$ whose outcomes are either in A or in B. The **intersection** of two events A and B is the event $A \cap B$ whose outcomes are outcomes of both events A and B. Two events A and B are said to be **mutually exclusive** if they have no outcomes in common. Clearly, for any event E, the events E and E^c are mutually exclusive.

Example 6.2
Consider the sample space of rolling a die. Let A be the event of rolling an even number, B the event of rolling an odd number, and C the event of rolling a 2. Find
(a) A^c, B^c and C^c.
(b) $A \cup B, A \cup C$, and $B \cup C$.
(c) $A \cap B, A \cap C$, and $B \cap C$.
(d) Which events are mutually exclusive?

Solution.
(a) We have $A^c = B,\ B^c = A$ and $C^c = \{1, 3, 4, 5, 6\}$.
(b) We have

$$\begin{aligned}
A \cup B &= \{1, 2, 3, 4, 5, 6\} \\
A \cup C &= \{2, 4, 6\} \\
B \cup C &= \{1, 2, 3, 5\}
\end{aligned}$$

(c)

$$A \cap B = \emptyset$$
$$A \cap C = \{2\}$$
$$B \cap C = \emptyset$$

(d) A and B are mutually exclusive as well as B and C ■

Remark 6.1
The above definitions of intersection, union, and mutually exclusive can be extended to any number of events.

Probability Axioms
Probability is the measure of occurrence of an event. It is a function $\Pr(\cdot)$ defined on the collection of all (subsets) events of a sample space Ω and which satisfies **Kolmogorov axioms**:
Axiom 1: For any event $E \subset \Omega$, $0 \le \Pr(E) \le 1$.
Axiom 2: $\Pr(\Omega) = 1$.
Axiom 3: For any sequence of mutually exclusive events $\{E_n\}_{n\ge1}$, that is $E_i \cap E_j = \emptyset$ for $i \neq j$, we have

$$\Pr\left(\cup_{n=1}^{\infty} E_n\right) = \textstyle\sum_{n=1}^{\infty} \Pr(E_n). \text{ (Countable Additivity)}$$

If we let $E_1 = \Omega$, $E_n = \emptyset$ for $n > 1$ then by Axioms 2 and 3 we have $1 = \Pr(\Omega) = \Pr\left(\cup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \Pr(E_n) = \Pr(\Omega) + \sum_{n=2}^{\infty} \Pr(\emptyset)$. This implies that $\Pr(\emptyset) = 0$. Also, if $\{E_1, E_2, \cdots, E_n\}$ is a finite set of mutually exclusive events, then by defining $E_k = \emptyset$ for $k > n$ and Axioms 3 we find

$$\Pr\left(\cup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \Pr(E_k).$$

Any function Pr that satisfies Axioms 1-3 will be called a **probability measure**.

Example 6.3
Consider the sample space $\Omega = \{1, 2, 3\}$. Suppose that $\Pr(\{1, 2\}) = 0.5$ and $\Pr(\{2, 3\}) = 0.7$. Find $\Pr(1), \Pr(2)$, and $\Pr(3)$. Is Pr a valid probability measure?

Solution.
For Pr to be a probability measure we must have $\Pr(1)+\Pr(2)+\Pr(3)=1$. But $\Pr(\{1,2\}) = \Pr(1)+\Pr(2) = 0.5$. This implies that $0.5+\Pr(3) = 1$ or $\Pr(3) = 0.5$. Similarly, $1 = \Pr(\{2,3\}) + \Pr(1) = 0.7 + \Pr(1)$ and so $\Pr(1) = 0.3$. It follows that $\Pr(2) = 1 - \Pr(1) - \Pr(3) = 1 - 0.3 - 0.5 = 0.2$. It can be easily seen that Pr satisfies Axioms 1-3 and so Pr is a probability measure ■

Probability Trees
For all multistage experiments, the probability of the outcome along any path of a tree diagram is equal to the product of all the probabilities along the path.

Example 6.4
In a state assembly, 35% of the legislators are Democrats, and the other 65% are Republicans. 70% of the Democrats favor raising sales tax, while only 40% of the Republicans favor the increase.
If a legislator is selected at random from this group, what is the probability that he or she favors raising sales tax?

Solution.
Figure 6.1 shows a tree diagram for this problem.

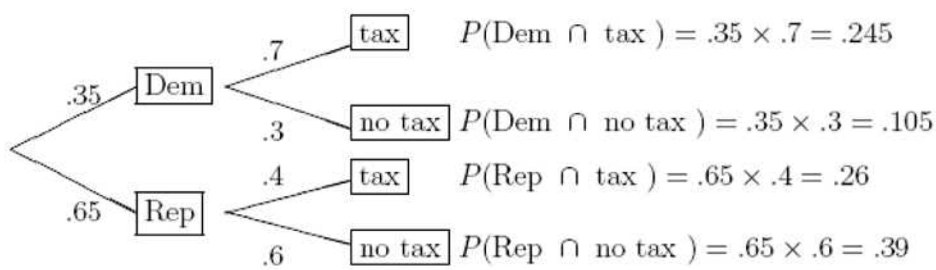


Figure 6.1

The first and third branches correspond to favoring the tax. We add their probabilities.

$$P(\text{tax}) = 0.245 + 0.26 = 0.505 \ ■$$

Conditional Probability and Bayes Theorem
We desire to know the probability of an event A conditional on the knowledge that another event B has occurred. The information the event B has occurred

causes us to update the probabilities of other events in the sample space.
To illustrate, suppose you cast two dice; one red, and one green. Then the probability of getting two ones is 1/36. However, if, after casting the dice, you ascertain that the green die shows a one (but know nothing about the red die), then there is a 1/6 chance that both of them will be one. In other words, the probability of getting two ones changes if you have partial information, and we refer to this (altered) probability as **conditional probability**.
If the occurrence of the event A depends on the occurrence of B then the conditional probability will be denoted by $P(A|B)$, read as the *probability of A given B*. Conditioning restricts the sample space to those outcomes which are in the set being conditioned on (in this case B). In this case,

$$P(A|B) = \frac{\text{number of outcomes corresponding to event A and B}}{\text{number of outcomes of B}}.$$

Thus,

$$P(A|B) = \frac{n(A \cap B)}{n(B)} = \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(B)}{n(S)}} = \frac{P(A \cap B)}{P(B)}$$

provided that $P(B) > 0$.

Example 6.5
Let A denote the event "student is female" and let B denote the event "student is French". In a class of 100 students suppose 60 are French, and suppose that 10 of the French students are females. Find the probability that if I pick a French student, it will be a female, that is, find $P(A|B)$.

Solution.
Since 10 out of 100 students are both French and female, $P(A \cap B) = \frac{10}{100} = 0.1$. Also, 60 out of the 100 students are French, so $P(B) = \frac{60}{100} = 0.6$. Hence, $P(A|B) = \frac{0.1}{0.6} = \frac{1}{6}$ ■

It is often the case that we know the probabilities of certain events conditional on other events, but what we would like to know is the "reverse". That is, given $P(A|B)$ we would like to find $P(B|A)$.
Bayes' formula is a simple mathematical formula used for calculating $P(B|A)$ given $P(A|B)$. We derive this formula as follows. Let A and B be two events. Then

$$A = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c).$$

Since the events $A \cap B$ and $A \cap B^c$ are mutually exclusive, we can write

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^c) \\ &= P(A|B)P(B) + P(A|B^c)P(B^c) \end{aligned} \tag{6.1}$$

Example 6.6
The completion of a construction job may be delayed because of a strike. The probabilities are 0.60 that there will be a strike, 0.85 that the construction job will be completed on time if there is no strike, and 0.35 that the construction will be completed on time if there is a strike. What is the probability that the construction job will be completed on time?

Solution.
Let A be the event that the construction job will be completed on time and B is the event that there will be a strike. We are given $P(B) = 0.60,\ P(A|B^c) = 0.85$, and $P(A|B) = 0.35$. From Equation (6.1) we find

$$P(A) = P(B)P(A|B) + P(B^c)P(A|B^c) = (0.60)(0.35) + (0.4)(0.85) = 0.55 \ \blacksquare$$

From Equation (6.1) we can get Bayes' formula:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}. \tag{6.2}$$

Example 6.7
A company has two machines A and B for making shoes. It has been observed that machine A produces 10% of the total production of shoes while machine B produces 90% of the total production of shoes. Suppose that 1% of all the shoes produced by A are defective while 5% of all the shoes produced by B are defective. What is the probability that a shoe taken at random from a day's production was made by the machine A, given that it is defective?

Solution.
We are given $P(A) = 0.1,\ P(B) = 0.9,\ P(D|A) = 0.01$, and $P(D|B) = 0.05$. We want to find $P(A|D)$. Using Bayes' formula we find

$$\begin{aligned} P(A|D) =& \frac{P(A \cap D)}{P(D)} = \frac{P(D|A)P(A)}{P(D|A)P(A) + P(D|B)P(B)} \\ =& \frac{(0.01)(0.1)}{(0.01)(0.1) + (0.05)(0.9)} \approx 0.0217 \ \blacksquare \end{aligned}$$

7 Classification of Random Variables

By definition, a **random variable** X is a function with domain the sample space of an experiment and range a subset of the real numbers. The range of X is sometimes called the **support** of X. The notation $X(s) = x$ means that x is the value associated with the outcome s by the random variable X. We consider three types of random variables: Discrete, continuous, and mixed random variables.
A random variable is said to be **discrete** if the range of the variable as a function is either finite or a subset of the set of non-negative integers. A discrete random variable can be either finite or infinite. An example of a finite discrete random variable is the variable X that represents the age of students in your class. An example of an infinite random variable is the variable X that represents the number of times you roll a die until you get a 6.(If you are extremely unlucky, it might take you a million rolls before you get a 6!).
A random variable X that takes an uncountable number of values is said to be **continuous**. An example of a continuous random variable is the random variable X that measures the depth of a randomly selected location in a specific lake. The range of X is the interval $[0, M]$ where M is the maximum depth of the lake.
A **mixed** random variable is partially discrete and partially continuous. An example of a mixed random variable is the random variable that represents the weight of a tumor when it is possible that there are no tumors (and consequently the weight would be zero).
We use upper-case letters X, Y, Z, etc. to represent random variables. We use small letters x, y, z, etc to represent possible values that the corresponding random variables X, Y, Z, etc. can take. The statement $X = x$ defines an event consisting of all outcomes with $X-$measurement equal to x which is the set $\{s \in \Omega : X(s) = x\}$.

Example 7.1
State whether the random variables are discrete, continuous, or mixed.
(a) A coin is tossed ten times. The random variable X is the number of tails that are noted.
(b) A light bulb is burned until it burns out. The random variable Y is its lifetime in hours.
(c) Z is the income of an individual.

Solution.
(a) X can only take the values 0, 1, ..., 10, so X is a discrete random variable.
(b) Y can take any positive real value, so Y is a continuous random variable.
(c) $Z(s) > 0$ for a working individual and $Z(s) = 0$ for a out-of-work individual so Z is a mixed random variable ■

Examples of actuarial related random variables that will be encountered in the text are:
• the age-at-death from birth
• the time-until-death from insurance policy issue.
• the number of times an insured automobile makes a claim in a one-year period.
• the amount of the claim of an insured automobile, given a claim is made (or an accident occurred).
• the value of a specific asset of a company at some future date.
• the total amount of claims in an insurance portfolio.

8 Discrete Random Variables

Because the value of a random variable is determined by the outcomes of the experiment, we may assign probabilities to the possible values of the random variable. The set of all probability values constitutes the **distribution** of the random variable.

Probability Mass Function (PMF)

For a discrete random variable X, the distribution of X is described by the **probability distribution** or the **probability mass function** given by the equation

$$p(x) = \Pr(X = x).$$

That is, a probability mass function (pmf) gives the probability that a discrete random variable is exactly equal to some value. The pmf can be an equation, a table, or a graph that shows how probability is assigned to possible values of the random variable.

Example 8.1

Suppose a variable X can take the values 1, 2, 3, or 4. The probabilities associated with each outcome are described by the following table:

x	1	2	3	4
$p(x)$	0.1	0.3	0.4	0.2

Draw the probability histogram.

Solution.

The probability histogram is shown in Figure 8.1 ■

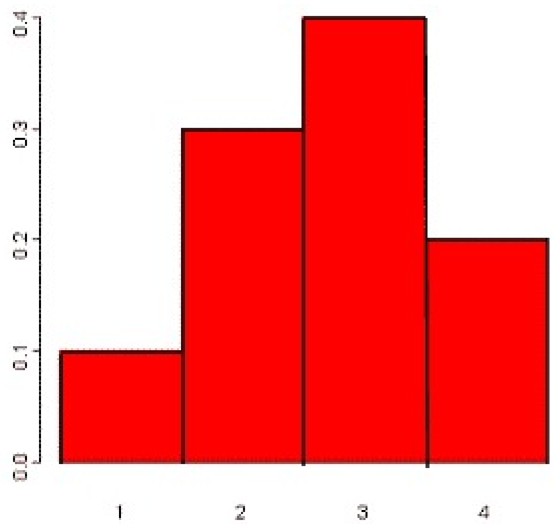


Figure 8.1

Example 8.2
A committee of 4 is to be selected from a group consisting of 5 men and 5 women. Let X be the random variable that represents the number of women in the committee. Create the probability mass distribution.

Solution.
For $x = 0, 1, 2, 3, 4$ we have

$$p(x) = \frac{\binom{5}{x}\binom{5}{4-x}}{\binom{10}{4}}.$$

The probability mass function can be described by the table

x	0	1	2	3	4
$p(x)$	$\frac{5}{210}$	$\frac{50}{210}$	$\frac{100}{210}$	$\frac{50}{210}$	$\frac{5}{210}$

■

Note that if the range of a random variable is Support $= \{x_1, x_2, \cdots\}$ then

$$\begin{aligned} p(x) &\geq 0, \quad x \in \text{Support} \\ p(x) &= 0, \quad x \notin \text{Support} \end{aligned}$$

Moreover,

$$\sum_{x \in \text{Support}} p(x) = 1.$$

Expected Value

With the previous sum and the definition of weighted average with the probabilities of Support being the weights, we define the **expected value** of X to be

$$E(X) = \sum_{x \in \text{Support}} x \cdot p(x).$$

The expected value of X is also known as the **mean** value.

Example 8.3

Suppose that an insurance company has broken down yearly automobile claims for drivers from age 16 through 21 as shown in the following table.

Amount of claim	Probability
\$ 0	0.80
\$ 2000	0.10
\$ 4000	0.05
\$ 6000	0.03
\$ 8000	0.01
\$ 10000	0.01

How much should the company charge as its average premium in order to break even on costs for claims?

Solution.

Let X be the random variable of the amount of claim. Finding the expected value of X we have

$$E(X) = 0(.80)+2000(.10)+4000(.05)+6000(.03)+8000(.01)+10000(.01) = 760.$$

Since the average claim value is \$760, the average automobile insurance premium should be set at \$760 per year for the insurance company to break even ■

Expectation of a Function of a Random Variable

If we apply a function $g(\cdot)$ to a random variable X, the result is another random variable $Y = g(X)$. For example, $X^2, \log X, \frac{1}{X}$ are all random variables derived from the original random variable X.

Example 8.4
Let X be a discrete random variable with range $\{-1, 0, 1\}$ and probabilities $\Pr(X = -1) = 0.2, \Pr(X = 0) = 0.5$, and $\Pr(X = 1) = 0.3$. Compute $E(X^2)$.

Solution.
Let $Y = X^2$. Then the range of Y is $\{0, 1\}$. Also, $\Pr(Y = 0) = \Pr(X = 0) = 0.5$ and $\Pr(Y = 1) = \Pr(X = -1) + \Pr(X = 1) = 0.2 + 0.3 = 0.5$ Thus, $E(X^2) = 0(0.5)+1(0.5) = 0.5$. Note that $E(X) = -1(0.2)+0(0.5)+1(0.3) = 0.1$ so that $E(X^2) \neq (E(X))^2$ ■

Now, if X is a discrete random variable and $g(X) = X$ then we know that

$$E(g(X)) = E(X) = \sum_{x \in \text{Support}} xp(x).$$

This suggests the following result for finding $E(g(X))$.

Theorem 8.1
If X is a discrete random variable with range D and pmf $p(x)$, then the expected value of any function $g(X)$ is computed by

$$E(g(X)) = \sum_{x \in D} g(x)p(x).$$

As a consequence of the above theorem we have the following result.

Corollary 8.1
If X is a discrete random variable, then for any constants a and b we have

$$E(aX + b) = aE(X) + b.$$

and

$$E(aX^2 + bX + c) = aE(X^2) + bE(X) + c.$$

Example 8.5
If X is the number of points rolled with a balanced die, find the expected value of $g(X) = 2X^2 + 1$.

Solution.
Since each possible outcome has the probability $\frac{1}{6}$, we get

$$\begin{aligned}E[g(X)] =&\sum_{i=1}^{6}(2i^2+1)\cdot\frac{1}{6}\\ =&\frac{1}{6}\left(6+2\sum_{i=1}^{6}i^2\right)\\ =&\frac{1}{6}\left(6+2\frac{6(6+1)(2\cdot 6+1)}{6}\right)=\frac{94}{3}\ \blacksquare\end{aligned}$$

Now, if $g(X)=X^n$ then we call $E(X^n)$ the n^{th} **moment** about the origin of X. Thus, $E(X)$ is the first moment of X. For a discrete random variable we have

$$E(X^n)=\sum_{x\in\text{Support}}x^np(x).$$

Example 8.6
Let X be a random variable with probability mass function given below

x	0	10	20	50	100
$p(x)$	0.4	0.3	0.15	0.1	0.05

Calculate the third moment of X.

Solution.
We have

$$\begin{aligned}E(X^3)=&\sum_x x^3p(x)\\ =&0^3\cdot 0.4+10^3\cdot 0.3+20^3\cdot 0.15+50^3\cdot 0.1+100^3\cdot 0.05\\ =&64000\end{aligned}$$

Variance and Standard Deviation
From the above, we learned how to find the expected values of various functions of random variables. The most important of these are the variance and the standard deviation which give an idea about how spread out the probability mass function is about its expected value.
The expected squared distance between the random variable and its mean is called the **variance** of the random variable. The positive square root of the variance is called the **standard deviation** of the random variable. If σ_X denotes the standard deviation then the variance is given by the formula

$$\text{Var(X)} = \sigma_X^2 = E\left[(X - E(X))^2\right]$$

The variance of a random variable is typically calculated using the following formula

$$\text{Var}(X) = E(X^2) - (E(X))^2.$$

Example 8.7
We toss a fair coin and let $X = 1$ if we get heads, $X = -1$ if we get tails. Find the variance of X.

Solution.
Since $E(X) = 1 \times \frac{1}{2} - 1 \times \frac{1}{2} = 0$ and $E(X^2) = 1^2\frac{1}{2} + (-1)^2 \times \frac{1}{2} = 1$ we find $\text{Var}(X) = 1 - 0 = 1$ ■

A useful identity is given in the following result

Theorem 8.2
If X is a discrete random variable then for any constants a and b we have

$$\text{Var}(aX + b) = a^2\text{Var}(X)$$

Example 8.8
This year, Toronto Maple Leafs tickets cost an average of \$80 with a variance of 105 square dollar. Toronto city council wants to charge a 3% tax on all tickets(i.e., all tickets will be 3% more expensive). If this happens, what would be the variance of the cost of Toranto Maple Leafs tickets?

Solution.
Let X be the current ticket price and Y be the new ticket price. Then $Y = 1.03X$. Hence,

$$\text{Var}(Y) = \text{Var}(1.03X) = 1.03^2\text{Var}(X) = (1.03)^2(105) = 111.3945 \blacksquare$$

Example 8.9
Roll one die and let X be the resulting number. Find the variance and standard deviation of X.

Solution.
We have

$$E(X) = (1 + 2 + 3 + 4 + 5 + 6) \cdot \frac{1}{6} = \frac{21}{6} = \frac{7}{2}$$

and

$$E(X^2) = (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) \cdot \frac{1}{6} = \frac{91}{6}.$$

Thus,

$$\text{Var}(X) = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

The standard deviation is

$$\sigma(X) = \sqrt{\frac{35}{12}} \approx 1.7078 \blacksquare$$

Moment Generating Functions (MGF)

One of the applications of Theorem 8.1 is when $g(X) = e^{tX}$. In this case, the expected value of $g(X)$ is called the **moment generating function** of X. For a discrete random variable, the moment generating function of X is given

$$M_X(t) = E[e^{tX}] = \sum_{x \in \text{Support}} e^{tx} p(x).$$

Example 8.10

Let X be a discrete random variable with pmf given by the following table

x	1	2	3	4	5
p(x)	0.15	0.20	0.40	0.15	0.10

Find $M_X(t)$.

Solution.

We have

$$M_X(t) = 0.15e^t + 0.20e^{2t} + 0.40e^{3t} + 0.15e^{4t} + 0.10e^{5t} \blacksquare$$

As the name suggests, the moment generating function can be used to generate moments $E(X^n)$ for $n = 1, 2, \cdots$. The next result shows how to use the moment generating function to calculate moments.

Theorem 8.3

$$E(X^n) = M_X^n(0)$$

where

$$M_X^n(0) = \frac{d^n}{dt^n} M_X(t) \mid_{t=0}$$

Example 8.11
Consider the random variable X with pmf given by

$$p(n) = \frac{\lambda^n}{n!}e^{-\lambda}, \quad n = 0, 1, 2, \cdots .$$

Find the second moment of X using the above theorem.

Solution.
We can write

$$\begin{aligned} M_X(t) =& E(e^{tX}) = \sum_{n=0}^{\infty} \frac{e^{tn}e^{-\lambda}\lambda^n}{n!} = e^{-\lambda}\sum_{n=0}^{\infty}\frac{e^{tn}\lambda^n}{n!} \\ =& e^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda e^t)^n}{n!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)} \end{aligned}$$

Differentiating for the first time we find

$$M_X'(t) = \lambda e^t e^{\lambda(e^t-1)}.$$

Differentiating a second time we find

$$M_X''(t) = (\lambda e^t)^2 e^{\lambda(e^t-1)} + \lambda e^t e^{\lambda(e^t-1)}.$$

Hence,

$$E(X^2) = M_X''(0) = \lambda^2 + \lambda \blacksquare$$

Cumulative Distribution Function (CDF)

All random variables (discrete and continuous) have a **distribution function** or **cumulative distribution function**, abbreviated cdf. It is a function giving the probability that the random variable X is less than or equal to x, for every value x. For a discrete random variable, the cumulative distribution function is found by summing up the probabilities. That is,

$$F(a) = \Pr(X \le a) = \sum_{x \le a} p(x).$$

Example 8.12
Given the following pmf

$$p(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases}$$

Find a formula for $F(x)$ and sketch its graph.

Solution.
A formula for $F(x)$ is given by

$$F(x) = \begin{cases} 0, & \text{if } x < a \\ 1, & \text{otherwise} \end{cases}$$

Its graph is given in Figure 8.2 ■

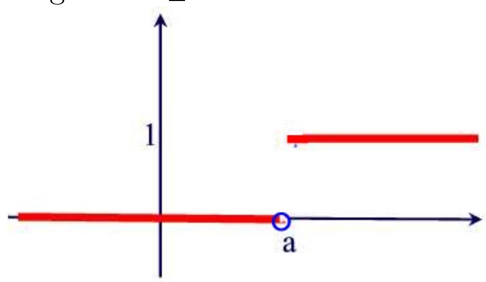


Figure 8.2

For discrete random variables the cumulative distribution function will always be a step function with jumps at each value of x that has probability greater than 0. At the jump points the function is right-continuous.

Example 8.13
Consider the following probability distribution

x	1	2	3	4
$p(x)$	0.25	0.5	0.125	0.125

Find a formula for $F(x)$ and sketch its graph.

Solution.
The cdf is given by

$$F(x) = \begin{cases} 0, & x < 1 \\ 0.25, & 1 \leq x < 2 \\ 0.75, & 2 \leq x < 3 \\ 0.875, & 3 \leq x < 4 \\ 1, & 4 \leq x. \end{cases}$$

Its graph is given in Figure 8.3 ■

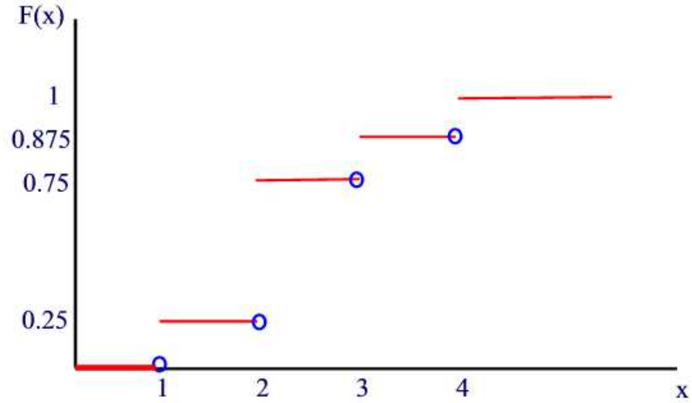


Figure 8.3

Note that the size of the step at any of the values 1,2,3,4 is equal to the probability that X assumes that particular value. That is, we have

Theorem 8.4
If the range of a discrete random variable X consists of the values $x_1 < x_2 < \cdots < x_n$ then $p(x_1) = F(x_1)$ and

$$p(x_i) = F(x_i) - F(x_{i-1}), \quad i = 2, 3, \cdots, n$$

Example 8.14
If the distribution function of X is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{16}, & 0 \le x < 1 \\ \frac{5}{16}, & 1 \le x < 2 \\ \frac{11}{16}, & 2 \le x < 3 \\ \frac{15}{16}, & 3 \le x < 4 \\ 1, & x \ge 4 \end{cases}$$

find the pmf of X.

Solution.
Making use of the previous theorem, we get $p(0) = \frac{1}{16}, p(1) = \frac{1}{4}, p(2) = \frac{3}{8}, p(3) = \frac{1}{4}$, and $p(4) = \frac{1}{16}$ and 0 otherwise ■

9 Continuous Random Variables

In this section, we consider continuous random variables.

Probability Density Function

We say that a random variable is **continuous** if there exists a nonnegative function f (not necessarily continuous) defined for all real numbers and having the property that for any set B of real numbers we have

$$\Pr(X \in B) = \int_B f(x)dx.$$

We call the function f the **probability density function** (abbreviated pdf) of the random variable X.
If we let $B = (-\infty, \infty) = \mathbb{R}$ then

$$\int_{-\infty}^{\infty} f(x)dx = \Pr[X \in (-\infty, \infty)] = 1.$$

Now, if we let $B = [a, b]$ then

$$\Pr(a \leq X \leq b) = \int_a^b f(x)dx.$$

That is, areas under the probability density function represent probabilities as illustrated in Figure 9.1.

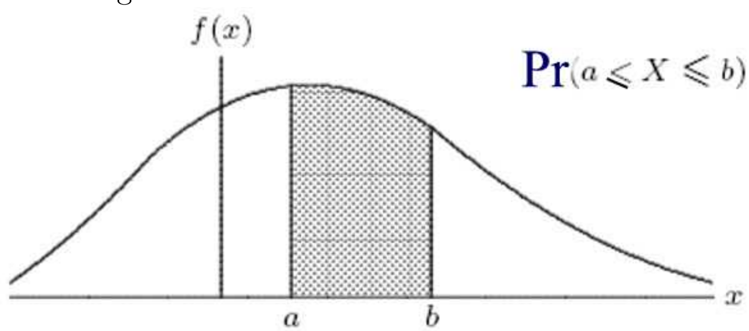


Figure 9.1

Now, if we let $a = b$ in the previous formula we find

$$\Pr(X = a) = \int_a^a f(x)dx = 0.$$

It follows from this result that

$$\Pr(a \le X < b) = \Pr(a < X \le b) = \Pr(a < X < b) = \Pr(a \le X \le b)$$

and

$$\Pr(X \le a) = \Pr(X < a) \text{ and } \Pr(X \ge a) = \Pr(X > a).$$

Cumulative Distribution Function

The **cumulative distribution function** (abbreviated cdf) $F(t)$ of the random variable X is defined as follows

$$F(t) = \Pr(X \le t)$$

i.e., $F(t)$ is equal to the probability that the variable X assumes values, which are less than or equal to t. From this definition we can write

$$F(t) = \int_{-\infty}^{t} f(y)dy.$$

Geometrically, $F(t)$ is the area under the graph of f to the left of t.

Example 9.1

Find the distribution functions corresponding to the following density functions:

$$\begin{aligned}
(a) f(x) =& \frac{1}{\pi(1+x^2)}, & -\infty < x < \infty.\\
(b) f(x) =& \frac{e^{-x}}{(1+e^{-x})^2}, & -\infty < x < \infty.\\
(c) f(x) =& \frac{a-1}{(1+x)^a}, & 0 < x < \infty.\\
(d) f(x) =& k\alpha x^{\alpha-1}e^{-kx^\alpha}, & 0 < x < \infty, k > 0, \alpha > 0.
\end{aligned}$$

Solution.

(a)

$$\begin{aligned}
F(x) =& \int_{-\infty}^{x} \frac{1}{\pi(1+y^2)}dy\\
=& \left[\frac{1}{\pi}\arctan y\right]_{-\infty}^{x}\\
=& \frac{1}{\pi}\arctan x - \frac{1}{\pi}\cdot\frac{-\pi}{2}\\
=& \frac{1}{\pi}\arctan x + \frac{1}{2}.
\end{aligned}$$

(b)

$$
\begin{aligned}
F(x) =& \int_{-\infty}^{x} \frac{e^{-y}}{(1+e^{-y})^2} dy \\
=& \left[\frac{1}{1+e^{-y}}\right]_{-\infty}^{x} \\
=& \frac{1}{1+e^{-x}}.
\end{aligned}
$$

(c) For $x \geq 0$

$$
\begin{aligned}
F(x) =& \int_{-\infty}^{x} \frac{a-1}{(1+y)^a} dy \\
=& \left[-\frac{1}{(1+y)^{a-1}}\right]_{0}^{x} \\
=& 1 - \frac{1}{(1+x)^{a-1}}.
\end{aligned}
$$

For $x < 0$ it is obvious that $F(x) = 0$, so we could write the result in full as

$$
F(x) = \begin{cases} 0, & x < 0 \\ 1 - \frac{1}{(1+x)^{a-1}}, & x \geq 0. \end{cases}
$$

(d) For $x \geq 0$

$$
\begin{aligned}
F(x) =& \int_{0}^{x} k\alpha y^{\alpha-1} e^{-ky^{\alpha}} dy \\
=& \left[-e^{-ky^{\alpha}}\right]_{0}^{x} \\
=& 1 - ke^{-kx^{\alpha}}.
\end{aligned}
$$

For $x < 0$ we have $F(x) = 0$ so that

$$
F(x) = \begin{cases} 0, & x < 0 \\ 1 - ke^{-kx^{\alpha}}, & x \geq 0 \end{cases} \blacksquare
$$

Next, we list the properties of the cumulative distribution function $F(t)$ for a continuous random variable X.

Theorem 9.1
The cumulative distribution function of a continuous random variable X satisfies the following properties:
(a) $0 \leq F(t) \leq 1$.
(b) $F'(t) = f(t)$ whenever the derivative exists.
(c) $F(t)$ is a non-decreasing function, i.e. if $a < b$ then $F(a) \leq F(b)$.
(d) $F(t) \to 0$ as $t \to -\infty$ and $F(t) \to 1$ as $t \to \infty$.
(e) $\Pr(a < X \leq b) = F(b) - F(a)$.
(f) F is right-continuous.

A pdf needs not be continuous, as the following example illustrates.

Example 9.2
(a) Determine the value of c so that the following function is a pdf.

$$f(x) = \begin{cases} \frac{15}{64} + \frac{x}{64}, & -2 \leq x \leq 0 \\ \frac{3}{8} + cx, & 0 < x \leq 3 \\ 0, & \text{otherwise.} \end{cases}$$

(b) Determine $\Pr(-1 \leq X \leq 1)$.
(c) Find $F(x)$.

Solution.
(a) Observe that f is discontinuous at the points -2 and 0, and is potentially also discontinuous at the point 3. We first find the value of c that makes f a pdf.

$$\begin{aligned} 1 &= \int_{-2}^{0} \left(\frac{15}{64} + \frac{x}{64}\right) dx + \int_{0}^{3} \left(\frac{3}{8} + cx\right) dx \\ &= \left[\frac{15}{64}x + \frac{x^2}{128}\right]_{-2}^{0} + \left[\frac{3}{8}x + \frac{cx^2}{2}\right]_{0}^{3} \\ &= \frac{30}{64} - \frac{2}{64} + \frac{9}{8} + \frac{9c}{2} \\ &= \frac{100}{64} + \frac{9c}{2} \end{aligned}$$

Solving for c we find $c = -\frac{1}{8}$.
(b) The probability $\Pr(-1 \leq X \leq 1)$ is calculated as follows.

$$\Pr(-1 \leq X \leq 1) = \int_{-1}^{0} \left(\frac{15}{64} + \frac{x}{64}\right) dx + \int_{0}^{1} \left(\frac{3}{8} - \frac{x}{8}\right) dx = \frac{69}{128}.$$

(c) For $-2 \le x \le 0$ we have

$$F(x) = \int_{-2}^{x} \left(\frac{15}{64} + \frac{t}{64}\right) dt = \frac{x^2}{128} + \frac{15}{64}x + \frac{7}{16}$$

and for $0 < x \le 3$

$$F(x) = \int_{-2}^{0} \left(\frac{15}{64} + \frac{x}{64}\right) dx + \int_{0}^{x} \left(\frac{3}{8} - \frac{t}{8}\right) dt = \frac{7}{16} + \frac{3}{8}x - \frac{1}{16}x^2.$$

Hence the full cdf is

$$F(x) = \begin{cases} 0, & x < -2 \\ \frac{x^2}{128} + \frac{15}{64}x + \frac{7}{16}, & -2 \le x \le 0 \\ \frac{7}{16} + \frac{3}{8}x - \frac{1}{16}x^2, & 0 < x \le 3 \\ 1, & x > 3. \end{cases}$$

Observe that at all points of discontinuity of the pdf, the cdf is continuous. That is, even when the pdf is discontinuous, the cdf is continuous ■

Remark 9.1
The intuitive interpretation of the p.d.f. is that for small $\epsilon > 0$ we have

$$\Pr(a \le X \le a + \epsilon) = F_X(a + \epsilon) - F_X(a) = \int_a^{a+\epsilon} f_X(x)dx \approx \epsilon f(a).$$

In particular,

$$\Pr\left(a - \frac{\epsilon}{2} \le X \le a + \frac{\epsilon}{2}\right) \approx \epsilon f(a).$$

This means that the probability that X will be contained in an interval of length ϵ around the point a is approximately $\epsilon f(a)$. Thus, $f(a)$ is a measure of how likely it is that the random variable will be near a.

Expected Value and Variance of X
We define the **expected value** of a continuous random variable X by the improper integral

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

provided that the improper integral converges.

Example 9.3
Find $E(X)$ when the density function of X is

$$f(x) = \begin{cases} 2x, & \text{if } 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Solution.
Using the formula for $E(X)$ we find

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 2x^2 dx = \frac{2}{3} \blacksquare$$

The expected value formula of X is a special case of the formula

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx.$$

Example 9.4 ‡
An insurance policy reimburses a loss up to a benefit limit of 10 . The policyholder's loss, X, follows a distribution with density function:

$$f(x) = \begin{cases} \frac{2}{x^3}, & x > 1 \\ 0, & \text{otherwise.} \end{cases}$$

What is the expected value of the benefit paid under the insurance policy?

Solution.
Let Y denote the claim payments. Then

$$Y = \begin{cases} X, & 1 < X \le 10 \\ 10, & X \ge 10. \end{cases}$$

It follows that

$$\begin{aligned} E(Y) &= \int_1^{10} x\frac{2}{x^3}dx + \int_{10}^{\infty} 10\frac{2}{x^3}dx \\ &= -\frac{2}{x}\Big|_1^{10} - \frac{10}{x^2}\Big|_{10}^{\infty} = 1.9 \blacksquare \end{aligned}$$

The variance of a continuous random variable is given by

$$\text{Var}(X) = \int_{-\infty}^{\infty} (x - E(X))^2 f(x)dx = E(X^2) - (E(X))^2.$$

Again, the positive square root of the variance is called the **standard deviation** of the random variable and is denoted by σ.

Example 9.5
Let X be a continuous random variable with pdf

$$f(x) = \begin{cases} 4xe^{-2x}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

For this example, you might find the identity $\int_0^\infty t^n e^{-t} dt = n!$ useful.
(a) Find $E(X)$.
(b) Find the variance of X.
(c) Find the probability that $X < 1$.

Solution.
(a) Using the substitution $t = 2x$ we find

$$E(X) = \int_0^\infty 4x^2 e^{-2x} dx = \frac{1}{2}\int_0^\infty t^2 e^{-t} dt = \frac{2!}{2} = 1.$$

(b) First, we find $E(X^2)$. Again, letting $t = 2x$ we find

$$E(X^2) = \int_0^\infty 4x^3 e^{-2x} dx = \frac{1}{4}\int_0^\infty t^3 e^{-t} dt = \frac{3!}{4} = \frac{3}{2}.$$

Hence,

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{3}{2} - 1 = \frac{1}{2}.$$

(c) We have

$$\Pr(X < 1) = \Pr(X \le 1) = \int_0^1 4xe^{-2x} dx = \int_0^2 te^{-t} dt = -(t+1)e^{-t}\Big|_0^2 = 1 - 3e^{-2} \blacksquare$$

Parallel to the n^{th} moment of a discrete random variable, the n^{th} moment of a continuous random variable is given by

$$E(X^n) = \int_{-\infty}^\infty x^n f(x) dx.$$

The moment generating function for a continuous random variable X is given by

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^\infty e^{tx} f(x) dx.$$

Example 9.6
Let X be the random variable on the interval $[a, b]$ with pdf $f(x) = \frac{1}{b-a}$ for $x > 0$ and 0 otherwise. Find $M_X(t)$.

Solution.
We have

$$M_X(t) = \int_a^b \frac{e^{tx}}{b-a} dx = \frac{1}{t(b-a)}[e^{tb} - e^{ta}] \blacksquare$$

10 Raw and Central Moments

Several quantities can be computed from the pdf that describe simple characteristics of the distribution. These are called **moments.** The most common is the mean, the first moment about the origin, and the variance, the second moment about the mean. The mean is a measure of the centrality of the distribution and the variance is a measure of the spread of the distribution about the mean.

The n^{th} moment $E(X^n)$ of a random variable X is also known as the n^{th} **moment about the origin** or the n^{th} **raw moment.**

By contrast, the quantity $\mu_n = E[(X - E(X))^n]$ is called the n^{th} **central moment** of X or the n^{th} **moment about the mean.** For a continuous random variable X we have

$$\mu_n = \int_{-\infty}^{\infty} (x - E(X))^n f(x)dx$$

and for a discrete random variable we have

$$\mu_n = \sum_x (x - E(X))^n p(x).$$

Note that $\text{Var}(X)$ is the second central moment of X.

Example 10.1
Let X be a continuous random variable with pdf given by $f(x) = \frac{3}{8}x^2$ for $0 \leq x \leq 2$ and 0 otherwise. Find the second central moment of X.

Solution.
We first find the mean of X. We have

$$E(X) = \int_0^2 xf(x)dx = \int_0^2 \frac{3}{8}x^3dx = \left.\frac{3}{32}x^4\right|_0^2 = 1.5.$$

The second central moment is

$$\begin{aligned}
\mu_2 =& \int_0^2 (x - 1.5)^2 f(x)dx \\
=& \int_0^2 \frac{3}{8}x^2(x - 1.5)^2dx \\
=& \frac{3}{8}\left[\frac{x^5}{5} - 0.75x^4 + 0.75x^3\right]_0^2 = 0.15 \blacksquare
\end{aligned}$$

Departure from Normality: Coefficient of Skewness

The third central moment, μ_3, is called the **skewness** and is a measure of the symmetry of the pdf. A distribution, or data set, is symmetric if it looks the same to the left and right of the mean.
A measure of skewness is given by the **coefficient of skewness** γ_1 :

$$\gamma_1 = \frac{\mu_3}{\sigma^3}.$$

That is, γ_1 is the ratio of the third central moment to the cube of the standard deviation. Equivalently, γ_1 is the third central moment of the standardized variable

$$X^* = \frac{X-\mu}{\sigma}.$$

If γ_1 is close to zero then the distribution is symmetric about its mean such as the normal distribution. A positively skewed distribution has a "tail" which is pulled in the positive direction. A negatively skewed distribution has a "tail" which is pulled in the negative direction (See Figure 10.1).

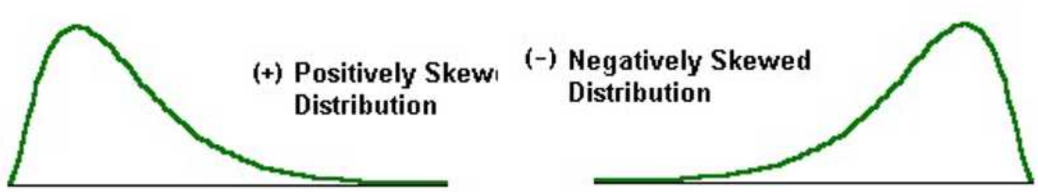


Figure 10.1

Example 10.2

A random variable X has the following pmf:

x	120	122	124	150	167	245
$p(x)$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{3}$

Find the coefficient of skewness of X.

Solution.

We first find the mean of X :

$$\mu = E(X) = 120\times\frac{1}{4}+122\times\frac{1}{12}+124\times\frac{1}{6}+150\times\frac{1}{12}+167\times\frac{1}{12}+245\times\frac{1}{3} = \frac{2027}{12}.$$

The second raw moment is

$$E(X^2) = 120^2 \times \frac{1}{4} + 122^2 \times \frac{1}{12} + 124^2 \times \frac{1}{6} + 150^2 \times \frac{1}{12} + 167^2 \times \frac{1}{12} + 245^2 \times \frac{1}{3} = \frac{379325}{12}.$$

Thus, the variance of X is

$$\text{Var}(X) = \frac{379325}{12} - \frac{4108729}{144} = \frac{443171}{144}$$

and the standard deviation is

$$\sigma = \sqrt{\frac{443171}{144}} = 55.475908183.$$

The third central moment is

$$\begin{aligned}\mu_3 =& \left(120 - \frac{2027}{12}\right)^3 \times \frac{1}{4} + \left(122 - \frac{2027}{12}\right)^3 \times \frac{1}{12} + \left(124 - \frac{2027}{12}\right)^3 \times \frac{1}{6} \\ &+ \left(150 - \frac{2027}{12}\right)^3 \times \frac{1}{12} + \left(167 - \frac{2027}{12}\right)^3 \times \frac{1}{12} + \left(245 - \frac{2027}{12}\right)^3 \times \frac{1}{3} \\ =& 93270.81134.\end{aligned}$$

Thus,

$$\gamma_1 = \frac{93270.81134}{55.475908183} = 0.5463016252 \blacksquare$$

Coefficient of Kurtosis

The fourth central moment, μ_4, is called the **kurtosis** and is a measure of peakedness/flatness of a distribution with respect to the normal distribution. A measure of kurtosis is given by the **coefficient of kurtosis:**

$$\gamma_2 = \frac{E[(X - \mu))^4]}{\sigma^4} - 3.$$

The coefficient of kurtosis of the normal distribution is 0. A negative value of γ_2 indicates that the distribution is flatter compared to the normal distribution, and a positive value indicates a higher peak (relative to the normal distribution) around the mean value.(See Figure 10.2)

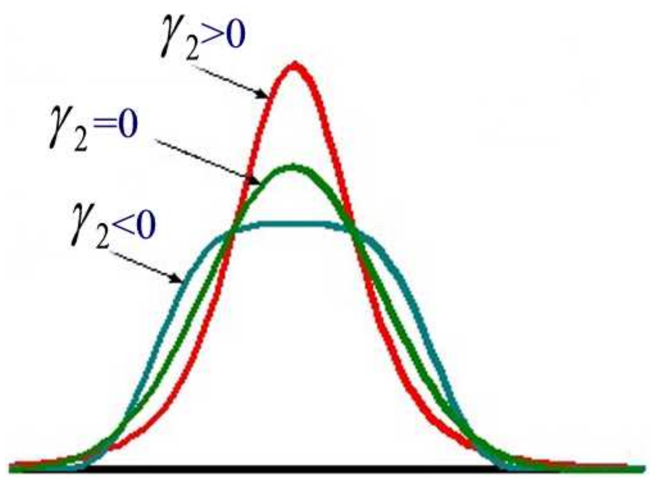


Figure 10.2

Example 10.3
A random variable X has the following pmf:

x	120	122	124	150	167	245
$p(x)$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{3}$

Find the coefficient of kurtosis of X.

Solution.
We first find the fourth central moment.

$$\begin{aligned}\mu_4 =&\left(120-\frac{2027}{12}\right)^4\times\frac{1}{4}+\left(122-\frac{2027}{12}\right)^4\times\frac{1}{12}+\left(124-\frac{2027}{12}\right)^4\times\frac{1}{6}\\&+\left(150-\frac{2027}{12}\right)^4\times\frac{1}{12}+\left(167-\frac{2027}{12}\right)^4\times\frac{1}{12}+\left(245-\frac{2027}{12}\right)^4\times\frac{1}{3}\\&=13693826.62.\end{aligned}$$

Thus,

$$\gamma_2=\frac{13693826.62}{55.475908183}-3=1.44579641-3=-1.55420359\ \blacksquare$$

<u>Coefficient of Variation</u>
Some combinations of the raw moments and central moments that are also

commonly used. One such combination is the **coefficient of variation** ($CV(X)$) of a random variable X which is defined as the ratio of the standard deviation to the mean:

$$CV(X) = \frac{\sigma}{\mu}, \quad \mu = E(X).$$

It is an indication of the size of the standard deviation relative to the mean, for the given random variable.
Often the coefficient of variation is expressed as a percentage. Thus, it expresses the standard deviation as a percentage of the sample mean and it is unitless. Statistically, the coefficient of variation is very useful when comparing two or more sets of data that are measured in different units of measurement.

Example 10.4
Let X be a random variable with mean of 4 meters and standard deviation of 0.7 millimeters. Find the coefficient of variation of X.

Solution.
The coefficient of variation is

$$CV(X) = \frac{0.7}{4000} = 0.0175\% \blacksquare$$

Example 10.5
A random variable X has the following pmf:

x	120	122	124	150	167	245
$p(x)$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{3}$

Find the coefficient of variation of X.

Solution.
We know that $\mu = \frac{2027}{12} = 168.9166667$ and $\sigma = 55.47590818$. Thus, the coefficient of variation of X is

$$CV(X) = \frac{55.47590818}{168.9166667} = 0.3284217754 \blacksquare$$

11 Median, Mode, Percentiles, and Quantiles

In addition to the information provided by the moments of a distribution, some other metrics such as the median, the mode, the percentile, and the quantile provide useful information.

Median of a Random Variable

The **median** of a discrete random variable X is the number M such that $\Pr(X \leq M) \geq 0.50$ and $\Pr(X \geq M) \geq 0.50$.

Example 11.1

Let the random variable X represent the number of telephone lines in use by the technical support center of a software manufacturer at a particular time of day. Suppose that the probability mass function (pmf) of X is given by:

x	0	1	2	3	4	5
$p(x)$	0.35	0.20	0.15	0.15	0.10	0.05

Find the median of X.

Solution.

Since $\Pr(X \leq 1) = 0.55$ and $\Pr(X \geq 1) = 0.65$, 1 is the median of X ■

In the case of a continuous random variable X, the median is the number M such that $\Pr(X \leq M) = \Pr(X \geq M) = 0.5$. Generally, M is found by solving the equation $F(M) = 0.5$ where F is the cdf of X.

Example 11.2

Let X be a continuous random variable with pdf $f(x) = \frac{1}{b-a}$ for $a < x < b$ and 0 otherwise. Find the median of X.

Solution.

We must find a number M such that $\int_a^M \frac{dx}{b-a} = 0.5$. This leads to the equation $\frac{M-a}{b-a} = 0.5$. Solving this equation we find $M = \frac{a+b}{2}$ ■

Remark 11.1

A discrete random variable might have many medians. For example, let X be the discrete random variable with pmf given by $p(x) = \left(\frac{1}{2}\right)^x, \ x = 1, 2, \cdots$. Then any number $1 < M < 2$ satisfies $\Pr(X \leq M) = \Pr(X \geq M) = 0.5$.

Mode of a Random Variable

The mode is defined as the value that maximizes the probability mass function $p(x)$ (discrete case) or the probability density function $f(x)$ (continuous case.)

Example 11.3

Let X be the discrete random variable with pmf given by $p(x) = \left(\frac{1}{2}\right)^x, \ x = 1, 2, \cdots$ and 0 otherwise. Find the mode of X.

Solution.

The value of x that maximizes $p(x)$ is $x = 1$. Thus, the mode of X is 1 ■

Example 11.4

Let X be the continuous random variable with pdf given by $f(x) = 0.75(1 - x^2)$ for $-1 \leq x \leq 1$ and 0 otherwise. Find the mode of X.

Solution.

The pdf is maximum for $x = 0$. Thus, the mode of X is 0 ■

Percentiles and Quantiles

For a random variable X and $0 < p < 1$, the 100p^{th} **percentile** (or the p^{th} **quantile**) is the number x such

$$\Pr(X < x) \leq p \leq \Pr(X \leq x).$$

For a continuous random variable, this is the solution to the equation $F(x) = p$. For example, the 90^{th} percentile separates the top 10% from the bottom 90%.

Example 11.5

An insurer's annual weather-related loss, X, is a random variable with density function

$$f(x) = \begin{cases} \frac{2.5(200)^{2.5}}{x^{3.5}} & x > 200 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the difference between the 25th and 75th percentiles of X.

Solution.

First, the cdf is given by

$$F(x) = \int_{200}^{x} \frac{2.5(200)^{2.5}}{t^{3.5}} dt$$

If Q_1 is the 25th percentile then it satisfies the equation

$$F(Q_1) = \frac{1}{4}$$

or equivalently

$$1 - F(Q_1) = \frac{3}{4}$$

This leads to

$$\frac{3}{4} = \int_{Q_1}^{\infty} \frac{2.5(200)^{2.5}}{t^{3.5}} dt = -\left(\frac{200}{t}\right)^{2.5}\Big|_{Q_1}^{\infty} = \left(\frac{200}{Q_1}\right)^{2.5}.$$

Solving for Q_1 we find $Q_1 = 200(4/3)^{0.4} \approx 224.4$. Similarly, the third quartile (i.e. 75th percentile) is given by $Q_3 = 348.2$, The interquartile range (i.e., the difference between the 25th and 75th percentiles) is $Q_3 - Q_1 = 348.2 - 224.4 = 123.8$ ■

Example 11.6
Let X be the random variable with pdf $f(x) = \frac{1}{b-a}$ for $a < x < b$ and 0 otherwise. Find the p^{th} quantile of X.

Solution.
We have

$$p = \Pr(X \leq x) = \int_a^x \frac{dt}{b-a} = \frac{x-a}{b-a}.$$

Solving this equation for x, we find $x = a + (b-a)p$ ■

12 Mixed Distributions

Thus far we have only considered distributions that are either discrete or continuous, the types encountered in most applications. However, on some occasions, combinations of the two types of random variables are found; that is, positive probability is assigned to each of certain points and also is spread over one or more intervals of outcomes, each point of which has zero probability.

Example 12.1
The distribution function for a random variable with a mixed distribution is given by

$$F(x) = \begin{cases} 0, & \text{for } x < 0 \\ 1 - 0.75e^{-x}, & \text{for } 0 \le x < 1 \\ 1, & \text{for } x \ge 1. \end{cases}$$

(a) Sketch the graph of $F(x)$.
(b) What is $\Pr(0 \le x < 1)$?

Solution.
(a) For a point x at which a random variable X has positive probability,

$$F(x) - F(x^-) = \Pr(X \le x) - \Pr(X < x) = \Pr(X = x) > 0$$

where

$$F(x^-) = \lim_{t \to x^-} F(t).$$

So to sketch the graph of $F(x)$, let's look at

$$\begin{aligned} F(0^-) =& \lim_{x \to 0^-} F(x) = 0 \\ F(0) =& 1 - 0.75 = 0.25 \\ F(1^-) =& \lim_{x \to 1^-} F(x) = 1 - 0.75e^{-1} \approx 0.72 \\ F(1) =& 1 \end{aligned}$$

The graph is shown in Figure 12.1.

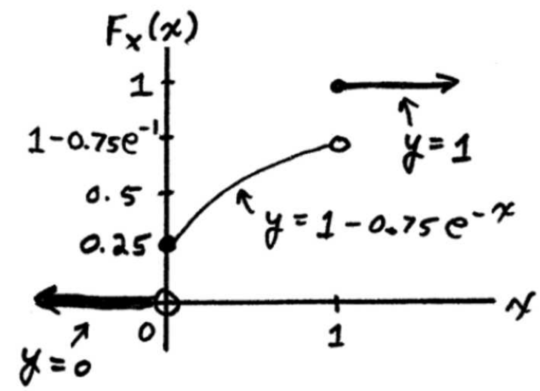


Figure 12.1

Note that for this random variable, X, positive probability is assigned to the points $x = 0$ and $x = 1$ [its discrete part] and to the interval $(0, 1)$ [its continuous part].

(b) We have

$$\Pr(0 \leq x < 1) = F(1^-) - F(0) = 1 - 0.75e^{-1} - 0.25 \approx 0.47 \blacksquare$$

Suppose that X has a mixed distribution. Specifically, assume that its range S is the union of disjoint sets D and C such that the conditional distribution of X given X in D is discrete with density $p_D(x)$ and the conditional distribution of X given X in C is continuous with density $f_C(x)$. Let $p = \Pr(X \text{ } in \text{ } D)$. The expected value of X is $E(X) = p \sum_{x \in D} x p_x(x) + (1-p) \int_C x f_C(x) dx = E[X|X \text{ } in \text{ } D]\Pr(X \text{ } in \text{ } D) + E[X|X \text{ } in \text{ } C]\Pr(X \text{ } in \text{ } C)$.

Example 12.2

The distribution function for a random variable with a mixed distribution is given by

$$F(x) = \begin{cases} 0, & \text{for } x < 0 \\ \frac{x^2}{4}, & \text{for } 0 \leq x < 1 \\ \frac{1}{2}, & \text{for } 1 \leq x < 2 \\ \frac{x}{3}, & \text{for } 2 \leq x < 3 \\ 1, & \text{for } x \geq 3. \end{cases}$$

(a) Sketch the graph of $F(x)$.
(b) Find the expected value of X.
(c) Find the variance of X.

Solution.
(a) The graph is shown in Figure 12.2.

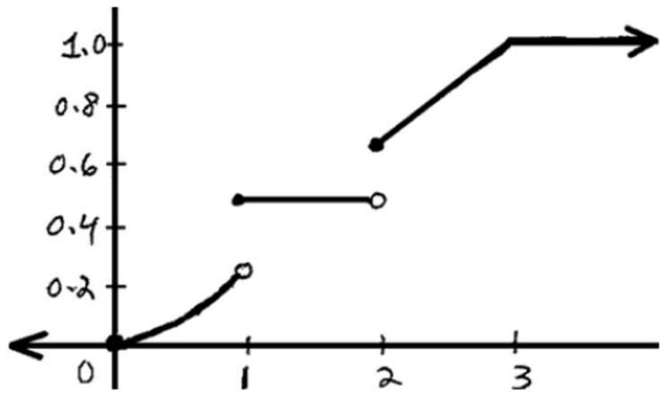


Figure 12.2

Note that for this random variable, X, positive probability is assigned to the points $x = 1$ and $x = 2$ [its discrete part] and to the intervals $(0, 1)$ and $(2, 3)$ [its continuous part].
(b) Basically we need to find the probability mass function of $X, p(x) = \Pr(X = x)$, where X is discrete, and the probability density function of $X, f(x) = F'(x)$, where X is continuous; multiply each by x; and then sum where X is discrete and integrate where X is continuous. Thus,

$$\begin{aligned}E(X) =& \int_0^1 x \cdot \frac{x}{2}dx + 1 \cdot p(1) + 2 \cdot p(2) + \int_2^3 x \cdot \frac{1}{3}dx \\ =& \left.\frac{x^3}{6}\right|_0^1 + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{6} + \left.\frac{x^2}{6}\right|_2^3 = \frac{19}{12}.\end{aligned}$$

(c) The second moment of X is

$$\begin{aligned}E(X^2) =& \int_0^1 x^2 \cdot \frac{x}{2}dx + 1 \cdot p(1) + 4 \cdot p(2) + \int_2^3 x^2 \cdot \frac{1}{3}dx \\ =& \left.\frac{x^4}{8}\right|_0^1 + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{6} + \left.\frac{x^3}{9}\right|_2^3 = \frac{227}{72}.\end{aligned}$$

Thus,

$$\mathrm{Var}(X) = \frac{227}{72} - \left(\frac{19}{12}\right)^2 = \frac{31}{48} \blacksquare$$

13 A List of Commonly Encountered Discrete R.V

Next, we review some important discrete random variables that the reader should be familiar with.

13.1 Discrete Uniform Distribution

Let X be a discrete random variable with support $\{x_1, x_2, \cdots, x_n\}$ such that each x_i has the same probability. Such a random variable is called a **discrete uniform distribution**. Its pmf is defined by

$$p(x_i) = \frac{1}{n},\ i = 1, 2, \cdots, n.$$

The expected value of this random variable is

$$E(X) = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

and the variance is

$$\text{Var}(X) = \frac{1}{n^2}\left[\sum_{i=1}^{n} x_i^2 + \left(\sum_{i=1}^{n} x_i\right)^2\right].$$

The moment generating function of a discrete uniform distribution is given by

$$M_X(t) = \frac{1}{n}\sum_{i=1}^{n} e^{tx_i}.$$

Example 13.1
Consider the random variable with support equals to $\{1, 2, \cdots, n\}$.
(a) Find the expected value and the variance of X.
(b) Find the moment generating function of X.

Solution.
(a) The expected value is

$$E(X) = \sum_{i=1}^{n} \frac{i}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}.$$

The second mooment of X is

$$E(X^2) = \sum_{i=1}^{n} \frac{i^2}{n} = \frac{(n+1)(2n+1)}{6}.$$

The variance of X is

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}.$$

(b) The moment generating function is

$$M_X(t) = \sum_{i=1}^{n} \frac{e^{it}}{n} = \frac{e^t}{n} \sum_{i=0}^{n-1} e^{it} = \frac{e^t}{n} \cdot \frac{1-e^{nt}}{1-e^t} \blacksquare$$

13.2 The Binomial Distribution

Binomial experiments are problems that consist of a fixed number of trials n, with each trial having exactly two possible outcomes: **Success** and **failure.** The probability of a success is denoted by p and that of a failure by q. Moreover, p and q are related by the formula

$$p + q = 1.$$

Also, we assume that the trials are **independent**, that is what happens to one trial does not affect the probability of a success in any other trial.
Let X represent the number of successes that occur in n trials. Then X is said to be a **binomial random variable** with parameters (n, p). If $n = 1$ then X is said to be a **Bernoulli random variable**.

Example 13.2
Privacy is a concern for many users of the Internet. One survey showed that 79% of Internet users are somewhat concerned about the confidentiality of their e-mail. Based on this information, we would like to find the probability that for a random sample of 12 Internet users, 7 are concerned about the privacy of their e-mail. What are the values of n, p, q, r?

Solutions.
This is a binomial experiment with 12 trials. If we assign success to an Internet user being concerned about the privacy of e-mail, the probability of success is 79%. We are interested in the probability of 7 successes. We have $n = 12, p = 0.79, q = 1 - 0.79 = 0.21, r = 7$ ■

The pmf of the binomial distribution is given by

$$p(x) = \Pr(X = r) = C(n, r)p^x q^{n-x}.$$

The expected value is

$$E(X) = np$$

and the variance

$$\mathrm{Var}(X) = np(1 - p).$$

The moment generating function of a binomial distribution is

$$M_X(t) = (q + pe^t)^n.$$

13.3 The Negative Binomial Distribution

Consider a statistical experiment where a success occurs with probability p and a failure occurs with probability $q = 1 - p$. If the experiment is repeated indefinitely and the trials are independent of each other, then the random variable X, the number of trials at which the x^{th} success occurs, has a **negative binomial** distribution with parameters x and p. The probability mass function of X is

$$p(n) = \Pr(X = n) = C(n-1, x-1)p^x(1-p)^{n-x},$$

where $n = x, x+1, \cdots$ (In order to have x successes there must be at least x trials.)

Example 13.3
A research scientist is inoculating rabbits, one at a time, with a disease until he finds two rabbits which develop the disease. If the probability of contracting the disease $\frac{1}{6}$, what is the probability that eight rabbits are needed?

Solution.
Let X be the number of rabbits needed until the first rabbit to contract the disease. Then X follows a negative binomial distribution with $x = 2$ and $p = \frac{1}{6}$. Thus,

$$\Pr(8 \text{ rabbits are needed}) = C(8-1, 2-1)\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^6 \approx 0.0651 \blacksquare$$

The expected value of X is

$$E(X) = \frac{x}{p}$$

and the variance

$$\text{Var}(X) = \frac{x(1-p)}{p^2}.$$

The moment generating function of a negative binomial distribution is given by

$$M_X(t) = \left(\frac{p}{1-(1-p)e^t}\right)^{n-x}.$$

13.4 The Geometric Distribution

A **geometric** random variable with parameter $p, 0 < p < 1$ has a probability mass function

$$p(n) = p(1-p)^{n-1}, \quad n = 1, 2, \cdots .$$

A geometric random variable models the number of successive Bernoulli trials that must be performed to obtain the first "success". For example, the number of flips of a fair coin until the first head appears follows a geometric distribution.

Example 13.4
If you roll a pair of fair dice, the probability of getting an 11 is $\frac{1}{18}$. If you roll the dice repeatedly, what is the probability that the first 11 occurs on the 8th roll?

Solution.
Let X be the number of rolls on which the first 11 occurs. Then X is a geometric random variable with parameter $p = \frac{1}{18}$. Thus,

$$\Pr(X=8) = \left(\frac{1}{18}\right)\left(1 - \frac{1}{18}\right)^7 = 0.0372 \blacksquare$$

The expected value is

$$E(X) = \frac{1}{p}$$

and the variance

$$\mathrm{Var}(X) = \frac{(1-p)}{p^2}.$$

The moment generating function of a geometric distribution is given by

$$M_X(t) = \frac{p}{1-(1-p)e^t}.$$

13.5 The Poisson Distribution

A random variable X is said to be a **Poisson** random variable with parameter $\lambda > 0$ if its probability mass function has the form

$$p(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \cdots$$

where λ indicates the average number of successes per unit time or space.

Example 13.5
The number of false fire alarms in a suburb of Houston averages 2.1 per day. Assuming that a Poisson distribution is appropriate, what is the probability that 4 false alarms will occur on a given day?

Solution.
The probability that 4 false alarms will occur on a given day is given by

$$\Pr(X = 4) = e^{-2.1}\frac{(2.1)^4}{4!} \approx 0.0992 \blacksquare$$

The expected value of X is

$$E(X) = \lambda$$

and the variance

$$\text{Var}(X) = \lambda.$$

The moment generating function of the Poisson distribution is given by

$$M_X(t) = e^{\lambda(e^t - 1)}.$$

Poisson Approximation to the Binomial Random Variable.

Theorem 13.1
Let X be a binomial random variable with parameters n and p. If $n \to \infty$ and $p \to 0$ so that $np = \lambda = E(X)$ remains constant then X can be approximated by a Poisson distribution with parameter λ.

14 A List of Commonly Encountered Continuous R.V

In this section, we discuss four important continuous distributions: Uniform distribution, normal distribution, exponential distribution, and Gamma distribution.

14.1 Continuous Uniform Distribution

The simplest continuous distribution is the uniform distribution. A continuous random variable X is said to be **uniformly** distributed over the interval $a \leq x \leq b$ if its pdf is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \leq x \leq b \\ 0, & \text{otherwise.} \end{cases}$$

Since $F(x) = \int_{-\infty}^{x} f(t)dt$, the cdf is given by

$$F(x) = \begin{cases} 0, & \text{if } x \leq a \\ \frac{x-a}{b-a}, & \text{if } a < x < b \\ 1, & \text{if } x \geq b. \end{cases}$$

Figure 14.1 presents a graph of $f(x)$ and $F(x)$.

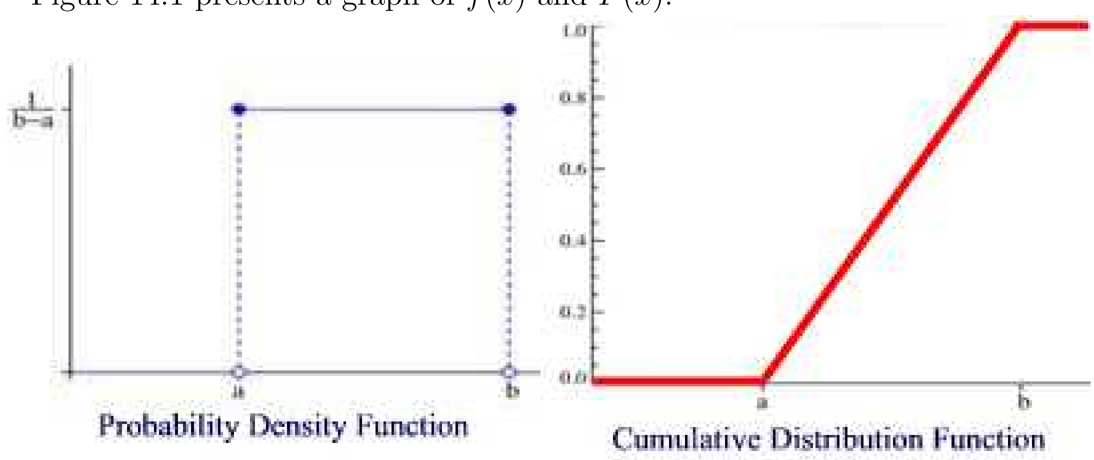


Figure 14.1

If $a = 0$ and $b = 1$ then X is called the **standard uniform** random variable.

Example 14.1
You are the production manager of a soft drink bottling company. You believe that when a machine is set to dispense 12 oz., it really dispenses 11.5 to 12.5 oz. inclusive. Suppose the amount dispensed has a uniform distribution. What is the probability that less than 11.8 oz. is dispensed?

Solution.
Since $f(x) = \frac{1}{12.5-11.5} = 1$,

$$\Pr(11.5 \leq X \leq 11.8) = \text{area of rectangle of base 0.3 and height } 1 = 0.3 \blacksquare$$

The expected value of X is

$$E(X) = \frac{a+b}{2}$$

and the variance

$$\text{Var}(X) = \frac{(b-a)^2}{12}.$$

The moment generating function of a continuous uniform distribution is given by

$$M_X(t) = \frac{e^{bt} - e^{at}}{t(b-a)}.$$

Note that the median of X is the same as the expected value (see Example 11.2). Since the pdf of X is constant, the uniform distribution does not have a mode.

14.2 Normal and Standard Normal Distributions

A **normal** random variable with parameters μ and σ^2 has a pdf

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

This density function is a bell-shaped curve that is symmetric about μ (See Figure 14.2).

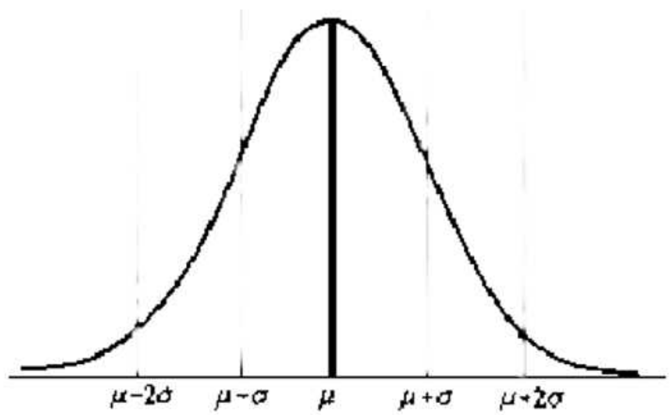


Figure 14.2

Example 14.2
The width of a bolt of fabric is normally distributed with mean 950mm and standard deviation 10 mm. What is the probability that a randomly chosen bolt has a width between 947 and 950 mm?

Solution.
Let X be the width (in mm) of a randomly chosen bolt. Then X is normally distributed with mean 950 mm and variation 100 mm. Thus,

$$\Pr(947 \le X \le 950) = \frac{1}{10\sqrt{2\pi}} \int_{947}^{950} e^{-\frac{(x-950)^2}{200}} dx \approx 0.118 \blacksquare$$

If X is a normal distribution with parameter (μ, σ^2) then its expected value is

$$E(X) = \mu$$

and its variance is

$$\text{Var}(X) = \sigma^2.$$

Example 14.3
A college has an enrollment of 3264 female students. Records show that the mean height of these students is 64.4 inches and the standard deviation is 2.4 inches. Since the shape of the relative histogram of sample college students approximately normally distributed, we assume the total population distribution of the height X of all the female college students follows the normal distribution with the same mean and the standard deviation. Find $\Pr(66 \le X \le 68)$.

Solution.
If you want to find out the percentage of students whose heights are between 66 and 68 inches, you have to evaluate the area under the normal curve from 66 to 68 as shown in Figure 14.3.

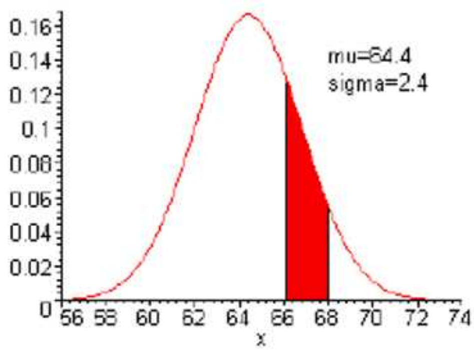


Figure 14.3

Thus,

$$\Pr(66 \le X \le 68) = \frac{1}{\sqrt{2\pi}(2.4)} \int_{66}^{68} e^{-\frac{(x-64.4)^2}{2(2.4)^2}} dx \approx 0.1846 \blacksquare$$

If X is a normal random variable with parameters (μ, σ^2) then the standerdized random variable $Z = \frac{X-\mu}{\sigma}$ is a normal distribution with expected value 0 and variance 1. Such a random variable is called the **standard normal random variable.** Its probability density function is given by

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Note that

$$f_X(x) = \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right). \tag{14.1}$$

Now, f_Z does not have a closed form antiderivative and as a result probability values can not be found by integrating f_z. Instead, numerical integration techniques are used to find values of $F_Z(x)$ that are listed in a table for look-up as needed. Thus, probabilities involving normal random variables are reduced to the ones involving standard normal variable. For example

$$\Pr(X \leq a) = P\left(\frac{X-\mu}{\sigma} \leq \frac{a-\mu}{\sigma}\right) = F_Z\left(\frac{a-\mu}{\sigma}\right).$$

Area under the Standard Normal Curve from $-\infty$ to x

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Example 14.4
Let X be a normal random variable with parameters μ and σ^2. Find
(a)$\Pr(\mu - \sigma \le X \le \mu + \sigma)$.
(b)$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$.
(c)$\Pr(\mu - 3\sigma \le X \le \mu + 3\sigma)$.

Solution.
(a) We have

$$\begin{aligned}\Pr(\mu - \sigma \le X \le \mu + \sigma) =&\Pr(-1 \le Z \le 1)\\ =&\Phi(1) - \Phi(-1)\\ =&2(0.8413) - 1 = 0.6826.\end{aligned}$$

Thus, 68.26% of all possible observations lie within one standard deviation to either side of the mean.
(b) We have

$$\begin{aligned}\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) =&\Pr(-2 \le Z \le 2)\\ =&\Phi(2) - \Phi(-2)\\ =&2(0.9772) - 1 = 0.9544.\end{aligned}$$

Thus, 95.44% of all possible observations lie within two standard deviations to either side of the mean.
(c) We have

$$\begin{aligned}\Pr(\mu - 3\sigma \le X \le \mu + 3\sigma) =&\Pr(-3 \le Z \le 3)\\ =&\Phi(3) - \Phi(-3)\\ =&2(0.9987) - 1 = 0.9974.\end{aligned}$$

Thus, 99.74% of all possible observations lie within three standard deviations to either side of the mean. The above are summarized in Figure 14.4 ■

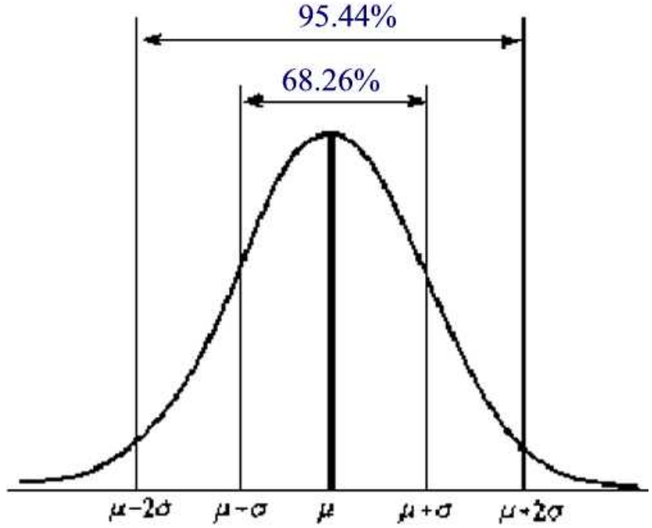


Figure 14.4

The moment generating function of a normal distribution with parameters (μ, σ^2) is given by

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

14.3 Exponential Distribution

An **exponential** random variable with parameter $\lambda > 0$ is a random variable with pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

and cdf

$$F(X) = 1 - e^{-\lambda x}, \quad x > 0.$$

The expected value is

$$E(X) = \frac{1}{\lambda}$$

and the variance is

$$\mathrm{Var}(X) = \frac{1}{\lambda^2}.$$

Exponential random variables are often used to model arrival times, waiting times, and equipment failure times.

Example 14.5
The time between machine failures at an industrial plant has an exponential distribution with an average of 2 days between failures. Suppose a failure has just occurred at the plant. Find the probability that the next failure won't happen in the next 5 days.

Solution.
Let X denote the time between accidents. The mean time to failure is 2 days. Thus, $\lambda = 0.5$. Now, $\Pr(X > 5) = 1 - \Pr(X \leq 5) = \int_5^\infty 0.5e^{-0.5x}dx \approx 0.082085$ ■

Example 14.6
The mileage (in thousands of miles) which car owners get with a certain kind of radial tire is a random variable having an exponential distribution with mean 40. Find the probability that one of these tires will last at most 30 thousand miles.

Solution.
Let X denote the mileage (in thousdands of miles) of one these tires. Then

X is an exponential distribution with paramter $\lambda = \frac{1}{40}$. Thus,

$$\begin{aligned}\Pr(X \leq 30) &= \int_0^{30} \frac{1}{40} e^{-\frac{x}{40}} dx \\ &= -e^{-\frac{x}{40}}\Big|_0^{30} = 1 - e^{-\frac{3}{4}} \approx 0.5276 \blacksquare\end{aligned}$$

The moment generating function of an exponential distribution with parameter λ is given by

$$M_X(t) = \frac{1}{1 - \lambda t}.$$

14.4 Gamma Distribution

The **Gamma** function is defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy, \quad \alpha > 0.$$

For example,

$$\Gamma(1) = \int_0^\infty e^{-y} dy = -e^{-y} |_0^\infty = 1.$$

For $\alpha > 1$ we can use integration by parts with $u = y^{\alpha-1}$ and $dv = e^{-y}dy$ to obtain

$$\begin{aligned}\Gamma(\alpha) =& - e^{-y} y^{\alpha-1} |_0^\infty + \int_0^\infty e^{-y}(\alpha-1)y^{\alpha-2}dy \\ =&(\alpha-1)\int_0^\infty e^{-y}y^{\alpha-2}dy \\ =&(\alpha-1)\Gamma(\alpha-1)\end{aligned}$$

If n is a positive integer greater than 1 then by applying the previous relation repeatedly we find

$$\begin{aligned}\Gamma(n) =&(n-1)\Gamma(n-1) \\ =&(n-1)(n-2)\Gamma(n-2) \\ =&\cdots \\ =&(n-1)(n-2)\cdots 3\cdot 2\cdot \Gamma(1) = (n-1)!\end{aligned}$$

A **Gamma** random variable with parameters $\alpha > 0$ and $\lambda > 0$ has a pdf

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

The expected value of the gamma distribution with parameters (λ, α) is

$$E(X) = \frac{\alpha}{\lambda}$$

and the variance

$$Var(X) = \frac{\alpha}{\lambda^2}.$$

It is easy to see that when the parameter set is restricted to $(\alpha, \lambda) = (1, \lambda)$ the gamma distribution becomes the exponential distribution.
The gamma random variable can be used to model the waiting time until a number of random events occurs. The number of random events sought is α in the formula of $f(x)$.

Example 14.7
In a certain city, the daily consumption of electric power in millions of kilowatt hours can be treated as a random variable having a gamma distribution with $\alpha = 3$ and $\lambda = 0.5$.
(a) What is the random variable? What is the expected daily consumption?
(b) If the power plant of this city has a daily capacity of 12 million kWh, what is the probability that this power supply will be inadequate on a given day? Set up the appropriate integral but do not evaluate.

Solution.
(a) The random variable is the daily consumption of power in kilowatt hours. The expected daily consumption is the expected value of a gamma distributed variable with parameters $\alpha = 3$ and $\lambda = \frac{1}{2}$ which is $E(X) = \frac{\alpha}{\lambda} = 6$.
(b) The probability is $\frac{1}{2^3\Gamma(3)} \int_{12}^{\infty} x^2 e^{-\frac{x}{2}} dx = \frac{1}{16} \int_{12}^{\infty} x^2 e^{-\frac{x}{2}} dx$ ■

The moment generating function of a gamma distribution with parameters (λ, α) is given by

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}.$$

15 Bivariate Random Variables

There are many situations which involve the presence of several random variables and we are interested in their joint behavior. For example:
(i) A meteorological station may record the wind speed and direction, air pressure and the air temperature.
(ii) Your physician may record your height, weight, blood pressure, cholesterol level and more.
This section is concerned with the joint probability structure of two or more random variables defined on the same sample space.

15.1 Joint CDFs

Suppose that X and Y are two random variables defined on the same sample space Ω. The **joint cumulative distribution function** of X and Y is the function

$$F_{XY}(x,y) = \Pr(X \leq x, Y \leq y) = \Pr(\{e \in \Omega : X(e) \leq x \textit{ and } Y(e) \leq y\}).$$

Example 15.1
Consider the experiment of throwing a fair coin and a fair die simultaneously. The sample space is

$$\Omega = \{(H,1),(H,2),\cdots,(H,6),(T,1),(T,2),\cdots,(T,6)\}.$$

Let X be the number of head showing on the coin, $X \in \{0,1\}$. Let Y be the number showing on the die, $Y \in \{1,2,3,4,5,6\}$. Thus, if $e = (H,1)$ then $X(e) = 1$ and $Y(e) = 1$. Find $F_{XY}(1,2)$.

Solution.

$$\begin{aligned}
F_{XY}(1,2) =& \Pr(X \leq 1, Y \leq 2) \\
=& \Pr(\{(H,1),(H,2),(T,1),(T,2)\}) \\
=& \frac{4}{12} = \frac{1}{3} \blacksquare
\end{aligned}$$

In what follows, individual cdfs will be referred to as **marginal distributions**. Relationships between the joint cdf and the marginal distributions as

well as properties of the joint cdf are listed next:
- $F_X(x) = \lim_{y\to\infty} F_{XY}(x,y) = F_{XY}(x,\infty)$.
- $F_Y(y) = \lim_{x\to\infty} F_{XY}(x,y) = F_{XY}(\infty,y)$.
- $F_{XY}(\infty,\infty) = \Pr(X < \infty, Y < \infty) = 1$.
- $F_{XY}(-\infty,y) = 0$.
- $F_{XY}(x,-\infty) = 0$.

All joint probability statements about X and Y can be answered in terms of their joint distribution functions. For example,

$$\begin{aligned}\Pr(X > x, Y > y) =& 1 - \Pr(\{X > x, Y > y\}^c)\\ =& 1 - \Pr(\{X > x\}^c \cup \{Y > y\}^c)\\ =& 1 - [\Pr(\{X \le x\} \cup \{Y \le y\})\\ =& 1 - [\Pr(X \le x) + \Pr(Y \le y) - \Pr(X \le x, Y \le y)]\\ =& 1 - F_X(x) - F_Y(y) + F_{XY}(x,y)\end{aligned}$$

Also, if $a_1 < a_2$ and $b_1 < b_2$ then

$$\begin{aligned}\Pr(a_1 < X \le a_2, b_1 < Y \le b_2) =& \Pr(X \le a_2, Y \le b_2) - \Pr(X \le a_2, Y \le b_1)\\ &- \Pr(X \le a_1, Y \le b_2) + \Pr(X \le a_1, Y \le b_1)\\ =& F_{XY}(a_2,b_2) - F_{XY}(a_1,b_2) - F_{XY}(a_2,b_1) + F_{XY}(a_1,b_1)\end{aligned}$$

15.2 Bivariate Distributions: The Discrete Case

If X and Y are both discrete random variables, we define the **joint probability mass function** of X and Y by

$$p_{XY}(x,y) = \Pr(X = x, Y = y).$$

The marginal probability mass function of X can be obtained from $p_{XY}(x,y)$ by

$$p_X(x) = \Pr(X = x) = \sum_{y:p_{XY}(x,y)>0} p_{XY}(x,y).$$

Similarly, we can obtain the marginal pmf of Y by

$$p_Y(y) = \Pr(Y = y) = \sum_{x:p_{XY}(x,y)>0} p_{XY}(x,y).$$

This simply means to find the probability that X takes on a specific value we sum across the row associated with that value. To find the probability that Y takes on a specific value we sum the column associated with that value as illustrated in the next example.

Example 15.2
A fair coin is tossed 4 times. Let the random variable X denote the number of heads in the first 3 tosses, and let the random variable Y denote the number of heads in the last 3 tosses.
(a) What is the joint pmf of X and Y?
(b) What is the probability 2 or 3 heads appear in the first 3 tosses and 1 or 2 heads appear in the last three tosses?
(c) What is the joint cdf of X and Y?
(d) What is the probability less than 3 heads occur in both the first and last 3 tosses?
(e) Find the probability that one head appears in the first three tosses.

Solution.
(a) The joint pmf is given by the following table

$X\backslash Y$	0	1	2	3	$p_X(.)$
0	1/16	1/16	0	0	2/16
1	1/16	3/16	2/16	0	6/16
2	0	2/16	3/16	1/16	6/16
3	0	0	1/16	1/16	2/16
$p_Y(.)$	2/16	6/16	6/16	2/16	1

(b) $\Pr((X,Y) \in \{(2,1),(2,2),(3,1),(3,2)\}) = p(2,1) + p(2,2) + p(3,1) + p(3,2) = \frac{3}{8}$
(c) The joint cdf is given by the following table

$X\backslash Y$	0	1	2	3
0	1/16	2/16	2/16	2/16
1	2/16	6/16	8/16	8/16
2	2/16	8/16	13/16	14/16
3	2/16	8/16	14/16	1

(d) $\Pr(X < 3, Y < 3) = F(2,2) = \frac{13}{16}$
(e) $\Pr(X = 1) = \Pr((X,Y) \in \{(1,0),(1,1),(1,2),(1,3)\}) = 1/16 + 3/16 + 2/16 = \frac{3}{8}$ ■

15.3 Bivariate Distributions: The Continuous Case

Two random variables X and Y are said to be **jointly continuous** if there exists a function $f_{XY}(x,y) \geq 0$ with the property that for every subset C of $\mathbb{R}^2$ we have

$$\Pr((X,Y) \in C) = \iint_{(x,y)\in C} f_{XY}(x,y)dxdy$$

The function $f_{XY}(x,y)$ is called the **joint probability density function** of X and Y.
If A and B are any sets of real numbers then by letting $C = \{(x,y) : x \in A, y \in B\}$ we have

$$\Pr(X \in A, Y \in B) = \int_B \int_A f_{XY}(x,y)dxdy$$

As a result of this last equation we can write

$$\begin{aligned} F_{XY}(x,y) =& \Pr(X \in (-\infty, x], Y \in (-\infty, y]) \\ =& \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(u,v)dudv \end{aligned}$$

It follows upon differentiation that

$$f_{XY}(x,y) = \frac{\partial^2}{\partial y \partial x} F_{XY}(x,y)$$

whenever the partial derivatives exist.

Example 15.3
The cumulative distribution function for the joint distribution of the continuous random variables X and Y is $F_{XY}(x,y) = 0.2(3x^3y + 2x^2y^2),\ 0 \leq x \leq 1, 0 \leq y \leq 1$. Find $f_{XY}(\frac{1}{2}, \frac{1}{2})$.

Solution.
Since

$$f_{XY}(x,y) = \frac{\partial^2}{\partial y \partial x} F_{XY}(x,y) = 0.2(9x^2 + 8xy)$$

we find $f_{XY}(\frac{1}{2}, \frac{1}{2}) = \frac{17}{20}$ ■

Now, if X and Y are jointly continuous then they are individually continuous, and their probability density functions can be obtained as follows:

$$\begin{aligned}\Pr(X \in A) =& \Pr(X \in A, Y \in (-\infty, \infty))\\ =& \int_A \int_{-\infty}^{\infty} f_{XY}(x,y)dydx\\ =& \int_A f_X(x,y)dx\end{aligned}$$

where

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$$

is thus the probability density function of X. Similarly, the probability density function of Y is given by

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx.$$

Example 15.4
Let X and Y be random variables with joint pdf

$$f_{XY}(x,y) = \begin{cases} \frac{1}{4}, & -1 \le x, y \le 1 \\ 0, & \text{Otherwise.} \end{cases}$$

Determine
(a) $\Pr(X^2 + Y^2 < 1)$,
(b) $\Pr(2X - Y > 0)$,
(c) $\Pr(|X + Y| < 2)$.

Solution.
(a)

$$\Pr(X^2 + Y^2 < 1) = \int_0^{2\pi} \int_0^1 \frac{1}{4} r dr d\theta = \frac{\pi}{4}.$$

(b)

$$\Pr(2X - Y > 0) = \int_{-1}^1 \int_{\frac{y}{2}}^1 \frac{1}{4} dx dy = \frac{1}{2}.$$

Note that $\Pr(2X - Y > 0)$ is the area of the region bounded by the lines $y = 2x, x = -1, x = 1, y = -1$ and $y = 1$. A graph of this region will help

you understand the integration process used above.
(c) Since the square with vertices $(1,1),(1,-1),(-1,1),(-1,-1)$ is completely contained in the region $-2 < x + y < 2$, we have

$$\Pr(|X+Y| < 2) = 1 \blacksquare$$

15.4 Independent Random Variables

Let X and Y be two random variables defined on the same sample space S. We say that X and Y are **independent** random variables if and only if for any two sets of real numbers A and B we have

$$\Pr(X \in A, Y \in B) = \Pr(X \in A)\Pr(Y \in B) \tag{15.1}$$

That is the events $E = \{X \in A\}$ and $F = \{Y \in B\}$ are independent.
The following theorem expresses independence in terms of pdfs.

Theorem 15.1
If X and Y are discrete random variables, then X and Y are independent if and only if

$$p_{XY}(x, y) = p_X(x)p_Y(y)$$

where $p_X(x)$ and $p_Y(y)$ are the marginal pmfs of X and Y respectively. Similar result holds for continuous random variables where sums are replaced by integrals and pmfs are replaced by pdfs.

Example 15.5
A month of the year is chosen at random (each with probability $\frac{1}{12}$). Let X be the number of letters in the month's name, and let Y be the number of days in the month (ignoring leap year).
(a) Write down the joint pdf of X and Y. From this, compute the pdf of X and the pdf of Y.
(b) Find $E(Y)$.
(c) Are the events "$X \leq 6$" and "$Y = 30$" independent?
(d) Are X and Y independent random variables?

Solution.
(a) The joint pdf is given by the following table

Y\ X	3	4	5	6	7	8	9	$p_Y(y)$
28	0	0	0	0	0	$\frac{1}{12}$	0	$\frac{1}{12}$
30	0	$\frac{1}{12}$	$\frac{1}{12}$	0	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{4}{12}$
31	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	0	$\frac{7}{12}$
$p_X(x)$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{1}{12}$	1

(b) $E(Y) = \left(\frac{1}{12}\right) \times 28 + \left(\frac{4}{12}\right) \times 30 + \left(\frac{7}{12}\right) \times 31 = \frac{365}{12}$
(c) We have $\Pr(X \leq 6) = \frac{6}{12} = \frac{1}{2}$, $\Pr(Y = 30) = \frac{4}{12} = \frac{1}{3}$, $\Pr(X \leq 6, Y = 30) = \frac{2}{12} = \frac{1}{6}$. Since, $\Pr(X \leq 6, Y = 30) = \Pr(X \leq 6)\Pr(Y = 30)$, the two events are independent.
(d) Since $p_{XY}(5, 28) = 0 \neq p_X(5)p_Y(28) = \frac{1}{6} \times \frac{1}{12}$, X and Y are dependent ■

In the jointly continuous case the condition of independence is equivalent to

$$f_{XY}(x, y) = f_X(x)f_Y(y).$$

It follows from the previous theorem, that if you are given the joint pdf of the random variables X and Y, you can determine whether or not they are independent by calculating the marginal pdfs of X and Y and determining whether or not the relationship $f_{XY}(x, y) = f_X(x)f_Y(y)$ holds.

Example 15.6
The joint pdf of X and Y is given by

$$f_{XY}(x, y) = \begin{cases} 4e^{-2(x+y)}, & 0 < x < \infty,\ 0 < y < \infty \\ 0, & \text{Otherwise.} \end{cases}$$

Are X and Y independent?

Solution.
Marginal density $f_X(x)$ is given by

$$f_X(x) = \int_0^\infty 4e^{-2(x+y)}dy = 2e^{-2x}\int_0^\infty 2e^{-2y}dy = 2e^{-2x},\ x > 0$$

Similarly, the mariginal density $f_Y(y)$ is given by

$$f_Y(y) = \int_0^\infty 4e^{-2(x+y)}dx = 2e^{-2y}\int_0^\infty 2e^{-2x}dx = 2e^{-2y},\ y > 0$$

Now since

$$f_{XY}(x, y) = 4e^{-2(x+y)} = [2e^{-2x}][2e^{-2y}] = f_X(x)f_Y(y)$$

X and Y are independent ■

The following theorem provides a necessary and sufficient condition for two random variables to be independent.

Theorem 15.2
Two continuous random variables X and Y are independent if and only if their joint probability density function can be expressed as

$$f_{XY}(x,y) = h(x)g(y), \quad -\infty < x < \infty, -\infty < y < \infty.$$

The same result holds for discrete random variables.

Example 15.7
The joint pdf of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} xye^{-\frac{(x^2+y^2)}{2}}, & 0 \le x, y < \infty \\ 0, & \text{Otherwise.} \end{cases}$$

Are X and Y independent?

Solution.
We have

$$f_{XY}(x,y) = xye^{-\frac{(x^2+y^2)}{2}} = xe^{-\frac{x^2}{2}}ye^{-\frac{y^2}{2}}$$

By the previous theorem, X and Y are independent ■

Example 15.8
The joint pdf of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} x+y, & 0 \le x, y < 1 \\ 0, & \text{Otherwise.} \end{cases}$$

Are X and Y independent?

Solution.
Let

$$I(x,y) = \begin{cases} 1, & 0 \le x < 1,\ 0 \le y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$f_{XY}(x,y) = (x+y)I(x,y)$$

which clearly does not factor into a part depending only on x and another depending only on y. Thus, by the previous theorem X and Y are dependent ■

15.5 Conditional Distributions: The Discrete Case

If X and Y are discrete random variables then we define the conditional probability mass function of X given that $Y = y$ by

$$p_{X|Y}(x|y) = \frac{p_{XY}(x,y)}{p_Y(y)} \tag{15.2}$$

provided that $p_Y(y) > 0$.

Example 15.9
Suppose you and me are tossing two fair coins independently, and we will stop as soon as each one of us gets a head.
(a) Find the chance that we stop simultaneously.
(b) Find the conditional distribution of the number of coin tosses given that we stop simultaneously.

Solution.
(a) Let X be the number of times I have to toss my coin before getting a head, and Y be the number of times you have to toss your coin before getting a head. So X and Y are independent identically distributed geometric random variables with parameter $p = \frac{1}{2}$. Thus,

$$\begin{aligned}
\Pr(X = Y) &= \sum_{k=1}^{\infty} P(X = k, Y = k) = \sum_{k=1}^{\infty} P(X = k)P(Y = k) \\
&= \sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{1}{3}
\end{aligned}$$

(b) Notice that given the event $[X = Y]$ the number of coin tosses is well defined and it is X (or Y). So for any $k \geq 1$ we have

$$\Pr(X = k|Y = k) = \frac{\Pr(X = k, Y = k)}{\Pr(X = Y)} = \frac{\frac{1}{4^k}}{\frac{1}{3}} = \frac{3}{4}\left(\frac{1}{4}\right)^{k-1}.$$

Thus given $[X = Y]$, the number of tosses follows a geometric distribution with parameter $p = \frac{3}{4}$ ■

Sometimes it is not the joint distribution that is known, but rather, for each y, one knows the conditional distribution of X given $Y = y$. If one also knows the distribution of Y, then one can recover the joint distribution using (15.2). We also mention one more use of (15.2):

$$p_X(x) = \sum_y p_{XY}(x,y)$$
$$= \sum_y p_{X|Y}(x|y)p_Y(y) \tag{15.3}$$

Thus, given the conditional distribution of X given $Y = y$ for each possible value y, and the (marginal) distribution of Y, one can compute the (marginal) distribution of X, using (15.3).
The conditional cumulative distribution of X given that $Y = y$ is defined by

$$F_{X|Y}(x|y) = \Pr(X \le x|Y = y)$$
$$= \sum_{a \le x} p_{X|Y}(a|y)$$

Note that if X and Y are independent, then the conditional mass function and the conditional distribution function are the same as the unconditional ones. This follows from the next theorem.

Theorem 15.3
If X and Y are independent, then

$$p_{X|Y}(x|y) = p_X(x).$$

Example 15.10
Given the following table.

X\ Y	Y=1	Y=2	Y=3	$p_X(x)$
X=1	.01	.20	.09	.3
X=2	.07	.00	.03	.1
X=3	.09	.05	.06	.2
X=4	.03	.25	.12	.4
$p_Y(y)$	.2	.5	.3	1

Find $p_{X|Y}(x|y)$ where $Y = 2$.

Solution.

$$
\begin{aligned}
p_{X|Y}(1|2) =& \frac{p_{XY}(1,2)}{p_Y(2)} = \frac{.2}{.5} = 0.4 \\
p_{X|Y}(2|2) =& \frac{p_{XY}(2,2)}{p_Y(2)} = \frac{0}{.5} = 0 \\
p_{X|Y}(3|2) =& \frac{p_{XY}(3,2)}{p_Y(2)} = \frac{.05}{.5} = 0.1 \\
p_{X|Y}(4|2) =& \frac{p_{XY}(4,2)}{p_Y(2)} = \frac{.25}{.5} = 0.5 \\
p_{X|Y}(x|2) =& \frac{p_{XY}(x,2)}{p_Y(2)} = \frac{0}{.5} = 0, \quad x > 4 \blacksquare
\end{aligned}
$$

15.6 Conditional Distributions: The Continuous Case

Let $f_{X|Y}(x|y)$ denote the probability density function of X given that $Y = y$. The **conditional density function** of X given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

provided that $f_Y(y) > 0$.

Example 15.11
Suppose X and Y have the following joint density

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2}, & |X| + |Y| < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the marginal distribution of X.
(b) Find the conditional distribution of Y given $X = \frac{1}{2}$.

Solution.
(a) Clearly, X only takes values in $(-1,1)$. So $f_X(x) = 0$ if $|x| \geq 1$. Let $-1 < x < 1$,

$$f_X(x) = \int_{-\infty}^{\infty} \frac{1}{2} dy = \int_{-1+|x|}^{1-|x|} \frac{1}{2} dy = 1 - |x|.$$

(b) The conditional density of Y given $X = \frac{1}{2}$ is then given by

$$f_{Y|X}(y|x) = \frac{f(\frac{1}{2}, y)}{f_X(\frac{1}{2})} = \begin{cases} 1, & -\frac{1}{2} < y < \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $f_{Y|X}$ follows a uniform distribution on the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ ■

Example 15.12
Suppose that X is uniformly distributed on the interval $[0,1]$ and that, given $X = x$, Y is uniformly distributed on the interval $[1-x, 1]$.
(a) Determine the joint density $f_{XY}(x,y)$.
(b) Find the probability $\Pr(Y \geq \frac{1}{2})$.

Solution.
Since X is uniformly distributed on $[0,1]$, we have $f_X(x) = 1, 0 \leq x \leq 1$. Similarly, since, given $X = x$, Y is uniformly distributed on $[1-x, 1]$, the

conditional density of Y given $X = x$ is $\frac{1}{1-(1-x)} = \frac{1}{x}$ on the interval $[1-x, 1]$; i.e., $f_{Y|X}(y|x) = \frac{1}{x}, 1 - x \leq y \leq 1$ for $0 \leq x \leq 1$. Thus

$$f_{XY}(x,y) = f_X(x)f_{Y|X}(y|x) = \frac{1}{x}, 0 < x < 1, 1-x < y < 1$$

(b) Using Figure 15.1 we find

$$\begin{aligned}
\Pr(Y \geq \frac{1}{2}) =& \int_0^{\frac{1}{2}} \int_{1-x}^{1} \frac{1}{x} dydx + \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} \frac{1}{x} dydx \\
=& \int_0^{\frac{1}{2}} \frac{1-(1-x)}{x} dx + \int_{\frac{1}{2}}^{1} \frac{1/2}{x} dx \\
=& \frac{1+\ln 2}{2} \blacksquare
\end{aligned}$$

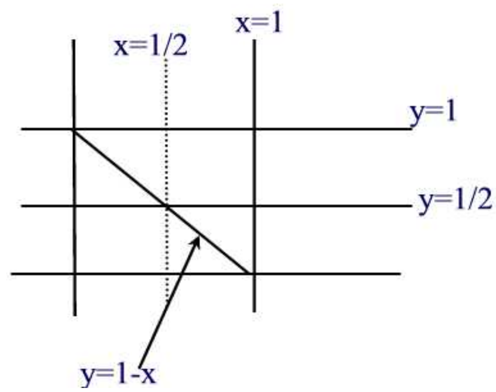


Figure 15.1

Note that

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = \int_{-\infty}^{\infty} \frac{f_{XY}(x,y)}{f_Y(y)} dx = \frac{f_Y(y)}{f_Y(y)} = 1.$$

The **conditional cumulative distribution function** of X given $Y = y$ is defined by

$$F_{X|Y}(x|y) = \Pr(X \leq x | Y = y) = \int_{-\infty}^{x} f_{X|Y}(t|y)dt.$$

From this definition, it follows

$$f_{X|Y}(x|y) = \frac{\partial}{\partial x} F_{X|Y}(x|y).$$

Example 15.13
The joint density of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} \frac{15}{2}x(2-x-y), & 0 \le x, y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Compute the conditional density of X, given that $Y = y$ for $0 \le y \le 1$.

Solution.
The marginal density function of Y is

$$f_Y(y) = \int_0^1 \frac{15}{2}x(2-x-y)dx = \frac{15}{2}\left(\frac{2}{3} - \frac{y}{2}\right).$$

Thus,

$$\begin{aligned} f_{X|Y}(x|y) =& \frac{f_{XY}(x,y)}{f_Y(y)} \\ =& \frac{x(2-x-y)}{\frac{2}{3} - \frac{y}{2}} \\ =& \frac{6x(2-x-y)}{4-3y} \quad \blacksquare \end{aligned}$$

Theorem 15.4
Continuous random variables X and Y are independent if and only if

$$f_{X|Y}(x|y) = f_X(x).$$

Example 15.14
Let X and Y be two continuous random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} c, & 0 \le y < x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $f_X(x), f_Y(y)$ and $f_{X|Y}(x|1)$.
(b) Are X and Y independent?

Solution.
(a) We have

$$f_X(x) = \int_0^x cdy = cx, \quad 0 \le x \le 2$$

$$f_Y(y) = \int_y^2 c dx = c(2-y), \quad 0 \le y \le 2$$

and

$$f_{X|Y}(x|1) = \frac{f_{XY}(x,1)}{f_Y(1)} = \frac{c}{c} = 1, \quad 0 \le x \le 1.$$

(b) Since $f_{X|Y}(x|1) \neq f_X(x)$, X and Y are dependent ■

15.7 The Expected Value of $g(X,Y)$

Suppose that X and Y are two random variables taking values in S_X and S_Y respectively. For a function $g : S_X \times S_Y \to \mathbb{R}$ the expected value of $g(X,Y)$ is

$$E(g(X,Y) = \sum_{x\in S_X}\sum_{y\in S_Y} g(x,y)p_{XY}(x,y).$$

if X and Y are discrete with joint probability mass function $p_{XY}(x,y)$ and

$$E(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{XY}(x,y)dxdy$$

if X and Y are continuous with joint probability density function $f_{XY}(x,y)$.

Example 15.15
Let X and Y be two discrete random variables with joint probability mass function:

$$p_{XY}(1,1) = \tfrac{1}{3}, p_{XY}(1,2) = \tfrac{1}{8}, p_{XY}(2,1) = \tfrac{1}{2}, p_{XY}(2,2) = \tfrac{1}{24}$$

Find the expected value of $g(X,Y) = XY$.

Solution.
The expected value of the function $g(X,Y) = XY$ is calculated as follows:

$$\begin{aligned}
E(g(X,Y)) =& E(XY) = \sum_{x=1}^{2}\sum_{y=1}^{2} xyp_{XY}(x,y)\\
=&(1)(1)(\frac{1}{3}) + (1)(2)(\frac{1}{8}) + (2)(1)(\frac{1}{2}) + (2)(2)(\frac{1}{24})\\
=&\frac{7}{4} \blacksquare
\end{aligned}$$

The following result provides a test of independence for two random variables: If X and Y are independent random variables then for any function h and g we have

$$E(g(X)h(Y)) = E(g(X))E(h(Y)).$$

In particular, $E(XY) = E(X)E(Y)$.

Example 15.16
Suppose a box contains 10 green, 10 red and 10 black balls. We draw 10 balls from the box by sampling with replacement. Let X be the number of green balls, and Y be the number of black balls in the sample.
(a) Find $E(XY)$.
(b) Are X and Y independent? Explain.

Solution.
First we note that X and Y are binomial with $n = 10$ and $p = \frac{1}{3}$.
(a) Let X_i be 1 if we get a green ball on the i^{th} draw and 0 otherwise, and Y_j be the event that in j^{th} draw we got a black ball. Trivially, X_i and Y_j are independent if $1 \le i \ne j \le 10$. Moreover, $X_iY_i = 0$ for all $1 \le i \le 10$. Since $X = X_1 + X_2 + \cdots X_{10}$ and $Y = Y_1 + Y_2 + \cdots Y_{10}$ we have

$$XY = \sum \sum_{1 \le i \ne j \le 10} X_iY_j.$$

Hence,

$$E(XY) = \sum \sum_{1 \le i \ne j \le 10} E(X_iY_j) = \sum \sum_{1 \le i \ne j \le 10} E(X_i)E(Y_j) = 90 \times \frac{1}{3} \times \frac{1}{3} = 10.$$

(b) Since $E(X) = E(Y) = \frac{10}{3}$, we have $E(XY) \ne E(X)E(Y)$ so X and Y are dependent ■

Covariance
Now, when two random variables are dependent one is interested in the strength of relationship between the two variables. One measure is given by the **covariance**:

$$Cov(X, Y) = E[(X - E(X))(Y - E(Y))].$$

An alternative expression that is sometimes more convenient is

$$Cov(X, Y) = E(XY) - E(X)E(Y).$$

Recall that for independent X, Y we have $E(XY) = E(X)E(Y)$ and so $Cov(X, Y) = 0$. However, the converse statement is false as there exists random variables that have covariance 0 but are dependent. For example, let X be a random variable such that

$$\Pr(X = 0) = \Pr(X = 1) = \Pr(X = -1) = \frac{1}{3}$$

and define

$$Y = \begin{cases} 0, & \text{if } X \neq 0 \\ 1, & \text{otherwise.} \end{cases}$$

Thus, Y depends on X.
Clearly, $XY = 0$ so that $E(XY) = 0$. Also,

$$E(X) = (0 + 1 - 1)\frac{1}{3} = 0$$

and thus

$$Cov(X, Y) = E(XY) - E(X)E(Y) = 0.$$

Useful facts are collected in the next result.

Theorem 15.5
(a) $Cov(X, Y) = Cov(Y, X)$ (Symmetry)
(b) $Cov(X, X) = Var(X)$
(c) $Cov(aX, Y) = aCov(X, Y)$
(d) $Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} Cov(X_i, Y_j)$

Example 15.17
Given that $E(X) = 5, E(X^2) = 27.4, E(Y) = 7, E(Y^2) = 51.4$ and $\text{Var}(X + Y) = 8$, find $\text{Cov}(X + Y, X + 1.2Y)$.

Solution.
By definition,

$$\text{Cov}(X + Y, X + 1.2Y) = E((X + Y)(X + 1.2Y)) - E(X + Y)E(X + 1.2Y)$$

Using the properties of expectation and the given data, we get

$$\begin{aligned} E(X + Y)E(X + 1.2Y) =& (E(X) + E(Y))(E(X) + 1.2E(Y)) \\ =& (5 + 7)(5 + (1.2) \cdot 7) = 160.8 \\ E((X + Y)(X + 1.2Y)) =& E(X^2) + 2.2E(XY) + 1.2E(Y^2) \\ =& 27.4 + 2.2E(XY) + (1.2)(51.4) \\ =& 2.2E(XY) + 89.08 \end{aligned}$$

Thus,

$$\text{Cov}(X + Y, X + 1.2Y) = 2.2E(XY) + 89.08 - 160.8 = 2.2E(XY) - 71.72$$

To complete the calculation, it remains to find $E(XY)$. To this end we make use of the still unused relation $\text{Var}(X+Y) = 8$

$$\begin{aligned} 8 =& \text{Var}(X+Y) = E((X+Y)^2) - (E(X+Y))^2 \\ =& E(X^2) + 2E(XY) + E(Y^2) - (E(X) + E(Y))^2 \\ =& 27.4 + 2E(XY) + 51.4 - (5+7)^2 = 2E(XY) - 65.2 \end{aligned}$$

so $E(XY) = 36.6$. Substituting this above gives $\text{Cov}(X+Y, X+1.2Y) = (2.2)(36.6) - 71.72 = 8.8$ ■

15.8 Conditional Expectation

Since conditional probability measures are probabilitiy measures (that is, they possess all of the properties of unconditional probability measures), conditional expectations inherit all of the properties of regular expectations. Let X and Y be random variables. We define **conditional expectation** of X given that $Y = y$ by

$$\begin{aligned} E(X|Y=y\} &= \sum_x xP(X=x|Y=y) \\ &= \sum_x xp_{X|Y}(x|y) \end{aligned}$$

where $p_{X|Y}$ is the conditional probability mass function of X, given that $Y = y$ which is given by

$$p_{X|Y}(x|y) = P(X=x|Y=y) = \frac{p(x,y)}{p_Y(y)}.$$

In the continuous case we have

$$E(X|Y=y) = \int_{-\infty}^{\infty} xf_{X|Y}(x|y)dx$$

where

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}.$$

Example 15.18
Suppose X and Y are discrete random variables with values 1, 2, 3, 4 and joint p.m.f. given by

$$f(x,y) = \begin{cases} \frac{1}{16}, & \text{if } x = y \\ \frac{2}{16}, & \text{if } x < y \\ 0, & \text{if } x > y. \end{cases}$$

for $x, y = 1, 2, 3, 4$.
(a) Find the joint probability distribution of X and Y.
(b) Find the conditional expectation of Y given that $X = 3$.

Solution.
(a) The joint probability distribution is given in tabular form

X \ Y	1	2	3	4	$p_X(x)$
1	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{7}{16}$
2	0	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{5}{16}$
3	0	0	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$
4	0	0	0	$\frac{1}{16}$	$\frac{1}{16}$
$p_Y(y)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$	1

(b) We have

$$\begin{aligned}
E(Y|X=3) =& \sum_{y=1}^{4} y p_{Y|X}(y|3) \\
=& \frac{p_{XY}(3,1)}{p_X(3)} + \frac{2p_{XY}(3,2)}{p_X(3)} + \frac{3p_{XY}(3,3)}{p_X(3)} + \frac{4p_{XY}(3,4)}{p_X(3)} \\
=& 3 \cdot \frac{1}{3} + 4 \cdot \frac{2}{3} = \frac{11}{3} \blacksquare
\end{aligned}$$

Example 15.19
Suppose that the joint density of X and Y is given by

$$f_{XY}(x,y) = \frac{e^{-\frac{x}{y}} e^{-y}}{y}, \quad x, y > 0.$$

Compute $E(X|Y=y)$.

Solution.
The conditional density is found as follows

$$\begin{aligned}
f_{X|Y}(x|y) =& \frac{f_{XY}(x,y)}{f_Y(y)} \\
=& \frac{f_{XY}(x,y)}{\int_{-\infty}^{\infty} f_{XY}(x,y)dx} \\
=& \frac{(1/y)e^{-\frac{x}{y}} e^{-y}}{\int_0^{\infty} (1/y)e^{-\frac{x}{y}} e^{-y}dx} \\
=& \frac{(1/y)e^{-\frac{x}{y}}}{\int_0^{\infty} (1/y)e^{-\frac{x}{y}}dx} \\
=& \frac{1}{y} e^{-\frac{x}{y}}
\end{aligned}$$

Hence,

$$E(X|Y=y) = \int_0^\infty \frac{x}{y}e^{-\frac{x}{y}}dx = -\left[xe^{-\frac{x}{y}}\Big|_0^\infty - \int_0^\infty e^{-\frac{x}{y}}dx\right]$$
$$= -\left[xe^{-\frac{x}{y}} + ye^{-\frac{x}{y}}\right]_0^\infty = y \blacksquare$$

Notice that if X and Y are independent then $p_{X|Y}(x|y) = p(x)$ so that $E(X|Y=y) = E(X)$.
Next, let $\phi_X(y) = E(X|Y=y)$ denote the function of the random variable Y whose value at $Y=y$ is $E(X|Y=y)$. Clearly, $\phi_X(y)$ is a random variable. We denote this random variable by $E(X|Y)$. The expectation of this random variable is just the expectation of X as shown in the following theorem.

Theorem 15.6 *(Double Expectation Property)*

$$E(X) = E(E(X|Y))$$

Example 15.20
Suppose that $E(X|Y) = 18 - \frac{3}{5}Y$ and $E(Y|X) = 10 - \frac{1}{3}X$. Find $E(X)$ and $E(Y)$.

Solution.
Take the expectation on both sides and use the double expectation theorem we find $E(X) = 18 - \frac{3}{5}E(Y)$ and $E(Y) = 10 - \frac{1}{3}E(X)$. Solving this system of two equations in two unknowns we find $E(X) = 15$ and $E(Y) = 5$. $\blacksquare$

Example 15.21
Let X be an exponential random variable with $\lambda = 5$ and Y a uniformly distributed random variable on $(-3, X)$. Find $E(Y)$.

Solution.
First note that for given $X = x$, Y in uniformly distributed on $(-3, x)$ so that $E(Y|X=x) = \frac{x-3}{2}$. Now, using the double expectation identity we find

$$E(Y) = \int_0^\infty E(Y|X=x)f_X(x)dx = \int_0^\infty \frac{x-3}{2} \cdot 5e^{-5x}dx = -1.4 \blacksquare$$

<u>The Conditional Variance</u>
Next, we introduce the concept of conditional variance. Just as we have

defined the conditional expectation of X given that $Y = y$, we can define the conditional variance of X given Y as follows

$$\text{Var}(X|Y = y) = E[(X - E(X|Y))^2|Y = y].$$

Note that the conditional variance is a random variable since it is a function of Y.

Theorem 15.7
Let X and Y be random variables. Then
(a) $\text{Var}(X|Y) = E(X^2|Y) - [E(X|Y)]^2$
(b) $E(\text{Var}(X|Y)) = E[E(X^2|Y) - (E(X|Y))^2] = E(X^2) - E[(E(X|Y))^2]$.
(c) $\text{Var}(E(X|Y)) = E[(E(X|Y))^2] - (E(X))^2$.
(d) Law of Total Variance: $\text{Var}(X) = E[\text{Var}(X|Y)] + \text{Var}(E(X|Y))$.

Example 15.22
Suppose that X and Y have joint distribution

$$f_{XY}(x,y) = \begin{cases} \frac{3y^2}{x^3}, & 0 < y < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find $E(X),\ E(X^2), Var(X), E(Y|X), Var(Y|X), E[Var(Y|X)], Var[E(Y|X)]$, and $Var(Y)$.

Solution.
First we find marginal density functions.

$$f_X(x) = \int_0^x \frac{3y^2}{x^3} dy = 1, \quad 0 < x < 1$$

$$f_Y(y) = \int_y^1 \frac{3y^2}{x^3} dx = \frac{3}{2}(1 - y^2), \quad 0 < y < 1$$

Now,

$$E(X) = \int_0^1 x dx = \frac{1}{2}$$

$$E(X^2) = \int_0^1 x^2 dx = \frac{1}{3}$$

Thus,

$$Var(X) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Next, we find conditional density of Y given $X = x$

$$f_{Y|X}(x|y) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{3y^2}{x^3}, \quad 0 < x < y < 1$$

Hence,

$$E(Y|X) = \int_0^x \frac{3y^3}{x^3} dx = \frac{3}{4}x$$

and

$$E(Y^2|X) = \int_0^x \frac{3y^4}{x^3} dx = \frac{3}{5}x^2$$

Thus,

$$Var(Y|X) = E(Y^2|X) - [E(Y|X)]^2 = \frac{3}{5}x^2 - \frac{9}{16}x^2 = \frac{3}{80}x^2$$

Also,

$$Var[E(Y|X)] = Var\left(\frac{3}{4}x\right) = \frac{9}{16}Var(X) = \frac{9}{16} \times \frac{1}{12} = \frac{3}{64}$$

and

$$E[Var(Y|X)] = E\left(\frac{3}{80}X^2\right) = \frac{3}{80}E(X^2) = \frac{3}{80} \times \frac{1}{3} = \frac{1}{80}.$$

Finally,

$$Var(Y) = Var[E(Y|X)] + E[Var(Y|X)] = \frac{19}{320} \blacksquare$$

16 Sums of Independent Random Variables

In actuarial science one encounters models where a random variable is the sum of a finite number of mutually independent random variables. For example, in risk theory one considers two types of risk models:
(i) The **collective risk model** represents the total loss random variable S as a sum of a random number N of the individual payment amounts. That is,

$$S = X_1 + X_2 + \cdots + X_N,$$

where the $X_i's$ are called the **individual** or **single-loss random variables**. For this model, it is assumed that the subscripted Xs are independent and identically distributed.
(ii) The **individual risk model** represents the aggregate loss as a sum, $S = X_1 + X_2 + \cdots + X_n$ of a fixed number of insurance contracts, N. The random variables $X_1, X_2, \cdots, X_N$ are assumed to be independent but are not necessarily identically distributed.
In what follows, we will review related properties of the random variable S from probability theory. We will assume that S is the sum of N mutually independent random variables.

16.1 Moments of S

Because of independence, the following results hold:

$$E(X_1 + X_2 + \cdots + X_N) = E(X_1) + E(X_2) + \cdots + E(X_N)$$

$$\text{Var}(X_1 + X_2 + \cdots + X_N) = \text{Var}(X_1) + \text{Var}(X_2) + \cdots + \text{Var}(X_N)$$

$$M_{X_1+X_2+\cdots+X_N} = M_{X_1} \cdot M_{X_2} \cdots M_{X_N}.$$

16.2 Distributions Closed Under Convolution

When the subscripted independent random variables Xs belong to a same family of distributions and the resulting sum S belongs to the same family, we say that the family is **closed under convolution.**
One way to show that a family is closed under convolutions is to use an important property of moment generating functions which asserts the existence of a one-to-one correspondence of cdfs and mgfs. That is, if random variables X and Y both have moment generating functions $M_X(t)$ and $M_Y(t)$ that exist in some neighborhood of zero and if $M_X(t) = M_Y(t)$ for all t in this neighborhood, then X and Y have the same distributions.
We next list examples of families of distributions closed under convolution.

Normal Random Variables
Suppose that $X_1, X_2, \cdots, X_N$ are independent normal random variables with parameters $(\mu_1, \sigma_1^2), (\mu_2, \sigma_2^2), \cdots, (\mu_N, \sigma_N^2)$. Then

$$M_S(t) = e^{0.5\sigma_1^2t^2+\mu_1t} \cdots e^{0.5\sigma_N^2t^2+\mu_Nt} = e^{0.5(\sigma_1^2+\cdots+\sigma_N^2)t^2+(\mu_1+\cdots+\mu_N)t}.$$

Hence, S is a normal random variable with parameters $(\mu_1 + \cdots + \mu_N, \sigma_1^2 + \cdots + \sigma_N^2)$.

Poisson Random Variables
Suppose that $X_1, X_2, \cdots, X_N$ are independent Poisson random variables with parameters $\lambda_1, \cdots, \lambda_N$. Then

$$M_S(t) = e^{\lambda_1(e^t-1)} \cdots e^{\lambda_N(e^t-1)} = e^{(\lambda_1+\cdots+\lambda_N)(e^t-1)}.$$

Hence, S is a Poisson random variable with parameter $\lambda_1 + \cdots + \lambda_N$.

Binomial Random Variables
Suppose that $X_1, X_2, \cdots, X_N$ are independent binomial random variables with parameters $(n_1, p), (n_2, p), \cdots, (n_N, p)$. Then

$$M_S(t) = (q + pe^t)^{n_1} \cdots (q + pe^t)^{n_N} = (q + pe^t)^{n_1+n_2+\cdots n_N}.$$

Hence, S is a binomial random variable with parameters $(n_1 + \cdots + n_N, p)$.

Negative Binomial Random Variables

Suppose that $X_1, X_2, \cdots, X_N$ are independent negative binomial random variables with parameters $(r_1, p), (r_2, p), \cdots, (r_N, p)$. Then

$$M_S(t) = \left(\frac{p}{1-(1-p)e^t}\right)^{n-r_1} \cdots \left(\frac{p}{1-(1-p)e^t}\right)^{n-r_2} = \left(\frac{p}{1-(1-p)e^t}\right)^{n-(r_1+r_2+\cdots r_N)}$$

Hence, S is a negative binomial random variable with parameters $(r_1 + \cdots + r_N, p)$. In particular, the family of geometric random variables is closed under convolution.

Gamma Random Variables

Suppose that $X_1, X_2, \cdots, X_N$ are independent gamma random variables with parameters $(\lambda, \alpha_1), (\lambda, \alpha_2), \cdots, (\lambda, \alpha_N)$. Then

$$M_S(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1} \cdots \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_N} = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1 + \cdots \alpha_N}$$

Hence, S is a gamma random variable with parameters $(\lambda, \alpha_1 + \cdots + \alpha_N)$. In particular, the family of exponential random variables is closed under convolution.

16.3 Distribution of S : Convolutions

In this section we turn to the important question of determining the distribution of a sum of independent random variables in terms of the distributions of the individual constituents.

Discrete Case

In this subsection we consider only sums of discrete random variables, reserving the case of continuous random variables for the next subsection. We consider here only discrete random variables whose values are nonnegative integers. Their distribution mass functions are then defined on these integers.

Suppose X and Y are two independent discrete random variables with pmf $p_X(x)$ and $p_Y(y)$ respectively. We would like to determine the pmf of the random variable $X+Y$. To do this, we note first that for any nonnegative integer n we have

$$\{X+Y=n\}=\bigcup_{k=0}^{n}A_k$$

where $A_k=\{X=k\}\cap\{Y=n-k\}$. Note that $A_i\cap A_j=\emptyset$ for $i\neq j$. Since the A_i's are pairwise disjoint and X and Y are independent, we have

$$P(X+Y=n)=\sum_{k=0}^{n}P(X=k)P(Y=n-k).$$

Thus,

$$p_{X+Y}(n)=p_X(n)*p_Y(n)$$

where $p_X(n)*p_Y(n)$ is called the **convolution** of p_X and p_Y.

Example 16.1

A die is rolled twice. Let X and Y be the outcomes, and let $Z=X+Y$ be the sum of these outcomes. Find the probability mass function of Z.

Solution. Note that X and Y have the common pmf :

x	1	2	3	4	5	6
p_X	1/6	1/6	1/6	1/6	1/6	1/6

The probability mass function of Z is then the convolution of p_X with itself. Thus,

$$\begin{aligned}
\Pr(Z=2)=&p_X(1)p_X(1)=\frac{1}{36}\\
\Pr(Z=3)=&p_X(1)p_X(2)+p_X(2)p_X(1)=\frac{2}{36}\\
\Pr(Z=4)=&p_X(1)p_X(3)+p_X(2)p_X(2)+p_X(3)p_X(1)=\frac{3}{36}
\end{aligned}$$

Continuing in this way we would find $\Pr(Z=5)=4/36, \Pr(Z=6)=5/36, \Pr(Z=7)=6/36, \Pr(Z=8)=5/36, \Pr(Z=9)=4/36, \Pr(Z=10)=3/36, \Pr(Z=11)=2/36$, and $\Pr(Z=12)=1/36$ ■

Continuous Case

In this subsection we consider the continuous version of the problem posed in the discrete case: How are sums of independent continuous random variables distributed?

Let X and Y be two independent continuous random variables with probability density functions $f_X(x)$ and $f_Y(y)$, respectively. Assume that both $f_X(x)$ and $f_Y(y)$ are defined for all real numbers. Then the probability density function of $X+Y$ is the convolution of $f_X(x)$ and $f_Y(y)$. That is,

$$f_{X+Y}(a)=\int_{-\infty}^{\infty}f_X(a-y)f_Y(y)dy=\int_{-\infty}^{\infty}f_Y(a-x)f_X(x)dx.$$

Example 16.2

Let X and Y be two independent random variables uniformly distributed on $[0,1]$. Compute the distribution of $X+Y$.

Solution.

Since

$$f_X(a)=f_Y(a)=\begin{cases}1, & 0\le a\le 1\\ 0, & \text{otherwise.}\end{cases}$$

we have

$$f_{X+Y}(a)=\int_0^1 f_X(a-y)dy.$$

Now the integrand is 0 unless $0\le a-y\le 1$(i.e. unless $a-1\le y\le a$) and then it is 1. So if $0\le a\le 1$ then

$$f_{X+Y}(a)=\int_0^a dy=a.$$

If $1 < a < 2$ then

$$f_{X+Y}(a) = \int_{a-1}^{1} dy = 2 - a.$$

Hence,

$$f_{X+Y}(a) = \begin{cases} a, & 0 \le a \le 1 \\ 2 - a, & 1 < a < 2 \\ 0, & \text{otherwise} \ \blacksquare \end{cases}$$

16.4 Estimating the Distribution of S : The Central Limit Theorem

If the distribution of S is hard to evaluate analytically, we can use estimation technique thanks to the Central Limit Theorem which says that the sum of large number of independent identically distributed random variables is well-approximated by a normal random variable. More formally, we have

Theorem 16.1
Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables, each with mean μ and variance σ^2. Then,

$$\Pr\left(\frac{\sqrt{n}}{\sigma}\left(\frac{X_1 + X_2 + \cdots + X_n}{n} - \mu\right) \leq a\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{x^2}{2}} dx$$

as $n \to \infty$.

The Central Limit Theorem says that regardless of the underlying distribution of the variables X_i, so long as they are independent, the distribution of $\frac{\sqrt{n}}{\sigma}\left(\frac{X_1+X_2+\cdots+X_n}{n} - \mu\right)$ converges to the same, normal, distribution.
The central limit theorem suggests approximating the random variable

$$\frac{\sqrt{n}}{\sigma}\left(\frac{X_1 + X_2 + \cdots + X_n}{n} - \mu\right)$$

with a standard normal random variable. This implies that the sample mean has approximately a normal distribution with mean μ and variance $\frac{\sigma^2}{n}$.
Also, a sum of n independent and identically distributed random variables with common mean μ and variance σ^2 can be approximated by a normal distribution with mean $n\mu$ and variance $n\sigma^2$.

Example 16.3
The weight of an arbitrary airline passenger's baggage has a mean of 20 pounds and a variance of 9 pounds. Consider an airplane that carries 200 passengers, and assume every passenger checks one piece of luggage. Estimate the probability that the total baggage weight exceeds 4050 pounds.

Solution.
Let X_i =weight of baggage of passenger i. Thus,

$$\begin{aligned}\Pr\left(\sum_{i=1}^{200} X_i > 4050\right) =&\Pr\left(\frac{\sum_{i=1}^{200} X_i - 200(20)}{3\sqrt{200}} > \frac{4050 - 20(200)}{3\sqrt{200}}\right)\\ \approx&\Pr(Z > 1.179) = 1 - \Pr(Z \le 1.179)\\ =&1 - \Phi(1.179) = 1 - 0.8810 = 0.119 \blacksquare\end{aligned}$$

17 Compound Probability Distributions

Let N be a random variable. Let $X_1, X_2, \cdots, X_N$ be mutually independent identically distributed (i.e., have the same distribution) random variable with common expected value $E(X)$ and common variance $\text{Var}(X)$. We also assume that N and all the X_i are independent. Define

$$S = X_1 + X_2 + \cdots + X_N.$$

In probability theory, S is said to have a **compound distribution.** The variable N is called the **primary distribution** and the X_i's are called the **secondary distributions.**
In the context of risk theory, S will be referred to as a **collective risk model.** The variable S stands for the total claim amount of a portfolio regarded as a collective that produces a random number N of claims in a certain time period. The X_i is the i^{th} claim.

17.1 Mean and Variance of S

We first calculate the expected value of S by using the double expectation property: $E_N[E(S|N)] = E(S)$. Using the assumptions on the X_i's and N, we can write

$$E(S|N) = E(X_1|N) + \cdots + E(X_N|N) = E(X_1) + \cdots + E(X_N) = N \cdot E(X)$$

and likewise

$$\text{Var}(S|N) = N \cdot \text{Var}(X).$$

It follows that

$$E(S) = E_N[E(S|N)] = E[N \cdot E(X)] = E(N) \cdot E(X)$$

since $E(X)$ is constant with respect to the random variable N. Likewise, we have

$$\begin{aligned}\text{Var}(S) =& E_N[\text{Var}(S|N)] + \text{Var}_N[E(S|N)] \\ =& E_N[N \cdot \text{Var}(X)] + \text{Var}_N[N \cdot E(X)] \\ =& E(N) \cdot \text{Var}(X) + \text{Var}(N) \cdot [E(X)]^2\end{aligned}$$

since $\text{Var}(X)$ and $E(X)$ are constant with respect to the random variable N.

Example 17.1
Suppose that N has a geometric distribution with parameter p, where $0 < p < 1$ and X has an exponential distribution with parameter 1. Find the mean and the variance of S.

Solution.
We are given that $E(N) = \frac{1}{p}$ and $E(X) = 1$ so that $E(S) = \frac{1}{p}$. Also, we know that $\text{Var}(N) = \frac{1-p}{p^2}$ and $\text{Var}(X) = 1$. Thus, $\text{Var}(S) = \frac{1}{p} \cdot 1 + \frac{1-p}{p^2} \cdot 1 = \frac{1}{p^2}$ ■

17.2 Moment Generating Function of S

Using the double expectation property we can find the moment generating function of S. Indeed, we have

$$\begin{aligned}M_S(t) =&E(e^{tS}) = E_N[E(e^{tS}|N)]\\ =&E_N[M_X(t)^N] = E_N[e^{N\ln M_X(t)}]\\ =&E_N[e^{rN}] = M_N(r)\end{aligned}$$

where $r = \ln M_X(t)$. Thus, the monent generating function of S can be expressed in terms of the moment generating function of N and the monent generating function of X.

Example 17.2
Suppose that N has a geometric distribution with parameter p, where $0 < p < 1$ and X has an exponential distribution with parameter 1. Find the distribution S.

Solution.
We are given that $M_N(t) = \frac{p}{1-qe^t}$, $q = 1-p$, and $M_X(t) = \frac{1}{1-t}$. Then

$$M_S(t) = M_N[\ln M_X(t)] = \frac{p}{1-qe^{\ln M_X(t)}} = \frac{p}{1-qM_X(t)} = p + q\frac{p}{p-t},$$

so S is a combination of the mgfs of the constant 0 and the exponential distribution with parameter $\frac{1}{p}$. Because of the one-to-one correspondence of cdfs and mgfs, we conclude that

$$F_S(t) = p + q(1-e^{-px}) = 1 - qe^{-px}, \quad x \geq 0 \blacksquare$$

Actuarial Survival Models

There are insurance policies that provide a benefit on the death of the policyholder. Since the death date of the policyholder is unknown, the insurer when issuing the policy does not know exactly when the death benefit will be payable. Thus, an estimate of the time of death is needed. For that, a model of human mortality is needed so that the probability of death at a certain age can be calculated. Survival models provide such a framework.
A **survival model** is a special kind of a probability distribution. In the actuarial context, a survival model can be the random variable that represents the future lifetime of an entity that existed at time 0. In reliability theory, an example of a survival model is the distribution of the random variable representing the lifetime of a light bulb. The light is said to survive as long as it keeps burning, and fails at the instant it burns out.
In Sections 18 and 20, we develop the basic nomenclature for describing two actuarial survival models: the distribution age-at-death random variable and the corresponding time-until-death random variable. We will define these variables, introduce some actuarial notation, and discuss some properties of these distributions.
Parametric survival models are models for which the survival function is given by a mathematical formula. In Section 19, we explore some important parametric survival models. In Section 21, the central-death-rate is introduced.

18 Age-At-Death Random Variable

The central difficulty in issuing life insurance is that of determining the time of death of the insured. In this section, we introduce the first survival model: The age-at-death distribution. Let time 0 denote the time of birth of an individual. We will always assume that everyone is alive at birth . The **age-at-death**[1] of the individual can be modeled by a positive continuous random variable X.

18.1 The Cumulative Distribution Function of X

The **cumulative distribution function** (CDF) of X is given by

$$F(x) = \Pr(X \le x) = \Pr(X < x).$$

That is, $F(x)$ is the probability that death will occur prior to (or at) age x. Since X is positive, the event $\{s : X(s) \le 0\}$ is impossible so that $F(0) = 0$. Also, from Theorem 9.1, we know that $F(\infty) = 1$, $F(x)$ is nondecreasing, $F(X)$ is right-continuous, and that $F(x) = 0$ for $x < 0$.

Example 18.1
Can the function $F(x) = \frac{1}{x+1},\ x \ge 0$ be a legitimate cumulative distribution function of an age-at-death random variable?

Solution.
Since $F(0) = 1 \neq 0$, the given function can not be a CDF of an age-at-death random variable ■

Example 18.2
Express the following probability statement in terms of $F(x)$: $\Pr(a < X \le b|X > a)$.

Solution.
By Bayes' formula, we have

$$\begin{aligned}\Pr(a < X \le b|X > a) =& \frac{\Pr[(a < X \le b) \cap (X > a)]}{\Pr(X > a)}\\ =& \frac{\Pr(a < X \le b)}{\Pr(X > a)} = \frac{F(b) - F(a)}{1 - F(a)}\ ■\end{aligned}$$

[1]We will consider the terms death and failure to be synonymous. The age-at-death is used in biological organisms whereas the age-at-failure is used in mechanical systems (known as **reliability theory** in engineering.)

Example 18.3
Let

$$F(x) = 1 - \left(1 - \frac{x}{120}\right)^{\frac{1}{6}},\ 0 \le x \le 120.$$

Determine the probability that a life aged 35 dies before the age of 55.

Solution.
The probability is given by

$$\Pr(35 < X \le 55|X > 35) = \frac{F(55) - F(35)}{1 - F(35)} = 0.0437 \blacksquare$$

An alternative notation for $F(x)$ used by actuaries is

$${}_xq_0 = F(x) = \Pr(X \le x).$$

Example 18.4
Given that ${}_xq_0 = 1 - e^{-0.008x},\ x \ge 0$. Find the probability that a newborn baby dies between age 60 and age 70.

Solution.
We have

$$\Pr(60 < X \le 70) = {}_{70}q_0 - {}_{60}q_0 = e^{-0.48} - e^{-0.56} = 0.04757 \blacksquare$$

Practice Problems

Problem 18.1
Can the function $F(x) = \frac{x}{3x+1},\ x \geq 0$ be a legitimate cumulative distribution function of an age-at-death random variable?

Problem 18.2
Which of the following formulas could serve as a cumulative distribution function?
(I) $F(x) = \frac{x+1}{x+3},\ x \geq 0.$
(II) $F(x) = \frac{x}{2x+1},\ x \geq 0.$
(III) $F(x) = \frac{x}{x+1},\ x \geq 0.$

Problem 18.3
Determine the constants A and B so that $F(x) = A + Be^{-0.5x},\ x \geq 0$ is a CDF.

Problem 18.4
The CDF of a continuous random variable is given by

$$F(x) = 1 - e^{-0.34x},\ x \geq 0.$$

Find $\Pr(10 < X \leq 23)$.

Problem 18.5
Let

$$F(x) = 1 - \left(1 - \frac{x}{120}\right)^{\frac{1}{6}},\ 0 \leq x \leq 120.$$

Determine the probability that a newborn survives beyond age 25.

Problem 18.6
The CDF of an age-at-death random variable is given by $F(x) = 1 - e^{-0.008x},\ x \geq 0$. Find the probability that a newborn baby dies between age 60 and age 70.

18.2 The Survival Distribution Function of X

In many life insurance problems, one is interested in the probability of survival rather than death, and so we define the **survival distribution function** (abbreviated SDF) by

$$s(x) = 1 - F(x) = \Pr(X > x),\ x \geq 0.$$

Thus, $s(x)$ is the probability that a newborn will survive to age x. From the properties of F we see that $s(0) = 1$, (everyone is alive at birth), $s(\infty) = 0$ (everyone dies eventually), $s(x)$ is right continuous, and that $s(x)$ is nonincreasing. These four conditions are necessary and sufficient so that any nonnegative function $s(x)$ that satisfies these conditions serves as a survival function.
In actuarial notation, the survival function is denoted by ${}_xp_0 = s(x) = \Pr(X > x) = 1 - {}_xq_0$.

Example 18.5
A survival distribution is defined by $s(x) = ax^2 + b$ for $0 \leq x \leq \omega$. Determine a and b.

Solution.
We have $1 = s(0) = b$ and $0 = s(\omega) = a\omega^2 + 1$. Solving for a we find $a = -\frac{1}{\omega^2}$. Thus, $s(x) = -\frac{x^2}{\omega^2} + 1$ ■

Example 18.6
Consider an age-at-death random variable X with survival distribution defined by

$$s(x) = \frac{1}{10}(100 - x)^{\frac{1}{2}},\ 0 \leq x \leq 100.$$

(a) Explain why this is a suitable survival function.
(b) Find the corresponding expression for the cumulative probability function.
(c) Compute the probability that a newborn with survival function defined above will die between the ages of 65 and 75.

Solution.
(a) We have that $s(0) = 1, s'(x) = -\frac{1}{20}(100 - x)^{-\frac{1}{2}} \leq 0$, $s(x)$ is right continuous, and $s(100) = 0$. Thus, s satisfies the properties of a survival function.

(b) We have $F(x) = 1 - s(x) = 1 - \frac{1}{10}(100 - x)^{\frac{1}{2}}$.
(c) We have

$$\Pr(65 < X \leq 75) = s(65) - s(75) = \frac{1}{10}(100 - 65)^{\frac{1}{2}} - \frac{1}{10}(100 - 75)^{\frac{1}{2}} \approx 0.092 \blacksquare$$

Example 18.7
The survival distribution function for an individual is determined to be

$$s(x) = \frac{75 - x}{75}, \quad 0 \leq x \leq 75.$$

(a) Find the probability that the person dies before reaching the age of 18.
(b) Find the probability that the person lives more than 55 years.
(c) Find the probability that the person dies between the ages of 25 and 70.

Solution.
(a) We have

$$\Pr(X < 18) = \Pr(X \leq 18) = F(18) = 1 - s(18) = 0.24.$$

(b) We have

$$\Pr(X > 55) = s(55) = 0.267.$$

(c) We have

$$\Pr(25 < X < 70) = F(70) - F(25) = s(25) - s(70) = 0.60 \blacksquare$$

Practice Problems

Problem 18.7
Show that the function $s(x) = e^{-0.34x}$ can serve as a survival distribution function, where $x \geq 0$.

Problem 18.8
Consider an age-at-death random variable X with survival distribution defined by
$$s(x) = e^{-0.34x},\ x \geq 0.$$
Compute $\Pr(5 < X < 10)$.

Problem 18.9
Consider an age-at-death random variable X with survival distribution
$$_xp_0 = e^{-0.34x},\ x \geq 0.$$
Find $_xq_0$.

Problem 18.10
Find the cumulative distribution function corresponding to the survival function $s(x) = 1 - \frac{x^2}{100}$ for $x \geq 0$ and 0 otherwise.

Problem 18.11
Which of the following is a SDF?
(I) $s(x) = (x+1)e^{-x},\ x \geq 0$.
(II) $s(x) = \frac{x}{2x+1},\ x \geq 0$.
(III) $s(x) = \frac{x+1}{x+2},\ x \geq 0$.

Problem 18.12
The survival distribution is given by $s(x) = 1 - \frac{x}{100}$ for $0 \leq x \leq 100$ and 0 otherwise.
(a) Find the probability that a person dies before reaching the age of 30.
(b) Find the probability that a person lives more than 70 years.

Problem 18.13
The mortality pattern of a certain population may be described as follows: Out of every 108 lives born together one dies annually until there are no survivors. Find a simple function that can be used as $s(x)$ for this population.

18.3 The Probability Density Function of X

The **probability density function** of an age-at-death random variable X measures the relative likelihood for death to occur for a given age. Values of the age-at-death random variable in regions with higher density values are more likely to occur than those in regions with lower values. See Figure 18.1.

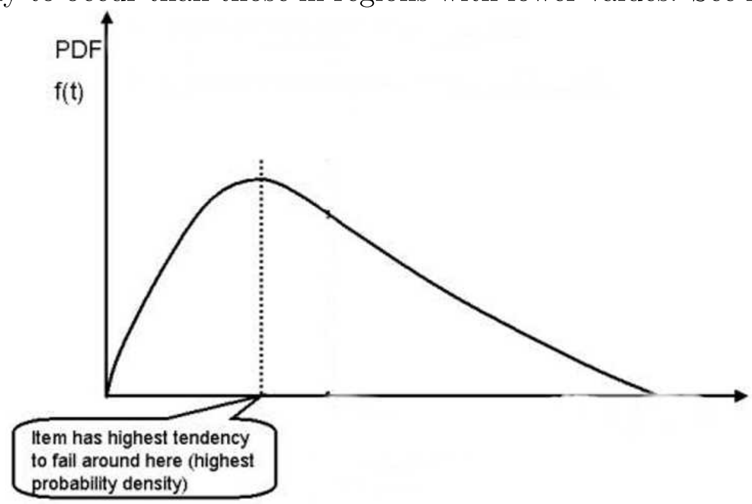


Figure 18.1

The PDF of X relates to a point and is considered as an instantaneous measure of death at age x compared to $F(x)$ and $s(x)$ which are probabilities over time intervals so they are considered as time-interval measures.
Mathematically, the probability density function, denoted by $f(x)$, is given by

$$f(x) = \frac{d}{dx}F(x) = -\frac{d}{dx}s(x), \ x \geq 0$$

whenever the derivative exists. Note that $f(x) \geq 0$ for $x \geq 0$ since F is nondecreasing for $x \geq 0$.
The PDF can assume values greater than 1 and therefore is not a probability. While the density function does not directly provide probabilities, it does provide relevant information. Probabilities for intervals, the cumulative and survival functions can be recovered by integration. That is, when the density

function is defined over the relevant interval, we have

$$\Pr(a < X \leq b) = \int_a^b f(x)dx, \quad F(x) = \int_{-\infty}^x f(t)dt = \int_0^x f(t)dt$$
$$s(x) = 1 - F(x) = \int_0^\infty f(t)dt - \int_0^x f(t)dt = \int_x^\infty f(t)dt.$$

Using laws of probability, we can use either $F(x)$ or $s(x)$ to answer probability statements.

Example 18.8
Explain why the function $f(x) = \frac{1}{(x+1)^{\frac{5}{2}}}$, $x \geq 0$ and 0 otherwise can not be a PDF.

Solution.
If $f(x)$ is a PDF then we must have $\int_0^\infty f(x)dx = 1$. But

$$\int_0^\infty \frac{dx}{(x+1)^{\frac{5}{2}}} = -\left.\frac{2}{3(x+1)^{\frac{3}{2}}}\right|_0^\infty = \frac{2}{3} \neq 1 \blacksquare$$

Example 18.9
Suppose that the survival function of a person is given by

$$s(x) = \frac{80 - x}{80}, \ 0 \leq x \leq 80.$$

Calculate
(a) $F(x)$.
(b) $f(x)$.
(c) $\Pr(20 < X < 50)$.

Solution.
(a) We have

$$F(x) = 1 - s(x) = \frac{x}{80}, \ x \geq 0.$$

(b) We have

$$f(x) = -s'(x) = \frac{1}{80}, \ x > 0.$$

(c) We have

$$\Pr(20 < X < 50) = s(20) - s(50) = 0.375 \blacksquare$$

Example 18.10
Find a formula both in terms of $F(x)$ and $s(x)$ for the conditional probability that an entity will die between the ages of x and z, given that the entity is survived to age x.

Solution.
We have

$$\begin{aligned}\Pr(x < X \leq z|X > x) =&\frac{\Pr[(x < X \leq z) \cap (X > x)]}{\Pr(X > x)}\\ =&\frac{\Pr(x < X \leq z)}{\Pr(X > x)}\\ =&\frac{F(z) - F(x)}{1 - F(x)}\\ =&\frac{s(x) - s(z)}{s(x)} \blacksquare\end{aligned}$$

Example 18.11
An age-at-death random variable is modeled by an exponential random variable with PDF $f(x) = 0.34e^{-0.34x}, \ x \geq 0$. Use the given PDF to estimate $\Pr(1 < X < 1.02)$.

Soltution.
We have (See Remark 2.2)

$$\Pr(1 < X < 1.02) \approx 0.02f(1) = 0.005 \blacksquare$$

Practice Problems

Problem 18.14
Determine which of the following functions is a probability density function of an age-at-death:
(a) $f(x) = \frac{2}{(2x+1)^2},\ x \geq 0$.
(b) $f(x) = \frac{1}{(x+1)^4},\ x \geq 0$.
(c) $f(x) = (2x-1)e^{-x},\ x \geq 0$.

Problem 18.15
The density function of a random variable X is given by $f(x) = xe^{-x}$ for $x \geq 0$. Find the survival distribution function of X.

Problem 18.16
Consider an age-at-death random variable X with survival distribution defined by

$$s(x) = e^{-0.34x},\ x \geq 0.$$

Find the PDF and CDF of X.

Problem 18.17
Find the probability density function of a continuous random variable X with survival function $s(x) = e^{-\lambda x},\ \lambda > 0,\ x \geq 0$.

Problem 18.18
The cumulative distribution function of an age-at-death random variable X is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{7x}{16}, & 0 \leq x < 1 \\ \frac{3x^2+4}{16}, & 1 \leq x < 2 \\ 1, & 2 \leq x. \end{cases}$$

Find the PDF of X.

18.4 Force of Mortality of X

The probability density function at x can be used to estimate the probability that death occurs in the itnerval $(x, x+dx]$ for small dx given only that the entity existed at age $x = 0$. That is, $f(x)$ can be considered as the unconditional death rate at age x. In this section, we want to define the conditional death rate at age x. That is, the conditional death rate at age x conditioned on the survival to age x. Let us denote it by $\mu(x)$.
By Example 18.10, the probability of a newborn will die between the ages of x and $x+h$ (where h is small) given that the newborn survived to age x is given by

$$\Pr(x < X \leq x+h|X > x) = \frac{F(x+h) - F(x)}{1 - F(x)}.$$

Since $f(x) = F'(x)$, whenever the derivative exists, we can write

$$\frac{F(x+h) - F(x)}{h} \approx f(x).$$

Thus,

$$\frac{\Pr(x < X \leq x+h|X > x)}{h} \approx \frac{f(x)}{s(x)}.$$

The death rate at age x given survival to age x is defined by

$$\mu(x) = \lim_{h \to 0} \frac{\Pr(x < X \leq x+h|X > x)}{h} = \frac{f(x)}{s(x)}.$$

In demography theory and actuarial science, $\mu(x)$ is called the **force of mortality** or **death rate**; in reliability theory the term **hazard rate function**[2] is used. Using the properties of $f(x)$ and $s(x)$ we see that $\mu(x) \geq 0$ for every $x \geq 0$.

Example 18.12
A life aged 50 has a force of mortality at age 50 equal to 0.0044. Estimate the probability that the person dies on his birthday.

Solution.
From the above definition with $x = 50, \mu(50) = 0.0044$, and $h = \frac{1}{365} =$

[2] Also knwon as the **hazard rate** or **failure rate.**

0.00274 we can write

$$\begin{aligned}\Pr(50 < X < 50 + 0.00274|X > 50) \approx & \mu(50) \times 0.0044 \\ = & (0.00274)(0.0044) = 1.2 \times 10^{-5} \blacksquare\end{aligned}$$

We can relate the mortality function to the survival function from birth as shown in the next example.

Example 18.13
Show that

$$\mu(x) = -\frac{s'(x)}{s(x)} = -\frac{d}{dx}[\ln s(x)]. \tag{18.1}$$

Solution.
The equation follows from $f(x) = -s'(x)$ and $\frac{d}{dx}[\ln s(x)] = \frac{s'(x)}{s(x)}$ ■

Example 18.14
Find the hazard rate function of an exponential random variable with parameter μ.

Solution.
We have

$$\mu(x) = \frac{f(x)}{s(x)} = \frac{\mu e^{-\mu x}}{e^{-\mu x}} = \mu.$$

The exponential random variable is an example of a **constant force model** ■

Example 18.15
Show that

$$s(x) = e^{-\Lambda(x)}$$

where

$$\Lambda(x) = \int_0^x \mu(s) ds.$$

Solution.
Integrating equation (18.1) from 0 to x, we have

$$\int_0^x \mu(s) ds = -\int_0^x \frac{d}{ds}[\ln s(s)] ds = \ln s(0) - \ln s(x) = \ln 1 - \ln s(x) = -\ln s(x).$$

Now the result follows upon exponentiation ■

The function $\Lambda(x)$ is called the **cumulative hazard function** or the **integrated hazard function** (CHF). The CHF can be thought of as the accumulation of hazard up to time x.

Example 18.16
Find the cumulative hazard function of the exponential random variable with parameter μ.

Solution.
We have
$$\Lambda(x) = \int_0^x \mu(s)ds = \int_0^x \mu dx = \mu x \blacksquare$$
Summarizing, if we know any one of the functions $\mu(x), \Lambda(x), s(x)$ we can derive the other two functions:
•$\mu(x) = -\frac{d}{dx} \ln s(x)$
•$\Lambda(x) = -\ln(s(x))$
•$s(x) = e^{-\Lambda(x)}$.

Now, for a function $\mu(x)$ to be an acceptable force of mortality, the function $s(x) = e^{-\Lambda(x)}$ must be an acceptable survival function.

Example 18.17
Which of the following functions can serve as a force of mortality?
(I) $\mu(x) = BC^x,\ B > 0,\ 0 < C < 1,\ x \geq 0$.
(II) $\mu(x) = B(x+1)^{-\frac{1}{2}},\ B > 0,\ x \geq 0$.
(III) $\mu(x) = k(x+1)^n,\ n > 0,\ k > 0,\ x \geq 0$.

Solution.
(I) Finding $s(x)$, we have
$$s(x) = e^{-\int_0^x \mu(t)dt} = e^{\frac{B}{\ln C}(1-C^x)}.$$
We have $s(x) \geq 0, s(0) = 1, s(\infty) = e^{\frac{B}{\ln C}} \neq 0$. Thus, the given function cannot be a force of mortality.
(II) Finding $s(x)$, we have
$$s(x) = e^{-\int_0^x \mu(t)dt} = e^{-2B[(x+1)^{\frac{1}{2}}-1]}.$$
We have, $s(x) \geq 0, s(0) = 1, s(\infty) = 0, s'(x) < 0$, and $s(x)$ is right-continuous. Thus, $\mu(x)$ is a legitimate force of mortality.
(III) Finding $s(x)$, we have
$$s(x) = e^{-\int_0^x \mu(t)dt} = e^{-\frac{kx^{n+1}}{n+1}}.$$
We have, $s(x) \geq 0, s(0) = 1, s(\infty) = 0, s'(x) < 0$, and $s(x)$ is right-continuous. Thus, $\mu(x)$ is a legitimate force of mortality $\blacksquare$

Practice Problems

Problem 18.19
Describe in words the difference between the two functions $f(x)$ and $\mu(x)$?

Problem 18.20
An age-at-death random variable has a survival function

$$s(x) = \frac{1}{10}(100-x)^{\frac{1}{2}},\ 0 \le x \le 100.$$

Find the force of mortality of this random variable.

Problem 18.21
Show that $f(x) = \mu(x)e^{-\Lambda(x)}$.

Problem 18.22
Show that the improper integral $\int_0^\infty \mu(x)dx$ is divergent.

Problem 18.23
Consider an age-at-death random variable X with survival distribution defined by

$$s(x) = e^{-0.34x},\ x \ge 0.$$

Find $\mu(x)$.

Problem 18.24
Consider an age-at-death random variable X with force of mortality $\mu(x) = \mu > 0$. Find $s(x), f(x)$, and $F(x)$.

Problem 18.25
Let

$$F(x) = 1 - \left(1 - \frac{x}{120}\right)^{\frac{1}{6}},\ 0 \le x \le 120.$$

Find $\mu(40)$.

Problem 18.26
Find the cumulative hazard function if $\mu(x) = \frac{1}{x+1},\ x \ge 0$.

Problem 18.27
The cumulative hazard function for an age-at-death random variable X is given by $\Lambda(x) = \ln\left(\frac{4}{4-x^2}\right),\ 0 \le x < 2$. Find the hazard rate function $\mu(x)$.

Problem 18.28
The cumulative hazard function for an age-at-death random variable X is given by $\Lambda(x) = \mu x,\ x \geq 0, \mu > 0$. Find $s(x), F(x)$, and $f(x)$.

Problem 18.29 ‡
Given: The survival function $s(x)$, where

$$s(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 1 - \frac{e^x}{100}, & 1 \leq x < 4.5 \\ 0, & 4.5 \leq x. \end{cases}$$

Calculate $\mu(4)$.

Problem 18.30 ‡
Given the following information:
(i) $\mu(x) = F + e^{2x}$
(ii) ${}_{0.4}p_0 = 0.5$.
Determine the value of F.

Problem 18.31 ‡
Which of the following formulas could serve as a force of mortality?
(I) $\mu(x) = BC^x,\ B > 0,\ C > 1,\ x \geq 0$
(II) $\mu(x) = a(b+x)^{-1}, a > 0,\ b > 0,\ x \geq 0$
(III) $\mu(x) = (1+x)^{-3},\ x \geq 0$.

Problem 18.32 ‡
For a loss distribution X where $x \geq 2$, you are given:

$$\text{(i) } \mu(x) = \frac{A^2}{2x},\ x \geq 0 \text{ and (ii) } F(5) = 0.84.$$

Determine the value of A.

Problem 18.33
The force of mortality of a survival model is given by $\mu(x) = \frac{3}{x+2},\ x \geq 0$. Find $\Lambda(x), s(x), F(x)$, and $f(x)$.

18.5 The Mean and Variance of X

Recall from Chapter 2 that for a continuous random variable X, the mean of X, also known as the (unconditional) first moment of X, is given by

$$E(X) = \int_0^\infty xf(x)dx$$

provided that the integral is convergent.
The following technical result is required in finding the mean and variance of X.

Theorem 18.1
If both the integrals $\int_0^\infty xf(x)dx$ and $\int_0^\infty x^2f(x)dx$ are convergent then

$$\lim_{x\to\infty} x^n s(x) = 0, \ n = 1, 2.$$

Proof
A proof for the case $n = 1$ is given. The case when $n = 2$ is left for the reader. Since $\int_0^\infty xf(x)dx < \infty$, we have for all $x > 0$

$$\int_x^\infty xf(x)dx \leq \int_0^\infty xf(x)dx < \infty.$$

Thus,

$$\begin{aligned}
\lim_{x\to\infty} xs(x) &= \lim_{x\to\infty} x\int_x^\infty f(y)dy \\
&= \lim_{x\to\infty}\int_x^\infty xf(y)dy \\
&\leq \lim_{x\to\infty}\int_x^\infty yf(y)dy = 0.
\end{aligned}$$

Thus, $0 \leq \lim_{x\to\infty} xs(x) \leq 0$ and the Squeeze Rule of Calculus imply

$$\lim_{x\to\infty} xs(x) = 0 \ \blacksquare$$

For the random variable X representing the death-at-age, we can use integration by parts and Theorem 18.1 above to write

$$\begin{aligned}
E(X) &= \int_0^\infty xf(x)dx = -\int_0^\infty xs'(x)dx \\
&= - \left. xs(x)\right]_0^\infty + \int_0^\infty s(x)dx = \int_0^\infty s(x)dx.
\end{aligned}$$

In the actuarial context, $E(X)$ is known as the **life expectancy** or the **complete expectation of life at birth** and is denoted by $\overset{\circ}{e}_0$.
Likewise, the mean of X^2 or the (unconditional) second moment of X is given by

$$E(X^2) = \int_0^\infty x^2 f(x)dx$$

provided that the integral is convergent.
Using integration by parts and Theorem 18.1 we find

$$E(X^2) = 2\int_0^\infty xs(x)dx.$$

The variance of X is given by

$$\text{Var}(X) = E(X^2) - [E(X)]^2.$$

Example 18.18
An actuary models the lifetime in years of a random selected person as a random variable X with survival function $s(x) = \left(1 - \frac{x}{100}\right)^{\frac{1}{2}}$, $0 \le x \le 100$. Find $\overset{\circ}{e}_0$ and $\text{Var}(X)$.

Solution.
The mean is

$$\overset{\circ}{e}_0 = E(X) = \int_0^{100} \left(1 - \frac{x}{100}\right)^{\frac{1}{2}} dx = -\frac{2}{3}\left(1 - \frac{x}{100}\right)^{\frac{3}{2}}\Bigg|_0^{100} = \frac{200}{3}.$$

The second moment of X is

$$E(X^2) = \int_0^{100} x^2 f(x)dx = \frac{1}{200}\int_0^{100} x^2\left(1 - \frac{x}{100}\right)^{-\frac{1}{2}} dx = \frac{16000}{3}.$$

Thus,

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{16000}{3} - \frac{40000}{9} = \frac{80000}{9} \blacksquare$$

Two more important concepts about X are the median and the mode. In Chapter 2, the **median** of a continuous random variable X is defined to be the value m satisfying the equation

$$\Pr(X \le m) = \Pr(X \ge m) = \frac{1}{2}.$$

Thus, the median age-at-death m is the solution to

$$F(m) = 1 - s(m) = \frac{1}{2}.$$

We define the **mode** of X to be the value of x that maximizes the PDF $f(x)$.

Example 18.19
Consider an age-at-death random variable X with cumulative distribution defined by

$$F(x) = 1 - \left(1 - \frac{x}{100}\right)^{\frac{1}{2}}, \ 0 \leq x \leq 100.$$

Find the median and the mode of X.

Solution.
To find the median, we have to solve the equation $\left(1 - \frac{m}{100}\right)^{\frac{1}{2}} = 0.5$ which gives $m = 75$.
To find the mode, we first find $f(x) = F'(x) = -\frac{1}{200}\left(1 - \frac{x}{100}\right)^{-\frac{1}{2}}$. The maximum of this function on the interval $[0, 100]$ occurs when $x = 0$ (you can check this either analytically or graphically) so that the mode of X is 0 ■

Practice Problems

Problem 18.34
Consider an age-at-death random variable X with survival function $s(x) = (1+x)e^{-x}, x \geq 0$. Calculate $E(X)$.

Problem 18.35
The age-at-death random variable has the PDF $f(x) = \frac{1}{k}, 0 \leq x \leq k$. Suppose that the expected age-at-death is 4. Find the median age-at-death.

Problem 18.36
Given $0 < p < 1$, the 100p−percentile (or p−th quantile) of a random variable X is a number α_p such that $\Pr(X \leq \alpha_p) = p$. Let X be an age-at-death random variable with PDF $f(x) = \frac{1}{k}, 0 \leq x \leq k$. Find the $p-$th quantile of X.

Problem 18.37
The first **quartile** of a random variable X is the 25−th percentile. Likewise, we define the second quartile to be the 50−th percentile, and the third quartile to be the 75−th percentile. Let X be an age-at-death random variable with PDF $f(x) = \frac{1}{k}, 0 \leq x \leq k$. Find the third quartile of X.

Problem 18.38
The survival distribution of an age-at-death random variable is given by $s(x) = -\frac{x^2}{k^2} + 1,\ 0 \leq x \leq k$. Given that $E(X) = 60$, find the median of X.

Problem 18.39
An actuary models the lifetime in years of a random selected person as a random variable X with survival function $s(x) = 1 - \frac{x^2}{8100},\ 0 \leq x \leq 90$. Calculate $\overset{\circ}{e}_0$ and $\mathrm{Var}(X)$.

Problem 18.40
A survival distribution has a force of mortality given by $\mu(x) = \frac{1}{720-6x}, 0 \leq x < 120$.
(a) Calculate $\overset{\circ}{e}_0$.
(b) Find $\Pr(30 \leq X \leq 60)$.

Problem 18.41
The PDF of a survival model is given by $f(x) = \frac{24}{2+x}$. Find the median and the mode of X.

19 Selected Parametric Survival Models

Parametric survival models are models for which the survival function $s(x)$ is given by a mathematical formula. In this section, we explore some important parametric survival models.

19.1 The Uniform or De Moivre's Model

Let X be a uniform random variable on the interval $[a, b]$. It is easy to see[3] that the PDF of this random variable is $f(x) = \frac{1}{b-a}$ for $a \le x \le b$ and zero elsewhere. This model is an example of a two-parameter model with parameters a and b.
If X is the age-at-death random variable, we take $a = 0$ and $b = \omega$ where ω is the maximum or **terminal age** by which all people have died then the PDF is $f(x) = \frac{1}{\omega}$ for $0 \le x \le \omega$ and 0 otherwise. In the actuarial context, this survival model is known as **De Moivre's Law**.

Example 19.1
Consider the uniform distribution model as defined above. Find $F(x), s(x)$, and $\mu(x)$.

Solution.
We have

$$\begin{aligned}
F(x) =& \int_0^x f(s)ds = \frac{x}{\omega} \\
s(x) =& 1 - F(x) = \frac{\omega - x}{\omega} \\
\mu(x) =& \frac{f(x)}{s(x)} = \frac{1}{\omega - x} \blacksquare
\end{aligned}$$

Example 19.2
Consider the uniform distribution model as defined above. Find $\overset{\circ}{e}_0$ and $\text{Var}(X)$.

Solution.
We have

$$\overset{\circ}{e}_0 = E(X) = \int_0^\omega x f(x) dx = \frac{\omega}{2}$$

[3]See Section 24 of [2].

and

$$E(X^2) = \int_0^{\omega} x^2 f(x)dx = \frac{\omega^2}{3}.$$

Thus,

$$\text{Var}(X) = \frac{\omega^2}{3} - \frac{\omega^2}{4} = \frac{\omega^2}{12} \blacksquare$$

Example 19.3
Suppose that X is uniformly distributed in $[0, \omega]$. For $0 \leq x \leq \omega$, define $T(x) = X - x$. Let $F_{T(x)}(t) = \Pr(T(x) \leq t | X > x)$. That is, $F_{T(x)}(t)$ is the probability that a person alive at age x to die by the age of $x + t$.
(a) Find a formula for $F_{T(x)}(t)$.
(b) Find $s_{T(x)}(t), f_{T(x)}(t)$, and $\mu_{T(x)}(t)$. Conclude that $T(x)$ is uniform on $[0, \omega - x]$.

Solution.
(a) We have

$$\begin{aligned} F_{T(x)}(t) =& \Pr(T(x) \leq t | X > x) = \frac{\Pr(x < X \leq x + t)}{\Pr(X > x)} \\ =& \frac{F(x + t) - F(x)}{1 - F(x)} = \frac{t}{\omega - x}. \end{aligned}$$

(b) We have

$$\begin{aligned} s_{T(x)}(t) =& 1 - F_{T(x)}(t) = \frac{(\omega - x) - t}{\omega - x} \\ f_{T(x)}(t) =& \frac{d}{dt}[F_{T(x)}(t)] = \frac{1}{\omega - x} \\ \mu_{T(x)}(t) =& -\frac{s'_{T(x)}(t)}{s_{T(x)}(t)} = \frac{1}{(\omega - x) - t}. \end{aligned}$$

It follows that $T(x)$ is uniform on $[0, \omega - x]$ $\blacksquare$

Practice Problems

Problem 19.1
Suppose that the age-at-death random variable X is uniform in $[0, \omega]$. Find $\text{Var}(X)$ if $\overset{\circ}{e}_0 = 45$.

Problem 19.2
Suppose that the age-at-death random variable X is uniform in $[0, \omega]$ with $\text{Var}(X) = \frac{625}{3}$. Find ω.

Problem 19.3
The survival function in De Moivre's Law is given by $s(x) = 1 - \frac{x}{90}, 0 \le x \le 90$. Calculate
(a) $\mu(x)$ (b) $F(x)$ (c) $f(x)$ (d) $\Pr(20 < X < 50)$.

Problem 19.4
A modified De Moivre's Law is defined by the survival function

$$s(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha}, \ 0 \le x \le \omega, \alpha > 0.$$

Find (a) $F(x)$ (b) $f(x)$ and (c) $\mu(x)$.

Problem 19.5
A mortality model is uniformly distributed in $[0, \omega]$. Find the probability that a life aged x will die within t years beyond x. That is, find $\Pr(x < X \le x + t | X > x)$.

Problem 19.6
A mortality model is uniformly distributed in $[0, \omega]$. Find the probability that a life aged x will attain age $x + t$. That is, find $\Pr(X > x + t | X > x)$.

Problem 19.7
A mortality model is uniformly distributed in $[0, 80]$. Find the median of an age-at-death.

Problem 19.8 ‡
The actuarial department for the SharpPoint Corporation models the lifetime of pencil sharpeners from purchase using a generalized De Moivre model with

$$s(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha}, \ \alpha > 0, \ 0 \le x \le \omega.$$

A senior actuary examining mortality tables for pencil sharpeners has determined that the original value of α must change. You are given:
(i) The new complete expectation of life at purchase is half what it was previously.
(ii) The new force of mortality for pencil sharpeners is 2.25 times the previous force of mortality for all durations.
(iii) ω remains the same.

Calculate the value of α.

Problem 19.9
A survival random variable is uniform on $[0, \omega]$. Find the cumulative hazard function.

19.2 The Exponential Model

In this survival model, the age-at-death random variable follows an exponential distribution with survival function given by

$$s(x) = e^{-\mu x},\ x \geq 0, \mu > 0.$$

This is an example of one-parameter model. This model is also known as the **constant force model** since the force of mortality is constant as shown in the next example.

Example 19.4
Consider the exponential model. Find $F(x), f(x)$, and $\mu(x)$.

Solution.
We have

$$\begin{aligned}
F(x) =&1 - s(x) = 1 - e^{-\mu x}\\
f(x) =&F'(x) = \mu e^{-\mu x}\\
\mu(x) =&\frac{f(x)}{s(x)} = \mu \blacksquare
\end{aligned}$$

Example 19.5
Consider the exponential model. Find $\overset{\circ}{e}_0$ and $\mathrm{Var}(X)$.

Solution.
We have

$$\overset{\circ}{e}_0 = E(X) = \int_0^\infty xf(x)dx = \int_0^\infty \mu x e^{-\mu x}dx = \frac{1}{\mu}$$

and

$$E(X^2) = \int_0^\infty x^2 f(x)dx = \int_0^\infty \mu x^2 e^{-\mu x}dx = \frac{2}{\mu^2}.$$

Thus,

$$\mathrm{Var}(X) = \frac{2}{\mu^2} - \frac{1}{\mu^2} = \frac{1}{\mu^2} \blacksquare$$

Example 19.6 *(Memory less property)*
Show that

$$\Pr(X > x + t|X > x) = \Pr(X > t).$$

Solution.
Let A denote the event $X > x+t$ and B the event $X > x$. Using the definition of conditional probability, we have

$$\begin{aligned}\Pr(X > x+t|X > x) =&\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}\\ =&\frac{\Pr(A)}{\Pr(B)} = \frac{s(x+t)}{s(x)}\\ =&\frac{e^{-\mu(x+t)}}{e^{-\mu x}} = e^{-\mu t}\\ =&s(t) = \Pr(X > t).\end{aligned}$$

In words, given the survival to time x, the chance of surviving a further time t is the same as the chance of surviving to time t in the first place ∎

Example 19.7
Consider an exponentail model with density function $f(x) = 0.01e^{-0.01x},\ x > 0$. Calculate ${}_xq_0$ and ${}_xp_0$.

Solution.
We have

$$\begin{aligned}{}_xq_0 =&\Pr(X \le x) = F(x) = 1 - e^{-0.01x}\\ {}_xp_0 =&1 - {}_xq_0 = e^{-0.01x}\ \blacksquare\end{aligned}$$

Example 19.8
Let the age-at-death be exponential with density function $f(x) = 0.01e^{-0.01x}$. Find $\Pr(1 < x < 2)$.

Solution.
We have

$$\Pr(1 < X < 2) = s(1) - s(2) = {}_1p_0 - {}_2p_0 = e^{-0.01} - e^{-0.02} = 0.00985\ \blacksquare$$

Example 19.9 ‡
Given the survival function

$$s(x) = \begin{cases} 1, & 0 \le x \le 1\\ 1 - 0.01e^x & 1 < x < 4.5\\ 0. & x \ge 4.5\end{cases}$$

Find $\mu(4)$.

Solution.
We have

$$\mu(4) = -\frac{s'(4)}{s(4)} = \frac{0.01e^4}{1 - 0.01e^4} = 1.203 \blacksquare$$

Practice Problems

Problem 19.10
Consider an exponential survival model with $s(x) = e^{-0.04x}$ where x in years. Find the probability that a newborn survives beyond 20 years.

Problem 19.11
An exponential model has a force of mortality equals to 0.04. Find the probability that a newborn dies between the age of 20 and 30.

Problem 19.12
An exponential model has a survival function $s(x) = e^{-\frac{x}{2}}, x \geq 0$. Calculate $\mu(40)$.

Problem 19.13
Find the value of μ and the median survival time for an exponential survival function if $s(3) = 0.4$.

Problem 19.14
A mortality model is exponentially distributed with parameter μ. Find the probability that a life aged x will die within t years beyond x. That is, find $\Pr(x < X \leq x + t | X > x)$.

Problem 19.15
The lifetime X of a device is described as follows:
- For $0 \leq x < 40$, X is uniformly distributed on $[0, 80)$.
- For $X \geq 40$, X is exponentially distributed with parameter 0.03.

Find the expected lifetime of the device.

Problem 19.16
The age-at-death random variable is described by the PDF $f(x) = \frac{1}{60}e^{-\frac{x}{60}}$. Find $\overset{\circ}{e}_0$ and $\text{Var}(X)$.

19.3 The Gompertz Model

One of the very first attempt to develop a parametric model was Gompertz model (1825). He conjectured that the hazard function for human mortality should increase at a rate proportional to the function itself. That is,

$$\frac{d\mu(x)}{dx} = K\mu(x).$$

Solving this differential equation, we find that the force of mortality of this model is given by

$$\mu(x) = Bc^x,$$

where $B > 0, c > 1, x \geq 0$. This model is a two-parameter model.

Example 19.10
Find the survival function of Gompertz model.

Solution.
We have

$$s(x) = e^{-\int_0^x \mu(s)ds} = e^{-\int_0^x Bc^s ds} = e^{\frac{B}{\ln c}(1-c^x)} \blacksquare$$

Example 19.11
Under Gompertz' Law, evaluate $e^{-\int_x^{x+t} \mu(y)dy}$.

Solution.
We have

$$\int_x^{x+t} \mu(y)dy = \int_x^{x+t} Bc^y dy = \left.\frac{B}{\ln c}c^y\right|_x^{x+t} = \frac{Bc^x}{\ln c}(c^t - 1)$$

and

$$e^{-\int_x^{x+t} \mu(y)dy} = e^{\frac{Bc^x}{\ln c}(1-c^t)} \blacksquare$$

Under Gompertz' Law, the parameter values B and c can be determined given the value of the force of mortality at any two ages.

Example 19.12
Under Gompertz' Law, you are given that $\mu(20) = 0.0102$ and $\mu(50) = 0.025$. Find $\mu(x)$.

Solution.
The given hypotheses lead to the two equations $Bc^{20} = 0.0102$ and $Bc^{50} = 0.025$. Thus, taking ratios we find $c^{30} = \frac{0.025}{0.0102}$. Solving for c we find $c = 1.03$. Thus, $B = \frac{0.0102}{1.03^{20}} = 0.0056$. The force of mortality is given by $\mu(x) = 0.0056(1.03)^x$ ■

Example 19.13
You are given that mortality follows Gompertz with $B = 0.01$ and $c = 1.1$. Calculate
(i) $\mu(10)$
(ii) the probability of a life aged 20 to attain age 30
(iii) the probability of a life aged 20 to die within 10 years.

Solution.
(i) We have
$$\mu(10) = BC^{10} = 0.01(1.1)^{10} = 0.025937$$
(ii) We have
$$\begin{aligned}\Pr(X > 30|X > 20) =&\frac{\Pr(X > 30)}{\Pr(X > 20)} = \frac{s(30)}{s(20)}\\ =&\frac{e^{\frac{0.01}{\ln 1.1}(1-1.1^{30})}}{e^{\frac{0.01}{\ln 1.1}(1-1.1^{20})}} = 0.32467.\end{aligned}$$
(iii) We have
$$\Pr(X < 30|X > 20) = 1 - \Pr(X > 30|X > 20) = 1 - 0.32467 = 0.67533 \blacksquare$$

Practice Problems

Problem 19.17
Which of the following statement is true about Gompertz' Law?
(I) $\mu(x)$ is linear
(II) $\mu(x)$ is logarithmic
(III) $\mu(x)$ is log-linear.

Problem 19.18
Consider a Gompertz model with the following information:
- $c = 1.03$.
- $s(40) = 0.65$.

Calculate $\mu(40)$.

Problem 19.19
Find the survival and the cumulative distribution functions for the Gompertz model.

Problem 19.20
Find the probability density function for the Gompertz model.

Problem 19.21
Find the cumulative hazard function for the Gompertz model.

Problem 19.22
Suppose that the lives of a certain species follow Gompertz's Law. It is given that $\mu(0) = 0.43$ and $\mu(1) = 0.86$. Determine $\mu(4)$.

Problem 19.23
Suppose that Gompertz' Law applies with $\mu(30) = 0.00013$ and $\mu(50) = 0.000344$. Find $\frac{s'(x)}{s(x)}$.

Problem 19.24
A survival model follows Gompertz' Law with parameters $B = 0.0004$ and $c = 1.07$. Find the cumulative distribution function.

19.4 The Modified Gompertz Model: The Makeham's Model

The Gompertz function is an age-dependent function. That is, age was considered the only cause of death. It has been observed that this model was not a good fit for a certain range of ages. In 1867, Makeham proposed a modification of the model by adding a positive constant that covers causes of deaths that were age-independent, such as accidents. In this model, the death rate or the force of mortality is described by

$$\mu(x) = A + Bc^x,$$

where $B > 0, A \geq -B, c > 1, x \geq 0$. In words, the model states that death rate or the force of mortality is the sum of an age-independent component (the **Makeham term** or the **accident hazard**) and an age-dependent component (the **Gompertz function** or the **hazard of aging**) which increases exponentially with age. Note that the Makeham's model is a three-parameter model.

Example 19.14
Find the survival function of Makeham's model.

Solution.
We have

$$s(x) = e^{-\int_0^x \mu(s)ds} = e^{-\int_0^x (A+Bc^s)ds} = e^{-Ax-\frac{B}{\ln c}(c^x-1)} \blacksquare$$

Example 19.15
Using Makeham's model, find an expression for ${}_tp_x = e^{-\int_x^{x+t} \mu(y)dy}$.

Solution.
We have

$$\int_x^{x+t} \mu(y)dy = \int_x^{x+t} (A + Bc^y)dy = \left[Ay + \frac{B}{\ln c}c^y\right]_x^{x+t} = At + \frac{Bc^x}{\ln c}(c^t - 1).$$

Thus,

$${}_tp_x = e^{-\int_x^{x+t} \mu(y)dy} = e^{-At-\frac{Bc^x}{\ln c}(c^t-1)} \blacksquare$$

Assuming that the underlying force of mortality follows Makeham's Law, the parameter values A, B and c can be determined given the value of the force of mortality at any three ages.

Example 19.16
A survival model follows Makeham's Law. You are given the following information:
- ${}_5p_{70} = 0.70$.
- ${}_5p_{80} = 0.40$.
- ${}_5p_{90} = 0.15$.

Determine the parameters A, B, and c.

Solution.
Let $\alpha = e^{-A}$ and $\beta = e^{\frac{B}{\ln c}}$ in the previous problem so that ${}_tp_x = \alpha^t \beta^{c^x(1-c^t)}$. From the given hypotheses we can write

$$\begin{aligned}\alpha^5\beta^{c^{70}(1-c^5)} =&0.70\\ \alpha^5\beta^{c^{80}(1-c^5)} =&0.40\\ \alpha^5\beta^{c^{90}(1-c^5)} =&0.15.\end{aligned}$$

Hence,

$$\ln\left(\frac{0.40}{0.70}\right) = c^{70}(1-c^5)(c^{10}-1)\ln\beta \tag{19.1}$$

and

$$\ln\left(\frac{0.15}{0.40}\right) = c^{80}(1-c^5)(c^{10}-1)\ln\beta.$$

From these two last equations, we find

$$c^{10} = \ln\left(\frac{0.15}{0.40}\right) \div \ln\left(\frac{0.40}{0.70}\right) \Longrightarrow c = 1.057719.$$

Substituting this into (19.1) we find

$$\ln\beta = \ln\left(\frac{0.40}{0.70}\right) \div c^{70}(1-c^5)(c^{10}-1) \Longrightarrow \beta = 1.04622.$$

Solving the equation

$$e^{\frac{B}{\ln 1.057719}} = 1.04622$$

we find $B = 0.002535$. Also,

$$A = -0.2\ln\left[0.70\beta^{c^{70}(c^5-1)}\right] = -0.07737.$$

Finally, the force of mortality is given by

$$\mu(x) = -0.07737 + 1.04622(1.057719)^x,\ x \ge 0 \blacksquare$$

Practice Problems

Problem 19.25
Suppose that the lives of a certain species follow Makeham's Law. Find a formula for the PDF $f(x)$.

Problem 19.26
Suppose that the lives of a certain species follow Makeham's Law. Find a formula for the CDF $F(x)$.

Problem 19.27
The force of mortality of Makeham's model is given by $\mu(x) = 0.31 + 0.43(2^x), x \geq 0$. Find $f(x)$ and $F(x)$.

Problem 19.28
The force of mortality of Makeham's model is given by $\mu(x) = 0.31 + 0.43(2^x), x \geq 0$. Calculate $s(3)$.

Problem 19.29
The following information are given about a Makeham's model:
• The accident hazard is 0.31.
• The hazard of aging is 1.72 for $x = 2$ and 3.44 for $x = 3$.
Find $\mu(x)$.

Problem 19.30
The force of mortality of a Makeham's Law is given by $\mu(x) = A+0.1(1.003)^x, x \geq 0$. Find $\mu(5)$ if $s(35) = 0.02$.

19.5 The Weibull Model

The Weibull law of mortality is defined by the hazard function

$$\mu(x) = kx^n,$$

where $k > 0, n > 0, x \geq 0$. That is, the death rate is proportional to a power of age. Notice that the exponential model is a special case of Weibull model where $n = 0$.

Example 19.17
Find the survival function corresponding to Weibull model.

Solution.
We have

$$s(x) = e^{-\int_0^x \mu(s)ds} = e^{-\int_0^x ks^n ds} = e^{-\frac{kx^{n+1}}{n+1}} \blacksquare$$

Example 19.18
Suppose that X follows an exponential model with $\mu = 1$. Define the random variable $Y = h(X) = X^{\frac{1}{3}}$. Show that Y follows a Weibull distribution. Determine the values of k and n.

Solution.
We need to find the force of mortality of Y using the formula $\mu_Y(y) = \frac{f_Y(y)}{s_Y(y)}$. We first find the CDF of Y. We have

$$F_Y(y) = \Pr(Y \leq y) = \Pr(X^{\frac{1}{3}} \leq y) = \Pr(X \leq y^3) = 1 - e^{-y^3}.$$

Thus,

$$s_Y(y) = 1 - F_Y(y) = e^{-y^3}$$

and

$$f_Y(y) = F'_Y(y) = 3y^2 e^{-y^3}.$$

The force of mortality of Y is

$$\mu_Y(y) = \frac{f_Y(y)}{s_Y(y)} = 3y^2.$$

It follows that Y follows a Weibull distribution with $n = 2$ and $k = 3$ $\blacksquare$

Example 19.19
You are given that mortality follows Weibull with $k = 0.00375$ and $n = 1.5$. Calculate
(i) $\mu(10)$
(ii) ${}_{10}p_{20} = e^{-\int_{20}^{30}\mu(x)dx}$.
(iii) ${}_{10}q_{20} = 1 - {}_{10}p_{20}$.

Solution.
(i) $\mu(10) = kx^n = 0.00375(10)^{1.5} = 0.11859$
(ii) ${}_{10}p_{20} = e^{-\int_{20}^{30} 0.00375x^{1.5}dx} = 0.089960$
(iii) ${}_{10}q_{20} = 1 - {}_{10}p_{20} = 1 - 0.089960 = 0.99100$ ■

Practice Problems

Problem 19.31
Find the PDF corresponding to Weibull model.

Problem 19.32
Suppose that X follows an exponential model with $\mu = 1$. Define the random variable $Y = h(X) = X^{\frac{1}{2}}$. Show that Y follows a Weibull distribution. Determine the values of k and n.

Problem 19.33
Consider a Weibull model with $k = 2$ and $n = 1$. Calculate $\mu(15)$.

Problem 19.34
Consider a Weibull model with $k = 2$ and $n = 1$. Calculate $s(15)$.

Problem 19.35
Consider a Weibull model with $k = 3$. It is given that $\mu(2) = 12$. Calculate $s(4)$.

Problem 19.36
Consider a Weibull model with $n = 1$. It is given that $s(20) = 0.14$. Determine the value of k.

Problem 19.37
A survival model follows a Weibull Law with mortality function $\mu(x) = kx^n$. It is given that $\mu(40) = 0.0025$ and $\mu(60) = 0.02$. Determine the parameters k and n

20 Time-Until-Death Random Variable

Up to this point, we explored the random variable that represents the age-at-death of a newborn that existed at time 0. In this section, we introduce the random variable representing the time-until-death past a certain age where the person is known to be alive.
First, we let (x) to indicate that a newborn is known to be alive at age x or simply a life aged x. We let $T(x)$ be the continuous random variable that represents the additional time (x) might survive beyond the age of x. We refer to this variable as the **time-until-death** or the **future-lifetime** random variable. From this definition, we have a relationship between the age-at-death random variable X and the time-until-death $T(x)$ given by

$$X = x + T(x).$$

A pictorial representation of X, x and $T(x)$ is shown in Figure 20.1.

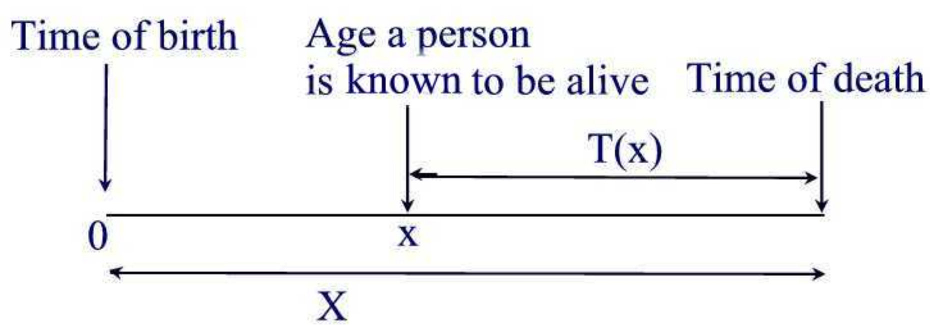


Figure 20.1

Note that if Support$(X) = [0, \omega)$ then Support$(T(x)) = [0, \omega - x]$.

20.1 The Survival Function of $T(x)$

We next calculate the survival function of $T(x)$. For $t \geq 0$, the actuarial notation of the survival function of $T(x)$ is ${}_tp_x$. Thus, ${}_tp_x$ is the probability of a life aged x to attain age $x + t$. That is, ${}_tp_x$ is a conditional probability. In symbol,

$${}_tp_x = s_{T(x)}(t) = \Pr(X > x + t | X > x) = \Pr[T(x) > t]. \tag{20.1}$$

In the particular case of a life aged 0, we have $T(0) = X$ and ${}_xp_0 = s(x), x \geq 0$. The next example explores a relationship between the survival functions of both X and $T(x)$.

Example 20.1
Show that

$$ {}_tp_x = \frac{s(x+t)}{s(x)} = \frac{{}_{x+t}p_0}{{}_xp_0}, \ t \geq 0. \tag{20.2}$$

Solution.
Using the fact that the event $\{X > x+t\}$ is a subset of the event $\{X > x\}$ and the conditional probability formula $\Pr(E|F) = \frac{\Pr(E\cap F)}{\Pr(F)}$ we find

$$\begin{aligned} {}_tp_x =& \Pr(X > x+t|X > x) = \frac{\Pr[(X > x+t)\cap(X > x)]}{\Pr(X > x)} \\ =& \frac{\Pr(X > x+t)}{\Pr(X > x)} = \frac{s(x+t)}{s(x)} \blacksquare \end{aligned}$$

Example 20.2
Find ${}_tp_x$ in the case X is exponentially distributed with parameter μ.

Solution.
We have

$$ {}_tp_x = \frac{s(x+t)}{s(x)} = \frac{e^{-\mu(x+t)}}{e^{-\mu x}} = e^{-\mu t} \blacksquare $$

Example 20.3
Suppose that ${}_tp_x = \frac{75-x-t}{75-x}, \ 0 \leq t \leq 75 - x$. Find the probability that a 35-year-old reaches age 70.

Solution.
We have

$$ {}_{70-35}p_{35} = \frac{75-35-35}{75-35} = \frac{5}{35} = 0.143 \blacksquare $$

Example 20.4 ‡
You are given:
(i) $R = 1 - e^{-\int_0^1 \mu(x+t)dt}$
(ii) $S = 1 - e^{-\int_0^1 (\mu(x+t)+k)dt}$
(iii) k is a constant such that $S = 0.75R$.
Determine an expression for k.

Solution.
We have

$$\begin{aligned}
R =& 1 - e^{-\int_0 1\mu(x+t)dt} = 1 - p_x \\
S =& 1 - e^{-\int_0 1(\mu(x+t)+k)dt} = 1 - e^{-\int_0 1\mu(x+t)dt} e^{-\int_0 1kdt} = 1 - p_x e^{-k}.
\end{aligned}$$

Hence,

$$\begin{aligned}
S =& 0.75R \\
1 - p_x e^{-k} =& 0.75(1 - p_x) \\
p_x e^{-k} =& 0.25 + 0.75p_x \\
e^{-k} =& \frac{0.25 + 0.75p_x}{p_x} \\
k =& \ln\left[\frac{p_x}{0.25 + 0.75p_x}\right] \blacksquare
\end{aligned}$$

Practice Problems

Problem 20.1
Show that if X is uniform on $[0,\omega]$ then $T(x)$ is uniform in $[0,\omega - x]$.

Problem 20.2
You are given the survival function $s(x) = 1 - \frac{x}{75}, 0 \le x \le 75$. Find the survival function and the probability density function of $T(x)$.

Problem 20.3
Show: ${}_{m+n}p_x = {}_mp_x \cdot {}_np_{x+m}$.

Problem 20.4
Suppose that the probability of a 50-year old person to reach age 60 is 0.97 and the probability of a 45-year old to reach 60 is 0.95. Find the probability that a 45-year old to reach 70.

Problem 20.5
Show that ${}_np_x = p_xp_{x+1}p_{x+2}\cdots p_{x+n-1}$, where n is a positive integer.

Problem 20.6
The survival function is given by $s(x) = 1 - \frac{x}{100},\ x \ge 0$.
(a) Express the probability that a person aged 35 will die between the ages of 52 and 73 using the p notation.
(b) Calculate the probability in (a).

Problem 20.7
The PDF of X is given by $f(x) = \frac{1}{(x+1)^2},\ x \ge 0$. Find ${}_tp_3$.

Problem 20.8
You are given the survival function $s(x) = \frac{1}{x}, x > 1$. What is the probability that a newborn that has survived to age 65 will survive to age 68?

Problem 20.9
Show that ${}_tp_x = e^{-\int_x^{x+t}\mu(y)dy}$. That is, all death probabilities can be expressed in terms of the force of mortality.

Problem 20.10
Given that $\mu(x) = kx$ and ${}_{10}p_{35} = 0.81$. Determine the value of k and find ${}_{20}p_{40}$.

Problem 20.11
The mortality pattern of a certain population may be described as follows: Out of every 108 lives born together one dies annually until there are no survivors.
(a) Find ${}_{10}p_{25}$.
(b) What is the probability that a life aged 30 will survive to attain age 35?

Problem 20.12
Show that $\frac{\partial}{\partial t}{}_tp_x = -{}_tp_x\mu(x+t)$.

Problem 20.13
Assume that a survival random variable obeys Makeham's Law $\mu(x) = A + Bc^x$. You are given that ${}_5p_{70} = 0.73$, ${}_5p_{75} = 0.62$, and ${}_5p_{80} = 0.50$. Determine the parameters A, B, and c.

20.2 The Cumulative Distribution Function of $T(x)$

From the survival function we can find the cumulative distribution function of $T(x)$ which we denote by ${}_tq_x = F_{T(x)}(t)$. Thus,

$$ {}_tq_x = \Pr[T(x) \le t] = 1 - {}_tp_x. $$

Note that ${}_tq_x$ is the probability of (x) does not survive beyond age $x+t$ or the conditional probability that death occurs not later than age $x+t$, given survival to age x. It follows from (20.2) that

$$ {}_tq_x = 1 - \frac{s(x+t)}{s(x)}. $$

Example 20.5
Express ${}_tq_x$ in terms of the cumulative distribution function of X.

Solution.
Since $s(x) = 1 - F(x)$, we obtain

$$ {}_tq_x = \frac{s(x) - s(x+t)}{s(x)} = \frac{F(x+t) - F(x)}{1 - F(x)} \blacksquare $$

In actuarial notation, when $t = 1$ year we will suppress the subscript to the left of p and q and use the notation p_x and q_x. Thus, the probability that (x) will die within one year beyond age x is q_x and the probability that (x) will survive an additional year is p_x.

Example 20.6
Find ${}_3p_5$ and ${}_4q_7$ if $s(x) = e^{-0.12x}$.

Solution.
We have

$$ {}_3p_5 = \frac{s(3+5)}{s(5)} = \frac{e^{-0.12(8)}}{e^{-0.12(5)}} = 0.69768. $$

Likewise,

$$ {}_4q_7 = 1 - \frac{s(4+7)}{s(7)} = 1 - \frac{e^{-0.12(11)}}{e^{-0.12(7)}} = 0.38122 \blacksquare $$

Example 20.7
Consider the two events $A = [T(x) \leq t]$ and $B = [t < T(x) \leq 1]$ where $0 < t < 1$. Rewrite the identity

$$\Pr(A \cup B) = \Pr(A) + \Pr(A^c)\Pr(B|A^c)$$

in actuarial notation.

Solution.
We have

$$\begin{aligned}
\Pr(A \cup B) =& \Pr(T(x) \leq 1) = q_x \\
\Pr(A) =& \Pr(T(x) \leq t) = {}_tq_x \\
\Pr(A^c) =& 1 - \Pr(A) = 1 - {}_tq_x = {}_tp_x \\
\Pr(B|A^c) =& \frac{\Pr(B \cap A^c)}{\Pr(A^c)} \\
=& \frac{\Pr(B)}{\Pr(A^c)} \\
=& \frac{{}_tp_x - {}_1p_x}{{}_tp_x} \\
=& 1 - \frac{\frac{s(x+1)}{s(x)}}{\frac{s(x+t)}{s(x)}} \\
=& \frac{s(x+t) - s(x+1)}{s(x+t)} \\
=& {}_{1-t}q_{x+t}.
\end{aligned}$$

Thus, we have

$$q_x = {}_tq_x + {}_tp_x {}_{1-t}q_{x+t} \blacksquare$$

Practice Problems

Problem 20.14
The probability that (x) will survive additional t years and die within the following u years is denoted by ${}_{t|u}q_x$. That is, ${}_{t|u}q_x$ is the probability that (x) will die between the ages of $x+t$ and $x+t+u$. In the case $u=1$ we will use the notation ${}_{t|}q_x$. Show that

$${}_{t|u}q_x = {}_{t+u}q_x - {}_tq_x = {}_tp_x - {}_{t+u}p_x.$$

Problem 20.15
With the notation of the previous problem, show that ${}_{t|u}q_x = {}_tp_x \cdot {}_uq_{x+t}$.

Problem 20.16
An age-at-death random variable has the survival function $s(x) = 1 - \frac{x}{100}, 0 \leq x < 100$. Calculate ${}_{20}q_{65}$.

Problem 20.17
You are given the cumulative distribution function $F(x) = (0.01x)^2, 0 \leq x \leq 100$. What is the probability that a newborn who survived to age 55 will survive to age 56?

Problem 20.18
You are given the survival function $s(x) = 1 - \frac{x}{75}, 0 \leq x < 75$. What is the probability of a newborn who lived to age 35 to die within one year?

Problem 20.19
Show that $\frac{\partial}{\partial t}{}_tq_x = {}_tp_x\mu(x+t)$.

Problem 20.20
The survival function for (x) is given by

$$s_{T(x)}(t) = \left(\frac{\alpha}{\alpha+t}\right)^{\beta}, \ \alpha, \beta > 0.$$

Find an expression for ${}_tq_x$.

Problem 20.21
Given the survival function $s(x) = \left(\frac{100}{100+x}\right)^{1.1}$. What is the probability of a newborn who lived to age 25 to survive an additional year?

Problem 20.22
Write in words the meaning of ${}_{2|}q_1$.

Problem 20.23
The survival function is given by

$$s(x) = (1 - 0.01x)^{0.5}, \quad 0 \le x \le 1.$$

What is the probability that a life aged 40 to die between the ages of 60 and 80?

Problem 20.24
You are given the hazard rate function

$$\mu(x) = \frac{1.1}{100 + x}, \; x \ge 0.$$

Find a formula for ${}_tq_{20}$.

Problem 20.25
The probability that a life aged 20 to live to age 30 is 0.9157. The probability of a life aged 30 to live to age 50 is 0.7823. What is the probability for a life aged 20 to die between the ages of 30 and 50?

Problem 20.26 ‡
The graph of a piecewise linear survival function, $s(x)$, consists of 3 line segments with endpoints $(0, 1), (25, 0.50), (75, 0.40), (100, 0)$.

Calculate $\frac{{}_{20|55}q_{15}}{{}_{55}q_{35}}$.

20.3 Probability Density Function of $T(x)$

Now, from the CDF of $T(x)$ we can find an expression for the probability density function of $T(x)$. Indeed, we have

$$f_{T(x)}(t) = \frac{d}{dt}[{}_tq_x] = -\frac{d}{dt}\left(\frac{s(x+t)}{s(x)}\right) = \frac{f(x+t)}{s(x)}.$$

In words, $f_{T(x)}(t)$ is the conditional density of death at time t given survival to age x or the conditional density at age $x+t$, given survival to age x.

Example 20.8
An age-at-death random variable has the survival function $s(x) = 1 - \frac{x^2}{10000}$ for $0 \le x < 100$.
(a) Find the survival function of $T(x)$.
(b) Find the PDF of $T(x)$.

Solution.
(a) The survival function of $T(x)$ is

$${}_tp_x = \frac{s(x+t)}{s(x)} = \frac{1 - \frac{(x+t)^2}{10000}}{1 - \frac{x^2}{10000}} = \frac{10000 - (x+t)^2}{10000 - x^2}, \ 0 \le t \le 100 - x.$$

(b) The PDF of $T(x)$ is

$$f_{T(x)}(t) = -\frac{d}{dt}\left(\frac{s(x+t)}{s(x)}\right) = \frac{2(x+t)}{10000 - x^2}, \ 0 \le t \le 100 - x \ \blacksquare$$

Example 20.9 ‡
A Mars probe has two batteries. Once a battery is activated, its future lifetime is exponential with mean 1 year. The first battery is activated when the probe lands on Mars. The second battery is activated when the first fails. Battery lifetimes after activation are independent. The probe transmits data until both batteries have failed.
Calculate the probability that the probe is transmitting data three years after landing.

Solution.
Let X and Y denote the future liftimes of the first and second batteries respectively. We are told that X and Y are independent exponential random variables with common parameter 1. Thus,

$$f_X(x) = e^{-x} \text{ and } f_Y(x) = e^{-y}.$$

Let $Z = X + Y$. Since both X and Y are positive random variables, we have $0 < X < Z$ and $0 < Y < Z$. By Section 16.3, the distribution of Z is given by

$$f_Z(z) = \int_0^z f_X(z-y)f_Y(y)dy = \int_0^z e^{y-z}e^{-y}dy = \int_0^z e^{-z}dy = ze^{-z}.$$

Hence,

$$\begin{aligned}\Pr(X+Y>3) =& \int_3^\infty f_Z(z)dz = \int_3^\infty ze^{-z}dz\\ =& \left[-ze^{-z} + e^{-z}\right]_3^\infty\\ =& 3e^{-3} + e^{-3} = 4e^{-3} = 0.20 \ \blacksquare\end{aligned}$$

Example 20.10 ‡
For the future lifetimes of (x) and (y) :
(i) With probability 0.4, $T(x) = T(y)$ (i.e., deaths occur simultaneously).
(ii) With probability 0.6, the joint density function is

$$f_{T(x),T(y)}(t,s) = 0.0005, \ 0 < t < 40, \ 0 < s < 50.$$

Calculate $\Pr[T(x) < T(y)]$.

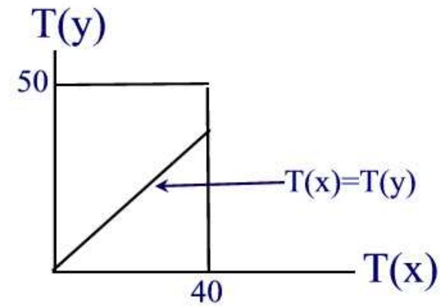
Solution.
Define the indicator random variable

$$I = \begin{cases} 0 & T(x) = T(y) \\ 1 & T(x) \neq T(y) \end{cases}$$

Then

$$\begin{aligned}\Pr[T(x) < T(y)] =& \Pr[T(x) < T(y)|I=0]\Pr(I=0) + \Pr[T(x) < T(y)|I=1]\Pr(I=1)\\ =& (0)(0.4) + (0.6)\Pr[T(x) < T(y)|I=1] = 0.6\Pr[T(x) < T(y)|I=1].\end{aligned}$$

Using the figure below, we can write



$$\begin{aligned}\Pr[T(x) < T(y)|I = 1] =& \int_0^{50}\int_0^y 0.0005dxdy + \int_{40}^{50}\int_0^{40} 0.0005dxdy\\ =& \frac{0.0005}{2}y^2\Big|_0^{50} + 0.02y|_{40}^{50}\\ =&0.40 + 0.20 = 0.60.\end{aligned}$$

Hence,

$$\Pr[T(x) < T(y)] = (0.6)(0.6) = 0.36 \blacksquare$$

Practice Problems

Problem 20.27
Consider the survival function $s(x) = 1 - \frac{x}{90},\ 0 \leq x \leq 90$. Find the survival function and the probability density function of $T(x)$.

Problem 20.28
Suppose that ${}_tp_x = 1 - \frac{t}{90-x},\ 0 \leq t \leq 90 - x$. Find the density function of $T(x)$.

Problem 20.29
$T(x)$, the future lifetime of (x), has the following distribution:

$$f_{T(x)}(t) = \begin{cases} kf_1(t), & 0 \leq t \leq 50 \\ 1.2f_2(t), & t > 50 \end{cases}$$

where $f_1(t)$ is exponentially distributed with parameter 0.02 and $f_2(t)$ follows DeMoivre's Law of mortality. Determine the value of k.

Problem 20.30
An age-at-death random variable has the CDF $F(x) = 1 - 0.10(100 - x)^{\frac{1}{2}}$ for $0 \leq x \leq 100$. Find $f_{T(36)}(t)$.

Problem 20.31
Let age-at-death random variable X have density function:

$$f(x) = \frac{x}{50},\ 0 \leq x \leq 10.$$

Find $f_{T(2)}(t)$.

Problem 20.32
A survival model is described by the mortality $\mu(x) = 0.015$. Find $f_{T(20)}(3)$.

20.4 Force of Mortality of $T(x)$

Recall that the hazard rate function $\mu(x)$ is the death rate at age x given survival to age x. That is,

$$\mu(x) = \lim_{h\to 0} \frac{\Pr(x < X \le x+h|X > x)}{h}.$$

Likewise, we have

$$\begin{aligned}\mu_{T(x)}(t) =& \lim_{h\to 0} \frac{\Pr(t < T(x) \le t+h|T(x) > t)}{h}\\ =& \lim_{h\to 0} \frac{\Pr(x+t < X \le x+t+h|X > x+t)}{h}\\ =&\mu(x+t).\end{aligned}$$

That is, $\mu_{T(x)}(t)$ is the death rate at age $x+t$ given survival to age $x+t$. The PDF of $T(x)$ can be expressed in terms of the distribution of X as shown in the next example.

Example 20.11
Show that $f_{T(x)}(t) = {}_tp_x\mu(x+t),\ x \ge 0, t \ge 0.$

Solution.
We have

$$\begin{aligned}f_{T(x)}(t) =&\frac{d}{dt}F_{T(x)}(t) = \frac{d}{dt}{}_tq_x\\ =&\frac{d}{dt}\left[1 - \frac{s(x+t)}{s(x)}\right]\\ =&\frac{s(x+t)}{s(x)}\left[-\frac{s'(x+t)}{s(x+t)}\right] = {}_tp_x\mu(x+t) \ \blacksquare\end{aligned}$$

Example 20.12
Find the hazard rate function of $T(x)$ if X is uniformly distributed in $[0, \omega]$.

Solution.
The hazard rate function of X is given by

$$\mu(x) = \frac{1}{\omega - x}.$$

Thus,

$$\mu(x+t) = \frac{1}{\omega - t - x} \ \blacksquare$$

Example 20.13
You are given

$$\mu(x) = \begin{cases} 0.06, & 35 \le x \le 45 \\ 0.07, & 45 < x. \end{cases}$$

Find $s_{T(35)}(t), F_{T(35)}(t)$, and $f_{T(35)}(t)$.

Solution.
Look at Figure 20.2.

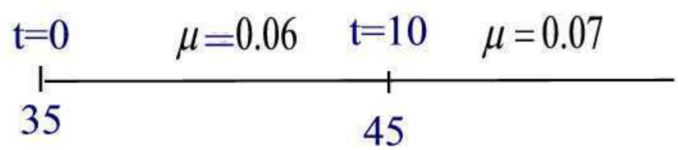


Figure 20.2

If $0 \le t \le 10$ we have

$$s_{T(35)}(t) = {}_tp_{35} = e^{-\int_0^t 0.06ds} = e^{-0.06t}.$$

If $10 < t$ then

$$s_{T(35)}(t) = {}_{10}p_{35} \cdot {}_{t-10}p_{45} = e^{-10(0.06)}e^{-0.07(t-10)}.$$

Hence,

$$s_{T(35)}(t) = \begin{cases} e^{-0.06t}, & 0 \le t \le 10 \\ e^{-0.6}e^{-0.07(t-10)}, & t > 10. \end{cases}$$

$$F_{T(35)}(t) = \begin{cases} 1 - e^{-0.06t}, & 0 \le t \le 10 \\ 1 - e^{-0.6}e^{-0.07(t-10)}, & t > 10. \end{cases}$$

$$f_{T(35)}(t) = \begin{cases} 0.06e^{-0.06t}, & 0 \le t < 10 \\ 0.07e^{-0.6}e^{-0.07(t-10)}, & t > 10 \; \blacksquare \end{cases}$$

Example 20.14 ‡
For a population of individuals, you are given: (i) Each individual has a constant force of mortality. (ii) The forces of mortality are uniformly distributed over the interval (0,2). Calculate the probability that an individual drawn at random from this population dies within one year.

Solution.
This is a mixed distribution problem. Let M be the force of mortality of an individual drawn at random; and T the future lifetime of the individual. Then we have

$$f_{T|M}(t|\mu) = \mu e^{-\mu t}, t > 0$$

$$f_M(\mu) = \frac{1}{2},\ 0 < \mu < 2.$$

$$f_{T,M}(t,\mu) = f_{T|M}(t|\mu)f_M(\mu) = \frac{1}{2}\mu e^{-\mu t},\ t > 0,\ 0 < \mu < 2.$$

$$f_T(t) = \int_0^2 \frac{1}{2}\mu e^{-\mu t} d\mu.$$

Hence,

$$\begin{aligned}\Pr(T \le 1) = \int_0^1 t f_T(t)dt = & \int_0^1 \int_0^2 \frac{1}{2}\mu e^{-\mu t} d\mu dt \\ =& \frac{1}{2}\int_0^2 \int_0^1 \mu e^{-\mu t} dt d\mu \\ =& \frac{1}{2}\int_0^2 [-e^{-\mu t}]_0^1 d\mu = \frac{1}{2}\int_0^2 (1 - e^{-\mu})d\mu \\ =& \frac{1}{2}[\mu + e^{-\mu}]_0^2 = 0.568 \blacksquare\end{aligned}$$

Example 20.15 ‡
A population has 30% who are smokers (s) with a constant force of mortality 0.2 and 70% who are non-smokers (ns) with a constant force of mortality 0.1. Calculate the 75th percentile of the distribution of the future lifetime of an individual selected at random from this population.

Solution.
This is a mixed distribution problem. Let X be an individual random variable with domain $\{s, ns\}$ such that $\Pr(X = s) = 0.3$ $\Pr(X = ns) = 0.7$; and let T denote the future lifetime of the individual. Then we have

$$f_{T|X}(t|s) = 0.2e^{-0.2t} \text{ and } f_{T|X}(t|ns) = 0.1e^{-0.1t}$$

$$f_X(s) = 0.3 \text{ and } f_X(ns) = 0.7.$$

$$f_{T,X}(t,s) = f_{T|X}(t|s)f_X(s) = 0.06e^{-0.2t} \text{ and}$$
$$f_{T,X}(t,ns) = f_{T|X}(t|ns)f_X(ns) = 0.07e^{-0.1t}$$

$$f_T(t) = 0.06e^{-0.2t} + 0.07e^{-0.1t}.$$

$$F_T(t) = \Pr(T \le t) = \int_0^t [0.06e^{-0.2s} + 0.07e^{-0.1s}]ds = 1 - 0.3e^{-0.2t} - 0.7e^{-0.1t}.$$

Let t denote the 75th percentile. Then t satisfies the equation $F_T(t) = 0.75$ or $0.3e^{-0.2t} + 0.7e^{-0.1t} = 0.25$. Letting $x = e^{-0.1t}$ we obtain the quadratic equation $0.3x^2 + 0.7x - 0.25 = 0$ whose solutions are $x = 0.3147$ and $x = -2.3685$. The negative root is to be discarded since $x > 0$. Hence,

$$e^{-0.1t} = 0.3147 \Longrightarrow t = \frac{\ln 0.3147}{-0.1} = 11.56 \blacksquare$$

Example 20.16 ‡
For a group of lives aged 30, containing an equal number of smokers and non-smokers, you are given:
(i) For non-smokers, $\mu^N(x) = 0.08,\ x \ge 30$
(ii) For smokers, $\mu^S(x) = 0.16,\ x \ge 30$
Calculate q_{80} for a life randomly selected from *those surviving to age 80.*

Solution.
We have

$$S_{T(30)}(t) = \Pr(T(30) > t|S)\Pr(S) + \Pr(T(30) > t|N)\Pr(N) = 0.5e^{-0.16t} + 0.5e^{-0.08t}.$$

Hence,

$$\begin{aligned} q_{80} =& 1 - P_{80} = 1 - \frac{S_{T(30)}(51)}{S_{T(30)}(50)} \\ =& 1 - \frac{0.5e^{-0.16(51)} + 0.5e^{-0.08(51)}}{0.5e^{-0.16(50)} + 0.5e^{-0.08(50)}} \\ =& 1 - 0.922 = 0.078 \blacksquare \end{aligned}$$

Practice Problems

Problem 20.33
Show that $\frac{d}{dt}(1 - {}_tp_x) = {}_tp_x\mu(x+t)$.

Problem 20.34
Show that $\int_0^\infty {}_tp_x\mu(x+t)dx = 1$.

Problem 20.35
Show that

$$\mu_{T(x)}(t) = \frac{F'_{T(x)}(t)}{1 - F_{T(x)}(t)}.$$

Problem 20.36
The CDF of $T(x)$ is given by

$$F_{T(x)}(t) = \begin{cases} \frac{t}{100-x}, & 0 \le t < 100 - x \\ 1, & t \ge 100 - x. \end{cases}$$

Calculate $\mu(x+t)$ for $0 \le t < 100 - x$.

Problem 20.37
Find the hazard rate function of $T(x)$ if X is exponentially distributed with parameter μ.

Problem 20.38
You are given the following information: $f_{T(x)}(t) = 0.015e^{-0.015t}$ and ${}_tp_x = e^{-0.015t}$. Find $\mu(x+t)$.

Problem 20.39 ‡
You are given $\mu(x) = (80 - x)^{-\frac{1}{2}},\ 0 \le x < 80$. Calculate the median future lifetime of (20).

Problem 20.40 ‡
You are given:

$$\mu(x) = \begin{cases} 0.05 & 50 \le x < 60 \\ 0.04 & 60 \le x < 70. \end{cases}$$

Calculate ${}_{4|14}q_{50}$.

20.5 Mean and Variance of $T(x)$

In this section, we find expressions for the expected value and the variation of the time-until-death random variable $T(x)$. Like any random variable, the expected value is defined (using actuarial notation) by

$$\overset{\circ}{e}_x = E[T(x)] = \int_0^\infty t f_{T(x)}(t)dt.$$

$\overset{\circ}{e}_x$ is called the **complete expectation of life at age** x. We next derive an expression for $\overset{\circ}{e}_x$ in terms of the survival function of X. Indeed, we have

$$\begin{aligned}\overset{\circ}{e}_x =& \int_0^\infty t f_{T(x)}(t)dt = \int_0^\infty t\frac{f(x+t)}{s(x)}dt\\ =&\frac{1}{s(x)}\int_0^\infty tf(x+t)dt\\ =&\frac{1}{s(x)}\left[-ts(x+t)|_0^\infty + \int_0^\infty s(x+t)dt\right]\\ =&\int_0^\infty \frac{s(x+t)}{s(x)}dt = \int_0^\infty {}_tp_x dt\end{aligned}$$

where we used integration by parts and Theorem 18.1.

Example 20.17
The age-at-death random variable is uniformly distributed in $[0, 90]$. Find $\overset{\circ}{e}_{30}$.

Solution.
Since X is uniform on $[0, 90]$, $T(30)$ is uniform on $[0, 60]$ and

$$\overset{\circ}{e}_{30} = E(T(30)) = \frac{0+60}{2} = 30 \blacksquare$$

Next, the variance of $T(x)$ is given by

$$\text{Var}(T(x)) = E[(T(x)^2] - [E(T(x)]^2$$

where

$$E[T(x)] = \int_0^\infty \frac{s(x+t)}{s(x)}dt = \int_0^\infty {}_tp_x dt$$

and

$$\begin{aligned}
E[(T(x))^2] =& \int_0^\infty t^2 f_{T(x)}(t)dt = \int_0^\infty t^2 \frac{f(x+t)}{s(x)} dt \\
=& \frac{1}{s(x)} \int_0^\infty t^2 f(x+t)dt \\
=& \frac{1}{s(x)} \left[-t^2 s(x+t)\Big|_0^\infty + \int_0^\infty 2ts(x+t)dt \right] \\
=& \int_0^\infty \frac{2ts(x+t)}{s(x)} dt = 2\int_0^\infty t \cdot {}_tp_x dt.
\end{aligned}$$

Example 20.18
Let the age-at-death X be exponential with density function $f(x) = 0.05e^{-0.05x}$, $x \geq 0$. Calculate the variance of $T(x)$.

Solution.
The CDF is given by

$$F(x) = 1 - e^{-0.05x}$$

and the SDF

$$s(x) = e^{-0.05x}.$$

Thus,

$$\begin{aligned}
E(T(x)) =& \int_0^\infty e^{-0.05t} dt = 20 \\
E[(T(x))^2] =& \int_0^\infty 2te^{-0.05t} dt = 80
\end{aligned}$$

Hence,

$$\text{Var}(T(x)) = 80 - 20^2 = 40 \blacksquare$$

Example 20.19 ‡
For a given life age 30, it is estimated that an impact of a medical breakthrough will be an increase of 4 years in $\mathring{e}_{30}$, the complete expectation of life. Prior to the medical breakthrough, $s(x)$ followed de Moivre's Law with $\omega = 100$ as the limiting age.
Assuming De Moivre's Law still applies after the medical breakthrough, calculate the new limiting age.

Solution.
The complete expectation of life for age 30 under De Moivre's Law is given by

$$\begin{aligned}\mathring{e}_{30} =& \int_0^{\omega-30}\left(1-\frac{t}{\omega-30}\right)dt\\ =& \left[t-\frac{t^2}{2(\omega-30)}\right]_0^{\omega-30} = \frac{\omega-30}{2}.\end{aligned}$$

Prior to medical breakthrough, the limiting age is $\omega = 100$ so that the complete life expectancy of life aged 30 is

$$\mathring{e}_{30} = \frac{100-30}{2} = 35.$$

This qunatity increases by 4 after a medical breakthrough so that the new complete life expectancy is

$$\frac{\omega'-30}{2} = \mathring{e}_{30} + 4 = 39 \Longrightarrow \omega' = 108 \blacksquare$$

Example 20.20 ‡
You are given:
(i) T is the future lifetime random variable.
(ii) $\mu(x+t) = \mu,\ t \geq 0$
(iii) $\mathrm{Var}[T] = 100$.
(iv) $X = \min\{(T, 10)\}$.
Calculate $E[X]$.

Solution.
By (ii), T is an exponential distribution with parameter μ. Thus, $E(T) = \frac{1}{\mu}$ and $E(T^2) = \frac{2}{\mu^2}$. Hence, $\mathrm{Var}(T) = \frac{2}{\mu^2} - \frac{1}{\mu^2} = \frac{1}{\mu^2} = 100$. Solving for μ we find $\mu = 0.1$. Now, we have

$$\begin{aligned}E[X] =& \int_0^{10} tf_T(t)dt + \int_{10}^{\infty} 10f_T(t)dt\\ =& \int_0^{10} te^{-0.1t}(0.1)dt + \int_{10}^{\infty} 10e^{-0.1t}(0.1)dt\\ =& \left[-te^{-0.1t} - 10e^{-0.1t}\right]_0^{10} - 10e^{-0.1t}\Big|_{10}^{\infty} = 6.3 \blacksquare\end{aligned}$$

Practice Problems

Problem 20.41
The SDF of an age-at-death random variable X is given by

$$s(x) = 0.1(100 - x)^{\frac{1}{2}},\ 0 \le x \le 100.$$

Find the expected value of $T(25)$.

Problem 20.42
The CDF of an age-at-death random variable X is given by

$$F(x) = 1 - 0.1(100 - x)^{\frac{1}{2}},\ 0 \le x \le 100.$$

Find the variance of $T(25)$.

Problem 20.43
The age-at-death random variable is uniformly distributed in $[0, 90]$. Find $\text{Var}(T(30))$.

Problem 20.44 ‡
For T, the future lifetime random variable for (0):
(i) $\omega > 70$
(ii) ${}_{40}p_0 = 0.6$
(iii) $E(T) = 62$
(iv) $E[\min(T, t)] = t - 0.005t^2,\ 0 < t < 60$
Calculate the complete expectation of life at 40.

20.6 Curtate-Future-Lifetime

For a given (x), there is a positive integer k such that $k-1 < T(x) \leq k$. We define the discrete random variable $K_x = k$ which we call the **time interval of death** of a life aged x. Thus, $K_x = k$ means that death of a life aged x occurred in the k^{th} interval. That is the interval $(k-1, k]$.

Example 20.21
Suppose that a life aged 30 dies at age 67.25. Find K_{30}.

Solution.
Since $T(x) = 67.25 - 30 = 37.25$, we have $K_{30} = 38$ ■

The **curtate-future-lifetime** of a life aged x, denoted by $K(x)$, is the random variable representing the number of full years lived after age x. That is, $K(x)$ is the integer part of $T(x)$, in symbol, $K(x) = \lfloor T(x) \rfloor$. Thus, if $T(x) = 67.35$ then $K(x) = 67$. Clealry, $K(x)$ is a discrete random variable taking values in the set $\{0, 1, 2, \cdots\}$. Also, note that $K(x) = K_x - 1$.
We can use the distribution function of $T(x)$ to derive the probability mass function of $K(x)$ as follows:

$$\begin{aligned}
p_{K(x)}(k) =& \Pr(K(x) = k) = \Pr(k \leq T(x) < k+1) \\
=& F_{T(x)}(k+1) - F_{T(x)}(k) = (1 - {}_{k+1}p_x) - (1 - {}_kp_x) \\
=& {}_kp_x - {}_{k+1}p_x = {}_kp_x \cdot (1 - p_{x+k}) = {}_kp_x \cdot q_{x+k} = {}_{k|}q_x
\end{aligned}$$

where we used Problems 20.3 and 20.9. This result is intuitive. If the random variable $K(x)$ takes the value k, then a life aged x must live for k complete years after age x. Therefore, the life must die in the year of age $x+k$ to $x+k+1$. But the probability of death in the year of age $x+k$ to $x+k+1$ is just ${}_{k|}q_x$.
The CDF of $K(x)$ is given by

$$\begin{aligned}
F_{K(x)}(y) =& \Pr(K(x) \leq y) = \sum_{k=0}^{\lfloor y \rfloor} p_{K(x)}(k) \\
=& \sum_{k=0}^{\lfloor y \rfloor} ({}_kp_x - {}_{k+1}p_x) = 1 - {}_{\lfloor y \rfloor + 1}p_x \\
=& {}_{\lfloor y \rfloor + 1}q_x.
\end{aligned}$$

In particular, for $k = 0, 1, 2, \cdots$, we have

$$s_{K(x)}(k) = \Pr(K(x) > k) = 1 - \Pr(K(x) \le k) = 1 - {}_{k+1}q_x = {}_{k+1}p_x.$$

Example 20.22
A survival model has the survival function $s(x) = 0.01(10-x)^2,\ 0 \le x \le 10$. Find the probability mass function of $K(x)$.

Solution.
We have

$$ {}_tp_x = \frac{s(x+t)}{s(x)} = \left(\frac{10-x-t}{10-x}\right)^2,\ 0 \le t \le 10-x.$$

Now, for $k = 0, \cdots, 10-x$, we have

$$p_{K(x)}(k) = {}_kp_x - {}_{k+1}p_x = \left(\frac{10-x-k}{10-x}\right)^2 - \left(\frac{10-x-k-1}{10-x}\right)^2 \blacksquare$$

Now, the expected value of the random variable $K(x)$, denoted by e_x, is known as the **curtate expectation of life** for a life of age x. Thus, we have:

$$\begin{aligned} e_x =& E[K(x)] = \sum_{k=0}^{\infty} k\Pr(K(x) = k) \\ =&({}_1p_x - {}_2p_x) + 2({}_2p_x - {}_3p_x) + 3({}_3p_x - {}_4p_x) + \cdots \\ =&{}_1p_x + {}_2p_x + {}_3p_x + \cdots = \sum_{k=1}^{\infty} {}_kp_x. \end{aligned}$$

Example 20.23
Suppose that $s(x) = e^{-\mu x},\ x \ge 0$. Find e_x and $\overset{\circ}{e}_x$.

Solution.
We have

$$\begin{aligned} {}_tp_x =& \frac{s(x+t)}{s(x)} = e^{-\mu t} \\ \overset{\circ}{e}_x =& \int_0^{\infty} {}_tp_x dt = \int_0^{\infty} e^{-\mu t} dt = \frac{1}{\mu} \\ e_x =& \sum_{k=1}^{\infty} {}_kp_x = \sum_{k=1}^{\infty} e^{-\mu k} \\ =& \frac{e^{-\mu}}{1-e^{-\mu}} \blacksquare \end{aligned}$$

Example 20.24
Using the trapezoid rule, show that

$$\overset{\circ}{e}_x \approx \frac{1}{2} + e_x.$$

Solution.
We have

$$\begin{aligned}
\overset{\circ}{e}_x =& \int_0^\infty {}_kp_x dt \\
\approx& \sum_{k=0}^\infty \int_k^{k+1} {}_kp_x dt \\
=& \frac{1}{2}({}_0p_x + {}_1p_x) + \frac{1}{2}({}_1p_x + {}_2p_x) + \frac{1}{2}({}_2p_x + {}_3p_x) + \cdots \\
=& \frac{1}{2}{}_0p_x + \sum_{k=1}^\infty {}_tp_x \\
=& \frac{1}{2} + e_x.
\end{aligned}$$

Thus, the complete expectation of life at age x is approximately equal to the curtate expectation of life plus one-half of a year. This is equivalent to the assumption that lives dying in the year of age $x+k$ to $x+k+1$ do so, on average, half-way through the year at age $x+k+\frac{1}{2}$ ■

To get the variance of $K(x)$, we need an expression for the second moment of $K(x)$. Thus, we have

$$\begin{aligned}
E[K(x)^2] =& \sum_{k=0}^\infty k^2 \Pr(K(x) = k) \\
=& ({}_1p_x - {}_2p_x) + 4({}_2p_x - {}_3p_x) + 9({}_3p_x - {}_4p_x) + \cdots \\
=& {}_1p_x + 3{}_2p_x + 5{}_3p_x + 7{}_4p_x + \cdots \\
=& \sum_{k=1}^\infty (2k-1){}_kp_x.
\end{aligned}$$

Example 20.25
Consider a mortality model with the property $p_k = \frac{1}{2}$ for all $k = 1, 2, \cdots$. Find $E[K(x)^2]$.

Solution.
First, notice that ${}_kp_x = p_xp_{x+1}\cdots p_{x+k-1} = \left(\frac{1}{2}\right)^k$.
We have,

$$\begin{aligned}
E[K(x)^2] =& \sum_{k=1}^{\infty}(2k-1){}_kp_x = \sum_{k=1}^{\infty}(2k-1)0.5^k\\
=&2\sum_{k=1}^{\infty}k(0.5)^k - \sum_{k=1}^{\infty}(0.5)^k\\
=&2\cdot\frac{0.5}{(1-0.5)^2} - \frac{0.5}{1-0.5} = 3
\end{aligned}$$

where we use the fact that for $|x|<1$ we have

$$x\left(\sum_{n=0}^{\infty}x^n\right)' = x\left(\frac{1}{1-x}\right)' = \frac{x}{(1-x)^2} \blacksquare$$

Finally, the variance of $K(x)$ is given by

$$\mathrm{Var}(K(x)) = \sum_{k=1}^{\infty}(2k-1){}_kp_x - \left(\sum_{k=0}^{\infty}{}_kp_x\right)^2.$$

Example 20.26
You are given $p_x = \frac{99-x}{100},\ x = 90, 91, \cdots, 99$. Calculate $\mathrm{Var}[K(96)]$.

Solution.
First note that $q_{99} = 1$ so that the limiting age is 99. We have

$$\begin{aligned}
E[K(96)] =& e_{96} = \sum_{k=1}^{99-96}{}_kp_{96} = p_{96} + {}_2p_{96} + {}_3p_{96}\\
=&p_{96} + p_{96}p_{97} + p_{96}p_{97}p_{98}\\
=&0.3 + 0.3\times 0.2 + 0.3\times 0.20\times 0.10 = 0.366\\
E[K(96)^2] =& \sum_{k=1}^{99-96}(2k-1){}_kp_{96} = 1({}_1p_{96}) + 3({}_2p_{96}) + 5({}_3p_{96})\\
=&0.3 + 3(0.3\times 0.2) + 5(0.3\times 0.20\times 0.10) = 0.51\\
\mathrm{Var}[K(96)] =& E[K(96)^2] - (E[K(96)])^2\\
=&0.51 - 0.366^2 = 0.376044 \blacksquare
\end{aligned}$$

Practice Problems

Problem 20.45
Find the probability mass function of the random variable K_x.

Problem 20.46
Show that $\Pr(K_x \geq k) = {}_{k-1}p_x$.

Problem 20.47
A survival model has the survival function $s(x) = 0.1(100 - x)^{0.5},\ 0 \leq x \leq 100$. Find the probability mass function of $K(x)$.

Problem 20.48
A survival model has the survival function $s(x) = 1 - \frac{x}{100},\ 0 \leq x \leq 100$. Find e_x and $\overset{\circ}{e}_x$.

Problem 20.49
Consider a mortality model with the property $p_k = \frac{1}{2}$ for all $k = 1, 2, \cdots$. Find e_x.

Problem 20.50
Consider a mortality model with the property $p_k = \frac{1}{2}$ for all $k = 1, 2, \cdots$. Find $\mathrm{Var}(K(x))$.

Problem 20.51
Show that $e_x = p_x(1 + e_{x+1})$.

Problem 20.52
Define the continous random variable S_x to represent the fractional part of the time interval lived through in the interval of death of an entity alive at age x. Thus, for a life aged 30 if death occur at 56.5, then $S_x = 0.5$.. Express S_x in terms of K_x and T_x.

Problem 20.53 ‡
For (x) :
(i) K is the curtate future lifetime random variable.
(ii) $q_{x+k} = 0.1(k + 1),\ k = 0, 1. \cdots, 9$.
(iii) $X = \min(K, 3)$.
Calculate $\mathrm{Var}(X)$.

Problem 20.54 ‡
Given:
(i) Superscripts M and N identify two forces of mortality and the curtate expectations of life calculated from them. (ii)

$$\mu^N(25+t) = \begin{cases} \mu^M(25+t) + 0.1(1-t) & 0 \le t \le 1 \\ \mu^M(25+t) & t > 1 \end{cases}$$

(iii) $e_{25}^M = 10.0$
Calculate e_{25}^N. Hint: Problem 20.50.

21 Central Death Rates

We study central death rates because they play an important role in the construction of life tables. In order to understand the concept of central rate of mortality, an understanding of the concept of weighted average is necessary. The weighted average of a discrete set of data $\{x_1, x_2, \cdots, x_n\}$ with nonnegative weights $\{w_1, w_2, \cdots, w_n\}$ is the number

$$\overline{x} = \frac{w_1 x_1 + w_2 x_2 + \cdots + w_n x_n}{w_1 + w_2 + \cdots + w_n}.$$

Therefore data elements with a high weight contribute more to the weighted mean than do elements with a low weight.
In the continuous setting, the continuous weighted average of a function $f(x)$ on the interval $[a, b]$ with a nonnegative weight function $w(x)$ is the number

$$\overline{x} = \frac{\int_a^b w(x) f(x) dx}{\int_a^b w(x) dx}.$$

The **central rate of death** or the **average hazard** on the interval $[x, x+1]$ is the continuous weighted average of the force of mortality $\mu(y)$ with weight function

$$w(y) = \frac{s(y)}{\int_x^{x+1} s(t) dt}.$$

We denote the weighted average by

$$m_x = \frac{\int_x^{x+1} s(y)\mu(y) dy}{\int_x^{x+1} s(y) dy}.$$

Equivalently we can express m_x in terms of $f(x)$ and $s(x)$ by using the fact that $f(x) = \mu(x)s(x)$:

$$m_x = \frac{\int_x^{x+1} f(y) dy}{\int_x^{x+1} s(y) dy}.$$

Example 21.1
Find m_x if X is exponentially distributed with parameter μ.

Solution.
For the type of distribution under consideration we have

$$s(x) = e^{-\mu x} \text{ and } \mu(x) = \mu.$$

Thus,

$$m_x = \frac{\int_x^{x+1} \mu e^{-\mu t} dt}{\int_x^{x+1} e^{-\mu t} dt} = \mu \blacksquare$$

Example 21.2
Assume that the force of mortality follows the DeMoivre's Law, where $\mu(x) = \frac{1}{80-x}$ for $0 \le x < 80$. Calculate m_{20}.

Solution.
We have

$$\begin{aligned} m_{20} =& \frac{\int_{20}^{21} s(y)\mu(y)dy}{\int_{20}^{21} s(y)dy} = \frac{\int_{20}^{21} \frac{dy}{80}}{\int_{20}^{21} \left(1 - \frac{y}{80}\right) dy} \\ =& \frac{1}{80y - 0.5y^2|_{20}^{21}} = 0.01681 \blacksquare \end{aligned}$$

The central death rate can be extended to an interval of the form $[x, x+n]$. In this case, we define the central death rate to be

$${}_n m_x = \frac{\int_x^{x+n} s(y)\mu(y)dy}{\int_x^{x+n} s(y)dy} = \frac{\int_0^n s(x+t)\mu(x+t)dt}{\int_0^n s(x+t)dt}$$

where the second equality is the result of the change of variable $y = x + t$. Note that $m_x = {}_1 m_x$.

Example 21.3
Show that

$${}_n m_x = \frac{\int_0^n {}_t p_x \mu(x+t)dt}{\int_0^n {}_t p_x dt}.$$

Solution.
We have

$$\begin{aligned} {}_n m_x =& \frac{\int_0^n s(x+t)\mu(x+t)dt}{\int_0^n s(x+t)dt} = \frac{\int_0^n \frac{s(x+t)}{s(x)} \mu(x+t)dt}{\int_0^n \frac{s(x+t)}{s(x)} dt} \\ =& \frac{\int_0^n {}_t p_x \mu(x+t)dt}{\int_0^n {}_t p_x dt} \blacksquare \end{aligned}$$

Example 21.4
You are given that $\mu = 0.02$. Calculate ${}_{10}m_{75}$.

Solution.
We have

$$ {}_{10}m_{75} = \mu = 0.02 \blacksquare $$

Example 21.5
You are given that mortality follows De Moivre's Law with limiting age ω. Calculate ${}_nm_x$.

Solution.
We have

$$ s(x) = 1 - \tfrac{x}{\omega} \text{ and } \mu(x) = \tfrac{1}{\omega - x}. $$

Thus,

$$ \begin{aligned} {}_nm_x =& \frac{\int_0^n s(x+t)\mu(x+t)dt}{\int_0^n s(x+t)dt} = \frac{\int_0^n \frac{dt}{\omega}}{\int_0^n \frac{(\omega-x-t)}{\omega}dt} \\ =& \frac{2}{2(\omega-x)-n} \blacksquare \end{aligned} $$

Example 21.6
You are given ${}_xq_0 = \frac{x^2}{10{,}000}, \ 0 < x < 100$. Calculate ${}_nm_x$.

Solution.
We have

$$ s(x) = 1 - \tfrac{x^2}{10{,}000} \text{ and } \mu(x) = \tfrac{2x}{10{,}000 - x^2} $$

Thus,

$$ \begin{aligned} {}_nm_x =& \frac{\int_0^n \frac{x+t}{5{,}000}dt}{\int_0^n \frac{(10{,}000-(x+t)^2)}{10{,}000}dt} \\ =& \frac{\int_0^n 2(x+t)dt}{\int_0^n [10,000 - (x+t)^2]dt} \\ =& \frac{(x+n)^2 - x^2}{10,000n - \frac{1}{3}(x+n)^3 + \frac{x^3}{3}} \\ =& \frac{2x+n}{10,000 - \frac{1}{3}(3x^2 + 3nx + n^2)} \blacksquare \end{aligned} $$

Further discussion of central death rates will be considered in Section 23.9. There a verbal interpretation of ${}_nm_x$ is given: the central rate of death is the rate of deaths per life year lived on the interval from x to $x+n$.

Practice Problems

Problem 21.1
Suppose that X is uniform on $[0, 100]$. Find m_x.

Problem 21.2
Show that

$$ {}_nm_x = \frac{s(x) - s(x+n)}{\int_x^{x+n} s(t)dt}. $$

Problem 21.3
For the survival function $s(x) = 1 - \frac{x}{100},\ 0 \leq x \leq 100$, find ${}_nm_x$.

Problem 21.4
You are given $\mu(x) = \frac{3}{2+x}$. Find ${}_2m_3$.

Problem 21.5
An age-at-death random variable follows De Moivre's Law on the interval $[0, 100]$. Find ${}_{10}m_{20}$.

Problem 21.6
You are given ${}_xq_0 = \frac{x^2}{10{,}000},\ 0 < x < 100$. Calculate m_{40} and ${}_{10}m_{75}$.

The Life Table Format

Life tables, also known as mortality tables, existed in the actuarial world before the survival model theory discussed previously in this book were fully developed. Mortality functions were represented in tabular form from which probabilities and expectations were derived. In this chapter, we describe the nature of the traditional life table and show how such a table has the same properties of the survival models discussed in the previous chapter.

22 The Basic Life Table

To create the basic life table, we start with a group of newborns known in the actuarial terminology as the **cohort**. The original number of individuals ℓ_0 in the cohort is called the **radix.** Let $\mathscr{L}(x)$ denote the number of survivors from the cohort at age x. For $j = 1, 2, \cdots, \ell_0$ define the indicator function

$$\mathbf{I}_j = \begin{cases} 1 & \text{if life } j \text{ survives to age } x \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mathscr{L}(x) = \sum_{j=1}^{\ell_0} \mathbf{I}_j$$

and $E(\mathbf{I}_j) = s(x)$.
Now, the expected number of individuals alive at that age x from the ℓ_0 newborns is defined by

$$\ell_x = E[\mathscr{L}(x)] = \sum_{j=1}^{\ell_0} E(\mathbf{I}_j) = \ell_0 s(x).$$

Likewise, let ${}_n\mathscr{D}_x$ denote the number of deaths between ages x and $x+n$ from among the initial ℓ_0 lives. In a similar fashion to ℓ_x we have

$${}_nd_x = E[{}_n\mathscr{D}_x] = \ell_0 \times \text{ probability that a new born dies between the ages of } x \text{ and } x+n = (s(x) - s(x+n))\ell_0 = \ell_x - \ell_{x+n}.$$

That is, ${}_nd_x$ is the expected number of deaths in the interval $[x, x+n)$. When $n = 1$, we will use the notation d_x.
A portion of a typical life table is shown below.

Age	ℓ_x	d_x
0	100,000	501
1	99,499	504
2	98,995	506
3	98,489	509
4	97,980	512
5	97,468	514

We can convert the life table into a survival function table by using the formula $\ell_x = \ell_0 \cdot s(x)$. Thus, obtaining

Age	$s(x)$
0	1.00000
1	0.99499
2	0.98995
3	0.98489
4	0.97980
5	0.97468

Example 22.1
Consider a survival model with survival function $s(x) = e^{-0.005x}$. Create a life table showing the survivorship of a cohort of 1000 newborns up to age 7.

Solution.
We generate the entries of ℓ_x using the formula $\ell_x = 1000e^{-0.005x}$.

Age	0	1	2	3	4	5	6	7
ℓ_x	1000	995	990	985	980	975	970	965 ■

We next redefine probabilities explored in the previous chapter in terms of the terminology of life tables introduced above. Namely,

$$ {}_tq_x = \frac{s(x) - s(x+t)}{s(x)} = \frac{\ell_0(s(x) - s(x+t))}{\ell_0 s(x)} = \frac{{}_td_x}{\ell_x}. $$

In particular,

$$ q_x = \frac{d_x}{l_x}. $$

Likewise,

$$ {}_tp_x = 1 - {}_tq_x = \frac{\ell_x - {}_td_x}{\ell_x} = \frac{\ell_{x+t}}{\ell_x} $$

and

$$ p_x = \frac{\ell_{x+1}}{\ell_x}. $$

Example 22.2
Consider the life table

Age	ℓ_x	d_x
0	100,000	501
1	99,499	504
2	98,995	506
3	98,489	509
4	97,980	512
5	97,468	514

(a) Find the number of individuals who die between ages 2 and 5.
(b) Find the probability of a life aged 2 to survive to age 4.

Solution.
(a) ${}_2d_3 = \ell_2 - \ell_5 = 98,995 - 97,468 = 1527.$
(b) ${}_2p_2 = \frac{\ell_4}{\ell_2} = \frac{97,980}{98,995} = 0.98975$ ■

Example 22.3
Suppose the radix is 100. Draw a possible sketch of ℓ_x.

Solution.
Since $\ell_x = \ell_0 s(x) = 100s(x)$, we see that ℓ_x decreases from 100 to 0. Thus, a possible sketch of ℓ_x is given in Figure 22.1.

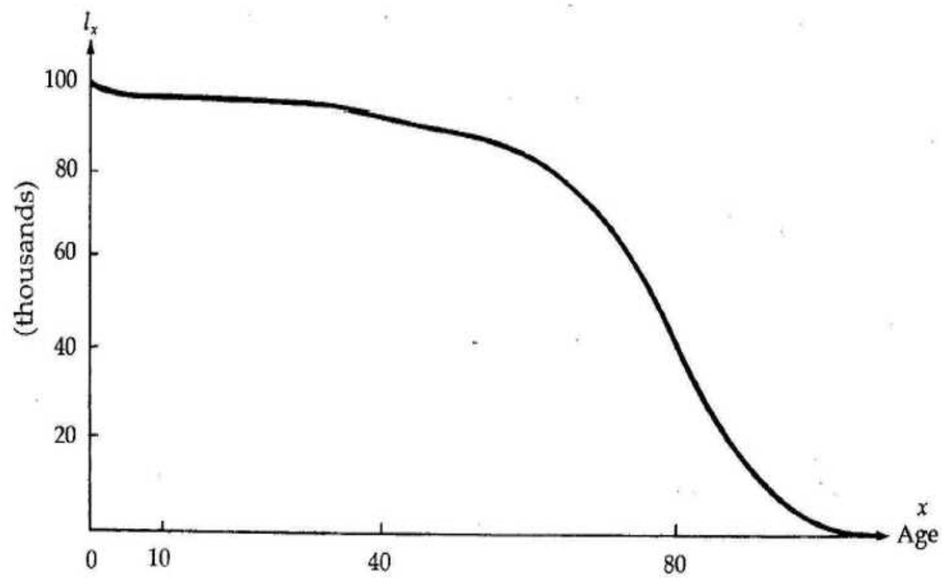


Figure 22.1

Example 22.4
You are given $p_x = \frac{99-x}{10}$, $x = 90, 91, \cdots, 99$. Calculate d_{92} if $\ell_{90} = 100,000$.

Solution.
We have

$$\begin{aligned} d_{92} =& \ell_{92} - \ell_{93} = (\ell_{90})({}_2p_{90}) - (\ell_{90})({}_3p_{90}) \\ =& \ell_{90}[p_{90}p_{91} - p_{90}p_{91}p_{92}] \\ =& 100000(0.9 \times 0.8 - 0.9 \times 0.8 \times 0.7) = 21,600 \end{aligned}$$ ■

Example 22.5
Assume that mortality follows the Illustrative Life Table. Calculate the probability that a life (65) will die between ages 80 and 90.

Solution.
We have

$$\begin{aligned}
{}_{15|10}q_{65} =& {}_{15}p_{65} - {}_{25}p_{65} = \frac{\ell_{80}}{\ell_{65}} - \frac{\ell_{90}}{\ell_{65}} \\
=& \frac{\ell_{80} - \ell_{90}}{\ell_{65}} = \frac{3914365 - 1058491}{7533964} = 0.3791 \blacksquare
\end{aligned}$$

Example 22.6
You are given:

$$\ell_x = 10000(100 - x)^2, \ 0 \le x \le 100.$$

Calculate the probability that a person now aged 20 will reach retirement age of 65.

Solution.
We have

$${}_{45}p_{20} = \frac{\ell_{65}}{\ell_{20}} = \frac{10000(100-65)^2}{10000(100-20)^2} = 0.1914 \blacksquare$$

Practice Problems

Problem 22.1
Suppose that $s(x) = 1 - \frac{x}{10},\ 0 \le x \le 10$.
(a) Find ℓ_x.
(b) Using life table terminology, find $p_2, q_3, {}_3p_7$, and ${}_2q_7$.

Problem 22.2
Let X be an age-at-death random variable. Express the CDF of X in terms of life table terminology.

Problem 22.3
Express ${}_{t|u}q_x$ in terms of life table terminology.

Problem 22.4
Complete the entries in the following table:

x	ℓ_x	d_x	p_x	q_x
0	100,000			
1	99,499			
2	98,995			
3	98,489			
4	97,980			
5	97,468			

Problem 22.5
Simplify $\ell_t p_t p_{t+1} \cdots p_{t+x-1}$ where $k, x \ge 0$.

Problem 22.6
The following is an extract from a life table.

Age	ℓ_x
30	10,000
31	9965
32	9927
33	9885
34	9839
35	9789
36	9734
37	9673
38	9607
39	9534

Using the table above, find
(a) ℓ_{36}.
(b) d_{34}.
(c) ${}_3d_{36}$.
(d) ${}_5q_{30}$.
(e) the probability that a life aged 30 dies between the ages 35 and 36.

23 Mortality Functions in Life Table Notation

In Section 22, we have seen how to express some survival models functions such as $p_x, q_x, {}_tp_x$, etc. in terms of life table terms such as ℓ_x and d_x. In this section, we continue to express more functions in the life table form. A word of caution must be made first regarding the function ℓ_x. Although, a life table does not show values of ℓ_x for non-integer numbers, we are going to make the assumption that the values of ℓ_x listed in a life table are produced by a *continuous* and *differentiable* ℓ_x. Thus, ℓ_x is defined for any nonnegative real number and not just integers.

23.1 Force of Mortality Function

With the differentiability assumption of ℓ_x we can express the force of mortality in terms of ℓ_x. Indeed, we have

$$\mu(x) = -\frac{s'(x)}{s(x)} = -\frac{\ell_0 s'(x)}{\ell_0 s(x)} = -\frac{\frac{d\ell_x}{dx}}{\ell_x}. \tag{23.1}$$

Example 23.1
Suppose you are given $\ell_x = 1000(100-x)^{0.95}$. Find $\mu(x)$.

Solution.
We first find the derivative of ℓ_x :

$$\frac{d\ell_x}{dx} = 1000(0.95)(-1)(100-x)^{0.95-1} = -950(100-x)^{-0.05}.$$

Thus,

$$\mu(x) = -\frac{-950(100-x)^{-0.05}}{1000(100-x)^{0.95}} = 0.95(100-x)^{-1} \blacksquare$$

By integrating (23.1) from 0 to x, we can express ℓ_x in terms of the force of mortality as follows:

$$\begin{aligned}
\int_0^x \frac{\frac{d\ell_t}{dt}}{\ell_t} dt =& -\int_0^x \mu(t)dt \\
\ln \ell_t|_0^x =& -\int_0^x \mu(t)dt \\
\ln\left(\frac{\ell_x}{\ell_0}\right) =& -\int_0^x \mu(t)dt \\
\ell_x =& \ell_0 e^{-\int_0^x \mu(t)dt}
\end{aligned}$$

Thus, the exponential term, known as a **decremental factor**, reduces the original size of the cohort to size ℓ_x at age x.

Example 23.2
Show that

$$\ell_{x+n} = \ell_x e^{-\int_x^{x+n} \mu(y)dy}.$$

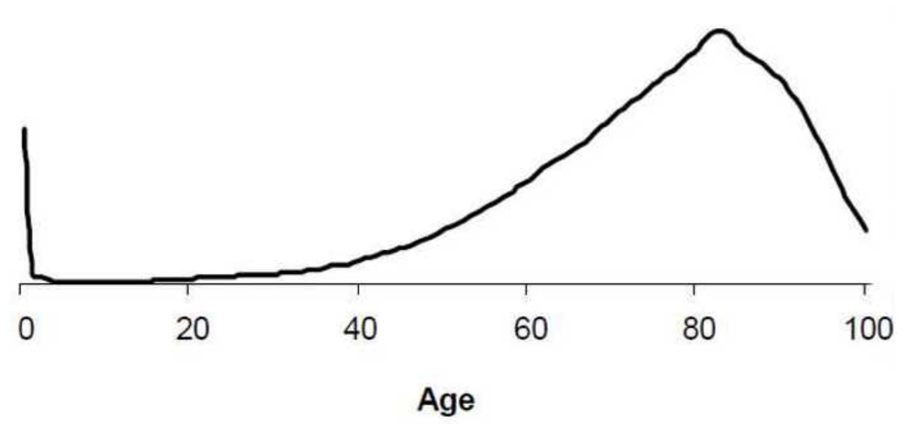
Solution.
This follows from Problem 20.9 and ${}_np_x = \frac{\ell_{x+n}}{\ell_x}$ ■

Example 23.3 ‡
For a population which contains equal numbers of males and females at birth:
(i) For males, $\mu^{(m)}(x+t) = 0.10, x \geq 0$
(ii) For females, $\mu^{(f)}(x+t) = 0.08, x \geq 0$
Calculate q_{60} for this population.

Solution.
We have

$$\begin{aligned}
q_{60} =&\frac{d_{60}^{(m)} + d_{60}^{(f)}}{\ell_{60}^{(m)} + \ell_{60}^{(f)}} = \frac{\ell_{60}^{(m)} - \ell_{61}^{(m)} + \ell_{60}^{(f)} - \ell_{61}^{(f)}}{\ell_{60}^{(m)} + \ell_{60}^{(f)}}\\
=&1 - \frac{\ell_{61}^{(m)} + \ell_{61}^{(f)}}{\ell_{60}^{(m)} + \ell_{60}^{(f)}}\\
=&1 - \frac{\ell_0^{(m)} e^{-0.10(61)} + \ell_0^{(f)} e^{-0.08(61)}}{\ell_0^{(m)} e^{-0.10(60)} + \ell_0^{(f)} e^{-0.08(60)}}\\
=&1 - \frac{e^{-0.10(61)} + e^{-0.08(61)}}{e^{-0.10(60)} + e^{-0.08(60)}} = 0.08111477 \blacksquare
\end{aligned}$$

Example 23.4 ‡
The following graph is related to current human mortality:



Which of the following functions of age does the graph most likely show?
(a) $\mu(x+t)$ (b) $\ell_x\mu(x+t)$ (c) $\ell_x p_x$ (d) ℓ_x (e) ℓ_x^2

Solution.
Since $\ell_x = \ell_0 s(x)$, $\ell_x p_x = \ell_{x+1}$ and $s(x)$ is decreasing, we find that ℓ_x and $\ell_x p_x$ are everywhere decreasing. Since $\frac{d}{dx}(\ell_x^2) = 2\ell_0 s(x)s'(x) < 0$, ℓ_x^2 is everywhere decreasing. Since $\mu(x+t) = -\frac{s'(x)}{s(x)} > 0$, $\mu(x+t)$ is increasing. According to Figure 22.1 and the fact that $\ell_x\mu(x+t) = -\frac{d}{dx}\ell_x$, we conclude that the graph most likely represents the function $\ell_x\mu(x+t)$ ■

Example 23.5
The mortality in a certain life table is such that:

$$\ell_x = \ell_0\left(1 - \frac{x}{110}\right)^{\frac{1}{2}}.$$

(a) Determe the limiting age, ω.
(b) Obtain an expression for $\mu(x)$.

Solution.
(a) ω is the lowest age for which $\ell_x = 0$. Solving this equation, we find $\omega = 110$.
(b) We have

$$\begin{aligned}\mu(x) =& -\frac{1}{\ell_x}\frac{d}{dx}(\ell_x) = -\frac{1}{\ell_0}\left(1-\frac{x}{110}\right)^{-\frac{1}{2}}\ell_0\left(-\frac{1}{220}\right)\left(1-\frac{x}{110}\right)^{-\frac{1}{2}}\\ =&\frac{1}{2(110-x)}\end{aligned}$$ ■

Practice Problems

Problem 23.1
Let $\ell_x = \omega - x,\ 0 \leq x \leq \omega$. Find $\mu(x)$.

Problem 23.2
The radix of a cohort of newborns is 100,000 and mortality is described by an exponential distribution with parameter 0.05. Find the mortality hazard function using the life table notation.

Problem 23.3
Show that

$$\mu(x+t) = -\frac{\frac{d\ell_{x+t}}{dt}}{\ell_{x+t}}.$$

Problem 23.4
Given the force of mortality $\mu(x) = \frac{1}{100-x},\ 0 \leq x \leq 100$. Find ℓ_x.

Problem 23.5
Suppose you are given

$$\ell_x = 1000(27 - 0.3x)^{\frac{1}{3}},\ 0 \leq x \leq 90.$$

Find the mortality function $\mu(x)$.

Problem 23.6
Show that

$$\ell_x - \ell_{x+n} = \int_x^{x+n} \ell_y \mu(y) dy.$$

Problem 23.7
Show that the local extreme points of $\ell_x\mu(x)$ correspond to points of inflection of ℓ_x.

23.2 The Probability Density Function of X

In Section 18.4, we established the following formula for the HRF

$$\mu(x) = \frac{f(x)}{s(x)}.$$

This and the actuarial notation $s(x) = {}_xp_0$ lead to

$$f(x) = s(x)\mu(x) = {}_xp_0\mu(x) = \left(\frac{\ell_x}{\ell_0}\right)\mu(x) = -\frac{\frac{d\ell_x}{dx}}{\ell_0}.$$

Example 23.6
Consider a survival model defined by

$$\ell_x = 1000(x+1)^{-3},\ x \geq 0.$$

Derive an expression for $f(x)$.

Solution.
We have

$$f(x) = -\frac{\frac{d\ell_x}{dx}}{\ell_0} = \frac{3000(1+x)^{-4}}{1000} = 3(1+x)^{-4} \blacksquare$$

Example 23.7
Show that $\int_0^\infty f(x)dx = 1$.

Solution.
We have

$$\int_0^\infty f(x)dx = -\frac{1}{\ell_0}\int_0^\infty \ell_x' dx = -\frac{1}{\ell_0}(\ell_\infty - \ell_0) = 1$$

since

$$\ell_\infty = \ell_0 s(\infty) = \ell_0 \times 0 = 0 \blacksquare$$

Example 23.8
Suppose $\ell_x = \ell_0\left(1 - \frac{x^2}{\omega^2}\right),\ 0 \leq x\omega$. Find $f(x)$.

Solution.
We have

$$f(x) = -\frac{1}{\ell_0}\frac{d}{dx}\ell_x = \frac{2x}{\omega^2} \blacksquare$$

Practice Problems

Problem 23.8
Consider a survival model defined by

$$\ell_x = 1000(100 - x)^{0.95},\ 0 \le x \le 100.$$

Derive an expression for $f(x)$.

Problem 23.9
You are given $\ell_x = 100,000e^{-0.05x},\ x \ge 0$. Find the value of $f(0)\ell_0$ without finding the expression for $f(x)$.

Problem 23.10
Show that $\int_0^\infty xf(x) = \frac{1}{\ell_0}\int_0^\infty \ell_x dx$.

Problem 23.11
Show that $\int_0^\infty x^2 f(x) = \frac{2}{\ell_0}\int_0^\infty x\ell_x dx$.

Problem 23.12
Let $T(x)$ denote the future lifetime random variable of (x). Express $f_{T(x)}(t)$ in terms of the function ℓ_x.

23.3 Mean and Variance of X

Up to this point, we have seen that the mean of an age-at-death random variable is given by

$$\overset{\circ}{e}_0 = E(X) = \int_0^\infty s(x)dx = \int_0^\infty {}_xp_0 dx.$$

Now, we can express the mean in terms of life table terms. By Problem 23.10, we can write

$$\overset{\circ}{e}_0 = \frac{1}{\ell_0}\int_0^\infty \ell_x dx. \tag{23.2}$$

Example 23.9
Let X be uniform in $[0, \omega]$.
(a) Find ℓ_x.
(b) Find $\overset{\circ}{e}_0$ using (23.2).

Solution.
(a) The survival function of X is given by $s(x) = 1 - \frac{x}{\omega}$ so that

$$\ell_x = \ell_0 s(x) = \ell_0 - \frac{\ell_0}{\omega}x.$$

(b) We have

$$\overset{\circ}{e}_0 = \frac{1}{\ell_0}\int_0^\infty \ell_x dx = \int_0^\omega \left(1 - \frac{x}{\omega}\right) dx = \frac{\omega}{2} \blacksquare$$

Now, define

$$T_x = \int_x^\infty \ell_y dy.$$

For the special case $x = 0$ we have

$$T_0 = \int_0^\infty \ell_x dx$$

so that (23.2) can be expressed in the form

$$\overset{\circ}{e}_0 = \frac{T_0}{\ell_0}.$$

In actuarial term, T_x is known as the **exposure**. The actuarial interpretation of T_x is that it represents the expected total life years lived after age x by the ℓ_x individuals alive at age x. Thus, the average lifetime $\overset{\circ}{e}_0$ is obtained by dividing expected total life years lived by the number of individuals born at time $x = 0$.

Example 23.10
Let X be exponential with $\ell_x = 100,000e^{-0.05x}$. Find T_x.

Solution.
We have

$$\begin{aligned} T_x =& \int_x^\infty \ell_y dy = \int_x^\infty 100,000e^{-0.05y} dy \\ =& -2,000,000\, e^{-0.05y}\Big|_x^\infty = 2,000,000e^{-0.05x} \blacksquare \end{aligned}$$

Example 23.11
Given $T_x = x^3 - 300x + 2000,\ 0 \le x \le 10$. Find ℓ_x.

Solution.
We have

$$\ell_x = -\frac{d}{dx}T_x = 300 - 3x^2 \blacksquare$$

Our next task for this section is finding the variance. Since we already have an expression for the first moment of X, we need to find the expression for the second moment. For this, we have

$$\begin{aligned} E(X^2) =& \int_0^\infty x^2 f(x)dx = -\frac{1}{\ell_0}\int_0^\infty x^2 \ell'_x dx \\ =& -\frac{1}{\ell_0}\left[x^2\ell_x\Big|_0^\infty - \int_0^\infty 2x\ell_x dx\right] \\ =& \frac{2}{\ell_0}\int_0^\infty x\ell_x dx \end{aligned}$$

Example 23.12
Let X be exponential with $\ell_x = 100,000e^{-0.05x}$. Find $E(X^2)$.

Solution.
We have

$$\begin{aligned}
E(X^2) =&\frac{2}{\ell_0}\int_0^\infty x\ell_x dx\\
=&\frac{2}{100,000}\int_0^\infty 100,000xe^{-0.05x}dx\\
=&-2\ 20xe^{-0.05x}+400e^{-0.05x}\Big|_0^\infty\\
=&800 \blacksquare
\end{aligned}$$

Now, with the expressions of $E(X)$ and $E(X^2)$, the variance of X is

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{2}{\ell_0}\int_0^\infty x\ell_x dx - \overset{\circ}{e}_0^{\,2}.$$

Example 23.13
Let X be exponential with $\ell_x = 100,000e^{-0.05x}$. Find Var$(X)$.

Solution.
We have

$$\overset{\circ}{e}_0 = \frac{T_0}{\ell_0} = \frac{2,000,000}{100,000} = 20$$

and

$$E(X^2) = 800.$$

Thus,

$$\text{Var}(X) = 800 - 20^2 = 400 \blacksquare$$

Practice Problems

Problem 23.13
Let $\ell_x = 100(10-x)^{0.85},\ 0 \le x \le 10$. Find T_x.

Problem 23.14
Let $\ell_x = 100(10-x)^{0.85}$. Find $\overset{\circ}{e}_0$.

Problem 23.15
Let $\ell_x = 100 - x^2,\ 0 \le x \le 10$. Find $\mathrm{Var}(X)$.

Problem 23.16
A survival function is defined by $\ell_x = 10,000(1+x)^{-3},\ x \ge 0$. Find T_x.

Problem 23.17
A survival function is defined by $\ell_x = 10,000(1+x)^{-3},\ x \ge 0$. Find $E(X^2)$.

Problem 23.18
A survival function is defined by $\ell_x = 10,000(1+x)^{-3},\ x \ge 0$. Find $\mathrm{Var}(X)$.

23.4 Conditional Probabilities

The probability of a life aged x to die between the ages of $x+n$ and $x+m+n$ is denoted by ${}_{n|m}q_x$. That is,

$$ {}_{n|m}q_x = \Pr(x+n < X < x+m+n | X > x). $$

Several formulas has been established for ${}_{n|m}q_x$, namely

$$ {}_{n|m}q_x = {}_{n+m}q_x - {}_nq_x = {}_np_x - {}_{n+m}p_x = {}_np_x \cdot {}_mq_{x+n}. $$

See Problems 20.14 and 20.15.

Example 23.14
Interpret the meaning of ${}_np_x \cdot {}_mq_{x+n}$

Solution.
${}_np_x \cdot {}_mq_{x+n}$ is the probability of a life aged x will survive n years, but then die within the next m years ■

Example 23.15
Interpret the meaning of ${}_{n+m}q_x - {}_nq_x$

Solution.
${}_{n+m}q_x$ is the probability of dying within $x+n+m$ years. If we remove ${}_nq_x$, which is the probability of dying within $x+n$ years, then we have the probability of surviving to age $x+n$ but dying by the age of $x+n+m$ which is ${}_{n|m}q_x$ ■

Now, we can express ${}_{n|m}q_x$ in terms of life table functions as follows:

$$ \begin{aligned} {}_{n|m}q_x =& {}_np_x \cdot {}_mq_{x+n} \\ =& \frac{\ell_{x+n}}{\ell_x} \cdot \frac{{}_md_{x+n}}{\ell_{x+n}} \\ =& \frac{{}_md_{x+n}}{\ell_x} \\ =& \frac{\ell_{x+n} - \ell_{x+n+m}}{\ell_x}. \end{aligned} $$

In the special case when $m = 1$ we have

$$ {}_{n|}q_x = \frac{d_{x+n}}{\ell_x} $$

which is the probability of a life aged x to survive to age $x+n$ but die within the following year.

Example 23.16
Consider a survival model defined by

$$\ell_x = 1000(x+1)^{-3},\ x \geq 0.$$

Find ${}_{5|4}q_{20}$.

Solution.
We have

$${}_{5|4}q_{20} = \frac{\ell_{25} - \ell_{29}}{\ell_{20}} = \frac{26^{-3} - 30^{-3}}{21^{-3}} = 0.1472 \blacksquare$$

Practice Problems

Problem 23.19
Interpret the meaning of ${}_np_x - {}_{n+m}p_x$.

Problem 23.20
Using ${}_nq_x$ notation, write down the probability of a life aged 64 to live to age 70 but not beyond the age of 80.

Problem 23.21
You are given $\ell_x = \omega - x,\ 0 \le x \le \omega$. Find ${}_{n|m}q_x$.

Problem 23.22
You are given $\ell_x = \ell_0 e^{-\mu x},\ x \ge 0$. Find ${}_{n|m}q_x$.

Problem 23.23
Given the survival function $s(x) = e^{-0.05x},\ x \ge 0$. Calculate ${}_{5|10}q_{30}$.

23.5 Mean and Variance of $T(x)$

In this section, we find expressions for the expected value and the variation of the time-until-death random variable $T(x)$ in terms of life table functions. Recall from Section 20.5 the formula for the mean of $T(x)$:

$$\overset{\circ}{e}_x = \int_0^\infty {}_tp_x dt.$$

Now substituting

$${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$$

we find

$$\overset{\circ}{e}_x = \frac{1}{\ell_x}\int_0^\infty \ell_{x+t}dt = \frac{1}{\ell_x}\int_x^\infty \ell_y dy = \frac{T_x}{\ell_x}.$$

In words, this says that the average remaining lifetime is the expected total number of years lived by the ℓ_x individuals divided by the expected number of individuals alive at age x.

Example 23.17
You are given $\ell_x = \omega - x,\ 0 \le x \le \omega$. Find $\overset{\circ}{e}_x$.

Solution.
We have

$$T_x = \int_x^\omega (\omega - y)dy = \frac{\omega^2}{2} - \omega x + \frac{x^2}{2} = \frac{1}{2}(x-\omega)^2.$$

Thus,

$$\overset{\circ}{e}_x = \frac{T_x}{\ell_x} = \frac{0.5(x-\omega)^2}{\omega - x} = 0.5(\omega - x) \blacksquare$$

Next, we recall from Section 20.5 the second moment of $T(x)$ given by

$$E[T(x)^2] = 2\int_0^\infty t \cdot {}_tp_x dt.$$

Again, substituting

$${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$$

and using integration by parts and the change of variable $y = x + t$, we find

$$E[T(x)^2] = \frac{2}{\ell_x}\int_0^\infty t\ell_{x+t}dt = \frac{2}{\ell_x}\int_x^\infty T_y dy.$$

Thus, the variance of $T(x)$ is given by

$$\text{Var}(T(x)) = \frac{2}{\ell_x}\int_x^\infty T_y dy - \left(\frac{T_x}{\ell_x}\right)^2.$$

Example 23.18
You are given $\ell_x = \omega - x,\ 0 \le x \le \omega$. Find Var$(T(x))$.

Solution.
We have

$$\int_x^\omega \frac{1}{2}(\omega - y)^2 dy = \frac{1}{6}(\omega - x)^3$$

and

$$E[T(x)^2] = 2\left(\frac{\frac{1}{6}(\omega - x)^3}{\omega - x}\right) = \frac{1}{3}(\omega - x)^2.$$

Hence,

$$\text{Var}(T(x)) = \frac{1}{3}(\omega - x)^2 - \frac{1}{4}(\omega - x)^2 = \frac{1}{12}(\omega - x)^2 \blacksquare$$

Example 23.19
Assume that the force of mortality follows the DeMoivre's Law, where $\mu(x) = \frac{1}{80-x},\ 0 \le x < 80$ and $\ell_0 = 800$. Calculate $\mathring{e}_{20}$ and T_{20}.

Solution.
We have

$$\begin{aligned}
s(x) =& 1 - \frac{x}{80}\\
{}_tp_x =& 1 - \frac{t}{80 - x}\\
\mathring{e}_{20} =& \int_0^{60} {}_tp_{20} dt = \int_0^{60}\left(1 - \frac{t}{60}\right)dt\\
=& 30\\
T_{20} =& \ell_0 \mathring{e}_{20} = 800(30) = 24000 \blacksquare
\end{aligned}$$

Practice Problems

Problem 23.24
Consider a survival model described by $\ell_x = 100e^{-0.05x},\ x \geq 0$. Find $\overset{\circ}{e}_x$.

Problem 23.25
Consider a survival model described by $\ell_x = 100e^{-0.05x},\ x \geq 0$. Find $E[T(x)^2]$.

Problem 23.26
Consider a survival model described by $\ell_x = 100e^{-0.05x},\ x \geq 0$. Find $\mathrm{Var}(T(x))$.

Problem 23.27
Suppose that a lifetime model is defined in a life-table form, with $\ell_x = 3000(1+x)^{-4},\ x \geq 0$. Find $\overset{\circ}{e}_x$.

Problem 23.28
Suppose that a lifetime model is defined in a life-table form, with $\ell_x = 3000(1+x)^{-4},\ x \geq 0$. Find $E[T(x)^2]$.

Problem 23.29
Suppose that a lifetime model is defined in a life-table form, with $\ell_x = 3000(1+x)^{-4},\ x \geq 0$. Find $\mathrm{Var}(T(x))$.

Problem 23.30 ‡
You are given the following information:
(i) $\overset{\circ}{e}_0 = 25$
(ii) $\ell_x = \omega - x,\ 0 \leq x \leq \omega$
(iii) $T(x)$ is the future lifetime random variable.
Calculate $\mathrm{Var}(T(10))$.

23.6 Temporary Complete Life Expectancy

Now, recall that $\overset{\circ}{e}_x$ is the expected future lifetime of a life aged x where the number of years of survival beyond age x is unknown. In some instances, one is interested in knowing the expected future lifetime over a determined number of years beyond x. That is, what is the expected number of years lived between age x and age $x+n$ by a life aged x from the ℓ_x group? Let us denote such expected value by $\overset{\circ}{e}_{x:\overline{n}|}$. A formula of $\overset{\circ}{e}_{x:\overline{n}|}$ is derived next.

$$\begin{aligned}
\overset{\circ}{e}_{x:\overline{n}|}=&\int_0^n s_{T(x)}(t)dt\\
=&\int_0^n {}_tp_x dt\\
=&\frac{1}{\ell_x}\int_0^n \ell_{x+t}dt\\
=&\frac{1}{\ell_x}\int_x^{x+n} \ell_y dy\\
=&\frac{1}{\ell_x}\left[\int_x^\infty \ell_y dy-\int_{x+n}^\infty \ell_y dy\right]\\
=&\frac{T_x-T_{x+n}}{\ell_x}.
\end{aligned}$$

We call $\overset{\circ}{e}_{x:\overline{n}|}$ the $n-$year **temporary expectation of life** of (x).

Example 23.20
Let $\ell_x=\omega-x,\ 0\le x\le\omega$. Find $\overset{\circ}{e}_{x:\overline{n}|}$.

Soltuion.
We have

$$\begin{aligned}
\overset{\circ}{e}_{x:\overline{n}|}=&\int_0^n \frac{\omega-x-t}{\omega-x}dt=\int_0^n\left(1-\frac{t}{\omega-x}\right)dt\\
=&t-\frac{t^2}{2(\omega-x)}\Big|_0^n=n-\frac{n^2}{2(\omega-x)}\ \blacksquare
\end{aligned}$$

Example 23.21
You are given:
(i) $\overset{\circ}{e}_{30:\overline{40}|}=27.692$
(ii) $s(x)=1-\frac{x}{\omega},\ 0\le x\le\omega$.
Determine the value of ω.

Soltuion.
Using the previous example with $n = 40$ and $x = 30$, we can write

$$40 - \frac{40^2}{2(\omega - 30)} = 27.692.$$

Solving this equation for ω and rounding the answer we find

$$\omega \approx 95 \blacksquare$$

Example 23.22 ‡
Mortality for Audra, age 25, follows De Moivre's Law with $\omega = 100$. If she takes up hot air ballooning for the coming year, her assumed mortality will be adjusted so that for the coming year only, she will have a constant force of mortality of 0.1.
Calculate the decrease in the 11-year temporary complete life expectancy for Audra if she takes up hot air ballooning.

Soltuion.
In case Audra did not take hot air ballooning, the standard 11-year temporary complete life expectancy is

$$\mathring{e}_{25:\overline{11}|} = \int_0^{11} \left(1 - \frac{t}{75}\right) dt = \left[t - \frac{t^2}{150}\right]_0^{11} = 10.1933.$$

If Audra decides to take the hot air ballooning the coming year only, then revised 11-year temporary complete life expectancy is (See Problem 23.33)

$$\begin{aligned}
\mathring{e}^{\text{Revised}}_{25:\overline{11}|} =& \mathring{e}_{25:\overline{1}|} + p_{25}\mathring{e}_{26:\overline{10}|} \\
=& \int_0^1 {}_tp_{25} dt + p_{25} \int_0^{10} \left(1 - \frac{t}{75}\right) dt \\
=& \int_0^1 e^{-0.1t} dt + e^{-0.1} \int_0^{10} \left(1 - \frac{t}{75}\right) dt \\
=& 0.95163 + 0.90484(9.32432) = 9.3886.
\end{aligned}$$

Hence, the original value of the 11-year temporary complete life expectancy decreased by

$$10.1933 - 9.3886 = 0.8047 \blacksquare$$

Example 23.23 ‡
You are given the survival function

$$s(x) = 1 - (0.01x)^2,\ \ 0 \le x \le 100.$$

Calculate $\mathring{e}_{30:\overline{50|}}$, the 50-year temporary complete expectation of life of (30).

Soltuion.
We have

$$\begin{aligned}
\mathring{e}_{30:\overline{50|}} =& \frac{1}{s(30)}\int_0^{50} s(30+t)dt = \frac{1}{s(30)}\int_{30}^{80} s(t)dt\\
=&\frac{\int_{30}^{80}\left(1 - \frac{t^2}{10{,}000}\right)dt}{1 - 0.3^2}\\
=&\frac{x - \frac{x^3}{30{,}000}\Big|_{30}^{80}}{0.91} = \frac{33.833}{0.91} = 37.18 \ \blacksquare
\end{aligned}$$

Practice Problems

Problem 23.31 ‡
You are given:
(i) $\overset{\circ}{e}_{30:\overline{40|}} = 27.692$
(ii) $s(x) = 1 - \frac{x}{\omega},\ 0 \leq x \leq \omega$
(iii) $T(x)$ is the future lifetime random variable for (x).
Determine the value of $\text{Var}(T(10))$.

Problem 23.32
You are given the survival function

$$s(x) = \left(1 - \frac{x}{\omega}\right)^{\frac{3}{2}},\ 0 \leq x \leq \omega.$$

If $\mu(70) = 0.03$, calculate $\overset{\circ}{e}_{50:\overline{25|}}$ and interpret its value.

Problem 23.33
Show that: $\overset{\circ}{e}_{x:\overline{m+n|}} = \overset{\circ}{e}_{x:\overline{m|}} + {}_mp_x \cdot \overset{\circ}{e}_{x+m:\overline{n|}}$.

Problem 23.34
Show that: $\overset{\circ}{e}_x = \overset{\circ}{e}_{x:\overline{n|}} + {}_np_x \cdot \overset{\circ}{e}_{x+n}$.

Problem 23.35
A survival model is defined by $\ell_x = 3000(1+x)^{-4},\ x \geq 0$. Find an expression for $\overset{\circ}{e}_{40:\overline{12|}}$.

Problem 23.36 ‡
You are given:

$$\mu(x) = \begin{cases} 0.04, & 25 < x < 40 \\ 0.05, & 40 < x. \end{cases}$$

Calculate $\overset{\circ}{e}_{25:\overline{25|}}$. Hint: Problem 23.33.

23.7 The Curtate Expectation of Life

Recall from Section 20.6 the discrete random variable $K(x)$ which represents the number of full years lived by (x). In terms of life-table functions the probability mass function can be written as

$$p_{K(x)}(k) = {}_kp_xq_{x+k} = \frac{\ell_{x+k}}{\ell_x} \cdot \frac{d_{x+k}}{\ell_{x+k}} = \frac{d_{x+k}}{\ell_x} = \frac{\ell_{x+k} - \ell_{x+k+1}}{\ell_x}.$$

The expected number of whole years of future live for an individual aged x is given by the **curtate expectation of life at age** x

$$e_x = \sum_{k=1}^{\infty} {}_kp_x = \frac{1}{\ell_x}\sum_{k=1}^{\infty} \ell_{x+k}.$$

Restricting the sum to a certain number of years n we get

$$e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_kp_x = \frac{1}{\ell_x}\sum_{k=1}^{n} \ell_{x+k}$$

which represents the expected (average) number of full years lived by (x) in the interval $(x, x+n]$.

Example 23.24
Consider the extract from a life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Calculate
(a) d_x for $x = 80, \cdots, 86$.
(b) The p.m.f. of the curtate life $K(80)$.
(c) The expected curtate life e_{80}.
(d) The expected number of whole years lived in the interval $(80, 83]$ by (80).

Solution.
(a) We have

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0
d_x	33	56	54	45	34	28	0

(b) We have

k	0	1	2	3	4	5
$p_{K(80)}(k)$	$\frac{33}{250}$	$\frac{56}{250}$	$\frac{54}{250}$	$\frac{45}{250}$	$\frac{34}{250}$	$\frac{28}{250}$

(c) We have

$$e_{80} = (0)\frac{33}{250} + (1)\frac{56}{250} + (2)\frac{54}{250} + (3)\frac{45}{250} + (4)\frac{34}{250} + (5)\frac{28}{250} = 2.3$$

(d) We have

$$e_{80:\overline{3}|} = \sum_{k=1}^{3} \frac{\ell_{80+k}}{\ell_{80}} = \frac{217 + 161 + 107}{250} = 1.64 \blacksquare$$

Example 23.25 ‡
You are given 3 mortality assumptions:
(i) Illustrative Life Table (ILT),
(ii) Constant force model (CF), where $sx) = e^{-\mu x}, x \geq 0$
(iii) DeMoivre model (DM), where $s(x) = 1 - \frac{x}{\omega}, 0 \leq x \leq \omega, \omega \geq 72$
For the constant force and DeMoivre models, ${}_2p_{70}$ is the same as for the Illustrative Life Table.
Rank $e_{70:\overline{2}|}$ for these models from smallest to largest.

Solution.
For the ILT model, we have

$$\begin{aligned} e^{ILT}_{70:\overline{2}|} =& p_{70} + {}_2p_{70} = \frac{\ell_{71}}{\ell_{70}} + \frac{\ell_{72}}{\ell_{70}} \\ =& \frac{6,396,609}{6,616,155} + \frac{6,164,663}{6,616,155} = 0.96682 + 0.93176 = 1.89858. \end{aligned}$$

For the CF model, we have

$$e^{CF}_{70:\overline{2}|} = p_{70} + {}_2p_{70} = e^{-\mu} + e^{-2\mu}.$$

But

$$e^{-2\mu} = \frac{6,164,663}{6,616,155} = 0.91376 \Longrightarrow \mu = 0.03534.$$

Thus,

$$e^{CF}_{70:\overline{2}|} = e^{-0.03534} + 0.91376 = 1.89704.$$

For the DM model, we have

$$0.93176 = {}_2p_{70} = \frac{\ell_{72}^{DM}}{\ell_{71}^{DM}} = \frac{\omega - 72}{\omega - 70} \Longrightarrow \omega = 99.30796.$$

Thus,

$$\begin{aligned} e_{70:\overline{2}|}^{ILT} =& p_{70} + {}_2p_{70} = \frac{\omega - 71}{\omega - 70} + \frac{\omega - 72}{\omega - 70} \\ =& \frac{99.30796 - 71}{99.30796 - 70} + 0.93176 = 1.89763. \end{aligned}$$

Hence,

$$e_{70:\overline{2}|}^{CF} < e_{70:\overline{2}|}^{DML} < e_{70:\overline{2}|}^{ILT} \blacksquare$$

Practice Problems

Problem 23.37
Express the second moment of $K(x)$ in life-table terms.

Problem 23.38
Consider the extract from a life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Calculate $E[K(80)^2]$.

Problem 23.39
Consider the extract from a life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Calculate $\text{Var}(K(80))$.

Problem 23.40
For a survival model, we have $\ell_x = 100e^{-0.05x},\ x \geq 0$. Find an expression for $p_{K(20)}(k)$.

Problem 23.41
For a survival model, we have $\ell_x = 100e^{-0.05x},\ x \geq 0$. What is the probability that the curtate-lifetime of (20) exceeds 1?

23.8 The ${}_nL_x$ Notation

In Section 23.3, we introduced the exposure T_x defined by

$$T_x = \int_x^\infty \ell_y dy.$$

In words, T_x is the expected total life years lived after age x by the ℓ_x individuals alive at age x. Thus, the difference $T_x - T_{x+n}$ is the expected total life years lived over the next n years *only* by the ℓ_x individuals alive at age x. We denote such a number by the symbol ${}_nL_x$. That is,

$${}_nL_x = T_x - T_{x+n}.$$

But

$$\mathring{e}_{x:\overline{n|}} = \frac{T_x - T_{x+n}}{\ell_x}$$

from which it follows

$${}_nL_x = \ell_x \, \mathring{e}_{x:\overline{n|}} \, .$$

Example 23.26
Show that ${}_nL_x = \int_0^n \ell_{x+t} dt$.

Solution.
We have

$$\begin{aligned}
{}_nL_x =& T_x - T_{x+n} = \int_x^\infty \ell_y dy - \int_{x+n}^\infty \ell_y dy \\
=& \int_x^{x+n} \ell_y dy \\
=& \int_0^n \ell_{x+t} dt
\end{aligned}$$

where we used the change of variable $t = y - x$ ■

Example 23.27
Let $\ell_x = \omega - x, \ 0 \le x \le \omega$. Calculate ${}_nL_x$.

Solution.
We have

$$T_x = \int_x^\omega (\omega - y) dy = \frac{1}{2}(x - \omega)^2.$$

Thus,

$$ {}_nL_x = T_x - T_{x+n} = \frac{1}{2}(x-\omega)^2 - \frac{1}{2}(x+n-\omega)^2 = -n(x-\omega) - \frac{n^2}{2} \blacksquare $$

Example 23.28
Assume that the force of mortality follows the DeMoivre's Law, where $\mu(x) = \frac{1}{80-x},\ 0 \le x < 80$ and $\ell_0 = 800$. Calculate ${}_{10}L_{20}$.

Solution.
We have

$$ \begin{aligned} s(x) =& 1 - \frac{x}{80} \\ \ell_x =& \ell_0 s(x) = 800 - 10x \\ {}_{10}L_{20} =& \int_0^{10} \ell_{20+t} dt = \int_0^{10} (600 - 10t) dt \\ =& 5500 \blacksquare \end{aligned} $$

Example 23.29
Given $T_x = x^3 - 300x + 2000,\ 0 \le x \le 10$. Find ${}_3L_2$.

Solution.
We have

$$ {}_3L_2 = T_2 - T_5 = 1408 - 625 = 783 \blacksquare $$

Practice Problems

Problem 23.42
Show that $T_x = \sum_{k=x}^{\infty} L_k$ where $L_k = {}_1L_k = \int_0^1 \ell_{k+t}dt$.

Problem 23.43
Consider the survival model with $\ell_x = 100e^{-0.05x},\ x \geq 0$. Find ${}_nL_x$.

Problem 23.44
Consider the survival model with $T_x = 1000(1+x)^{-3}$. Find ${}_nL_x$.

Problem 23.45
You are given $\ell_x = 1000e^{-0.1x}$. Find L_5.

Problem 23.46
Consider the survival model with $\ell_x = \omega - x,\ 0 \leq x \leq \omega$. Show that $\sum_{k=1}^{n} L_k = n$.

23.9 Central Death Rate

In Section 21, we showed that the central death rate on the interval $[x, x+n]$ is given by

$${}_nm_x = \frac{\int_0^n {}_tp_x\mu(x+t)dt}{\int_0^n {}_tp_xdt}.$$

Using the fact that

$${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$$

we can write

$${}_nm_x = \frac{\int_0^n \ell_{x+t}\mu(x+t)dt}{\int_0^n \ell_{x+t}dt}.$$

Since

$$\int_0^n \ell_{x+t}\mu(x+t)dt = -\ell_{x+t}|_0^n = \ell_x - \ell_{x+n} = {}_nd_x$$

and

$${}_nL_x = \int_0^n \ell_{x+t}dt$$

we obtain

$${}_nm_x = \frac{{}_nd_x}{{}_nL_x}.$$

That is, the central rate of death is the rate of deaths per life year lived on the interval from x to $x+n$.

Example 23.30
You are given $\ell_x = 1000e^{-0.1x}$ and $L_5 = 577.190$. Find m_5.

Solution.
We have

$$m_5 = \frac{d_5}{L_5} = \frac{\ell_5 - \ell_6}{L_5} = \frac{1000(e^{-0.5} - e^{-0.6})}{577.190} = 0.1 \blacksquare$$

Example 23.31
Let $\ell_x = 90 - x,\ 0 \le x \le 90$.
(a) Find ${}_4L_{15}$ and ${}_4d_{15}$.
(b) Find ${}_4m_{15}$.

Solution.
(a) We have

$$ {}_4L_{15} = \int_0^4 \ell_{15+t}dt = \int_0^4 (90 - 15 - t)dt = 75t - \left.\frac{t^2}{2}\right|_0^4 = 292 $$

and

$$ {}_4d_{15} = \ell_{15} - \ell_{19} = 4. $$

(b) The answer is

$$ {}_4m_{15} = \frac{{}_4d_{15}}{{}_4L_{15}} = \frac{4}{292} = 0.0137 \blacksquare $$

Example 23.32
Assume that the force of mortality follows the DeMoivre's Law, where $\mu(x) = \frac{1}{80-x}$, $0 \leq x < 80$ and $\ell_0 = 800$. Find
(a) L_{20}
(b) ${}_1m_{20}$.

Solution.
We have

$$ \begin{aligned} s(x) =& 1 - \frac{x}{80} \\ \ell_x =& \ell_0 s(x) = 800 - 10x \\ \ell_{20} =& 800 - 10(20) = 600 \\ \ell_{21} =& 800 - 10(21) = 590 \end{aligned} $$

(a) We have $L_x = \int_0^1 \ell_{x+t}dt = 795 - 10x$ so that $L_{20} = 795 - 10(20) = 595$.
(b) We have

$$ {}_1m_{20} = \frac{\ell_{20} - \ell_{21}}{L_{20}} = \frac{600 - 590}{595} = 0.01681 \blacksquare $$

Example 23.33
Given that $\ell_x = ke^{-x}$. Show that m_x is constant for all x.

Solution.
We have

$$\begin{aligned}
d_x =&\ell_x - \ell_{x+1} = ke^{-x}(1-e^{-1})\\
L_x =&\int_0^1 \ell_{x+t}dt = \int_0^1 ke^{-x}e^{-t}dt = ke^{-x}(1-e^{-1})\\
m_x =&\frac{d_x}{L_x} = 1 \blacksquare
\end{aligned}$$

Example 23.34
Given $T_x = x^3 - 300x + 2000,\ 0 \le x \le 10$. Find ${}_3m_2$.

Solution.
We have

$$\ell_x = -\frac{d}{dx}T_x = 300 - 3x^2.$$

Thus,

$$\begin{aligned}
{}_3m_2 =&\frac{{}_3d_2}{{}_3L_2} = \frac{\ell_2 - \ell_5}{T_2 - T_5}\\
=&\frac{288-225}{1408-625} = 0.08046 \blacksquare
\end{aligned}$$

Practice Problems

Problem 23.47
Show that $\frac{d}{dx}L_x = -d_x$.

Problem 23.48
Show that $L_{x+1} = L_x e^{-\int_x^{x+1} m_y dy}$.

Problem 23.49
Let $\ell_x = \omega - x,\ 0 \le x \le \omega$.
(a) Find $d_x,\ L_x$, and m_x.
(b) Find $\mu(x)$.
(c) Show that $\mu(x) = \frac{m_x}{1+0.5m_x}$.

Problem 23.50
Black swans always survive until age 16. After age 16, the lifetime of a black swan can be modeled by the cumulative distribution function $F(x) = 1 - \frac{4}{\sqrt{x}},\ x > 16$. There is a cohort of 3511 newborn black swans.
(a) How many years will members of this group in aggregate live between the ages of 31 and 32?
(b) Find the central death rate at age 31.

Problem 23.51
Let X be the age-at-death random variable. Assume that X obeys DeMoivre's Law with $\omega = 100$. Also, assume a cohort of 100 newborn individuals.
(a) Find ${}_{10}L_{20}$ and ${}_{10}d_{20}$.
(b) Find ${}_{10}m_{20}$.

Problem 23.52
Let X be uniform on $[0, \omega]$. Given that $m_{50} = 0.0202$, find ω.

24 Fractional Age Assumptions

Life tables are tabulated at integer ages only (**aggregate tables**) and therefore probabilities such as ${}_np_x$ and ${}_nq_x$ can be calculated from ℓ_x for x a nonnegative integer. However, in many probability computations, one needs to know ℓ_x for each $x \geq 0$ and not just for integral values. Given ℓ_x at integer ages together with some additional assumptions (i.e. interpolations) we will be able to calculate probabilities for non-integral ages such as ${}_{0.75}q_{0.25}$. In this section, we describe three most useful assumptions.

24.1 Linear Interpolation: Uniform Distribution of Deaths (UDD)

The **uniform distribution of deaths** assumes a uniform distribution of deaths within each year of age, that is, between integer-valued years x and $x+1$ the function $s(x+t)$ is linear for $0 \leq t \leq 1$. See Figure 24.1. Hence, we have

$$s(x+t) = (1-t)s(x) + ts(x+1).$$

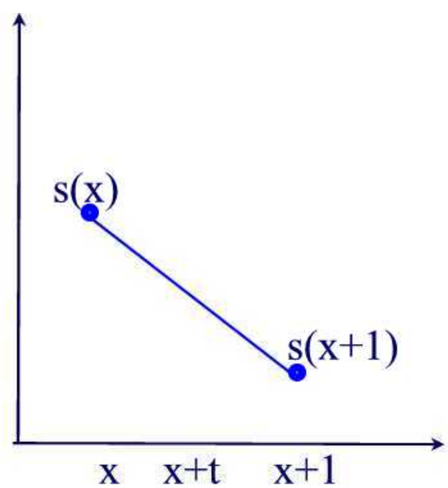


Figure 24.1

Thus, we can also write

$$\ell_{x+t} = (1-t)\ell_x + t\ell_{x+1}$$

or alternatively,

$$\ell_{x+t} = \ell_x - td_x.$$

This says, at time $x+t$, you have the ℓ_x individuals with which you started, less the deaths that occurred between time x and $x+t$. Thus, given ℓ_x and ℓ_{x+1} we can find ℓ_{x+t} by linear interpolation.
The UDD assumption gives us the helpful formulas below.

$$ {}_tp_x = s_{T(x)}(t) = \frac{\ell_{x+t}}{\ell_x} = \frac{\ell_x - td_x}{\ell_x} = 1 - t\frac{d_x}{\ell_x} = 1 - tq_x. $$

$$ {}_tq_x = F_{T(x)}(t) = 1 - {}_tp_x = tq_x. $$

$$ f_{T(x)}(t) = F'_{T(x)}(t) = q_x,\ 0 < t < 1. $$

$$ \mu(x+t) = \frac{-\frac{d}{dt}\ell_{x+t}}{\ell_{x+t}} = \frac{-\frac{d}{dt}{}_tp_x}{{}_tp_x} = \frac{q_x}{1-tq_x}. $$

$$ L_x = \int_0^1 \ell_{x+t}dt = \int_0^1 (\ell_x - td_x)dt = \ell_x - \frac{1}{2}d_x = \ell_{x+1} + \frac{1}{2}dx. $$

$$ T_x = \sum_{y=x}^{\infty} L_y = \sum_{y=x}^{\infty} \ell_{y+1} + \frac{1}{2}\ell_x. $$

$$ m_x = \frac{d_x}{L_x} = \frac{d_x}{\ell_x - \frac{1}{2}d_x} = \frac{q_x}{1 - \frac{1}{2}q_x}. $$

$$ \overset{\circ}{e}_x = \frac{T_x}{\ell_x} = \frac{\sum_{y=x}^{\infty} \ell_{y+1}}{\ell_x} + \frac{1}{2} = e_x + \frac{1}{2}. $$

Example 24.1
Given that $q_{80} = 0.02$, calculate ${}_{0.6}p_{80.3}$ under the assumption of a uniform distribution of deaths.

Solution.
We have

$$ \begin{aligned} {}_{0.6}p_{80.3} =& \frac{\ell_{80.9}}{\ell_{80.3}} \\ =& \frac{{}_{0.9}p_{80}}{{}_{0.3}p_{80}} \\ =& \frac{1 - 0.9q_{80}}{1 - 0.3q_{80}} \\ =& \frac{1 - 0.9(0.02)}{1 - 0.3(0.02)} = 0.9879 \ \blacksquare \end{aligned} $$

Example 24.2
Consider the following life table.

Age	ℓ_x	d_x
0	100,000	501
1	99,499	504
2	98,995	506
3	98,489	509
4	97,980	512
5	97,468	514

Under the uniform distribution of deaths assumption, calculate (a) ${}_{1.4}q_3$ and (b) ${}_{1.4}q_{3.5}$.

Solution.
(a) We have

$$\begin{aligned}
{}_{1.4}q_3 =& 1 - {}_{1.4}p_3 = 1 - (p_3)({}_{0.4}p_4) \\
=& 1 - \frac{\ell_4}{\ell_3} \cdot (1 - 0.4q_4) \\
=& 1 - \frac{\ell_4}{\ell_3} \cdot \left(1 - 0.4\frac{d_4}{\ell_4}\right) \\
=& 1 - \frac{97980}{98489} \cdot \left(1 - 0.4 \times \frac{512}{97980}\right) = 0.0072
\end{aligned}$$

(b) We have

$$\begin{aligned}
{}_{1.4}q_{3.5} =& 1 - {}_{1.4}p_{3.5} \\
=& 1 - \frac{\ell_{4.9}}{\ell_{3.5}} \\
=& 1 - \frac{\ell_4 - 0.9d_4}{\ell_3 - 0.5d_3} \\
=& 0.007281 \blacksquare
\end{aligned}$$

Example 24.3
Under the assumption of a uniform distribution of deaths, calculate
(a) $\lim_{t\to 1^-} \mu(80 + t)$ if $q_{80} = 0.02$
(b) $\lim_{t\to 0^+} \mu(81 + t)$ if $q_{81} = 0.04$

Solution.
(a) We have

$$\lim_{t\to 1^-}\mu(80+t)=\frac{q_{80}}{1-q_{80}}=0.0204$$

(b) We have

$$\lim_{t\to 0^+}\mu(81+t)=q_{81}=0.04 \blacksquare$$

Example 24.4 ‡
For a 4-year college, you are given the following probabilities for dropout from all causes:

$$\begin{aligned} q_0 =&0.15\\ q_1 =&0.10\\ q_2 =&0.05\\ q_3 =&0.01 \end{aligned}$$

Dropouts are uniformly distributed over each year.
Compute the temporary 1.5-year complete expected college lifetime of a student entering the second year, $\mathring{e}_{1:\overline{1.5}|}$.

Solution.
We have

$$\begin{aligned} \mathring{e}_{1:\overline{1.5}|} =& \int_0^{1.5} {}_tp_1 dt = \int_0^1 {}_tp_1 dt + \int_1^{1.5} {}_tp_1 dt\\ =& \int_0^1 {}_tp_1 dt + \int_0^{0.5} {}_{t+1}p_1 dt\\ =& \int_0^1 {}_tp_1 dt + p_1 \int_0^{0.5} {}_tp_2 dt\\ =& \int_0^1 (1-0.10t)dt + (1-0.10)\int_0^{0.5}(1-0.05t)dt\\ =&0.95+0.444=1.394 \blacksquare \end{aligned}$$

The following results are of theoretical importance.

Example 24.5
Assume that mortality follows the Illustrative Life Table for integral ages.

Assume that deaths are linearly distributed (UDD) between integral ages. Calculate:

$(a)_{0.5}q_{80}$ $(e)_{1.5}q_{80}$
$(b)_{0.5}p_{80}$ $(f)_{0.5}q_{80.5}$
$(c)\mu(80.5)$ $(g)_{0.5}q_{80.25}$
$(d)_{1.5}p_{80}$

Solution.
(a) ${}_{0.5}q_{80} = 0.5q_{80} = 0.5(0.08030) = 0.04015$
(b) ${}_{0.5}p_{80} = 1 - {}_{0.5}q_{80} = 1 - 0.04015 = 0.95985$
(c) $\mu(80.5) = \frac{q_{80}}{1-0.5q_{80}} = \frac{0.08030}{1-0.5(0.08030)} = 0.08366$
(d) ${}_{1.5}p_{80} = {}_1p_{80 0.5}p_{81} = (1 - 0.08030)(0.5)(1 - 0.08764) = 0.87940$
(e) ${}_{1.5}q_{80} = 1 - {}_{1.5}p_{80} = 1 - 0.87940 = 0.12060$
(f) We have

$$\begin{aligned}
{}_{0.5}q_{80.5} =&1 - {}_{0.5}p_{80.5} = 1 - \frac{\ell_{81}}{\ell_{80.5}}\\
=&1 - \frac{\ell_{81}}{0.5\ell_{80} + 0.5\ell_{81}} = 1 - \frac{3,600,038}{0.5(3,914,365) + 0.5(3,600,038)}\\
=&0.04183.
\end{aligned}$$

(g) We have

$$\begin{aligned}
{}_{0.5}q_{80.25} =&1 - {}_{0.5}p_{80.25} = 1 - \frac{\ell_{80.75}}{\ell_{80.25}}\\
=&1 - \frac{0.25\ell_{80} + 0.75\ell_{81}}{0.75\ell_{80} + 0.25\ell_{81}} = 1 - \frac{0.25(3,914,365) + 0.75(3,600,038)}{0.75(3,914,365) + 0.25(3,600,038)}\\
=&0.04097 \blacksquare
\end{aligned}$$

Theorem 24.1
Under the assumption of uniform distribution of deaths in the year of death,
(i) the random variable $S(x) = T(x) - K(x)$ has the uniform distribution on the interval $[0, 1)$. $S(x)$ is the random variable representing the fractional part of a year lived in the year of death. Clearly, $0 \leq S(x) \leq 1$.
(ii) $K(x)$ and $T(x) - K(x)$ are independent random variables.

Proof.
(i) We have

$$\Pr([K(x)=k]\cap[S(x)\le t]) = \Pr(k<T(x)\le k+t) = \frac{\ell_{x+t}-\ell_{x+t+k}}{\ell_x} = \frac{td_{x+k}}{\ell_x}.$$

Letting $t \to 1$ we find

$$\Pr(K(x)=k) = \frac{d_{x+k}}{\ell_x}.$$

Now, for $0 \le t < 1$ we have

$$\Pr(S(x)\le t]) = \sum_{k=0}^{\infty}\Pr([K(x)=k]\cap[S(x)\le t]) = \sum_{k=0}^{\infty}\frac{td_{x+k}}{\ell_x} = t.$$

This shows that $S(x)$ has a unitorm distribution on $[0,1)$.
(ii) For $0 \le t < 1$, we have

$$\begin{aligned}\Pr([K(x)=k]\cap[S(x)\le t]) =& \Pr(k\le T(x)\le k+t)\\ =& {}_kp_{xt}q_{x+k}\\ =& {}_kp_x tq_{x+k} = t\Pr(K(x)=k)\\ =& \Pr(K(x)=k)\Pr(S(x)\le t).\end{aligned}$$

This shows that the random variables $K(x)$ and $S(x)$ are independent ■

Practice Problems

Problem 24.1
Consider the following extract from a life table.

Age	ℓ_x	d_x
0	100,000	501
1	99,499	504
2	98,995	506
3	98,489	509
4	97,980	512
5	97,468	514

Under the uniform distribution of deaths assumption, calculate $\ell_t,\ 0 \leq t \leq 6$.

Problem 24.2
Using the previous problem and the UDD assumption, calculate ${}_tp_0,\ 0 \leq t \leq 6$.

Problem 24.3
Using the life table of Problem 24.1 and UDD assumption, calculate e_0 and $\overset{\circ}{e}_0$.

Problem 24.4
Show that under the uniform distribution of deaths assumption, we have ${}_tq_{x+s} = \frac{tq_x}{1-sq_x},\ 0 \leq s + t < 1$.

Problem 24.5 ‡
You are given $q_x = 0.1$. Find $\overset{\circ}{e}_{x:\overline{1}|}$ under the uniform distribution of deaths assumption.

Problem 24.6 ‡
You are given that deaths are uniformly distributed over each year of age, and ${}_{0.75}p_x = 0.25$. Which of the following are true?
(I) ${}_{0.25}q_{x+0.5} = 0.5$ (II) ${}_{0.5}q_x = 0.5$ (III) $\mu(x + 0.5) = 0.5$.

Problem 24.7 ‡
You are given:
(i) for $d_x = k, \quad x = 0, 1, , 2, \cdots , \omega - 1$

(ii) $\overset{\circ}{e}_{20:\overline{20|}} = 18$
(iii) Deaths are uniformly distributed over each year of age.

Calculate ${}_{30|10}q_{30}$.

Problem 24.8
Show that under UDD assumption, we have

$${}_{r|h}q_x = {}_rp_x {}_hq_{x+r} = hq_x, \quad r + h < 1.$$

Problem 24.9 ‡
T, the future lifetime of (0), has the following distribution.
(i) $f_1(t)$ follows the Illustrative Life Table, using UDD in each year.
(ii) $f_2(t)$ follows DeMoivre's law with $\omega = 100$.
(iii)

$$f_T(t) = \begin{cases} kf_1(t) & 0 \le t \le 50 \\ 1.2f_2(t) & 50 < t \end{cases}$$

Calculate ${}_{10}p_{40}$.

24.2 Constant Force of Mortality Assumption: Exponential Interpolation

The second fractional age assumption says that the force of mortality is constant between integer ages. That is, for integer $x \geq 0$ we have $\mu(x+t) = \mu_x$ for all $0 \leq t < 1$. The constant μ_x can be expressed in terms of p_x as follows:

$$p_x = e^{-\int_0^1 \mu(x+t)dt} = e^{-\mu_x} \Longrightarrow \mu_x = -\ln p_x.$$

Further, under the constant force of mortality assumption we can write

$$_tp_x = s_{T(x)}(t) = e^{-\int_0^t \mu(x+s)ds} = e^{-\mu_x t} = e^{\ln p_x^t} = p_x^t, \ 0 \leq t < 1.$$

$$_tq_x = F_{T(x)}(t) = 1 - {}_tp_x = 1 - p_x^t = 1 - (1-q_x)^t.$$

$$f_{T(x)}(t) = \frac{d}{dt}F_{T(x)}(t) = -\ln p_x p_x^t = \mu_x p_x^t = {}_tp_x\mu(x+t).$$

$$\ell_{x+t} = \ell_{x\,t}p_x = \ell_x p_x^t.$$

$$\mu(x+t) = \mu_x = -\ln p_x.$$

$$L_x = \int_0^1 \ell_{x+t}dt = \ell_x \left.\frac{p_x^t}{\ln p_x}\right|_0^1 = -\frac{\ell_x}{-\mu_x}q_x = \frac{d_x}{\mu_x}.$$

$$m_x = \frac{d_x}{L_x} = \mu_x.$$

$$T_x = \sum_{y=x}^{\infty} L_y = \sum_{y=x}^{\infty}\left(\frac{d_y}{\mu_y}\right).$$

$$\mathring{e}_x = \frac{T_x}{\ell_x} = \sum_{y=x}^{\infty}\frac{d_y}{\ell_x\mu_y}.$$

Example 24.6
Consider the following extract from a life table.

Age	ℓ_x	d_x
0	100,000	501
1	99,499	504
2	98,995	506
3	98,489	509
4	97,980	512
5	97,468	514

Under the constant force of mortality assumption, find (a) ${}_{0.75}p_2$ and (b) ${}_{1.25}p_2$.

Solution.
(a) We have

$$ {}_{0.75}p_2 = p_2^{0.75} = \left(\frac{\ell_2}{\ell_1}\right)^{0.75} = \left(\frac{98,995}{99,499}\right)^{0.75} = 0.9962. $$

(b) We have

$$ \begin{aligned} {}_{1.25}p_2 =& p_2 \cdot {}_{0.25}p_3 = \left(\frac{\ell_3}{\ell_2}\right)\left(\frac{\ell_4}{\ell_3}\right)^{0.25} \\ =& \left(\frac{98,489}{98,995}\right)\left(\frac{97,980}{98,489}\right)^{0.25} = 0.9936 \blacksquare \end{aligned} $$

Example 24.7
You are given
(i) $q_x = 0.02$
(ii) The force of mortality is constant between integer ages.
Calculate ${}_{0.5}q_{x+0.25}$.

Solution.
In general, for $s, t > 0$ and $s + t < 1$ we have

$$ {}_sp_{x+t} = e^{-\int_0^s \mu_x dy} = p_x^s. $$

Thus,

$$ {}_{0.5}q_{x+0.25} = 1 - {}_{0.5}p_{x+0.25} = 1 - p_x^{0.5} = 1 - (1 - 0.02)^{0.5} = 0.01 \blacksquare $$

Example 24.8 ‡
You are given the following information on participants entering a special 2-year program for treatment of a disease:
(i) Only 10% survive to the end of the second year.
(ii) The force of mortality is constant within each year.
(iii) The force of mortality for year 2 is three times the force of mortality for year 1.
Calculate the probability that a participant who survives to the end of month 3 dies by the end of month 21.

Solution.
If μ denote the force of mortality of the first year, then the force of mortality of the second year is 3μ. We have

$$0.10 = \Pr(\text{surviving two years}) = p_x p_{x+1} = e^{-4\mu} \Longrightarrow \mu = 0.5756.$$

The probability we are looking for is (time counted in years):

$$\begin{aligned}
{}_{1.5}q_{x+0.25} =& 1 - {}_{1.5}p_{x+0.25} = 1 - ({}_{0.75}p_{x+0.25})({}_{0.75}p_{x+1})\\
=& 1 - e^{-0.75\mu}e^{-0.75(3\mu)} = 1 - e^{-3\mu} = 1 - e^{-3(0.5756)}\\
=& 0.8221 \blacksquare
\end{aligned}$$

Example 24.9
Assume that mortality follows the Illustrative Life Table for integral ages. Assume constant force (CF) between integral ages. Calculate:

$(a)_{0.5}q_{80}$
$(b)_{0.5}p_{80}$
$(c)\mu(80.5)$
$(d)_{1.5}p_{80}$
$(e)_{1.5}q_{80}$
$(f)_{0.5}q_{80.5}$
$(g)_{0.5}q_{80.25}$

Solution.
(a) ${}_{0.5}q_{80} = 1 - p_{80}^{0.5} = 1 - (1 - 0.08030)^{0.5} = 0.04099.$
(b) ${}_{0.5}p_{80} = 1 - q_{80}^{0.5} = 1 - 0.04099 = 0.95901.$
(c) $\mu(80.5) = -\ln p_{80} = -\ln(1 - q_{80}) = -\ln(1 - 0.08030) = 0.08371.$
(d) ${}_{1.5}p_{80} = {}_1p_{80}{}_{0.5}p_{81} = (1 - q_{80})(1 - q_{81})^{0.5} = (1 - 0.08030)(1 - 0.08764)^{0.5} =$ 0.87847
(e) ${}_{1.5}q_{80} = 1 - {}_{1.5}p_{80} = 1 - 0.87847 = 0.12153.$
(f) We have

$$\begin{aligned}
{}_{0.5}q_{80.5} =& 1 - {}_{0.5}p_{80.5} = 1 - \frac{\ell_{81}}{\ell_{80.5}}\\
=& 1 - \frac{\ell_{81}}{\ell_{80}{}_{0.5}p_{80}} = 1 - \frac{3,600,038}{3,914,365(1 - 0.0803)^{0.5}}\\
=& 0.04099.
\end{aligned}$$

(g) We have

$$\begin{aligned}
{}_{0.5}q_{80.25} =& 1 - {}_{0.5}p_{80.25} = 1 - \frac{\ell_{80.75}}{\ell_{80.5}} \\
=& 1 - \frac{\ell_{80}p_{80}^{0.75}}{\ell_{80}p_{80}^{0.25}} = 1 - p_{80}^{0.5} \\
=& 1 - (1 - 0.8030)^{0.5} = 0.04099 \blacksquare
\end{aligned}$$

Practice Problems

Problem 24.10
You are given:
(i) $q_x = 0.16$
(ii) The force of mortality is constant between integer ages.
Calculate t such that ${}_tp_x = 0.95$.

Problem 24.11
You are given:
(i) $q_x = 0.420$
(ii) The force of mortality is constant between integer ages.
Calculate m_x.

Problem 24.12
Show that, under a constant force of mortality, we have ${}_{t-s}q_{x+s} = 1 - e^{-(t-s)\mu_x}$, $0 < s, t < 1$.

Problem 24.13
You are given:
(i) $p_{90} = 0.75$
(ii) a constant force of mortality between integer ages.
Calculate ${}_{\frac{1}{12}}q_{90}$ and ${}_{\frac{1}{12}}q_{90+\frac{11}{12}}$.

Problem 24.14
You are given:
(i) $q_x = 0.1$
(ii) a constant force of mortality between integer ages.
Calculate ${}_{0.5}q_x$ and ${}_{0.5}q_{x+0.5}$.

24.3 Harmonic (Balducci) Assumption

The **harmonic** or **Balducci** assumption consists of assuming the function $\frac{1}{\ell_{x+t}}$ to be linear in t and therefore ℓ_{x+t} is a hyperbolic function. Hence, $\ell_{x+t} = \frac{1}{a+bt}$. In this case, we have $\ell_x = \frac{1}{a}$ and $\ell_{x+1} = \frac{1}{a+b}$. Thus, $a = \frac{1}{\ell_x}$ and $b = \frac{1}{\ell_{x+1}} - \frac{1}{\ell_x}$. Therefore,

$$\frac{1}{\ell_{x+t}} = \frac{1}{\ell_x} + \left(\frac{1}{\ell_{x+1}} - \frac{1}{\ell_x}\right)t = t \cdot \frac{1}{\ell_{x+1}} + (1-t) \cdot \frac{1}{\ell_x}$$

or

$$\ell_{x+t} = \frac{\ell_{x+1}}{p_x + tq_x}.$$

Under the harmonic assumption, we derive the following:

$$\begin{aligned}
\frac{1}{{}_tp_x} =& \frac{\ell_x}{\ell_{x+t}} = t \cdot \frac{\ell_x}{\ell_{x+1}} + (1-t) \cdot \frac{\ell_x}{\ell_x} \\
=& \frac{t}{p_x} + (1-t) = \frac{t + (1-t)p_x}{p_x} \\
{}_tp_x =& \frac{p_x}{t + (1-t)p_x} \\
{}_tq_x =& 1 - \frac{p_x}{t + (1-t)p_x} = \frac{tq_x}{1-(1-t)q_x} \\
\mu(x+t) =& \frac{-\frac{d}{dt}\ell_{x+t}}{\ell_{x+t}} = \frac{q_x}{1-(1-t)q_x} \\
f_{T(x)}(t) =& \frac{p_x(1-p_x)}{(t + (1-t)p_x)^2}.
\end{aligned}$$

Example 24.10
Find a formula for L_x under the harmonic assumption.

Solution.
We have

$$\begin{aligned}
L_x = & \int_0^1 \ell_{x+t} dt = \int_0^1 \left[\frac{1}{\ell_x} + \left(\frac{1}{\ell_{x+1}} - \frac{1}{\ell_x}\right) t\right]^{-1} dt \\
= & \left.\frac{\ln\left(\frac{1}{\ell_x} + \left(\frac{1}{\ell_{x+1}} - \frac{1}{\ell_x}\right) t\right)}{\frac{1}{\ell_{x+1}} - \frac{1}{\ell_x}}\right|_0^1 = -\frac{\ln p_x}{\frac{1}{\ell_{x+1}} - \frac{1}{\ell_x}} \\
= & -\frac{\ell_{x+1} \ln p_x}{q_x} \blacksquare
\end{aligned}$$

Example 24.11
Find a formula for m_x under the harmonic assumption.

Solution.
We have

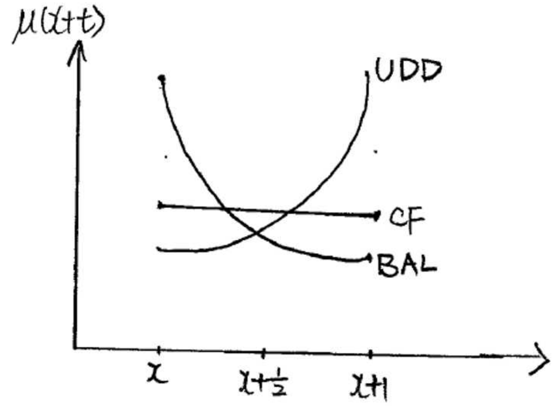
$$m_x = \frac{d_x}{L_x} = \frac{\ell_x - \ell_{x+1}}{-\frac{\ln p_x}{\frac{1}{\ell_{x+1}} - \frac{1}{\ell_x}}} = -\frac{(\ell_x - \ell_{x+1})^2}{\ell_x \ell_{x+1} \ln p_x} = -\frac{(q_x)^2}{p_x \ln p_x} \blacksquare$$

Example 24.12
Graph $\mu(x+t)$ under uniform distribution assumption, constant force of mortality assumption, and harmonic assumption.

Solution.
We have

	UD	CF	BAL
$\mu(x)$	q_x	$-\log(1-q_x)$	$\frac{q_x}{1-q_x}$
$\mu(x+\frac{1}{2})$	$\frac{q_x}{1-\frac{1}{2}q_x}$	$-\log(1-q_x)$	$\frac{q_x}{1-\frac{1}{2}q_x}$
$\mu(x+1)$	$\frac{q_x}{1-q_x}$	$-\log(1-q_x)$	q_x

The graph of $\mu(x+t)$ under the three assumptions is shown below ■

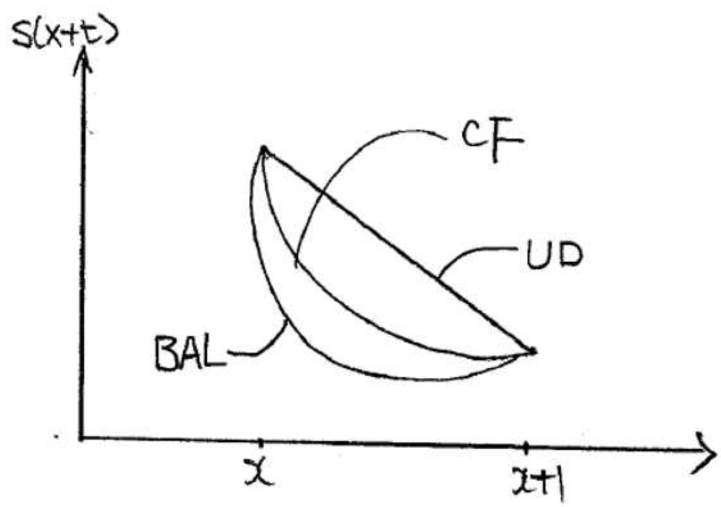


Example 24.13
Graph $s(x+t)$ under uniform distribution assumption, constant force of mortality assumption, and harmonic assumption.

Solution.
We have

$$s(x+t) \quad s(x)(1-{}_tq_x) \quad s(x)(1-q_x)^t \quad s(x)\frac{1-q_x}{1-(1-t)q_x}$$

The graph of $s(x+t)$ under the three assumptions is shown below ■



Example 24.14
The life table is given as follows

x	q_x
26	0.0213
27	0.0232
28	0.0254

Find the probability that (26.5) will survive to age 28.25 under
(i)) the UDD assumption
(ii) the constant-force assumption
(iii) the Balducci's assumption.

Solution.
We have

$${}_{28.25-26.5}p_{26.5} = {}_{1.75}p_{26.5} = {}_{0.5}p_{26.5}p_{27}{}_{0.25}p_{28}.$$

(i) Under the UDD assumption, we have

$$\begin{aligned}
{}_{0.5}p_{26.5} =& 1 - {}_{0.5}q_{26.5} = 1 - \frac{0.5q_{26}}{1-0.5q_{26}} = 1 - \frac{0.5(0.0213)}{1-0.5(0.0213)} = 0.989235\\
p_{27} =& 1 - q_{27} = 1 - 0.0232 = 0.9768\\
{}_{0.25}p_{28} =& 1 - {}_{0.25}q_{28} = 1 - 0.25q_{28} = 1 - 0.25(0.0254) = 0.99365\\
{}_{1.75}p_{26.5} =& (0.989235)(0.9768)(0.99365) = 0.960149.
\end{aligned}$$

(ii) Under the constant force of mortality, we have

$$\begin{aligned}
{}_{0.5}p_{26.5} =& 1 - {}_{0.5}q_{26.5} = 1 - (1 - p_{26}^{0.5}) = p_{26}^{0.5} = (1-q_{26})^{0.5} = 0.989293\\
p_{27} =& 1 - q_{27} = 1 - 0.0232 = 0.9768\\
{}_{0.25}p_{28} =& (1-q_{28})^{0.25} = (1-0.0254)^{0.25} = 0.993589\\
{}_{1.75}p_{26.5} =& (0.989293)(0.9768)(0.993589) = 0.960145.
\end{aligned}$$

(ii) Under the harmonic assumption, we have

$$\begin{aligned}
{}_{0.5}p_{26.5} =& 1 - {}_{0.5}q_{26.5} = \frac{1-0.5q_{26}}{1-(1-0.5-0.5)q_{26})} = 1 - 0.5q_{26} = 0.98935\\
p_{27} =& 1 - q_{27} = 1 - 0.0232 = 0.9768\\
{}_{0.25}p_{28} =& \frac{p_{28}}{1-(1-0.25)q_{28}} = \frac{1-0.0254}{1-0.75(0.0254)} = 0.993527\\
{}_{1.75}p_{26.5} =& (0.98935)(0.9768)(0.993527) = 0.960141 \blacksquare
\end{aligned}$$

Example 24.15
Assume that mortality follows the Illustrative Life Table for integral ages. Assume Balducci between integral ages. Calculate:

$(a)_{0.5}q_{80}$	$(e)_{1.5}q_{80}$
$(b)_{0.5}p_{80}$	$(f)_{0.5}q_{80.5}$
$(c)\mu(80.5)$	$(g)_{0.5}q_{80.25}$
$(d)_{1.5}p_{80}$	

Solution.
(a) ${}_{0.5}q_{80} = \frac{0.5q_{80}}{1-(1-0.5)q_{80}} = \frac{0.5(0.08030)}{1-0.5(0.08030)} = 0.04183.$
(b) ${}_{0.5}p_{80} = 1 - {}_{0.5}q_{80} = 1 - 0.04183 = 0.95817.$
(c) $\mu(80.5) = \frac{q_{80}}{1-(1-0.5)q_{80}} = \frac{0.08030}{1-0.5(0.08030)} = 0.08366$
(d) ${}_{1.5}p_{80} = {}_1p_{80\,0.5}p_{81} = (1-q_{80})\left(1 - \frac{0.5q_{81}}{1-(1-0.5)q_{81}}\right) = (1-0.08030)\left(1 - \frac{0.5(0.08764)}{1-0.5(0.08764)}\right) = 0.87755.$
(e) ${}_{1.5}q_{80} = 1 - {}_{1.5}p_{80} = 1 - 0.87755 = 0.12245.$
(f) We have

$$\begin{aligned}{}_{0.5}q_{80.5} =&1 - \frac{\ell_{81}}{\ell_{80.5}} = 1 - \frac{\ell_{81}}{\ell_{80\,0.5}p_{80}}\\ =&1 - \frac{3,600,038}{3,914,365\left(1 - \frac{0.5(0.08030)}{1-0.5(0.08030)}\right)}\\ =&0.04015.\end{aligned}$$

(g) We have

$$\begin{aligned}{}_{0.5}q_{80.25} =&1 - \frac{\ell_{80.75}}{\ell_{80.25}} = 1 - \frac{\ell_{80\,0.75}p_{80}}{\ell_{80\,0.25}p_{80}}\\ =&1 - \frac{1 - \frac{0.75q_{80}}{1-0.25q_{80}}}{1 - \frac{0.25q_{80}}{1-0.75q_{80}}}\end{aligned}$$

$$= 1 - \frac{1 - \frac{0.75(0.08030)}{1-0.25(0.08030)}}{1 - \frac{0.25(0.08030)}{1-0.75(0.08030)}}$$

$$=0.04097 \blacksquare$$

Practice Problems

Problem 24.15
Given $\ell_{95} = 800$ and $\ell_{96} = 600$. Calculate L_{95} and m_{95} under the harmonic assumption.

Problem 24.16
Show that under the Balduci assumption we have ${}_{s-t}q_{x+t} = \frac{(s-t)q_x}{1-(1-s)q_x}$.

Problem 24.17
You are given the following extract from a life table.

x	ℓ_x
80	53925
81	50987
82	47940
83	44803

Estimate ${}_{0.75}p_{80}$ and ${}_{2.25}p_{80}$ under the harmonic assumption.

Problem 24.18
If $\ell_x = 15,120$ and $q_x = \frac{1}{3}$, find $\ell_{x+0.25}$ under the harmonic assumption.

Problem 24.19 ‡
You are given:
(i) $\mu(x) = (80 - x)^{-\frac{1}{2}},\ 0 \le x < 80$.
(ii) F is the exact value of $s(10.5)$.
(iii) G is the value of $s(10.5)$ using the Balducci assumption.
Calculate $F - G$.

Problem 24.20
The life table is given as follows

x	q_x
26	0.0213
27	0.0232
28	0.0254

Find $\mathring{e}_{26:\overline{1.5}|}$ under
(i) the constant-force assumption
(ii) the Balducci's assumption.

25 Select-and-Ultimate Mortality Tables

A common practice of underwriters of life or health insurance policies is to request medical records of the potential policyholders in order to make sure that these candidates are in a satisfactory health conditions. As a result of this filtering, lives who have recently been accepted for cover can be expected to be in better health (and, thus, experience lighter mortality) than the general (or **ultimate**) population at the same age. Thus, a person who has just purchased life or health insurance has a lower probability of death than a person the same age in the general population. If we let $q_{[x]}$ denote the probability of death of the **select** individual and q_x the probability of death of the ultimate individual then we clearly have

$$q_{[x]} < q_x.$$

Now, the extent of the lighter mortality experienced by the select group of lives can be expected to reduce as the duration of time increases (as previously healthy individuals are exposed to the same medical conditions as the general population). In practice, select lives are often assumed to experience lighter mortality for a period of, say, k years (known as the **select period**). However, once the duration since selection exceeds the select period, the lives are assumed to experience the ultimate mortality rates appropriate for the general population at the same age. Thus, we can write

$$q_{[x]+k} = q_{x+k}.$$

Actuaries use special tables called **select-and-ultimate mortality tables** to reflect the effects of selection. An extract of such a table (known as AF80) with two-year selection period is shown in Table 25.1.

$[x]$	$1000q_{[x]}$	$1000q_{[x]+1}$	$1000q_{x+2}$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
30	0.222	0.330	0.422	9,907	9,905	9,901	32
31	0.234	0.352	0.459	9,903	9,901	9,897	33
32	0.250	0.377	0.500	9,899	9,896	9,893	34
33	0.269	0.407	0.545	9,894	9,892	9,888	35
34	0.291	0.441	0.596	9,889	9,887	9,882	36

Table 25.1

Example 25.1
Using Table 25.1, compute the following probabilities:
(i) ${}_2p_{[32]}$ (ii) ${}_2q_{[30]+1}$ (iii) ${}_{2|}q_{[31]}$ (iv) ${}_2q_{32}$ (v) ${}_{2|2}q_{[30]}$.

Solution.
We have
(i)
$$ {}_2p_{[32]} = \frac{\ell_{34}}{\ell_{[32]}} = \frac{9893}{9899} = 0.9994. $$
(ii)
$$ {}_2q_{[30]+1} = 1 - {}_2p_{[30]+1} = 1 - \frac{\ell_{33}}{\ell_{[30]+1}} = 1 - \frac{9897}{9905} = 8.0767 \times 10^{-4}. $$
(iii)
$$ {}_{2|}q_{[31]} = \frac{\ell_{33} - \ell_{34}}{\ell_{[31]}} = \frac{9897 - 9893}{9903} = 4.039 \times 10^{-4}. $$
(iv)
$$ {}_2q_{32} = 1 - {}_2p_{32} = 1 - \frac{\ell_{34}}{\ell_{32}} = 1 - \frac{9893}{9901} = 8.08 \times 10^{-4}. $$
(v)
$$ {}_{2|2}q_{[30]} = \frac{\ell_{32} - \ell_{34}}{\ell_{[30]}} = \frac{9901 - 9893}{9907} = 8.075 \times 10^{-4} \blacksquare $$

Example 25.2 ‡
For a 2-year select and ultimate mortality table, you are given:
(i) Ultimate mortality follows the Illustrative Life Table
(ii) $q_{[x]} = 0.5q_x$ for all x
(iii) $q_{[x]+1} = 0.5q_{x+1}$ for all x
(iv) $\ell_{[96]}10,000$.

Calculate $\ell_{[97]}$.

Solution.
The following is an extract from the Illustrative Life Table needed for our problem

x	ℓ_x	$1000q_x$
96	213,977	304.45
97	148,832	328.34
98	99,965	353.60
99	64,617	380.20
100	40,049	408.12

From the table, we have

$$\begin{aligned} q_{96} =&0.30445\\ q_{97} =&0.32834\\ q_{98} =&0.35360 \end{aligned}$$

Thus,

$$\begin{aligned} \ell_{[96]+1} =&\ell_{[96]}p_{[96]} = \ell_{[96]}(1-q_{[96]})\\ =&10,000(1-0.5\times 0.30445) = 8478\\ \ell_{98} =&\ell_{[96]+2} = \ell_{[96]+1}p_{[96]+1}\\ =&8478(1-0.5\times 0.32834) = 7086\\ \ell_{99} =&\ell_{98}p_{98} = 7086(1-0.5360) = 4580\\ \ell_{[97]+1} =&\frac{\ell_{99}}{p_{[97]+1}} = \frac{4580}{1-0.5\times 0.35360} = 5564\\ \ell_{[97]} =&\frac{\ell_{[97]+1}}{p_{[97]}} = \frac{5564}{1-0.5\times 0.32834} = 6657 \blacksquare \end{aligned}$$

Example 25.3 ‡
For a select-and-ultimate table with a 2-year select period:

$[x]$	$p_{[x]}$	$p_{[x]+1}$	$p_{[x]+2} = p_{x+2}$	$x+2$
48	0.9865	0.9841	0.9713	50
49	0.9858	0.9831	0.9698	51
50	0.9849	0.9819	0.9682	52
51	0.9838	0.9803	0.9664	53

Keith and Clive are independent lives, both age 50. Keith was selected at age 45 and Clive was selected at age 50.
Calculate the probability that exactly one will be alive at the end of three years.

Solution.
We have

$$\begin{aligned} \text{Pr(Only 1 survives)} =&1-\text{Pr(Both survive)} - \text{Pr(Neither survives)}\\ =&1 - {}_3p_{50}\,{}_3p_{[50]} - (1-{}_3p_{50})(1-{}_3p_{[50]})\\ =&1-(0.9713)(0.9698)(0.9682)(0.9849)(0.9819)(0.9682)\\ &-(1-(0.9713)(0.9698)(0.9682))(1-(0.9849)(0.9819)(0.9682))\\ =&0.140461. \end{aligned}$$

Example 25.4 ‡
You are given the following extract from a select-and-ultimate mortality table with a 2-year select period:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
60	80,625	79,954	78,839	62
61	79,137	78,402	77,252	63
62	77,575	76,770	75,578	64

Assume that deaths are uniformly distributed between integral ages.

Calculate $1000_{0.7}q_{[60]+0.8}$.

Solution.
We have

$$\begin{aligned}
\ell_{[60]+0.8} =&\ell_{[60]} + 0.8(\ell_{[60]+1} - \ell_{[60]})\\
=&80,625 + 0.8(79,954 - 80,625) = 80,088.20\\
\ell_{[60]+1.5} =&\ell_{[60]+1} + 0.5(\ell_{62} - \ell_{[60]+1})\\
=&79,954 + 0.5(78,839 - 79,954) = 79,396.50\\
1000_{0.7}q_{[60]+0.8} =&1000\frac{\ell_{[60]+0.8} - \ell_{[60]+1.5}}{\ell_{[60]+0.8}}\\
=&1000\frac{80,088.20 - 79,396.50}{80,088.20} = 8.637 \blacksquare
\end{aligned}$$

Practice Problems

Problem 25.1 ‡
For a 2-year select and ultimate mortality table, you are given:
(i) $q_{[x]} = (1-2k)q_x$
(ii) $q_{[x]+1} = (1-k)q_{x+1}$
(iii) $\ell_{[32]} = 90$
(iv) $\ell_{32} = 100$
(v) $\ell_{33} = 90$
(vi) $\ell_{34} = 63$.

Calculate $\ell_{[32]+1}$.

Problem 25.2 ‡
For a life table with a one-year select period, you are given:
(i)

$[x]$	$\ell_{[x]}$	$d_{[x]}$	ℓ_{x+1}	$\overset{\circ}{e}_{[x]}$
80	1000	90		8.5
81	920	90		

(ii) Deaths are uniformly distributed over each year of age.

Calculate $\overset{\circ}{e}_{[81]}$.

Problem 25.3 ‡
You are given the following extract from a 2-year select-and-ultimate mortality table:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
65			8200	67
66			8000	68
67			7700	69

The following relationships hold for all x :
(i) $3q_{[x]+1} = 4q_{[x+1]}$
(ii) $4q_{x+2} = 5q_{[x+1]+1}$.

Calculate $\ell_{[67]}$.

Problem 25.4 ‡
For a select-and-ultimate mortality table with a 3-year select period:
(i)

$[x]$	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
60	0.09	0.11	0.13	0.15	63
61	0.10	0.12	0.14	0.16	64
62	0.11	0.13	0.15	0.17	65
63	0.12	0.14	0.16	0.18	66
64	0.13	0.15	0.17	0.19	67

(ii) White was a newly selected life on 01/01/2000.
(iii) White's age on 01/01/2001 is 61.
(iv) P is the probability on 01/01/2001 that White will be alive on 01/01/2006.

Calculate P.

Problem 25.5 ‡
You are given the following extract from a select-and-ultimate mortality table with a 2-year select period:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
60	80,625	79,954	78,839	62
61	79,137	78,402	77,252	63
62	77,575	76,770	75,578	64

Assume that deaths are uniformly distributed between integral ages.

Calculate ${}_{0.9}q_{[60]+0.6}$.

Life Insurance: Contingent Payment Models

In this chapter we look at models where a single payment will occur in the future based on the occurrence of a defined random event. Such models are referred to as **contingent payment models.** For example, a life insurance policy has a face value to be paid to the policyholder upon his death. Thus, the single payment (benefit to the policyholder) is known as a **contingent payment** (since this payment is contingent on the death of the policyholder). If we let t denote the length of the time interval from issue to death, b_t the **benefit function**, and ν_t the interest discount factor from the time of payment back to the time of policy issue then the present value function $b_t\nu_t$ is known as the contingent payment random variable. Actuaries look at the expected value of this variable which is defined as a probability adjusted present value of the benefit. That is,

$$E(b_t\nu_t) = b_t\nu_t\text{Pr}(\text{benefit}).$$

We call this expected value as the **actuarial present value**[4] and we denote it by APV.

Example 26.1
Consider a policy which consists of a payment of $50,000 contingent upon retirement in 15 years (if the person is still alive). Suppose that the probability of a 45-year old to retire in 15 years is 0.82. That is, ${}_{15}p_{45} = 0.82$. Find the actuarial present value of the contingent payment. Assume, a 6% interest compounded annually.

[4] Also known as the **expected present value** or the **net single premium.**

Solution.
The actuarial present value is

$$\text{APV} = 50,000(1.06)^{-15}(0.82) = 17,107.87 \blacksquare$$

Example 26.2 ‡
A maintenance contract on a hotel promises to replace burned out light bulbs at the end of each year for three years. The hotel has 10,000 light bulbs. The light bulbs are all new. If a replacement bulb burns out, it too will be replaced with a new bulb.
You are given: (i) For new light bulbs, $q_0 = 0.10,\ q_1 = 0.30,\ q_2 = 0.50$
(ii) Each light bulb costs 1.
(iii) $i = 0.05$
Calculate the actuarial present value of this contract.

Solution.
At age 0, there are 10,000 new bulbs. At the end of year 1, there are 10,000(0.1) = 1000 bulbs to be replaced. So we have a total of 10,000 with 1,000 new and 9,000 1-year old.
At the end of year 2, there are $9,000(0.3) + 1,000(0.1) = 2,800$ bulbs to be replaced. Again, we have 10,000 bulbs with 2,800 new, 900 1-year old and 6,300 2-year old.
At the end of year 3, there are 6,300(0.50) + 900(0.30) + 2,800(0.10) = 3700 bulbs to be replaced. In this case, we have 10,000 bulbs with 3700 new, 2520 1-year old, 630 2-year old and 3150 3-year old.
The actuarial present value is

$$\frac{1,000}{1.05} + \frac{2,800}{1.05^2} + \frac{3,700}{1.05^3} = 6,688.26 \blacksquare$$

26 Insurances Payable at the Moment of Death

Benefit payments can be either made at the time the contingent event occurred or at some set time like the end of the year in which the contingent event occurred. In this section, we consider four conventional insurance models where the benefit payment is made at the time the contingent event occurred. In life insurance terms, this means the time of death of the insured. The models will be developed with a **benefit function**, b_t, and a **discount function**, ν^t, where t is length of time from issue to death. We define the **present-value function**, z_t, by

$$z_t = b_t \nu^t.$$

But the elapsed time from policy issue to the death of the insured is just the future-lifetime random variable $T = T(x)$. Thus, the present value , at policy issue, of the benefit payment is just the random variable

$$Z = z_T = b_T \nu^T.$$

We will follow the practice, used in the theory of interest, of assuming that the benefit amount is equal to 1 (called **unit insurance**). For a benefit of k dollars, one multiplies the present value of 1 dollar by k.

26.1 Level Benefit Whole Life Insurance

By a **whole life insurance** we mean an insurance that makes a benefit payment at the time of death of the insured person, no matter when that time might be. The whole life insurance is an example of **level benefit** insurance.[5]

For this type of life insurance, $b_t = 1$ and $\nu_t = \nu^t$ so that the present value randon variable is given by

$$\bar{Z}_x = \nu^T, \ T > 0.$$

The average cost of a whole life insurance is defined as the actuarial present value of the random variable $\bar{Z}_x$. For a life aged x, this average cost will be denoted by $\bar{A}_x = E(\bar{Z}_x)$.[6] A formula for $\bar{A}_x$ is derived next.

$$\bar{A}_x = E(\nu^T) = \int_0^\infty \bar{z}_x f_{\bar{Z}_x}(t)dt = \int_0^\infty \nu^t f_T(t)dt = \int_0^\infty \nu^t {}_tp_x \mu(x+t)dt.$$

[5]A **level benefit** insurance pays the same amount at death, regardless of the time.

[6]The bar over the A indicates that this average cost is calculated on a continuous basis. The bar is omitted for the discrete version.

The expression under the integral symbol is the benefit of 1 discounted to time $t = 0$ and multiplied by the probability of death in the interval $[t, t+dt]$ for small dt.

In order to compute the variability in cost of a whole life insurance, we need the second moment of $\bar{Z}_x$, denoted by ${}^2\bar{A}_x$, and is given by

$$ {}^2\bar{A}_x = \int_0^\infty \nu^{2t} f_T(t)dt = \int_0^\infty \nu^{2t} {}_tp_x \mu(x+t)dt. $$

Now, the variance of $\bar{Z}_x$ is given by

$$ \text{Var}(\bar{Z}_x) = {}^2\bar{A}_x - \bar{A}_x^2. $$

Example 26.3

The age-at-death random variable obeys De Moivre's Law on the interval $[0, \omega]$. Let $\bar{Z}_x$ be the contingent payment random variable for a life aged x. Assume a constant force of interest δ, find

(i) $\bar{A}_x$

(ii) ${}^2\bar{A}_x$

(iii) $\text{Var}(\bar{Z}_x)$.

Solution.

T obeys DeMoivre's Law on the interval $[0, \omega - x]$, we have $f_T(t) = \frac{1}{\omega - x}$.

(i)

$$ \begin{aligned} \bar{A}_x = & \int_0^{\omega-x} \frac{e^{-\delta t}}{\omega - x} dt = -\left.\frac{e^{-\delta t}}{\delta(\omega - x)}\right|_0^{\omega-x} \\ = & \frac{1}{\omega - x} \left[\frac{1 - e^{-\delta(\omega-x)}}{\delta} \right] = \frac{\bar{a}_{\overline{\omega-x}|}}{\omega - x}. \end{aligned} $$

(ii)

$$ \begin{aligned} {}^2\bar{A}_x = & \int_0^{\omega-x} \frac{e^{-2\delta t}}{\omega - x} dt \\ = & -\left.\frac{e^{-2\delta t}}{2\delta(\omega - x)}\right|_0^{\omega-x} \\ = & \left[\frac{1 - e^{-2\delta(\omega-x)}}{2\delta(\omega - x)} \right]. \end{aligned} $$

(iii)

$$\text{Var}(\bar{Z}_x) = \left[\frac{1 - e^{-2\delta(\omega - x)}}{2\delta(\omega - x)}\right] - \frac{\bar{a}^2_{\overline{\omega - x}|}}{(\omega - x)^2} \blacksquare$$

Example 26.4
The PDF of a future-lifetime random variable for a life aged x is given by

$$f_T(t) = \mu e^{-\mu t}, \ t \geq 0.$$

Assume a constant force of interest δ, find $\bar{A}_x$, ${}^2\bar{A}_x$, and $\text{Var}(\bar{Z}_x)$.

Solution.
We have

$$\begin{aligned}
\bar{A}_x =& \int_0^\infty e^{-\delta t} e^{-\mu t} \mu dt \\
=& -\mu \frac{e^{-(\mu+\delta)t}}{\mu + \delta}\Big|_0^\infty \\
=& \frac{\mu}{\mu + \delta}. \\
{}^2\bar{A}_x =& \int_0^\infty e^{-2\delta t} e^{-\mu t} \mu dt \\
=& -\mu \frac{e^{-(\mu+2\delta)t}}{\mu + 2\delta}\Big|_0^\infty \\
=& \frac{\mu}{\mu + 2\delta}. \\
\text{Var}(\bar{Z}_x) =& \frac{\mu}{\mu + 2\delta} - \left(\frac{\mu}{\mu + \delta}\right)^2 \blacksquare
\end{aligned}$$

Example 26.5
A whole life insurance of 1 issued to (20) is payable at the moment of death. You are given:
(i) The time-at-death random variable is exponential with $\mu = 0.02$.
(ii) $\delta = 0.14$.

Calculate $\bar{A}_{20}$, ${}^2\bar{A}_{20}$, and $\text{Var}(\bar{Z}_x)$.

Solution.
We have

$$\bar{A}_{20} = \frac{\mu}{\mu+\delta} = \frac{0.02}{0.02+0.14} = \frac{1}{8}$$

$$^2\bar{A}_{20} = \frac{\mu}{\mu+2\delta} = \frac{0.02}{0.02+0.28} = \frac{1}{15}$$

$$\text{Var}(\bar{Z}_x) = \frac{1}{15} - \left(\frac{1}{8}\right)^2 = 0.0510 \blacksquare$$

Finally, if the benefit payment is a constant $b \neq 1$ then $\bar{Z}_x = b\nu^t$ and one can easily derive the following moments

$$E(\bar{Z}_x) = b\bar{A}_x$$

$$E(\bar{Z}_x^2) = b^2\left({}^2\bar{A}_x\right).$$

Example 26.6
You are currently 45 years old. It is found that your mortality follows De Moivre's Law with $\omega = 90$. You purchase a whole life insurance policy that pays a benefit of \$1,000,000 at the moment of death. Calculate the actuarial present value of your death benefits. Assume an annual effective interest rate of 10%.

Solution.
Since X is uniform on $[0, 90]$, $T = T(45)$ is uniform on $[0, 45]$. Thus, $f_T(t) = \frac{1}{45}$. The actuarial present value of your death benefits is given by

$$\bar{A}_{45} = 1,000,000\int_0^{45}(1.1)^{-t}\cdot\frac{1}{45}dt = -1,000,000\frac{1}{45}\left.\frac{(1.1)^{-t}}{\ln 1.1}\right|_0^{45} = 229,958.13 \blacksquare$$

Example 26.7
A whole life insurance of 1 issued to (x) is payable at the moment of death. You are given:
(i) $\mu(x+t) = 0.02$
(ii) $\delta = 0.1$.
Calculate the 95^{th} percentile of the present value random variable for this insurance.

Solution.
We want to find c such that $\Pr(\bar{Z}_x \leq c) = 0.95$ where $\bar{Z}_x = e^{-0.1T}$. See Figure 26.1.

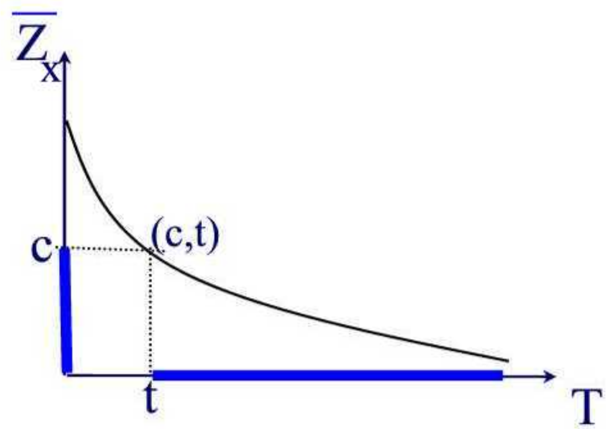


Figure 26.1

We have

$$0.95 = \Pr(\bar{Z}_x \leq c) = \Pr(T > t) = {}_tp_x = e^{-\int_0^t 0.02dt} = e^{-0.02t}.$$

Solving for t we find

$$t = -\frac{\ln 0.95}{0.02}.$$

Thus,

$$c = e^{0.1\times\frac{\ln 0.95}{0.02}} = 0.7738 \blacksquare$$

Example 26.8 ‡
For a group of individuals all age x, of which 30% are smokers (S) and 70% are non-smokers (NS), you are given:
(i) $\delta = 0.10$
(ii) $\bar{A}_x^S = 0.444$
(iii) $\bar{A}_x^{NS} = 0.286$
(iv) T is the future lifetime of (x).
(v) $\mathrm{Var}[\bar{a}_{\overline{T}|}^S] = 8.8818$
(vi) $\mathrm{Var}[\bar{a}_{\overline{T}|}^{NS}] = 8.503$
Calculate $\mathrm{Var}[\bar{a}_{\overline{T}|}]$ for an individual chosen from this group.

Solution.
We have

$$8.8818 = \mathrm{Var}[\bar{a}^S_{\overline{T}|}] = \frac{{}^2\bar{A}^S_x - [\bar{A}^S_x]^2}{\delta^2} \Longrightarrow {}^2\bar{A}^S_x = 0.285316.$$

Likewise,

$$8.503 = \mathrm{Var}[\bar{a}^{NS}_{\overline{T}|}] = \frac{{}^2\bar{A}^{NS}_x - [\bar{A}^{NS}_x]^2}{\delta^2} \Longrightarrow {}^2\bar{A}^{NS}_x = 0.166826.$$

Hence,

$$\bar{A}_x = \bar{A}^S_x \mathrm{Pr}(S) + \bar{A}^{NS}_x \mathrm{Pr}(NS) = (0.285316)(0.3) + (0.166826)(0.70) = 0.3334$$

and

$${}^2\bar{A}_x = {}^2\bar{A}^S_x \mathrm{Pr}(S) + {}^2\bar{A}^{NS}_x \mathrm{Pr}(NS) = (0.3)(0.285316) + (0.7)(0.166826) = 0.202373.$$

Finally,

$$\mathrm{Var}[\bar{a}_{\overline{T}|}] = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{\delta^2} = 9.121744 \blacksquare$$

Practice Problems

Problem 26.1 ‡
A whole life insurance of 1 issued to (20) is payable at the moment of death. You are given:
(i) $\mu(20+t) = 0.02$
(ii) $\delta = 0.1$.
Calculate the median of the present value random variable for this insurance.

Problem 26.2
A whole life insurance of 1 issued to (20) is payable at the moment of death. You are given:
(i) The time-at-death random variable is uniform with $\omega = 100$.
(ii) $\delta = 0.02$.

Calculate $\bar{A}_{20},\ {}^2\bar{A}_{20}$, and $\mathrm{Var}(\bar{Z}_{20})$.

Problem 26.3 ‡
$\bar{Z}_x$ is the present-value random variable for a whole life insurance of b payable at the moment of death of (x). You are given:
(i) $\delta = 0.04$
(ii) $\mu(x+t) = 0.02,\ t \geq 0$
(iii) The actuarial present value of the single benefit payment is equal to $\mathrm{var}(\bar{Z}_x)$.

Find the value of b.

Problem 26.4
$\bar{Z}_x$ is the present-value random variable for a whole life insurance of 1 payable at the moment of death of (x). You are given:
(i) $\delta = 0.06$
(ii) The force of mortality is constant,say μ.
(iii) ${}^2\bar{A}_x = 0.25$.

Calculate the value of μ.

Problem 26.5
For a whole life insurance of 1,000 on (x) with benefit payable at the moment of death, you are given:

(i) $\delta = 0.05$
(ii) $\mu(x+t) = 0.06$ for $0 \le t \le 10$ and $\mu(x+t) = 0.07$ otherwise.
Calculate the actuarial present value for this insurance.

Problem 26.6
For a group of individuals all age x, you are given:
(i) 30% are smokers and 70% are non-smokers.
(ii) The constant force of mortality for smokers is 0.06.
(iii) The constant force of mortality for non-smokers is 0.03.
(iv) $\delta = 0.08$
Calculate $\mathrm{Var}(\bar{a}_{\overline{T(x)}|})$ for an individual chosen at random from this group.

26.2 Finite Term Insurance Payable at the Moment of Death

A cheaper life insurance than the whole life insurance is a finite term life insurance. An $n-$**year term life insurance** is a policy where the payment is made only when the insured person dies within n years of the policy's issue. That is, if the insured survives n years, no benefit is paid.
For a unit benefit, the cost of the policy is the actuarial present value of the contingent payment random variable

$$\bar{Z}^1_{x:\overline{n}|} = \begin{cases} \nu^T, & 0 \leq T \leq n \\ 0, & T > n. \end{cases}$$

where T is the random variable for the time left to live. This cost is denoted by $\bar{A}^1_{x:\overline{n}|}$ for a life aged x. In the symbol $\bar{A}^1_{x:\overline{n}|}$, the superscript "1" means that 1 unit of money is paid for a life currently aged x if that life dies prior to reaching the age of $x+n$.
We next develop a formula for the actuarial present value. We have

$$\begin{aligned} \bar{A}^1_{x:\overline{n}|} =& E(\nu^T) = \int_0^\infty z f_{\bar{Z}^1_{x:\overline{n}|}}(t)dt \\ =& \int_0^n \nu^t f_T(t)dt = \int_0^n \nu^t {}_tp_x \mu(x+t)dt. \end{aligned}$$

In order to find the variance of $\bar{Z}^1_{x:\overline{n}|}$ we need also to find $E[(\bar{Z}^1_{x:\overline{n}|})^2]$ which we denote by ${}^2\bar{A}^1_{x:\overline{n}|}$ and is given by the following formula

$${}^2\bar{A}^1_{x:\overline{n}|} = \int_0^n \nu^{2t} {}_tp_x \mu(x+t)dt.$$

The variance of $\bar{Z}^1_{x:\overline{n}|}$ is

$$\text{Var}(\bar{Z}^1_{x:\overline{n}|}) = {}^2\bar{A}^1_{x:\overline{n}|} - (\bar{A}^1_{x:\overline{n}|})^2.$$

Example 26.9
You are given the following information:
(i) A unit benefit $n-$year term life insurance policy.
(ii) The age-at-death random variable is uniform on $[0, \omega]$.
(iii) The constant force of interest δ.
(iv) $\bar{Z}^1_{x:\overline{n}|}$ is the contingent payment random variable for a life aged x.
Find $\bar{A}^1_{x:\overline{n}|}$, ${}^2\bar{A}^1_{x:\overline{n}|}$, and $\text{Var}(\bar{Z}^1_{x:\overline{n}|})$.

Solution.
Since X is uniform on $[0,\omega]$, the future-lifetime random variable $T = T(x)$ is uniform on $[0,\omega - x]$. Thus, $f_T(t) = \frac{1}{\omega - x}$.
We have

$$\begin{aligned}
\bar{A}^1_{x:\overline{n|}} =& \int_0^n \frac{e^{-\delta t}}{\omega - x} dt = \frac{1}{\delta(\omega - x)}(1 - e^{-n\delta}) \\
{}^2\bar{A}^1_{x:\overline{n|}} =& \int_0^n \frac{e^{-2\delta t}}{\omega - x} dt = \frac{1}{2\delta(\omega - x)}(1 - e^{-2n\delta}) \\
\text{Var}(\bar{Z}^1_{x:\overline{n|}}) =& \frac{1}{2\delta(\omega - x)}(1 - e^{-2n\delta}) - \frac{1}{\delta^2(\omega - x)^2}(1 - e^{-n\delta})^2 \blacksquare
\end{aligned}$$

Example 26.10
You are given the following information:
(i) A unit benefit $n-$year term life insurance policy.
(ii) The age-at-death random variable is exponential with parameter μ.
(iii) The constant force of interest δ.
(iv) $\bar{Z}^1_{x:\overline{n|}}$ is the contingent payment random variable for a life aged x.
Find $\bar{A}^1_{x:\overline{n|}}$, ${}^2\bar{A}^1_{x:\overline{n|}}$, and $\text{Var}(\bar{Z}^1_{x:\overline{n|}})$.

Solution.
Since X is exponential with parameter μ, by the memoryless property, the future-lifetime random variable $T = T(x)$ is also exponential with parameter μ. Thus, $f_T(t) = \mu e^{-\mu t}$.
We have

$$\begin{aligned}
\bar{A}^1_{x:\overline{n|}} =& \int_0^n e^{-\delta t} e^{-\mu t} \mu dt = \frac{\mu}{\mu + \delta}(1 - e^{-(\mu+\delta)n}) \\
{}^2\bar{A}^1_{x:\overline{n|}} =& \int_0^n e^{-2\delta t} e^{-\mu t} \mu dt = \frac{\mu}{\mu + 2\delta}(1 - e^{-(\mu+2\delta)n}) \\
\text{Var}(\bar{Z}^1_{x:\overline{n|}}) =& \frac{\mu}{\mu + 2\delta}(1 - e^{-(\mu+2\delta)n}) - \frac{\mu^2}{(\mu + \delta)^2}(1 - e^{-(\mu+\delta)n})^2 \blacksquare
\end{aligned}$$

Example 26.11
The age-at-death random variable is exponential with parameter 0.05. A life aged 35 buys a 25-year life insurance policy that pays 1 upon death. Assume an annual effective interest rate of 7%, find the actuarial present value of this policy.

Solution.
We have

$$\bar{A}^{\,1}_{35:\overline{25}|} = \int_0^{25} (1.07)^{-t} e^{-0.05t}(0.05)dt = 0.4025 \blacksquare$$

Example 26.12
You are given the following information:
(i) A unit benefit $n-$year term life insurance policy.
(ii) The future-lifetime random variable is uniform on $[0, 80]$.
(iii) The constant force of interest δ.
(iv) $\bar{Z}^{1}_{x:\overline{n}|}$ is the contingent payment random variable for a life aged x.
Find the 90[th] percentile of the distribution of the present value of benefit payment.

Solution.
We are looking for m such that $\Pr(\bar{Z}^{1}_{x:\overline{n}|} \le m) = 0.90$. Since $\bar{Z}^{1}_{x:\overline{n}|}$ is a decreasing exponential function of T we can find time α such that $e^{-\delta\alpha} = m$ and

$$\Pr(\bar{Z}^{1}_{x:\overline{n}|} \le m) = \Pr(T > \alpha) = 0.90$$

which implies that α is the 10[th] percentile of T. In this case,

$$\frac{\alpha}{80} = 0.10 \Longrightarrow \alpha = 8 \Longrightarrow m = e^{-8\delta} \blacksquare$$

Practice Problems

Problem 26.7

You are given the following information:
(i) A unit benefit 10−year term life insurance policy.
(ii) The age-at-death random variable is uniform on $[0, 110]$.
(iii) The constant force of interest $\delta = 0.06$.
(iv) $\bar{Z}^{\,1}_{25:\overline{10}|}$ is the contingent payment random variable for a life aged 25.
Find $\bar{A}^{\,1}_{25:\overline{10}|}$, ${}^2\bar{A}^{\,1}_{25:\overline{10}|}$, and $\mathrm{Var}(\bar{Z}^{\,1}_{25:\overline{10}|})$.

Problem 26.8

You are given the following information:
(i) A unit benefit 20−year term life insurance policy.
(ii) The age-at-death random variable is exponential with $\mu = 0.05$.
(iii) The constant force of interest $\delta = 0.10$.
(iv) $\bar{Z}^{\,1}_{x:\overline{n}|}$ is the contingent payment random variable for a life aged 30.
Find $\bar{A}^{\,1}_{x:\overline{n}|}$, ${}^2\bar{A}^{\,1}_{x:\overline{n}|}$, and $\mathrm{Var}(\bar{Z}^{\,1}_{30:\overline{20}|})$.

Problem 26.9

Consider a 25-year term insurance for a life aged 40, with payment due upon death. Assume that this person belongs to a population, whose lifetime has a probability distribution ${}_tp_0 = 1 - \frac{t}{100},\ t \in [0, 100]$. Assume that the force of interest is $\delta = 0.05$. Calculate the actuarial present value of this policy.

Problem 26.10

Consider a 20-year term insurance with constant force of mortality $\mu = 0.01$ and force of interest $\delta = 0.08$. Find the 90^{th} percentile of the distribution of the present value benefit.

Problem 26.11

Simplify $\frac{\bar{A}^{\,1}_{x:\overline{1}|}}{\bar{A}^{\,1}_{x:\overline{2}|}}$ if the force of mortality μ is constant.

Problem 26.12

You are given the following information:
(i) $\bar{A}^{\,1}_{x:\overline{n}|} = 0.4275$
(ii) $\mu(x+t) = 0.045$ for all t.
(iii) $\delta = 0.055$.
Calculate $e^{-0.1n}$.

26.3 Endowments

An $n-$year term life insurance pays the insured only if death occurs before n years. In contrast, endowment policies pay the insured only if he is still alive after n years. In this section, we discuss two types of endowments: a pure endowment and endowment insurance.

26.3.1 Pure Endowments

An $n-$ **year pure endowment** makes a payment at the conclusion of n years if and only if the insured person is alive n years after the policy has been issued. For example, a 20-year pure endowment with face amount \$100,000, issued to (x), will pay \$100,000 in 20 years if (x) is still alive at that time, and will pay nothing if (x) dies before age $x+20$.
The pure endowment of \$1, issued to (x), with a term of n years has the present value given by

$$\bar{Z}_{x:\overset{\;1}{\overline{n}|}} = \begin{cases} 0, & T(x) \le n \\ \nu^n, & T(x) > n. \end{cases}$$

The expected value of $\bar{Z}_{x:\overset{\;1}{\overline{n}|}}$ is

$$E(\bar{Z}_{x:\overset{\;1}{\overline{n}|}}) = 0 \times \Pr(T \le n)) + \nu^n \times \Pr(T > n) = \nu^n {}_np_x.$$

There are two actuarial notations for the actuarial present value: It may be denoted by $A_{x:\overset{\;1}{\overline{n}|}}$. Note that there is no bar over the A, because this is a single discrete payment at time $T = n$. A more convenient notation is ${}_nE_x$. Thus, we can write

$$A_{x:\overset{\;1}{\overline{n}|}} = {}_nE_x = \nu^n {}_np_x.$$

In order to find the variance of $\bar{Z}_{x:\overset{\;1}{\overline{n}|}}$, we need to know the second monent of $\bar{Z}_{x:\overset{\;1}{\overline{n}|}}$ which is given by

$${}^2A_{x:\overset{\;1}{\overline{n}|}} = \nu^{2n} {}_np_x.$$

Hence, the variance is given by

$$\text{Var}(\bar{Z}_{x:\overset{\;1}{\overline{n}|}}) = {}^2A_{x:\overset{\;1}{\overline{n}|}} - (A_{x:\overset{\;1}{\overline{n}|}})^2 = \nu^{2n} {}_np_x - \nu^{2n}({}_np_x)^2 = \nu^{2n} {}_np_x \cdot {}_nq_x.$$

Example 26.13
Suppose that the age-at-death random variable is uniform on $[0,\omega]$. Let $\bar{Z}_{x:\overset{\;1}{\overline{n}|}}$ be the present value of an $n-$year pure endowment for a life aged (x) with benefit payment of 1. Assume a force of interest δ, find $A_{x:\overset{\;1}{\overline{n}|}}$, ${}^2A_{x:\overset{\;1}{\overline{n}|}}$, and $\text{Var}(\bar{Z}_{x:\overset{\;1}{\overline{n}|}})$.

Solution.
We have

$$
\begin{aligned}
{}_np_x =& \frac{s(x+n)}{s(x)} = \frac{1-\frac{x+n}{\omega}}{1-\frac{x}{\omega}} = \frac{\omega-x-n}{\omega-x} \\
A_{x:\overline{n|}}^{\;\;1} =& {}_nE_x = \nu^n {}_np_x = \left(\frac{\omega-x-n}{\omega-x}\right)e^{-n\delta} \\
{}^2A_{x:\overline{n|}}^{\;\;1} =& \left(\frac{\omega-x-n}{\omega-x}\right)e^{-2n\delta} \\
\mathrm{Var}(\bar{Z}_{x:\overline{n|}}^{\;\;1}) =& \left(\frac{n(\omega-x-n)}{(\omega-x)^2}\right)e^{-2n\delta} \blacksquare
\end{aligned}
$$

Example 26.14
Suppose that the age-at-death random variable is exponential with constant force of mortality μ. Let $\bar{Z}_{x:\overline{n|}}^{\;\;1}$ be the present value of an $n-$year pure endowment for a life aged (x) with benefit payment of 1. Assume a force of interest δ, find $A_{x:\overline{n|}}^{\;\;1}$, ${}^2A_{x:\overline{n|}}^{\;\;1}$, and $\mathrm{Var}(\bar{Z}_{x:\overline{n|}}^{\;\;1})$.

Solution.
We have

$$
\begin{aligned}
{}_np_x =& \frac{s(x+n)}{s(x)} = \frac{e^{-\mu(n+x)}}{e^{-\mu x}} = e^{-\mu n} \\
A_{x:\overline{n|}}^{\;\;1} =& {}_nE_x = \nu^n {}_np_x = e^{-n\delta}e^{-\mu n} = e^{-n(\mu+\delta)} \\
{}^2A_{x:\overline{n|}}^{\;\;1} =& e^{-n(\mu+2\delta)} \\
\mathrm{Var}(\bar{Z}_{x:\overline{n|}}^{\;\;1}) =& e^{-n(\mu+2\delta)}(1-e^{-n\mu}) \blacksquare
\end{aligned}
$$

Practice Problems

Problem 26.13
Suppose that the age-at-death random variable is uniform on $[0, 100]$. Let $\bar{Z}_{30:\overset{1}{\overline{20}|}}$ be the present value of a 20−year pure endowment for a life aged (30) with benefit payment of 1. Assume a force of interest $\delta = 0.05$ find $A_{30:\overset{1}{\overline{20}|}}$, ${}^2A_{30:\overset{1}{\overline{20}|}}$, and $\text{Var}(\bar{Z}_{30:\overset{1}{\overline{20}|}})$.

Problem 26.14
Suppose that the age-at-death random variable is exponential with constant force of mortality $\mu = 0.05$. Let $\bar{Z}_{30:\overset{1}{\overline{10}|}}$ be the present value of a 10−year pure endowment for a life aged (30) with benefit payment of 1. Assume a force of interest $\delta = 0.10$, find $A_{30:\overset{1}{\overline{10}|}}$, ${}^2A_{30:\overset{1}{\overline{10}|}}$, and $\text{Var}(\bar{Z}_{30:\overset{1}{\overline{10}|}})$.

Problem 26.15 ‡
Z is the present value random variable for a 15-year pure endowment of 1 on (x) :
(i) The force of mortality is constant over the 15-year period.
(ii) $\nu = 0.9$.
(iii) $\text{Var}(Z) = 0.065A_{x:\overset{1}{\overline{15}|}}$.
Calculate q_x.

Problem 26.16
Consider the following extract of a life table based on an annual effective rate of interest of 6%.

x	ℓ_x
30	9,501,381
31	9,486,854
32	9,471,591
33	9,455,522
34	9,438,571
35	9,420,657

Find the net single premium for a 5 year pure endowment policy for (30) assuming an interest rate of 6%.

Problem 26.17
Consider a pure endowment policy for a boy, effected at his birth, which

provides \$6,000 for the boy on his 18th birthday. You are given the following information:
(i) $\ell_0 = 100,000$ and $\ell_{18} = 96514$.
(ii) The annual effective interest rate of $i = 10\%$.
Write down the present value of the benefit payment under this policy, regarding it as a random variable. Calculate its mean value and standard deviation (the square root of variance) to 2 decimal places.

26.3.2 Endowment Insurance

A policy of $n-$**year endowment insurance** makes a payment either upon the beneficiary's death or upon the beneficiary's survival to the end of a term of n years. The earliest of these two times is the payment of death. For n-year endowment insurance that pays one unit in benefits upon death, the contingent payment random variable is given by

$$\bar{Z}_{x:\overline{n}|} = \begin{cases} \nu^T, & T(x) \le n \\ \nu^n, & T(x) > n. \end{cases}$$

Clealry, an $n-$year endowment insurance can be expressed as a sum of an $n-$year pure endowment and an $n-$year term life insurance policy. That is,

$$\bar{Z}_{x:\overline{n}|} = \bar{Z}^1_{x:\overline{n}|} + \bar{Z}_{x:\overset{1}{\overline{n}|}}.$$

Taking expectation of both sides we see

$$E(\bar{Z}_{x:\overline{n}|}) = E(\bar{Z}^1_{x:\overline{n}|}) + E(\bar{Z}_{x:\overset{1}{\overline{n}|}}).$$

If we denote the actuarial present value of $\bar{Z}_{x:\overline{n}|}$ by $\bar{A}_{x:\overline{n}|}$ we can write

$$\bar{A}_{x:\overline{n}|} = \bar{A}^1_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}.$$

Example 26.15
Find $\text{Cov}(\bar{Z}^1_{x:\overline{n}|}, \bar{Z}_{x:\overset{1}{\overline{n}|}})$.

Solution.
We have

$$\bar{Z}^1_{x:\overline{n}|} = \begin{cases} \nu^T, & T(x) \le n \\ 0, & T(x) > n. \end{cases} \quad and \quad \bar{Z}_{x:\overset{1}{\overline{n}|}} = \begin{cases} 0, & T(x) \le n \\ \nu^n, & T(x) > n. \end{cases}$$

so that

$$\text{Cov}(\bar{Z}^1_{x:\overline{n}|}, \bar{Z}_{x:\overset{1}{\overline{n}|}}) = E(\bar{Z}^1_{x:\overline{n}|}\bar{Z}_{x:\overset{1}{\overline{n}|}}) - E(\bar{Z}^1_{x:\overline{n}|})E(\bar{Z}_{x:\overset{1}{\overline{n}|}}) = 0 - \bar{A}^1_{x:\overline{n}|} \cdot A_{x:\overset{1}{\overline{n}|}} = -\bar{A}^1_{x:\overline{n}|} \cdot A_{x:\overset{1}{\overline{n}|}} \blacksquare$$

It follows from the above example,

$$\begin{aligned} \text{Var}(\bar{Z}_{x:\overline{n}|}) =& \text{Var}(\bar{Z}^1_{x:\overline{n}|}) + \text{Var}(\bar{Z}_{x:\overset{1}{\overline{n}|}}) + 2\text{Cov}(\bar{Z}^1_{x:\overline{n}|}, \bar{Z}_{x:\overset{1}{\overline{n}|}}) \\ =& {}^2\bar{A}^1_{x:\overline{n}|} - (\bar{A}^1_{x:\overline{n}|})^2 + {}^2A_{x:\overset{1}{\overline{n}|}} - (A_{x:\overset{1}{\overline{n}|}})^2 - 2\bar{A}^1_{x:\overline{n}|} \cdot A_{x:\overset{1}{\overline{n}|}} \\ =& {}^2\bar{A}^1_{x:\overline{n}|} + {}^2A_{x:\overset{1}{\overline{n}|}} - (\bar{A}^1_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}})^2. \end{aligned}$$

Example 26.16
The lifetime of a group of people has the following survival function associated with it: $s(x) = 1 - \frac{x}{100}$, $0 \le x \le 100$. Paul, a member of the group, is currently 40 years old and has a 15-year endowment insurance policy, which will pay him \$50,000 upon death. Find the actuarial present value of this policy. Assume an annual force of interest $\delta = 0.05$.

Solution.
Since X is uniform on $[0, 100]$, T is uniform on $[0, 60]$ so that the density function of T is $f_T(t) = \frac{1}{60}$. We have

$$\begin{aligned}
50,000\bar{A}_{40:\overline{15}|} =&50,000(\bar{A}^{\,1}_{40:\overline{15}|} + A_{40:\overset{\,1}{\overline{15}|}})\\
=&50,000\left[\int_0^{15}\frac{e^{-0.05t}}{60}dt + e^{-0.05\times 15}{}_{15}p_{40}\right]\\
=&50,000\left[\int_0^{15}\frac{e^{-0.05t}}{60}dt + e^{-0.05\times 15}\left(\frac{45}{60}\right)\right]\\
=&50,000(0.5302) = \$26,510 \blacksquare
\end{aligned}$$

Example 26.17
Find the standard deviation of the present value of the benefit of the life insurance policy discussed in the previous example.

Solution.
The variance is

$$50,000^2\mathrm{Var}(\bar{Z}_{x:\overline{n}|}) = 50,000^2[{}^2\bar{A}^{\,1}_{40:\overline{15}|} + {}^2A_{40:\overset{\,1}{\overline{15}|}} - (\bar{A}^{\,1}_{40:\overline{15}|} + A_{40:\overset{\,1}{\overline{15}|}})^2]$$

where

$${}^2\bar{A}^{\,1}_{40:\overline{15}|} = \frac{1}{2\delta(\omega - x)}(1 - e^{-2n\delta}) = \frac{1}{120(0.05)}(1 - e^{-30(0.05)}) = 0.1295$$

$${}^2A_{40:\overset{\,1}{\overline{15}|}} = \left(\frac{\omega - x - n}{\omega - x}\right)e^{-2n\delta} = \frac{45}{60}e^{-1.5} = 0.1673.$$

Thus, the standard deviation is

$$50,000\sqrt{\mathrm{Var}(\bar{Z}_{x:\overline{n}|})} = 50,000\sqrt{[0.1295 + 0.1673 - 0.5302^2]} = 6262.58 \blacksquare$$

Practice Problems

Problem 26.18
Consider a 20-year endowment insurance for a life aged 45 with face amount of 1000. Assume an annual effective interest rate $i = 0.05$.
(a) What is the net single premium if the policyholder's dies at age 55.8?
(b) What is the net single premium if the policyholder's dies at age 70.2?

Problem 26.19
Let the remaining lifetime at birth random variable X be exponential with $\mu = 0.05$. Let $\bar{Z}_{x:\overline{n}|}$ be the contingent payment random variable for a life aged x under a 15-year endowment insurance that pays 1 upon death. Find $\bar{A}_{x:\overline{15}|}$ if the force of interest is $\delta = 0.06$.

Problem 26.20
You are given the following information:
(i) $\bar{A}^{1}_{x:\overline{n}|} = 0.4275$
(ii) $\mu(x+t) = 0.045$ for all t.
(iii) $\delta = 0.055$.
Calculate $\bar{A}_{x:\overline{n}|}$.

Problem 26.21
Let the remaining lifetime at birth random variable X be exponential with $\mu = 0.02$. Let $\bar{Z}_{45:\overline{20}|}$ be the contingent payment random variable for a life aged 45 with a 20-year endowment insurance that pays 1000 upon death. Suppose that $\bar{A}_{45:\overline{20}|} = 0.46185$ and that the annual force of interest is $\delta = 0.05$. Find $F_{\bar{Z}_{45:\overline{20}|}}(E(\bar{Z}_{45:\overline{20}|}))$.

Problem 26.22
Let ${}^2\bar{A}_{x:\overline{n}|}$ be the second moment of $\bar{Z}_{x:\overline{n}|}$ under the $n-$year endowment insurance. Find a formula for ${}^2\bar{A}_{x:\overline{n}|}$.

26.4 Deferred Life Insurance

An $m-$**year deferred insurance** policy pays a benefit to the insured only if the insured person dies at least m years from the time the policy was issued. The random variable ${}_{m|}\bar{Z}_x$ for an $m-$year deferred insurance with a benefit of 1 payable at the moment of death is

$$
{}_{m|}\bar{Z}_x = \begin{cases} 0, & T(x) \leq m \\ \nu^T, & T(x) > m. \end{cases}
$$

The actuarial present value of an $m-$year deferred insurance policy that pays one unit in benefits is denoted by ${}_{m|}\bar{A}_x$ and can be found

$$
\begin{aligned}
{}_{m|}\bar{A}_x =& E({}_{m|}\bar{Z}_x) = E(\nu^T) = \int_m^\infty \nu^t f_T(t)dt \\
=& \int_m^\infty \nu^t {}_tp_x \mu(x+t)dt \\
=& \nu^m {}_mp_x \int_m^\infty \nu^{t-m} {}_{t-m}p_{x+m}\mu(x+t)dt \\
=& \nu^m {}_mp_x \int_0^\infty \nu^u {}_up_{x+m}\mu(x+u+m)du \\
=& \nu^m {}_mp_x \bar{A}_{x+m}
\end{aligned}
$$

where $u = t - m$.
The left subscript, m, indicates the length of the deferral period. This notation is misleading and one has to be careful: The blank after the "|" symbol does not mean the period of evaluation is 1 period immediately after the m periods of deferral. For ${}_{m|}\bar{A}_x$, the period of evaluation is the time until death after age $x + m$.
Now, recall that $A_{x:\overline{m|}}^{\;\,1} = {}_mE_x = \nu^m {}_mp_x$ so that

$$
{}_{m|}\bar{A}_x = A_{x:\overline{m|}}^{\;\,1} \bar{A}_{x+m}.
$$

Example 26.18
Let the remaining lifetime at birth random variable X be uniform on $[0, \omega]$. Let ${}_{m|}\bar{Z}_x$ be the contingent payment random variable for a life aged x. Find ${}_{m|}\bar{A}_x$ if the force of interest is δ.

Solution.
We have

$$\begin{aligned}
{}_mp_x =&\frac{\omega-x-m}{\omega-x}\\
\bar{A}_{x+m} =&\frac{1}{\omega-x-m}\left[\frac{1-e^{-\delta(\omega-x-m)}}{\delta}\right]\\
{}_{m|}\bar{A}_x =&\nu^m\frac{1-e^{-\delta(\omega-x-m)}}{\delta(\omega-x)} \blacksquare
\end{aligned}$$

Example 26.19
Let the remaining lifetime at birth random variable X be exponential with parameter μ. Let ${}_{m|}\bar{Z}_x$ be the contingent payment random variable for a life aged x. Find ${}_{m|}\bar{A}_x$ if the force of interest is δ.

Solution.
We have

$$\begin{aligned}
{}_mp_x =&e^{-m\mu}\\
\bar{A}_{x+m} =&\frac{\mu}{\mu+\delta}\\
{}_{m|}\bar{A}_x =&e^{-m\delta}\frac{\mu}{\mu+\delta}e^{-m\mu} \blacksquare
\end{aligned}$$

The second moment of ${}_{m|}\bar{Z}_x$ is denoted by ${}^2{}_{m|}\bar{A}_x$. The variance of ${}_{m|}\bar{Z}_x$ is given by

$$\mathrm{Var}({}_{m|}\bar{Z}_x) = {}^2{}_{m|}\bar{A}_x - ({}_{m|}\bar{A}_x)^2.$$

Example 26.20
Let Z_1 denote the present value random variable of an $m-$year term insurance of \$1, while Z_2 that of an $m-$year deferred insurance of \$1, with death benefit payable at the moment of death of (x).
You are given:
(i) $\bar{A}^1_{x:\overline{n}|} = 0.01$ and ${}^2\bar{A}^1_{x:\overline{n}|} = 0.0005$
(ii) ${}_{m|}\bar{A}_x = 0.10$ and ${}^2{}_{m|}\bar{A}_x = 0.0136$.
Calculate the coefficient of correlation given by

$$\mathrm{Corr}(Z_1, Z_2) = \frac{\mathrm{Cov}(Z_1, Z_2)}{\sqrt{\mathrm{Var}(Z_1)\mathrm{Var}(Z_2)}}.$$

Solution.
We have

$$Z_1 = \begin{cases} \nu^T, & T(x) \leq n \\ 0, & T(x) > n. \end{cases} \quad and \quad Z_2 = \begin{cases} 0, & T(x) \leq n \\ \nu^T, & T(x) > n. \end{cases}$$

so that

$$\text{Cov}(Z_1, Z_2) = E(Z_1 Z_2) - E(Z_1)E(Z_2) = 0 - \bar{A}^{1}_{x:\overline{n}|} \cdot {}_{m|}\bar{A}_x = -0.01 \times 0.10 = -0.001.$$

Also, we have

$$\text{Var}(Z_1) = {}^2\bar{A}^{1}_{x:\overline{n}|} - (\bar{A}^{1}_{x:\overline{n}|})^2 = 0.0005 - 0.01^2 = 0.0004$$

and

$$\text{Var}(Z_2) = {}^2{}_{m|}\bar{A}_x - ({}_{m|}\bar{A}_x)^2 = 0.0136 - 0.10^2 = 0.0036.$$

Thus,

$$\text{Corr}(Z_1, Z_2) = -\frac{0.0001}{\sqrt{0.0004}\sqrt{0.0036}} = -\frac{5}{6} \blacksquare$$

Example 26.21
Show that: ${}_{m|}\bar{A}_x + \bar{A}^{1}_{x:\overline{m}|} = \bar{A}_x$.

Solution.
We have

$${}_{m|}\bar{A}_x + \bar{A}^{1}_{x:\overline{m}|} = \int_m^\infty \nu^t {}_tp_x \mu(x+t)dt + \int_0^m \nu^t {}_tp_x \mu(x+t)dt = \int_0^\infty \nu^t {}_tp_x \mu(x+t)dt = \bar{A}_x \blacksquare$$

The above identity has the following verbal interpretation: Purchasing an $m-$year term policy and an $m-$year deferred policy is equivalent to purchasing a whole life policy.

Example 26.22 ‡
For a whole life insurance of 1000 on (x) with benefits payable at the moment of death: (i)

$$\delta_t = \begin{cases} 0.04 & 0 < t \leq 10 \\ 0.05 & 10 < t \end{cases}$$

(ii)

$$\mu(x+t) = \begin{cases} 0.06 & 0 < t \leq 10 \\ 0.07 & 10 < t \end{cases}$$

Calculate the single benefit premium for this insurance.

Solution.
We have

$$
\begin{aligned}
1000\bar{A}_x =&1000[\bar{A}^{1}_{x:\overline{10}|} + {}_{10|}\bar{A}_x]\\
=&1000\left[\int_0^{10} e^{-0.04t}e^{-0.06t}(0.06)dt + \int_{10}^{\infty} e^{-0.07t}e^{-0.05t}(0.05)dt\right]\\
=&1000\left[\int_0^{10} e^{-0.04t}e^{-0.06t}(0.06)dt + e^{-0.07}e^{-0.05}\int_0^{\infty} e^{-0.07t}e^{-0.05t}(0.07)dt\right]\\
=&1000\left[0.06\int_0^{10} e^{-0.10t}dt + 0.07e^{-0.12}\int_0^{\infty} e^{-0.12t}dt\right]\\
=&1000\left[0.06\left[\frac{-e^{-0.10t}}{0.10}\right]_0^{10} + 0.07e^{-12}\left[\frac{-e^{-0.12t}}{0.12}\right]_0^{\infty}\right]\\
=&593.87 \blacksquare
\end{aligned}
$$

Practice Problems

Problem 26.23
Give an expression for the second moment of ${}_{m|}\bar{Z}_x$, denoted by ${}^2{}_{m|}\bar{A}_x$.

Problem 26.24
Suppose you are 30 years old and you buy a 10-year deferred life insurance that pays 1 at the time of death. The remaining lifetime at birth random variable has a survival function $s(x) = 1 - \frac{x}{100},\ 0 \leq x \leq 100$. Find ${}_{10|}\bar{A}_{30}$ if the force of interest is $\delta = 0.05$.

Problem 26.25
Find the variance in the previous problem.

Problem 26.26
Let the remaining lifetime at birth random variable X be exponential with parameter $\mu = 0.05$. Let ${}_{m|}\bar{Z}_x$ be the contingent payment random variable for a life aged x. Find ${}_{15|}\bar{A}_x$ if the force of interest is $\delta = 0.06$.

Problem 26.27
Find the variance in the previous problem.

27 Insurances Payable at the End of the Year of Death

Although in practice insurance is payable at the moment of death, traditionally actuarial work has been done with a life table which in turn is based on annual census data. Therefore discrete formulas are needed for net single premiums of insurances payable at the end of the year of death.
The continuous analysis of Section 26 was based on the continuous random variable $T(x)$ that represents the future lifetime. The discrete version is based on the discrete random variable $K(x)$ which is the curtate future lifetime. As we progress you will notice that the discrete versions of insurance look just like their continuous counterparts with the integrals being replaced by summations.
Now, recall from Section 20.6, that the notation $K(x) = k$ means that death occurred in the interval $[x+k, x+k+1)$. The probability of death in that interval is given by $\Pr(K(x) = k) = {}_kp_xq_{x+k}$. We can use the curtate probabilities of death to define expected values of contingent payments made at year end.
Next, we develop expected values of the four life insurances discussed in Section 26.

Whole Life Insurance

Consider a whole life insurance with benefit 1 paid at the end of the year of death. The present value random variable for this benefit is

$$Z_x = \nu^{K+1},\ K = K(x) \geq 0.$$

The actuarial present value of this policy for life (x) with one unit in benefits payable at the end of the year of death is denoted by A_x and is given by the formula

$$A_x = \sum_{k=0}^{\infty} \nu^{k+1}{}_kp_xq_{x+k}.$$

The second moment of Z_x is denoted by ${}^2A_x$ and is given by

$${}^2A_x = \sum_{k=0}^{\infty} \nu^{2(k+1)}{}_kp_xq_{x+k}.$$

The variance of Z_x is then

$$\text{Var}(Z_x) = {}^2A_x - (A_x)^2.$$

Example 27.1
Let the remaining lifetime at birth random variable X be uniform on [0,100]. Let Z_{30} be the contingent payment random variable for a life aged $x = 30$. Find A_{30}, ${}^2A_{30}$, and $\text{Var}(Z_{30})$ if $\delta = 0.05$.

Solution.
For this problem, recall

$$s(x) = 1 - \tfrac{x}{100}.$$

We have

$$\begin{aligned}
{}_kp_{30} =& \frac{s(30+k)}{s(30)} = \frac{1 - \frac{30+k}{100}}{1 - \frac{30}{100}} = 1 - \frac{k}{70}\\
q_{30+k} =& 1 - \frac{s(31+k)}{s(30+k)} = \frac{1}{70-k}\\
A_{30} =& \sum_{k=0}^{\infty} \frac{e^{-0.05(k+1)}}{70} = \frac{e^{-0.05}}{70(1-e^{-0.05})} = 0.2786\\
{}^2A_{30} =& \sum_{k=0}^{\infty} \frac{e^{-0.10(k+1)}}{70} = \frac{e^{-0.10}}{70(1-e^{-0.10})} = 0.1358\\
\text{Var}(Z_{30}) =& 0.1358 - 0.2786^2 = 0.0582 \blacksquare
\end{aligned}$$

Term Life Insurance
Consider an $n-$year term life insurance with benefit 1 paid at the end of the year of death. The present value random variable for this benefit is

$$Z^1_{x:\overline{n}|} = \begin{cases} \nu^{K+1}, & K = 0, 1, \cdots, n-1 \\ 0, & K \geq n. \end{cases}$$

The actuarial present value of this policy for life (x) with one unit in benefits payable at the end of the year of death is denoted by $A^1_{x:\overline{n}|}$ and is given by the formula

$$A^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} \nu^{k+1} {}_kp_x q_{x+k}.$$

The second moment of $Z^1_{x:\overline{n}|}$ is denoted by ${}^2A^1_{x:\overline{n}|}$ and is given by

$${}^2A^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} \nu^{2(k+1)} {}_kp_x q_{x+k}.$$

The variance of $Z^1_{x:\overline{n}|}$ is then

$$\text{Var}(Z^1_{x:\overline{n}|}) = {}^2A^1_{x:\overline{n}|} - (A^1_{x:\overline{n}|})^2.$$

Example 27.2
Let the remaining lifetime at birth random variable X be uniform on [0,100]. Let $Z^1_{30:\overline{10}|}$ be the contingent payment random variable for a life aged $x = 30$. Find $A^1_{30:\overline{10}|}$, ${}^2A^1_{30:\overline{10}|}$, and $\text{Var}(Z^1_{30:\overline{10}|})$ if $\delta = 0.05$.

Solution.
For this problem, recall

$$s(x) = 1 - \tfrac{x}{100}.$$

We have

$$\begin{aligned}
{}_kp_{30} =& \frac{s(30+k)}{s(30)} = \frac{1 - \frac{30+k}{100}}{1 - \frac{30}{100}} = 1 - \frac{k}{70}\\
q_{30+k} =& 1 - \frac{s(31+k)}{s(30+k)} = \frac{1}{70-k}\\
A^1_{30:\overline{10}|} =& \sum_{k=0}^{9} \frac{e^{-0.05(k+1)}}{70} = \frac{e^{-0.05}(1 - e^{-0.50})}{70(1 - e^{-0.05})} = 0.1096\\
{}^2A^1_{30:\overline{10}|} =& \sum_{k=0}^{9} \frac{e^{-0.10(k+1)}}{70} = \frac{e^{-0.10}(1 - e^{-1})}{70(1 - e^{-0.10})} = 0.0859\\
\text{Var}(Z^1_{30:\overline{10}|}) =& 0.0859 - 0.1096^2 = 0.0739 \blacksquare
\end{aligned}$$

Deferred Life Insurance

Consider an $m-$year deferred life insurance with benefit 1 paid at the end of the year of death. The present value random variable for this benefit is

$$_{m|}Z_x = \begin{cases} 0, & K = 0, 1, \cdots, m-1 \\ \nu^{K+1}, & K = m, m+1, \cdots \end{cases}$$

The actuarial present value of this policy for life (x) with one unit in benefits payable at the end of the year of death is denoted by $_{m|}A_x$ and is given by the formula

$$_{m|}A_x = \sum_{k=m}^{\infty} \nu^{k+1} {}_kp_x q_{x+k}.$$

The second moment of ${}_{m|}Z_x$ is denoted by ${}^2{}_{m|}A_x$ and is given by

$$ {}^2{}_{m|}A_x = \sum_{k=m}^{\infty} \nu^{2(k+1)} {}_kp_x q_{x+k}. $$

The variance of ${}_{m|}Z_x$ is then

$$ \text{Var}({}_{m|}Z_x) = {}^2{}_{m|}A_x - ({}_{m|}A_x)^2. $$

As in the continuous case, we have the following identity

$$ A_x = A^1_{x:\overline{n}|} + {}_{m|}A_x $$

Example 27.3
Let the remaining lifetime at birth random variable X be uniform on [0,100]. Let ${}_{10|}Z_{30}$ be the contingent payment random variable for a life aged $x = 30$. Find ${}_{10|}A_{30}, {}^2{}_{10|}A_{30}$ and $\text{Var}({}_{10|}Z_{30})$ if $\delta = 0.05$.

Solution.
For this problem, recall

$$ s(x) = 1 - \tfrac{x}{100}. $$

We have

$$ \begin{aligned} {}_kp_{30} =& \frac{s(30+k)}{s(30)} = \frac{1 - \frac{30+k}{100}}{1 - \frac{30}{100}} = 1 - \frac{k}{70} \\ q_{30+k} =& 1 - \frac{s(31+k)}{s(30+k)} = \frac{1}{70-k} \\ {}_{10|}A_{30} =& \sum_{k=10}^{\infty} \frac{e^{-0.05(k+1)}}{70} = \frac{e^{-0.55}}{70(1-e^{-0.05})} = 0.1690 \\ {}^2{}_{10|}A_{30} =& \sum_{k=10}^{\infty} \frac{e^{-0.10(k+1)}}{70} = \frac{e^{-1.1}}{70(1-e^{-0.10})} = 0.04997 \\ \text{Var}({}_{10|}Z_{30}) =& 0.04997 - 0.1690^2 = 0.0214 \blacksquare \end{aligned} $$

Endowment Life Insurance

Consider an $n-$year endowment insurance with benefit 1 paid at the end of the year of death. The present value random variable for this benefit is

$$Z_{x:\overline{n|}} = \begin{cases} \nu^{K+1}, & K = 0, 1, \cdots, n-1 \\ \nu^n, & K = n, n+1, \cdots \end{cases}$$

The actuarial present value of this policy for life (x) with one unit in benefits payable at the end of the year of death is denoted by $A_{x:\overline{n|}}$ and is given by the formula

$$A_{x:\overline{n|}} = \sum_{k=0}^{n-1} \nu^{k+1} {}_kp_x q_{x+k} + \nu^n {}_np_x.$$

The second moment of $Z_{x:\overline{n|}}$ is denoted by ${}^2A_{x:\overline{n|}}$ and is given by

$${}^2A_{x:\overline{n|}} = \sum_{k=0}^{n-1} \nu^{2(k+1)} {}_kp_x q_{x+k} + \nu^{2n} {}_np_x.$$

The variance of $Z_{x:\overline{n|}}$ is then

$$\text{Var}(Z_{x:\overline{n|}}) = {}^2A_{x:\overline{n|}} - (A_{x:\overline{n|}})^2.$$

As in the continuous case, an endowment insurance is the sum of a pure endowment and a term insurance

$$A_{x:\overline{n|}} = A^1_{x:\overline{n|}} + {}_nE_x = A^1_{x:\overline{n|}} + A_{x:\overset{1}{\overline{n|}}}.$$

Example 27.4

Let the remaining lifetime at birth random variable X be uniform on [0,100]. Let $Z_{30:\overline{10|}}$ be the contingent payment random variable for a life aged $x = 30$. Find $A_{30:\overline{10|}}, {}^2A_{30:\overline{10|}}$ and $\text{Var}(Z_{30:\overline{10|}})$ if $\delta = 0.05$.

Solution.

For this problem, recall

$$s(x) = 1 - \tfrac{x}{100}.$$

We have

$$\begin{aligned}
{}_kp_{30} =& \frac{s(30+k)}{s(30)} = \frac{1-\frac{30+k}{100}}{1-\frac{30}{100}} = 1 - \frac{k}{70}\\
q_{30+k} =& 1 - \frac{s(31+k)}{s(30+k)} = \frac{1}{70-k}\\
A_{30:\overline{10}|} =& \sum_{k=0}^{9} \frac{e^{-0.05(k+1)}}{70} + e^{-10(0.05)}{}_{10}p_{30} = 0.1096 + 0.5199 = 0.6295\\
{}^2A_{30:\overline{10}|} =& \sum_{k=0}^{9} \frac{e^{-0.10(k+1)}}{70} + e^{-20(0.05)}{}_{10}p_{30} = 0.0859 + 0.3153 = 0.4012\\
\mathrm{Var}(Z_{30:\overline{10}|}) =& 0.4012 - 0.6295^2 = 0.0049 \blacksquare
\end{aligned}$$

Example 27.5
Complete the following extract of a life table:

k	ℓ_k	q_k	${}_kp_0$	${}_kp_0 \cdot q_k$
0	125			
1	100			
2	75			
3	50			
4	25			
5	0			

Find (i) A_0, (ii) ${}^2A_0$, (iii) $A^1_{0:\overline{2}|}$, (iv) ${}^2A^1_{0:\overline{2}|}$, (v) ${}_{3|}A_0$, (vi) ${}^2{}_{3|}A_0$, (vii) $A_{0:\overline{2}|}$, (viii) ${}^2A_{0:\overline{2}|}$. Assume an annual effective interest rate of 6%.

Solution.
We have

k	ℓ_k	q_k	${}_kp_0$	${}_kp_0 \cdot q_k$
0	125	0.200	1.000	0.200
1	100	0.250	0.800	0.200
2	75	0.333	0.600	0.200
3	50	0.500	0.400	0.200
4	25	1.000	0.200	0.200
5	0			

(i) $A_0 = \sum_{k=0}^{\infty} \nu^{k+1}{}_kp_0q_k = 0.2[(1.06)^{-1} + (1.06)^{-2} + (1.06)^{-3} + (1.06)^{-4} + (1.06)^{-5}] = 0.8425.$
(ii) ${}^2A_0 = \sum_{k=0}^{\infty} \nu^{2(k+1)}{}_kp_0q_k = 0.2[(1.06)^{-2} + (1.06)^{-4} + (1.06)^{-6} + (1.06)^{-8} + (1.06)^{-10}] = 0.7146.$
(iii) $A^1_{0:\overline{2|}} = 0.2[(1.06)^{-1} + (1.06)^{-2}] = 0.3667.$
(iv) ${}^2A^1_{0:\overline{2|}} = 0.2[(1.06)^{-2} + (1.06)^{-4}] = 0.3364.$
(v) ${}_{3|}A_0 = 0.2[(1.06)^{-4} + (1.06)^{-5}] = 0.3079.$
(vi) ${}^2{}_{3|}A_0 = 0.2[(1.06)^{-8} + (1.06)^{-10}] = 0.2372.$
(vii) $A_{0:\overline{2|}} = A^1_{0:\overline{2|}} + \nu^2{}_2p_0 = 0.3667 + 0.534 = 0.9007.$
(viii) ${}^2A_{0:\overline{2|}} = {}^2A^1_{0:\overline{2|}} + \nu^4{}_2p_0 = 0.3364 + 0.4753 = 0.8117$ ■

Example 27.6 ‡
For a whole life insurance of 1000 on (80), with death benefits payable at the end of the year of death, you are given:
(i) Mortality follows a select and ultimate mortality table with a one-year select period.
(ii) $q_{[80]} = 0.5q_{80}$
(iii) $i = 0.06$
(iv) $1000A_{80} = 679.80$
(v) $1000A_{81} = 689.52$
Calculate $1000A_{[80]}$.

Solution.
From the identity

$$A_{80} = \nu q_{80} + \nu p_{80}A_{81}$$

we find

$$q_{80} = \frac{(1+i)A_{80} - A_{81}}{1 - A_{81}} = \frac{1.06(0.67980) - 0.68952}{1 - 0.68952} = 0.1000644.$$

Using the fact that the select period is 1 year and (ii), we find

$$\begin{aligned}
1000A_{[80]} =&1000\nu q_{[80]} + 1000\nu p_{[80]}A_{[80]+1}\\
=&1000(1.06)^{-1}(0.5q_{80}) + 1000\nu(1 - 0.5q_{80})A_{81}\\
=&1000(1.06)^{-1}[0.5(0.1000644)] + (1.06)^{-1}[1 - 0.5 - 0.1000644](689.52)\\
=&665.15 \blacksquare
\end{aligned}$$

Practice Problems

Problem 27.1

Let the remaining lifetime at birth random variable X be exponential with $\mu = 0.05$. Let Z_{30} be the contingent payment random variable for a life aged $x = 30$. Find $\bar{A}_{30}, {}^2\bar{A}_{30}$, and $\text{Var}(Z_{30})$ if $\delta = 0.10$.

Problem 27.2

Let the remaining lifetime at birth random variable X be expomemtial with $\mu = 0.05$. Let $Z^{\ 1}_{30:\overline{10|}}$ be the contingent payment random variable for a life aged $x = 30$. Find $\bar{A}^{\ 1}_{30:\overline{10|}}, {}^2\bar{A}^{\ 1}_{30:\overline{10|}}$, and $\text{Var}(Z^{\ 1}_{30:\overline{10|}})$ if $\delta = 0.10$.

Problem 27.3

Let the remaining lifetime at birth random variable X be exponential with $\mu = 0.05$. Let ${}_{10|}Z_{30}$ be the contingent payment random variable for a life aged $x = 30$. Find ${}_{10|}\bar{A}_{30}, {}^2_{10|}\bar{A}_{30}$ and $\text{Var}({}_{10|}Z_{30})$ if $\delta = 0.10$.

Problem 27.4

Let the remaining lifetime at birth random variable X be exponential with $\mu = 0.05$. Let $Z_{30:\overline{10|}}$ be the contingent payment random variable for a life aged $x = 30$. Find $\bar{A}_{30:\overline{10|}}, {}^2\bar{A}_{30:\overline{10|}}$ and $\text{Var}(Z_{30:\overline{10|}})$ if $\delta = 0.10$.

Problem 27.5 ‡

For a group of individuals all age x, you are given:

(i) 25% are smokers (s); 75% are nonsmokers (ns).

(ii)

k	q^s_{x+k}	q^{ns}_{x+k}
0	0.10	0.05
1	0.20	0.10
2	0.30	0.15

(iii) $i = 0.02$.

Calculate $10,000A^{\ 1}_{x:\overline{2|}}$ for an individual chosen at random from this group.

Problem 27.6 ‡

You are given:

(i) the following select-and-ultimate mortality table with 3-year select period:

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	$q_{[x]+3} = q_{x+3}$	$x+3$
60	0.09	0.11	0.13	0.15	63
61	0.10	0.12	0.14	0.16	64
62	0.11	0.13	0.15	0.17	65
63	0.12	0.14	0.16	0.18	66
64	0.13	0.15	0.17	0.19	67

(ii) $i = 0.03$
Calculate ${}_{2|2}A_{[60]}$, the actuarial present value of a 2-year deferred 2-year term insurance on [60].

Problem 27.7 ‡
Oil wells produce until they run dry. The survival function for a well is given by:

t(years)	0	1	2	3	4	5	6	7
$S(t)$	1.00	0.90	0.80	0.60	0.30	0.10	0.05	0.00

An oil company owns 10 wells age 3. It insures them for 1 million each against failure for two years where the loss is payable at the end of the year of failure.
You are given:
(i) R is the present-value random variable for the insurers aggregate losses on the 10 wells.
(ii) The insurer actually experiences 3 failures in the first year and 5 in the second year.
(iii) $i = 0.10$
Calculate the ratio of the actual value of R to the expected value of R.

28 Recursion Relations for Life Insurance

By a **recursion relation** we mean an equation where the value of a function at a certain value of the independent variable can be determined from the value(s) of the function at different values of the variable. The application of recursion formulas in this book involves one of the two forms:

Backward Recursion Formula

$$u(x) = c(x) + d(x)u(x+1)$$

or

Forward Recursion Formula

$$u(x+1) = -\frac{c(x)}{d(x)} + \frac{1}{d(x)}u(x).$$

In this section, we consider recursive relations involving the actuarial present values and second moments of the types of life insurance covered in Section 27.

The first relation is for the actuarial present value of a discrete whole live insurance. We have

$$\begin{aligned}
A_x =& \sum_{k=0}^{\infty} \nu^{k+1}{}_kp_x q_{x+k} \\
=& \nu q_x + \nu \sum_{k=1}^{\infty} \nu^k{}_kp_x q_{x+k} \\
=& \nu q_x + \nu \sum_{k=1}^{\infty} \nu^k p_{x\,k-1}p_{x+1} q_{x+k} \\
=& \nu q_x + \nu p_x \sum_{k=0}^{\infty} \nu^{k+1}{}_kp_{x+1} q_{x+k+1} \\
=& \nu q_x + \nu p_x A_{x+1}.
\end{aligned}$$

This recursion relation says that a whole life insurance policy for (x) is the same thing as a 1-year term policy with payment at year end, plus, if x survives an additional year, a whole life policy starting at age $x+1$.

Next, we have

$$A_x = \sum_{k=0}^{\infty} \nu^{k+1}{}_kp_xq_{x+k} = \nu q_x + \sum_{k=1}^{\infty} \nu^{k+1}{}_kp_xq_{x+k} = A^1_{x:\overline{1}|} + {}_{1|}A_x.$$

This relation says that a whole life insurance policy for (x) is the same thing as a 1-year term policy with payment at year end, plus a 1-year deferred whole live insurance.
A more genereal relation of the previous one is the following:

$$A_x = \sum_{k=0}^{\infty} \nu^{k+1}{}_kp_xq_{x+k} = \sum_{k=0}^{n-1} \nu^{k+1}{}_kp_xq_{x+k} + \sum_{k=n}^{\infty} \nu^{k+1}{}_kp_xq_{x+k} = A^1_{x:\overline{n}|} + {}_{n|}A_x.$$

That is, a whole life insurance that pays 1 at the end year of death is equivalent to an $n-$year term policy plus an $n-$year deferred policy.

Example 28.1
Show that

$$A^1_{x:\overline{n}|} = \nu q_x + \nu p_x A^{\ \ 1}_{x+1:\overline{n-1}|}.$$

Interpret the result verbally.

Solution.
We have

$$\begin{aligned}
A^1_{x:\overline{n}|} =& \sum_{k=0}^{n-1} \nu^{k+1}{}_kp_xq_{x+k} \\
=& \nu q_x + \sum_{k=1}^{n-1} \nu^{k+1}{}_kp_xq_{x+k} \\
=& \nu q_x + \nu \sum_{k=1}^{n-1} \nu^{k} p_{x\,k-1}p_{x+1}q_{x+k} \\
=& \nu q_x + \nu p_x \sum_{k=0}^{n-2} \nu^{k+1}{}_kp_{x+1}q_{x+k+1} \\
=& \nu q_x + \nu p_x A^{\ \ 1}_{x+1:\overline{n-1}|}.
\end{aligned}$$

This relation says that an $n-$year term policy is the same thing as a 1-year term policy and an $(n-1)-$year term policy at age $x+1$, provided that the insured is still alive at $x+1$ ■

Example 28.2 ‡
For a given whole life insurance of 1 on (41) with death benefit payable at the end of year of death, you are given:
(i) $i = 0.05$.
(ii) $p_{40} = 0.9972$.
(iii) $A_{41} - A_{40} = 0.00822$.
(iv) Z is the present value random variable for this insurance.
Calculate A_{41}.

Solution.
From the recursion relation

$$A_x = \nu q_x + \nu p_x A_{x+1}$$

we obtain

$$A_{40} = \nu q_{40} + \nu p_{40} A_{41}.$$

Substituting, we find

$$A(41) - 0.00822 = (1.05)^{-1}(1 - 0.9972) + (1.05)^{-1}(0.9972)A_{41}.$$

Solving this equation for A_{41} we find $A_{41} = 0.2165$ ■

The following recursive relations about second moments can be easily established:

$${}^2A_x = \nu^2 q_x + \nu^2 p_x {}^2A_{x+1}$$

and

$${}^2A^{\,1}_{x:\overline{n|}} = \nu^2 q_x + \nu^2 p_x {}^2A^{\,1}_{x+1:\overline{n-1|}}.$$

Example 28.3 ‡
Using the information of the previous example together with the condition ${}^2A_{41} - {}^2A_{40} = 0.00433$, find $\text{Var}(Z)$.

Solution.
We use the recursive relation

$${}^2A_x = \nu^2 q_x + \nu^2 p_x {}^2A_{x+1}$$

to obtain

$${}^2A_{40} = \nu^2 q_{40} + \nu^2 p_{40} {}^2A_{41}.$$

After substitution, we find

$$^2A_{41} - 0.00433 = (1.05)^{-2}(1 - 0.9972) + (1.05)^{-2}(0.9972)^2\,{}^2A_{41}.$$

Solving this equation for ${}^2A_{41}$ we find ${}^2A_{41} = 0.0719$ and therefore

$$\text{Var}(Z) = 0.0719 - 0.2165^2 = 0.025 \blacksquare$$

Example 28.4
Consider the following extract from the Illustrative Life Table.

x	ℓ_x	$1000q_x$	$1000A_x$	$1000\,{}^2A_x$
36	9,401,688	2.14	134.70	37.26
37	9,381,566	2.28	140.94	39.81
38	9,360,184	2.43	147.46	42.55
39	9,337,427	2.60	154.25	45.48
40	9,313,166	2.78	161.32	48.63

Calculate the actuarial present value of a 3-year term policy for a life aged (36). The annual interest rate is $i = 6\%$.

Solution.
Using Problem 28.2 we have

$$A^{1}_{36:\overline{3}|} = A_{36} - \nu^3\,{}_3p_{36}A(39).$$

But

$${}_3p_{36} = \frac{\ell_{39}}{\ell_{36}} = \frac{9,337,427}{9,401,688} = 0.9932.$$

Therefore,

$$A^{1}_{36:\overline{3}|} = 0.1347 - (1.06)^{-3}(0.9932)(0.15425) = 0.0061 \blacksquare$$

Practice Problems

Problem 28.1
You are given:
(i) $i = 0.05$.
(ii) $p_x = 0.90$.
(iii) $A_x = 0.670$
Find A_{x+1}.

Problem 28.2
Show that

$$A_x = A^1_{x:\overline{n}|} + \nu^n {}_np_x A_{x+n}.$$

Problem 28.3
Consider the following extract from the Illustrative Life Table.

x	ℓ_x	$1000q_x$	$1000A_x$	1000^2A_x
36	9,401,688	2.14	134.70	37.26
37	9,381,566	2.28	140.94	39.81
38	9,360,184	2.43	147.46	42.55
39	9,337,427	2.60	154.25	45.48
40	9,313,166	2.78	161.32	48.63

Calculate the actuarial present value of a 4-year term policy for a life aged (36). The annual interest rate is $i = 6\%$.

Problem 28.4
Given $A_{38} = 0.1475,\ A_{39} = 0.1543$, find q_{38} if $i = 0.06$.

Problem 28.5 ‡
For a sequence, $u(k)$ is defined by the following recursion formula

$$u(k) = \alpha(k) + \beta(k) \times u(k-1) \text{ for } k = 1, 2, 3, \cdots$$

where
i) $\alpha(k) = \left(\frac{q_{k-1}}{p_{k-1}}\right)$.
ii) $\beta(k) = \frac{1+i}{p_{k-1}}$.
iii) $u(70) = 1.0$.
Show that $u(40) = A_{40:\overline{30}|}$.

Problem 28.6
Show that
$$ {}^2A_{x:\overline{n}|} = {}^2A_x - \nu^n {}_nE_x {}^2A_{x+n} + \nu^n {}_nE_x. $$

Problem 28.7 ‡
For a given whole life insurance of 1 on (41) with death benefit payable at the end of year of death, you are given:
(i) $i = 0.05$.
(ii) $p_{40} = 0.9972$.
(iii) $A_{41} - A_{40} = 0.00822$.
(iv) ${}^2A_{41} - {}^2A_{40} = 0.00433$.
(v) Z is the present value random variable for this insurance.
Calculate $\mathrm{Var}(Z)$.

Problem 28.8 ‡
Lee, age 63, considers the purchase of a single premium whole life insurance of 10,000 with death benefit payable at the end of the year of death.
The company calculates benefit premiums using:
(i) mortality based on the Illustrative Life Table,
(ii) $i = 0.05$
The company calculates contract premiums as 112% of benefit premiums.
The single contract premium at age 63 is 5233.
Lee decides to delay the purchase for two years and invests the 5233.
Calculate the minimum annual rate of return that the investment must earn to accumulate to an amount equal to the single contract premium at age 65.

29 Variable Insurance Benefit

So far we have been studying level-benefit life insurance policies, i.e., policies that pay the same amount either at the moment of death (continuous models) or at the end of year of death (discrete models). Symbolically, b_t is constant. In practice, the amount of benefit received at death is not \$1 nor constant, but often a function of time. In this section, we consider a variable b_t.

29.1 Non-level Payments: A Simple Example

The following is a simple example of a variable benefit problem that has only a few cash flows and therefore can be directly computed.

Example 29.1 ‡
For a special discrete 3-year term on (x) you are given:
(i) Z is the present value random variable for the death benefits
(ii)

k	q_{x+k}	b_{k+1}
0	0.02	300
1	0.04	350
2	0.06	400

(iii) $i = 0.06$.
Calculate $E(Z)$.

Solution.
We have

$$\begin{aligned}E(Z) =&300\nu q_x + 350\nu^2 p_x q_{x+1} + 400\nu^3 {}_2p_x q_{x+2}\\ =&300\nu q_x + 350\nu^2 p_x q_{x+1} + 400\nu^3 p_x p_{x+1} q_{x+2}\\ =&300\nu q_x + 350\nu^2(1-q_x)q_{x+1} + 400\nu^3(1-q_x)(1-q_{x+1})q_{x+2}\\ =&300(1.06)^{-1}(0.02) + 350(1.06)^{-2}(0.98)(0.04) + 400(1.06)^{-3}(0.98)(0.96)(0.06)\\ =&36.8291 \blacksquare\end{aligned}$$

Example 29.2 ‡
For a special whole life insurance on (x), payable at the moment of death:
(i) $\mu(x+t) = 0.05, t > 0$

(ii) $\delta_x = 0.08$
(iii) The death benefit at time t is $b_t = e^{0.06t}, \ t > 0$.
(iv) Z is the present value random variable for this insurance at issue.
Calculate $\text{Var}(Z)$.

Solution.
We have

$$\begin{aligned}
E[Z] =& \int_0^\infty b_t \nu^t {}_tp_x \mu(x+t)dt = \int_0^\infty e^{0.06t} e^{-0.08t} e^{-0.05t}(0.05)dt \\
=& 0.05 \int_0^\infty e^{-0.07t}dt = 0.05\left[-\frac{e^{-0.07t}}{0.07}\right]_0^\infty = \frac{5}{7} \\
E[Z^2] =& \int_0^\infty (b_t)^2 \nu^{2t} {}_tp_x \mu(x+t)dt = \int_0^\infty e^{0.12t} e^{-0.16t} e^{-0.05t}(0.05)dt \\
=& 0.05 \int_0^\infty e^{-0.09t}dt = 0.05\left[-\frac{e^{-0.09t}}{0.09}\right]_0^\infty = \frac{5}{9} \\
\text{Var}(Z) =& \frac{5}{9} - \left(\frac{5}{7}\right)^2 = 0.04535 \blacksquare
\end{aligned}$$

Practice Problems

Problem 29.1 ‡

For a special discrete 3-year term on (x) you are given:
(i) Z is the present value random variable for the death benefits
(ii)

k	q_{x+k}	b_{k+1}
0	0.02	300
1	0.04	350
2	0.06	400

(iii) $i = 0.06$.
Calculate $E(Z^2)$. Recall that $Z^2 = b_t^2 \nu^{2t}$.

Problem 29.2 ‡

Find $\text{Var}(Z)$ in the previous problem.

Problem 29.3

For a special discrete 3-year term on (x) you are given:
(i) Z is the present value random variable for the death benefits
(ii)

k	q_{x+k}	b_{k+1}
0	0.03	200
1	0.06	150
2	0.09	100

(iii) $i = 0.06$.
Calculate $E(Z)$, $E(Z^2)$, and $\text{Var}(Z)$.

Problem 29.4

For a special type of whole life insurance issued to (40), you are given:
(i) death benefits are 1,000 for the first 5 years and 500 thereafter.
(ii) death benefits are payable at the end of the year of death.
(iii) mortality follows the Illustrative Life table.
(iv) $i = 0.06$.
Calculate the actuarial present value of the benefits for this policy.

Problem 29.5
Two life insurance policies to be issued to (40) are actuarially equivalent:

(I) A whole life insurance of 10 payable at the end of the year of death.
(II) A special whole life insurance, also payable at the end of the year of death, that pays 5 for the first 10 years and B thereafter.

You are given:
(i) $i = 0.04$
(ii) $10A_{40} = 3.0$
(iii) $10A_{50} = 3.5$
(iv) $10A^{\,1}_{40:\overline{10|}} = 0.9$
(v) ${}_{10}E_{40} = 0.6$
Calculate the value of B.

29.2 Increasing or Decreasing Insurances Payable at the Moment of Death

In this section, we look at increasing and decreasing insurances with benefits made at the moment of death.
We first consider the increasing ones. The two types of increasing life insurance are: Annually increasing and continuously increasing life insurance.
An **annually increasing whole life insurance** for a life aged x pays n if death occurs in year n. It follows that the more a person lives the more valuable is the policy. The present value of the contingent payment is given by

$$Z = \lfloor T+1 \rfloor \nu^T, \ T \geq 0.$$

The actuarial present value of such a policy is denoted by $(I\bar{A})_x$ and is derived as follows:

$$\begin{aligned}(I\bar{A})_x =& E(Z) = \int_0^\infty \lfloor t+1 \rfloor \nu^t f_{T(x)}(t)dt \\ =& \int_0^\infty \lfloor t+1 \rfloor \nu^t {}_tp_x \mu(x+t)dt\end{aligned}$$

Example 29.3
You are given that $T(x)$ is uniform on $[0,3]$. Calculate $(I\bar{A})_x$ if the force of interest is $\delta = 0.05$.

Solution.
$T(x)$ is uniform on $[0,3]$, we have

$$f_T(t) = \frac{1}{3}, \ 0 \leq t \leq 3.$$

Also,

$$\lfloor t+1 \rfloor = \begin{cases} 1, & 0 < t < 1 \\ 2, & 1 \leq t < 2 \\ 3, & 2 \leq t < 3. \end{cases}$$

Thus,

$$(I\bar{A})_x = \frac{1}{3}\left[\int_0^1 e^{-0.05t}dt + 2\int_1^2 e^{-0.05t}dt + 3\int_2^3 e^{-0.05t}dt\right] = 1.8263 \ \blacksquare$$

Now, a **continuously increasing whole life insurance** for a life aged x pays t in benefits if death occurs at time $x+t$, irrespective of whether t is a whole number or not. The present value of the contingent payment random variable is given by

$$Z = T\nu^T,\ T \geq 0.$$

The actuarial present value of Z, denoted by $(\overline{IA})_x$ or $(\bar{I}\bar{A})_x$ is given via the formula

$$(\bar{I}\bar{A})_x = \int_0^\infty t\nu^t {}_tp_x\mu(x+t)dt.$$

Example 29.4
You are given that $T(x)$ is uniform on $[0,3]$. Calculate $(\bar{I}\bar{A})_x$ if the force of interest is $\delta = 0.05$.

Solution.
We have

$$(\bar{I}\bar{A})_x = \frac{1}{3}\int_0^3 te^{-0.05t}dt = \frac{1}{3}\left[-20te^{-0.05t} - 400e^{-0.05t}\right]_0^3 = 1.3581 \blacksquare$$

The above discussion works as well for an annually or continuously increasing $n-$year term life insurance. In the case of an annually increasing type, the present value of the contingent payment is

$$Z = \begin{cases} \lfloor T+1 \rfloor \nu^T & T \leq n \\ 0, & T > n \end{cases}$$

and the actuarial present value is

$$(I\bar{A})^1_{x:\overline{n}|} = \int_0^n \lfloor t+1 \rfloor \nu^t {}_tp_x\mu(x+t)dt.$$

In the continuous case, the present value of contingent payment is

$$Z = \begin{cases} T\nu^T & T \leq n \\ 0, & T > n \end{cases}$$

and the actuarial present value is and

$$(\bar{I}\bar{A})^1_{x:\overline{n}|} = \int_0^n t\nu^t {}_tp_x\mu(x+t)dt.$$

The I here stands for increasing and the bar over the I denotes that the increases are continuous.

Example 29.5
The age-at-death random variable X is described by the survival function $s(x) = 1 - \frac{x}{100}$ for $0 \le x \le 100$ and 0 otherwise. A life aged 65 buys a 3-year annually increasing life insurance. Find the actuarial present value of the contingent payment random variable if $\delta = 0.05$.

Solution.
We have

$$_tp_{65} = \frac{s(t+65)}{s(65)} = 1 - \frac{t}{35}.$$

$$\mu(65+t) = -\frac{s'(65+t)}{s(65+t)} = \frac{1}{35-t}$$

Thus,

$$_tp_{65}\mu(65+t) = \frac{1}{35}.$$

Hence,

$$(I\bar{A})^1_{65:\overline{3}|} = \frac{1}{35}\left[\int_0^1 e^{-0.05t}dt + 2\int_1^2 e^{-0.05t}dt + 3\int_2^3 e^{-0.05t}dt\right] = 0.1565 \blacksquare$$

As for decreasing insurances, first they must be term insurances, since they cannot decrease forever without becoming negative. An **annually decreasing $n-$year term life insurance policy** will pay n unit in benefits if death occurs during year 1, $n-1$ units in benefits if death occurs during year 2, and $n-k+1$ in benefits if death occurs in year k. For such a policy, the present value of the contingent payment is given by

$$Z = \begin{cases} (n - \lfloor T \rfloor)\nu^T, & 0 \le T \le n \\ 0, & T > n. \end{cases}$$

The actuarial present value of Z is given by

$$(D\bar{A})^1_{x:\overline{n}|} = \int_0^n (n - \lfloor t \rfloor)\nu^t {}_tp_x\mu(x+t)dt.$$

Example 29.6
Let the remaining life time for (5) be exponential with $\mu = 0.34$. Find $(D\bar{A})^1_{5:\overline{2}|}$ if $\delta = 0.09$.

Solution.
We have

$$\begin{aligned}(D\bar{A})^1_{5:\overline{2}|} =& \int_0^2 (2-\lfloor t \rfloor)e^{-0.09t}e^{-0.34t}(0.34)dt \\ =& \int_0^1 2e^{-0.43t}(0.34)dt + \int_1^2 e^{-0.43}(0.34)dt = 0.7324 \blacksquare\end{aligned}$$

For a continuously decreasing $n-$year term for (x), the present value of the death benefit is

$$Z = \begin{cases} (n-T)\nu^T, & 0 \le T \le n \\ 0, & T > n. \end{cases}$$

The actuarial present value of Z is given by

$$(\bar{D}\bar{A})^1_{x:\overline{n}|} = \int_0^n (n-t)\nu^t{}_tp_x\mu(x+t)dt.$$

Example 29.7
Show that: $(\bar{I}\bar{A})^1_{x:\overline{n}|} + (\bar{D}\bar{A})^1_{x:\overline{n}|} = n\bar{A}^1_{x:\overline{n}|}$.

Solution.
We have

$$\begin{aligned}(\bar{I}\bar{A})^1_{x:\overline{n}|} + (\bar{D}\bar{A})^1_{x:\overline{n}|} =& \int_0^n t\nu^t{}_tp_x\mu(x+t)dt + \int_0^n (n-t)\nu^t{}_tp_x\mu(x+t)dt \\ =& n\int_0^n \nu^t{}_tp_x\mu(x+t)dt = n\bar{A}^1_{x:\overline{n}|} \blacksquare\end{aligned}$$

Practice Problems

Problem 29.6 ‡
For a continuously increasing whole life insurance on (x), you are given:
(i) The force of mortality is constant.
(ii) $\delta = 0.06$.
(iii) ${}^2\bar{A}_x = 0.25$.
Calculate $(\bar{I}\bar{A})_x$.

Problem 29.7
The age-at-death random variable X is described by the survival function $s(x) = 1 - \frac{x}{94}$ for $0 \le x \le 94$ and 0 otherwise. A life aged 77 buys a 3-year annually increasing life insurance. Find $(\bar{I}\bar{A})^1_{77:\overline{3}|}$ if $\delta = 0.02$.

Problem 29.8
The remaining lifetime random variable for a life aged (40) has the following density function

$$f_T(t) = 0.05e^{-0.05t}.$$

Suppose that (40) buys an annually increasing whole life policy where the present discount factor is given by $\nu_t = \frac{1}{\lfloor t+1 \rfloor}$, $t > 0$. Find the actuarial present value of such a policy.

Problem 29.9
Write $(I\bar{A})_{30}$ as a sum if $\delta = 0.02$ and X follows De Moivre's Law with $\omega = 100$. Do not evaluate the sum.

Problem 29.10
Suppose that X is exponential with parameter μ. Assume a force of interest δ, show that $(\bar{I}\bar{A})_x = (\mu + \delta)^{-1}\bar{A}_x$.

Problem 29.11
Let the remaining life time for (91) be uniform on $[0, 3]$. Find $(D\bar{A})^1_{91:\overline{3}|}$ if $\delta = 0.02$.

Problem 29.12
Show that: $(\bar{I}\bar{A})_x = \int_0^\infty {}_{s|}\bar{A}_x ds$.

Problem 29.13
Show that: $(I\bar{A})^1_{x:\overline{n}|} + (D\bar{A})^1_{x:\overline{n}|} = (n+1)\bar{A}^1_{x:\overline{n}|}$.

29.3 Increasing and Decreasing Insurances Payable at the End of Year of Death

Parallel to the discussion of the previous section, we can define increasing and decreasing insurances with benefits payable at the end of year of death. Consider an $n-$year term insurance for a life aged x that pays $k+1$ if death occurs in the interval $(x+k, x+k+1]$ for $k=0,1,2,\cdots,n-1$. Then if the benefit is paid at time $k+1$ (i.e., at age $x+k+1$), the benefit amount is $k+1$. The discount factor is ν^{k+1} and the probability that the benefit is paid at that date is the probability that the policy holder died in the year $(k, k+1]$, which is ${}_{k|}q_x$, so the actuarial present value of the death benefit is

$$\text{APV} = \sum_{k=0}^{n-1}(k+1)\nu^{k+1}{}_{k|}q_x.$$

In actuarial notation the above APV is denoted by $(IA)^{1}_{x:\overline{n|}}$. Note that the present value of the random variable of the death benefit is defined by

$$Z = \begin{cases} (K+1)\nu^{K+1}, & K \le n-1 \\ 0, & K \ge n. \end{cases}$$

Example 29.8
Find the second moment and the variance of Z.

Solution.
We have

$$E(Z^2) = \sum_{k=0}^{n-1}(k+1)^2\nu^{2(k+1)}{}_{k|}q_x$$

and

$$\text{Var}(Z) = \sum_{k=0}^{n-1}(k+1)^2\nu^{2(k+1)}{}_{k|}q_x - \left((IA)^{1}_{x:\overline{n|}}\right)^2 \blacksquare$$

If the term n is infinite, so that this is a whole life version of the increasing annual policy, with benefit $k+1$ following death in the year k to $k+1$, the APV of the death benefit is denoted by $(IA)_x$ where

$$(IA)_x = \sum_{k=0}^{\infty}(k+1)\nu^{k+1}{}_{k|}q_x.$$

The corresponding present value random variable of the benefit is given by

$$Z = (K+1)\nu^{K+1}, \; K \geq 0.$$

Example 29.9
Show that

$$(IA)_x = A_x + \nu p_x (IA)_{x+1} = \nu q_x + \nu p_x[(IA)_{x+1} + A_{x+1}].$$

Solution.
We have

$$\begin{aligned}
(IA)_x =&\sum_{k=0}^{\infty}(k+1)\nu^{k+1}{}_{k|}q_x = \sum_{k=0}^{\infty}(k+1)\nu^{k+1}{}_kp_xq_{x+k}\\
=&\sum_{k=0}^{\infty}k\nu^{k+1}{}_kp_xq_{x+k} + \sum_{k=0}^{\infty}\nu^{k+1}{}_kp_xq_{x+k}\\
=&\sum_{k=0}^{\infty}\nu^{k+1}{}_kp_xq_{x+k} + \nu\sum_{k=1}^{\infty}k\nu^{k}p_{x\,k-1}p_{x+1}q_{x+k}\\
=&\sum_{k=0}^{\infty}\nu^{k+1}{}_kp_xq_{x+k} + \nu p_x\sum_{k=0}^{\infty}(k+1)\nu^{k+1}{}_kp_{x+1}q_{x+k+1}\\
=&\sum_{k=0}^{\infty}\nu^{k+1}{}_kp_xq_{x+k} + \nu p_x\sum_{k=0}^{\infty}(k+1)\nu^{k+1}{}_{k|}q_{x+1}\\
=&A_x + \nu p_x(IA)_{x+1} = \nu q_x + \nu p_x[(IA)_{x+1} + A_{x+1}].
\end{aligned}$$

In the last row, we used the recursion relation $A_x = \nu q_x + \nu p_x A_{x+1}$ ■

Also, one can show (See Problem 29.14) that

$$(IA)^1_{x:\overline{n|}} = A^1_{x:\overline{n|}} + \nu p_x (IA)^{\;1}_{x+1:\overline{n-1|}}.$$

Consider an $n-$year term insurance for a life aged x that pays $n-k$ if death occurs in the interval $(x+k, x+k+1]$ for $k = 0, 1, 2, \cdots, n-1$. That is, the death benefit is n at the end of year 1, $n-1$ at the end of year 2, etc., then if the benefit is paid at time $k+1$ (i.e., at age $x+k+1$), the benefit amount is $n-k$. The discount factor is ν^{k+1} and the probability that the benefit is paid at that date is the probability that the policy holder died in the year $(k, k+1]$, which is ${}_{k|}q_x$, so the actuarial present value of the death benefit is

$$\text{APV} = \sum_{k=0}^{n-1}(n-k)\nu^{k+1}{}_{k|}q_x.$$

In actuarial notation the above APV is denoted by $(DA)^1_{x:\overline{n|}}$. Note that the present value of the random variable of the death benefit is defined by

$$Z = \begin{cases} (n-K)\nu^{K+1}, & K \le n-1 \\ 0, & K \ge n. \end{cases}$$

Example 29.10
Show that: $(DA)^1_{x:\overline{n|}} = n\nu q_x + \nu p_x (DA)^{\;1}_{x+1:\overline{n-1|}}$.

Solution.
We have

$$\begin{aligned}(DA)^1_{x:\overline{n|}} =& \sum_{k=0}^{n-1}(n-k)\nu^{k+1}{}_{k|}q_x \\ =& n\nu q_x + \nu\sum_{k=1}^{n-1}(n-k)\nu^k{}_kp_xq_{x+k} \\ =& n\nu q_x + \nu p_x\sum_{k=1}^{n-1}(n-k)\nu^k{}_{k-1}p_{x+1}q_{x+k} \\ =& n\nu q_x + \nu p_x\sum_{k=0}^{n-2}(n-k-1)\nu^{k+1}{}_kp_{x+1}q_{x+1+k} \\ =& n\nu q_x + \nu p_x\sum_{k=0}^{n-2}(n-k-1)\nu^{k+1}{}_{k|}q_{x+1} \\ =& n\nu q_x + \nu p_x(DA)^{\;1}_{x+1:\overline{n-1|}} \blacksquare\end{aligned}$$

Practice Problems

Problem 29.14
Show that :

$$(IA)^{1}_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + \nu p_x (IA)^{\;1}_{x+1:\overline{n-1}|}.$$

Problem 29.15
You are given $(IA)_{50} = 4.99675$, $A^{\;1}_{50:\overline{1}|} = 0.00558$, $A_{51} = 0.25961$, $q_{50} = 0.00592$, and $i = 0.06$. Find $(IA)_{51}$.

Problem 29.16
Show that: $(IA)^{1}_{x:\overline{n}|} = \nu q_x + \nu p_x \left((IA)^{\;1}_{x+1:\overline{n-1}|} + A^{\;1}_{x+1:\overline{n-1}|} \right)$.

Problem 29.17
Show that: $(IA)^{1}_{x:\overline{n}|} + (DA)^{1}_{x:\overline{n}|} = (n+1)A^{1}_{x:\overline{n}|}$.

Problem 29.18 ‡
A decreasing term life insurance on (80) pays $(20-k)$ at the end of the year of death if (80) dies in year $k+1$, for $k = 0, 1, 2, \cdots, 19$. You are given:
(i) $i = 0.06$
(ii) For a certain mortality table with $q_{80} = 0.2$, the single benefit premium for this insurance is 13.
(iii) For this same mortality table, except that $q_{80} = 0.1$, the single benefit premium for this insurance is P.
Calculate P.

30 Expressing APV's of Continuous Models in Terms of Discrete Ones

In this section, we will discuss ways of evaluating continuous APVs using life tables. But first we look at a numerical example where the continuous APV is different from the corresponding discrete case.

Example 30.1
(a) Consider a whole life insurance where the remaining lifetime random variable has a continuous uniform distribution on the interval $[0, 5]$. Assume a pay benefit of \$1000 at the time of death and that the effective annual interest rate is $i = 0.06$. Find $1000\bar{A}_0$.
(b) Now consider the 5-year life table which is discrete uniform and uses an interest rate of 6%. Find $1000A_0$.

k	ℓ_k	q_k	${}_kp_0$	${}_kp_0 \cdot q_k$
0	125	0.200	1.000	0.200
1	100	0.250	0.800	0.200
2	75	0.333	0.600	0.200
3	50	0.500	0.400	0.200
4	25	1.000	0.200	0.200
5	0			

Solution.
(a) We have

$$1000\bar{A}_0 = 1000\frac{1-e^{-\delta\omega}}{\delta\omega} = 1000\frac{1-e^{-5\ln 1.06}}{5\ln 1.06} = 867.50.$$

(b) We have

$$\begin{aligned}1000A_0 =&1000\sum_{k=0}^{\infty}\nu^{k+1}{}_kp_0q_k\\ =&1000(0.2)[(1.06)^{-1}+(1.06)^{-2}+(1.06)^{-3}+(1.06)^{-4}+(1.06)^{-5}]\\ =&842.50.\end{aligned}$$

It follows that $1000\bar{A}_0 - 1000A_0 = 867.50 - 842.50 = 25$. Also, we have $\bar{A}_0 \approx \frac{i}{\delta}A_0$ ■

Now, for most insurances and under certain assumptions, it is possible to adjust the discrete calculation to model continuous payments. Under the uniform distribution of deaths (UDD) assumption formulas relating the net single premium for insurance payable at the time of death to the corresponding net single premium for insurance payable at the end of the year of death can be easily found. For example, in the case of a whole life policy, we have

Example 30.2
Assuming a uniform distribution of deaths between integral ages. Show that

$$\bar{A}_x = \frac{i}{\delta}A_x.$$

Solution.
We have

$$\begin{aligned}\bar{A}_x =&E[\nu^T] = E[\nu^{K+1}\nu^{S-1}]\\ =&E[\nu^{K+1}]E[e^{\delta(1-S)}]\\ =&E[\nu^{K+1}]\int_0^1 e^{\delta(1-s)}ds\\ =&\frac{i}{\delta}E[\nu^{K+1}] = \frac{i}{\delta}A_x\end{aligned}$$

where the third equality springs from the independence of $K(x)$ and $S(x)$ under UDD, and the fourth equality comes from the fact that under UDD the random variable $S(x)$ has the uniform distribution on the unit interval (See Theorem 24.1) ■

Example 30.3
Assuming a uniform distribution of deaths between integral ages. Show that

$$\bar{A}^1_{x:\overline{n}|} = \frac{i}{\delta}A^1_{x:\overline{n}|}.$$

Solution.
First, let's define the indicator function

$$\mathbf{I}(T \le n) = \begin{cases} 1, & 0 \le T \le n \\ 0, & T > n. \end{cases}$$

For $n-$year term insurance with payment paid at the moment of death we have

$$\begin{aligned}\bar{A}^{1}_{x:\overline{n}|} =& E[\mathbf{I}(T \leq n)\nu^T] = E[\mathbf{I}(T \leq n)\nu^{K+1}\nu^{S-1}]\\ =& E[\mathbf{I}(K \leq n-1)\nu^{K+1}]E[e^{\delta(1-S)}]\\ =& E[\mathbf{I}(K \leq n-1)\nu^{K+1}]\int_0^1 e^{\delta(1-s)}ds\\ =& \frac{i}{\delta}E[\mathbf{I}(K \leq n-1)\nu^{K+1}] = \frac{i}{\delta}A^{1}_{x:\overline{n}|} \blacksquare\end{aligned}$$

Example 30.4

Assuming a uniform distribution of deaths between integral ages. Show that

$$(I\bar{A})^{1}_{x:\overline{n}|} = \frac{i}{\delta}(IA)^{1}_{x:\overline{n}|}.$$

Solution.

The present value of the annually increasing $n-$year term insurance payable at the moment of death is

$$Z = \begin{cases} \lfloor T+1 \rfloor \nu^T, & T \leq n \\ 0, & T > n. \end{cases}$$

Since $\lfloor T+1 \rfloor = K+1$, we can write

$$Z = \begin{cases} (K+1)\nu^{K+1}\nu^{S-1}, & T \leq n \\ 0, & T > n. \end{cases}$$

If we let W be the annually increasing $n-$year term insurance with benefit payable at the end of the year of death, then

$$W = \begin{cases} (K+1)\nu^{K+1}\nu^{S-1}, & 0 \leq K \leq n-1 \\ 0, & K \geq n. \end{cases}$$

Thus,

$$Z = W\nu^{S-1}.$$

One can proceed as in the previous two examples to show that

$$(I\bar{A})^{1}_{x:\overline{n}|} = \frac{i}{\delta}(IA)^{1}_{x:\overline{n}|} \blacksquare$$

Likewise, one can show that

$$(D\bar{A})^{1}_{x:\overline{n}|} = \frac{i}{\delta}(DA)^{1}_{x:\overline{n}|}.$$

In the last three examples, the term $\frac{i}{\delta}$ is called the **adjustment factor.** The process in the above examples does not apply for endowment insurance.

Example 30.5
Show that $\bar{A}_{x:\overline{n}|} = \frac{i}{\delta}A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ \ 1}$.

Solution.
We know from Section 26.3.2 that

$$\bar{A}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ \ 1}.$$

But

$$\bar{A}^{1}_{x:\overline{n}|} = \frac{i}{\delta}A^{1}_{x:\overline{n}|}.$$

Thus,

$$\bar{A}_{x:\overline{n}|} = \frac{i}{\delta}A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ \ 1} \blacksquare$$

Practice Problems

Problem 30.1
For a special type of whole life insurance issued to (30), you are given:
- death benefits are 1,000;
- death benefits are payable at the moment of death;
- deaths are uniformly distributed between integral ages;
- $i = 0.05$; and
- the following table of actuarial present values:

x	$1000A_x$
30	112.31
35	138.72
40	171.93

Calculate the actuarial present value of the benefits for this policy.

Problem 30.2
For a special type of whole life insurance issued to (30), you are given:
- death benefits are 1,000 for the first 10 years and 5,000 thereafter;
- death benefits are payable at the moment of death;
- deaths are uniformly distributed between integral ages;
- $i = 0.05$; and
- the following table of actuarial present values:

x	$1000A_x$	$1000{}_5E_x$
30	112.31	779.79
35	138.72	779.20
40	171.93	777.14

Calculate the actuarial present value of the benefits for this policy.

Problem 30.3
You are given the following:
- a 30-year endowment with payment of 10,000, providing death benefit at the moment of death, issued to a life aged 35;
- $i = 0.05$;
- deaths are uniformly distributed between integral ages;
- $A^{1}_{35:\overline{30|}} = 0.0887$;
- $A_{35:\overset{\;\;1}{\overline{30|}}} = 0.1850$.

Calculate the net single premium of this contract.

Problem 30.4
Show algebraically that under UDD, we have: $(I\bar{A})_x = \frac{i}{\delta}(IA)_x$.

Problem 30.5
Show that $E[(S-1)\nu^{S-1}] = -\left(\frac{1+i}{\delta} - \frac{i}{\delta^2}\right)$.

Problem 30.6
Show algebraically that under UDD, we have: $(\bar{I}\bar{A})_x = \frac{i}{\delta}(IA)_x - \left(\frac{1+i}{\delta} - \frac{i}{\delta^2}\right) A_x$.

31 m^{thly} Contingent Payments

In this section, we consider models where benefit payments are made at the end of a period. We assume the year is divived into m periods and that benefits are paid m^{thly}.
Consider first a whole life insurance that is paid at the end of the m^{thly} time interval in which death occurs. See Figure 31.1.

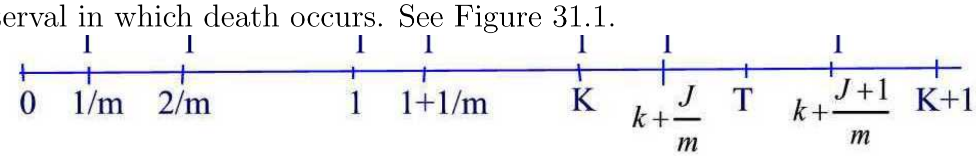


Figure 31.1

Recall that K is the number of complete insurance years lived prior to death. Let J be the number of complete $m-$ths of a year lived in the year of death. Then $J = \lfloor (T-K)m \rfloor$. For example, for quarterly payments with $T(x) = 36.82$, we have $J = \lfloor (36.82-36)(4) \rfloor = 3$.
The present value of a whole life insurance paid at the end of the $m$thly time interval in which death occurs is

$$Z_x^{(m)} = \nu^{K+\frac{J+1}{m}}.$$

Let $A_x^{(m)}$ denote the actuarial present value of such a policy. Then

$$\begin{aligned}
A_x^{(m)} =& \sum_{k=0}^{\infty} \nu^{k+1} \sum_{j=0}^{m-1} \nu^{\frac{j+1}{m}} {}_{k+\frac{j}{m}}p_x \, {}_{\frac{1}{m}}q_{x+k+\frac{j}{m}} \\
=& \sum_{k=0}^{\infty} \nu^{k+1} \sum_{j=0}^{m-1} \nu^{\frac{j+1}{m}} \nu^{k+\frac{j+1}{m}} {}_kp_x \cdot {}_{\frac{j}{m}}p_{x+k} \, {}_{\frac{1}{m}}q_{x+k+\frac{j}{m}} \\
=& \sum_{k=0}^{\infty} \nu^{k+1} {}_kp_x \sum_{j=0}^{m-1} \nu^{\frac{j+1}{m}-1} {}_{\frac{j}{m}}p_{x+k} \, {}_{\frac{1}{m}}q_{x+k+\frac{j}{m}}.
\end{aligned}$$

Under UDD assumption, we have from Problem 24.8

$${}_{\frac{j}{m}}p_{x+k} \, {}_{\frac{1}{m}}q_{x+k+\frac{j}{m}} = \frac{1}{m} q_{x+k}.$$

Thus,

$$\begin{aligned} A_x^{(m)} =& \sum_{k=0}^{\infty} \nu^{k+1} {}_k p_x q_{x+k} \sum_{j=0}^{m-1} \frac{\nu^{\frac{j+1}{m}-1}}{m} \\ =& \frac{i}{i^{(m)}} \sum_{k=0}^{\infty} \nu^{k+1} {}_k p_x q_{x+k} \\ =& \frac{i}{i^{(m)}} A_x. \end{aligned}$$

Now, for an $n-$year term insurance we have

$$Z^{(m)}{}^{1}_{x:\overline{n}|} = \begin{cases} \nu^{K+\frac{J+1}{m}}, & K + \frac{J+1}{m} \leq nm \\ 0, & \text{otherwise.} \end{cases}$$

and

$$A^{(m)}{}^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \nu^{k+1} {}_k p_x \sum_{j=0}^{m-1} \nu^{\frac{j+1}{m}-1} {}_{\frac{j}{m}} p_{x+k} {}_{\frac{1}{m}} q_{x+k+\frac{j}{m}}.$$

Under the UDD assumption, we obtain

$$A^{(m)}{}^{1}_{x:\overline{n}|} = \frac{i}{i^{(m)}} A^{1}_{x:\overline{n}|}.$$

For an $n-$year deferred insurance, we have

$${}_{n|}Z_x^{(m)} = \begin{cases} \nu^{K+\frac{J+1}{m}}, & K + \frac{J+1}{m} > nm \\ 0, & \text{otherwise.} \end{cases}$$

and

$${}_{n|}A_x^{(m)} = \sum_{k=n}^{\infty} \nu^{k+1} {}_k p_x \sum_{j=0}^{m-1} \nu^{\frac{j+1}{m}-1} {}_{\frac{j}{m}} p_{x+k} {}_{\frac{1}{m}} q_{x+k+\frac{j}{m}}$$

so that under the UDD assumption we find

$${}_{n|}A_x^{(m)} = \frac{i}{i^{(m)}} {}_{n|}A_x.$$

Finally, for an endowment insurance we have

$$Z_{x:\overline{n}|}^{(m)} = \begin{cases} \nu^{K+\frac{J+1}{m}}, & K + \frac{J+1}{m} \leq nm \\ \nu^n, & \text{otherwise.} \end{cases}$$

so that

$$A^{(m)}_{x:\overline{n}|} = A^{(m)}{}^{\,1}_{x:\overline{n}|} + {}_nE_x$$

and under the UDD assumption, we find

$$A^{(m)}_{x:\overline{n}|} = \frac{i}{i^{(m)}} A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;\,1}.$$

Example 31.1
Consider the following extract from the Illustrative Life Table

x	ℓ_x	$1000A_x$
30	9,501.381	102.48
31	9,486,854	107.27
32	9,471,591	112.28
33	9,455,522	117.51
34	9,438,571	122.99
35	9,420,657	128.72

Under the uniform distribution of deaths and with $i = 0.06$, find $A^{(4)}_{30}$.

Solution.
We have

$$A^{(4)}_{30} = \frac{i}{i^{(4)}} A_{30} = \frac{0.06}{4[(1.06)^{\frac{1}{4}} - 1]}(0.10248) = 0.1048 \blacksquare$$

Practice Problems

Problem 31.1
You are given:
- deaths are uniformly distributed between integral ages;
- $i = 0.06$;
- $q_{69} = 0.02$; and
- $\bar{A}_{70} = 0.53$.

Calculate $A_{69}^{(2)}$ and interpret this value.

Problem 31.2
- deaths are uniformly distributed between integral ages
- $i = 0.06$;
- $A_{40} = 0.16$
- ${}_{20}E_{20} = 0.30$
- $A_{20:\overline{20}|} = 0.32$.

Calculate $A_{20}^{(4)}$.

Problem 31.3
Consider the life table

x	95	96	97	98	99	100
ℓ_x	520	350	150	80	20	0

Assume uniform distribution of deaths between integral ages $i = 0.06$. Calculate $A_{95:\overline{3}|}^{(12)}$.

Problem 31.4
Suppose that $\bar{A}_{50} = 0.53$ and $i = 0.065$. Assume that death is uniformly distributed between integral ages. Find $A_{50}^{(2)}$.

Problem 31.5
Assume that death is uniformly distributed between integral ages. Suppose that x is an integer, $\mu(x + 0.3) = 0.02012$ and $\mu(x + 1.6) = 0.02538$. Find $A_{x:\overline{2}|}^{(12)}$ if $d = 0.08$.

32 Applications of Life Insurance

In this section, we consider some applications of the topics discussed in this chapter.

Sufficient Funds for Paying Claims

An insurance company writes 100 single premium whole life insurances of 10 payable at the moment of death. The insureds are assumed to have identically independent remaining lifetime random variable. Also, we assume that this random variable is exponential with $\mu = 0.04$ and the force of interest is $\delta = 0.06$. In this case, the contingent payment random variable is given by

$$Z_i = 10e^{-0.06t},\ i = 1, 2, \cdots, 100.$$

Let S be the present value of all future claims. Then

$$S = Z_1 + Z_2 + \cdots + Z_{100}.$$

We would like to know the amount of fund F that the insurer must have now to assure that the probability of having sufficient funds to pay all claims is 0.95. That is, we want F such that

$$\Pr(S \leq F) = 0.95.$$

In probability language, F is the 95$^{\text{th}}$ percentile of the random variable S. Now, using normal approximation, we can write

$$\Pr\left(Z_{0.95} \leq \frac{F - \mu_S}{\sigma_S}\right) = 0.95.$$

But $Z_{0.95} = 1.645$ from which we find

$$F = \mu_S + 1.645\sigma_S.$$

We find μ_S and σ_S as follows:

$$\begin{aligned}
E(Z_i) =& 10\bar{A}_x = 10\left(\frac{\mu}{\mu+\delta}\right) = 4\\
E(Z_i^2) =& [(10^2)]^2\bar{A}_x = 100\left(\frac{\mu}{\mu+2\delta}\right) = 25\\
\text{Var}(Z_i) =& 25 - 4^2 = 9\\
\mu_S =& E(S) = 100E(Z_i) = 400\\
\sigma_S =& \sqrt{100\text{Var}(Z_i)} = 30
\end{aligned}$$

Finally,

$$F = 400 + 1.645(30) = 449.35.$$

Example 32.1 ‡

A fund is established to pay annuities to 100 independent lives age x. Each annuitant will receive 10,000 per year continuously until death. You are given:

(i) $\delta = 0.06$

(ii) $\bar{A}_x = 0.40$

(iii) ${}^2\bar{A}_x = 0.25$

Calculate the amount (in millions) needed in the fund so that the probability, using the normal approximation, is 0.90 that the fund will be sufficient to provide the payments.

Solution.

Let

$$Y = 10,000\left(\frac{1-\nu^T}{\delta}\right)$$

denote the contingent random variable and S the present value of all future payments. We would like to know the amount of fund F that the insurer must have now to assure that the probability of having sufficient funds to make all payments is 0.90. That is, we want F such that

$$\Pr(S \le F) = 0.90.$$

We have

$$\begin{aligned}
E(Y) =&10,000\bar{a}_x = 10,000\left(\frac{1-\bar{A}_x}{\delta}\right) = 100,000\\
E(S) =&100E(Y) = 10,000,000\\
\text{Var}(Y) =&\frac{10,000^2}{\delta^2}({}^2\bar{A}_x - \bar{A}_x^2)) = 50,000^2\\
\text{Var}(S) =&100\text{Var}(Y) = 100(50,000)^2\\
\sigma_S =&500,000.
\end{aligned}$$

Thus, we have

$$\Pr\left(\frac{S-10,000,000}{500,000} \le \frac{F-10,000,000}{500,000}\right) = \Pr\left(Z_{0.90} \le \frac{F-10,000,000}{500,000}\right).$$

But $Z_{0.90} = 1.282$ from which we find

$$F = E(S) + 1.282\sigma_S = 10,000,000 + 1.282(500,000) = 10,641,000 \blacksquare$$

Distributions of Z

In this section, we try to find PDF, SF and CDF of actuarial present values. We will look at the case of a whole life insurance where the remaining future lifetime random variable is exponential.

Example 32.2

Let the remaining lifetime random variable be exponential with $\mu = 0.05$. Let $\bar{Z}_x$ be the contingent payment random variable for a continuous whole life insurance on (30). The force of interest is $\delta = 0.10$. Find the cumulative distribution function for $\bar{Z}_x$.

Solution.

Recall the survival function of T given by

$$s_{T(30)}(t) = e^{-0.05t},\ t \geq 0.$$

We are told that

$$\bar{Z}_x = e^{-0.10t},\ t \geq 0.$$

Now, we have

$$\begin{aligned}
F_{\bar{Z}_x}(t) =&\Pr(\bar{Z}_x \leq t)\\
=&\Pr(e^{-0.10T} \leq t)\\
=&\Pr\left(T > \frac{\ln t}{-0.10}\right)\\
=&S\left(\frac{\ln t}{-0.10}\right)\\
=&t^2 \blacksquare
\end{aligned}$$

Practice Problems

Problem 32.1 ‡
Each of 100 independent lives purchase a single premium 5-year deferred whole life insurance of 10 payable at the moment of death. You are given:
(i) $\mu = 0.04$;
(ii) $\delta = 0.06$;
(iii) F is the aggregate amount the insurer receives from the 100 lives.
Using the normal approximation, calculate F such that the probability the insurer has sufficient funds to pay all claims is 0.95.

Problem 32.2
Each of 100 independent lives purchase a single premium 10-year deferred whole life insurance of 10 payable at the moment of death. You are given:
(i) The remaining lifetime is uniform on $[0, 70]$;
(ii) $\delta = 0.05$;
(iii) F is the aggregate amount the insurer receives from the 100 lives.
Using the normal approximation, calculate F such that the probability the insurer has sufficient funds to pay all claims is 0.95.

Problem 32.3 ‡
For a whole life insurance of 1 on (x) with benefits payable at the moment of death:
(i)

$$\delta_t = \begin{cases} 0.02, & 0 < t < 12 \\ 0.03, & t \geq 12. \end{cases}$$

(ii)

$$\mu(x+t) = \begin{cases} 0.04, & 0 < t < 5 \\ 0.05, & t \geq 5. \end{cases}$$

Calculate the actuarial present value of this insurance.

Problem 32.4
Let the remaining lifetime random variable be uniform with $[0, 80]$. Let $\bar{Z}_x$ be the contingent payment random variable for a continuous whole life insurance on (20). The force of interest is $\delta = 0.05$. Find the cumulative distribution function for $\bar{Z}_x$.

Problem 32.5
Let the remaining lifetime random variable be uniform with $[0, 80]$. Let $\bar{Z}_x$ be the contingent payment random variable for a continuous whole life insurance on (20). The force of interest is $\delta = 0.05$. Find the 95^{th} percentile of $\bar{Z}_x$.

Problem 32.6 ‡
For a group of 250 individuals age x, you are given:
(i) The future lifetimes are independent.
(ii) Each individual is paid 500 at the beginning of each year, if living.
(iii) $A_x = 0.369131$
(iv) ${}^2A_x = 0.1774113$
(v) $i = 0.06$
Using the normal approximation, calculate the size of the fund needed at inception in order to be 90% certain of having enough money to pay the life annuities.

Contingent Annuity Models

Life insurances discussed in the previous chapter provide a single payment upon death of the insured. In contrast, **life annuities** are annuities that provide a sequence of payments contingent on the survival of the insured. That is, the insured will stop receiving payments at the time of death. Life annuities are also known as **contingent annuities**. The goal of this chapter is to understand the concept of life annuities and to become familiar with the related actuarial notation.
Three different life annuity models will be introduced in this section: whole life, temporary and deferred whole life annuities.

33 Continuous Whole Life Annuities

A **whole life annuity** is an annuity with payments made while the insured is alive. The payments can be made either at the beginning of the year (**whole life due annuity**), the end of the year (**whole life discrete immediate annuity**), or continuously (**whole life continuous annuity**). In this section, we consider continuous whole life annuity with a continuous flow of payments with constant rate of 1 per year made while an individual is alive. Recall from Section 3.3, that the present value of an annuity payable continuously for n interest conversion periods so that 1 is the total amount paid during each interest conversion period is given by the formula

$$\bar{a}_{\overline{n}|} = \frac{1-\nu^n}{\delta}.$$

For a whole life continuous annuity for (x) with unit rate, the length of time over which the payments are made is a random variable T, which is the remaining lifetime. Thus, the present value of the life annuity is also a random variable

$$\bar{Y}_x = \bar{a}_{\overline{T}|} = \frac{1-\nu^T}{\delta}.$$

The actuarial present value of a whole life continuous annuity for (x) with unit rate is denoted by $\bar{a}_x$. That is,

$$\bar{a}_x = E(\bar{Y}_x) = \int_0^\infty \bar{a}_{\overline{T}|} f_T(t)dt = \int_0^\infty \bar{a}_{\overline{t}|}\,{}_tp_x\mu(x+t)dt.$$

An alternative formula for finding $\bar{a}_x$ is given in the next example.

Example 33.1
Show that $\bar{a}_x = \int_0^\infty {}_tE_x dt = \int_0^\infty \nu^t{}_tp_x dt$. Interpret this result.

Solution.
First notice that

$$\frac{d}{dt}\bar{a}_t = e^{-\delta t} \text{ and } \frac{d}{dt}({}_tp_x) = -{}_tp_x\mu(x+t).$$

Now, using integration by parts we find

$$\begin{aligned}\bar{a}_x =& \int_0^\infty \bar{a}_{\overline{t|}} \frac{d}{dt}(-{}_tp_x)dt\\ =& -\bar{a}_{\overline{t|}}\,{}_tp_x|_0^\infty + \int_0^\infty e^{-\delta t}{}_tp_x dt\\ =& \int_0^\infty e^{-\delta t}{}_tp_x dt = \int_0^\infty {}_tE_x dt\end{aligned}$$

where we used the fact that $\int_0^\infty \mu(x+t)dt = \infty$.
This integral says that a continuous annuity of 1 pays $1dt$ in the interval $[t, t+dt]$ if the annuitant is alive in that interval. The actuarial present value of that $1dt$ is $\nu^t{}_tp_xdt = {}_tE_xdt$. The integral continuously adds up all those present values to give the total actuarial present value of the annuity ■

Example 33.2
Suppose that $\delta = 0.05$ and ${}_tp_x = e^{-0.34t},\ t \geq 0$. Calculate $\bar{a}_x$.

Solution.
We have

$$\bar{a}_x = \int_0^\infty e^{-0.05t}e^{-0.34t}dt = \int_0^\infty e^{-0.39t}dt = -\left.\frac{e^{-0.39t}}{0.39}\right|_0^\infty = \frac{100}{39} \blacksquare$$

If $\bar{A}_x$ is known then we can use the following formula for determining $\bar{a}_x$.

Example 33.3
Show that $\bar{a}_x = \frac{1-\bar{A}_x}{\delta}$.

Solution.
We have

$$\bar{Y}_x = \frac{1-\bar{Z}_x}{\delta}.$$

Taking expectation of both sides, we find

$$\bar{a}_x = E\left(\frac{1-\nu^T}{\delta}\right) = \frac{1}{\delta}[E(1) - E(\bar{Z}_x)] = \frac{1-\bar{A}_x}{\delta} \blacksquare$$

Example 33.4
Suppose that the remaining lifetime random variable is exponential with parameter μ. Find an expression for $\bar{a}_x$ if the force of interest is δ.

Solution.
Recall that $\bar{A}_x = \frac{\mu}{\mu+\delta}$. Therefore,

$$\bar{a}_x = \frac{1-\bar{A}_x}{\delta} = \frac{1-\frac{\mu}{\mu+\delta}}{\delta} = \frac{1}{\mu+\delta} \blacksquare$$

Example 33.5
Find the variance of $\bar{Y}_x$.

Solution.
We have

$$\begin{aligned}\text{Var}(\bar{Y}_x) =& \text{Var}\left(\frac{1-\bar{Z}_x}{\delta}\right) = \text{Var}\left(\frac{1}{\delta} - \frac{\bar{Z}_x}{\delta}\right) \\ =& \frac{1}{\delta^2}\text{Var}(\bar{Z}_x) = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{\delta^2} \blacksquare\end{aligned}$$

Example 33.6
For a continuous whole life annuity of 1 on (x) :
• $T(x)$ is the future lifetime random variable for (x).
• The force of interest is δ.
• The constant force of mortality μ.
Find an expression for $F_{\bar{Y}_x}(y)$.

Solution.
We have the following calculation

$$\begin{aligned}F_{\bar{Y}_x}(y) =& \Pr(\bar{Y}_x \le y) \\ =& \Pr\left(\frac{1-e^{-\delta T}}{\delta} \le y\right) \\ =& \Pr(1-e^{-\delta T} \le \delta y) \\ =& \Pr(e^{-\delta T} \ge 1-\delta y) \\ =& \Pr\left(T \le -\frac{\ln(1-\delta y)}{\delta}\right) \\ =& 1 - s_{T(x)}\left(-\frac{\ln(1-\delta y)}{\delta}\right) \\ =& 1 - e^{\frac{\mu}{\delta}\ln(1-\delta y)} \\ =& 1-(1-\delta y)^{\frac{\mu}{\delta}} \blacksquare\end{aligned}$$

Example 33.7 ‡
For a disability insurance claim:
(i) The claimant will receive payments at the rate of 20,000 per year, payable continuously as long as she remains disabled.
(ii) The length of the payment period in years is a random variable with pdf $f(t) = te^{-t},\ t > .$
(iii) Payments begin immediately.
(iv) $\delta = 0.05$
Calculate the actuarial present value of the disability payments at the time of disability.

Solution.
We have

$$\begin{aligned}
20,000\bar{a}_x =& 20,000\int_0^\infty \bar{a}_{\overline{T}|}f_T(t)dt\\
=& 20,000\int_0^\infty \left(\frac{1-\nu^t}{\delta}\right)te^{-t}dt\\
=& 400,000\int_0^\infty (te^{-t} - te^{-1.05t})dt\\
=& 400,000\left[-(t+1)e^{-t} + \left(\frac{t}{1.05} + \frac{1}{1.05^2}\right)e^{-1.05t}\right]_0^\infty\\
=& 37,188.20 \blacksquare
\end{aligned}$$

Example 33.8 ‡
For a group of lives age x, you are given:
(i) Each member of the group has a constant force of mortality that is drawn from the uniform distribution on $[0.01, 0.02]$.
(ii) $\delta = 0.01$
For a member selected at random from this group, calculate the actuarial present value of a continuous lifetime annuity of 1 per year.

Solution.
We have

$$\begin{aligned}\bar{a}_x =&E_Y[\bar{Y}_x] = E_\mu[E_Y[\bar{Y}_x|\mu]] = \int_{0.01}^{0.02} E[\bar{Y}_x|\mu]f_\mu(\mu)d\mu\\ =&\int_{0.01}^{0.02} \frac{1}{\mu+\delta}\frac{1}{0.02-0.01}d\mu\\ =&100\ln{(0.01+\mu)}\big|_{0.01}^{0.02} = 40.55 \blacksquare\end{aligned}$$

Practice Problems

Problem 33.1
You are given that $\delta = 0.05$ and $\bar{A}_x = 0.4$. Calculate $\bar{a}_x$.

Problem 33.2
Find an expression for $E(\bar{Y}_x^2)$.

Problem 33.3
Find $\text{Var}(\bar{Y}_{30})$ if $\bar{A}_{30} = 0.4, {}^2\bar{A}_{30} = 0.3$, and $\delta = 0.05$.

Problem 33.4 ‡
For a continuous whole life annuity of 1 on (x) :
(i) $T(x)$ is the future lifetime random variable for (x).
(ii) The force of interest and force of mortality are equal and constant.
(iii) $\bar{a}_x = 12.50$.
Calculate the standard deviation of $\bar{a}_{\overline{T(x)}|}$.

Problem 33.5
For a continuous whole life annuity of 1 on (x) :
- $T(x)$ is the future lifetime random variable for (x).
- The force of interest is δ.
- The constant force of mortality is μ.

Find $\Pr(\bar{Y}_x > \bar{a}_x)$.

Problem 33.6 ‡
You are given:
(i)

$$\mu(x+t) = \begin{cases} 0.01, & 0 \le t < 5 \\ 0.02, & 5 \le t. \end{cases}$$

(ii) $\delta = 0.06$.
Calculate $\bar{a}_x$.

Problem 33.7 ‡
You are given:
(i) $T(x)$ is the random variable for the future lifetime of (x);
(ii) $\mu(x+t) = \mu$;
(iii) $\delta = \mu$.
Determine $\text{Var}(\bar{a}_T)$ in terms of μ.

Problem 33.8 ‡
You are given:
(i) $T(x)$ is the random variable for the future lifetime of (x);
(ii) $\mu(x+t) = \mu$;
(iii) $\delta = 0.08$.
(iv) $\bar{A}_x = 0.3443$.
Determine $\mathrm{Var}(\bar{a}_{T})$.

Problem 33.9 ‡
You are given:
(i) $\mu(x+t) = 0.03$;
(ii) $\delta = 0.05$;
(iii) $T(x)$ is the future lifetime random variable;
(iv) $g = \sqrt{\mathrm{Var}(\bar{a}_{\overline{T}|}}$.
Calculate $\Pr(\bar{a}_{\overline{T}|} > \bar{a}_x - g)$.

Problem 33.10 ‡
For a whole life insurance of 1 on (x), you are given:
(i) The force of mortality is $\mu(x+t)$.
(ii) The benefits are payable at the moment of death.
(iii) $\delta = 0.06$
(iv) $\bar{A}_x = 0.60$.
Calculate the revised actuarial present value of this insurance assuming $\mu(x+t)$ is increased by 0.03 for all t and δ is decreased by 0.03.

Problem 33.11 ‡
Your company sells a product that pays the cost of nursing home care for the remaining lifetime of the insured.
(i) Insureds who enter a nursing home remain there until death.
(ii) The force of mortality, μ, for each insured who enters a nursing home is constant.
(iii) μ is uniformly distributed on the interval $[0.5, 1]$.
(iv) The cost of nursing home care is 50,000 per year payable continuously.
(v) $\delta = 0.045$
Calculate the actuarial present value of this benefit for a randomly selected insured who has just entered a nursing home.

Problem 33.12 ‡
You are given:

(i) Y is the present value random variable for a continuous whole life annuity of 1 per year on (40).
(ii) Mortality follows DeMoivre's Law with $\omega = 120$.
(iii) $\delta = 0.05$
Calculate the 75^{th} percentile of the distribution of Y.

34 Continuous Temporary Life Annuities

A **(continuous) n-year temporary life annuity** pays 1 per year continuously while (x) survives during the next n years. If the annuitant dies before time n then the total money collected is $\bar{a}_{\overline{T}|}$. If the annuitant dies at time n the total money collected is $\bar{a}_{\overline{n}|}$.
The present value of this annuity is the random variable

$$\bar{Y}_{x:\overline{n}|} = \begin{cases} \bar{a}_{\overline{T}|}, & 0 \leq T < n \\ \bar{a}_{\overline{n}|}, & T \geq n. \end{cases}$$

The actuarial present value of this annuity is

$$\begin{aligned}\bar{a}_{x:\overline{n}|} =&E(\bar{Y}_{x:\overline{n}|}) = \int_0^n \bar{a}_{\overline{t}|} f_T(t)dt + \int_n^\infty \bar{a}_{\overline{n}|} f_T(t)dt \\ =&\int_0^n \bar{a}_{\overline{t}|}\, {}_tp_x \mu(x+t)dt + \bar{a}_{\overline{n}|} \int_n^\infty {}_tp_x \mu(x+t)dt \\ =&\int_0^n \bar{a}_{\overline{t}|}\, {}_tp_x \mu(x+t)dt + \bar{a}_{\overline{n}|}\, {}_np_x.\end{aligned}$$

An alternative formula for finding $\bar{a}_{x:\overline{n}|}$ is given in the next example.

Example 34.1
Show that $\bar{a}_{x:\overline{n}|} = \int_0^n {}_tE_x dt = \int_0^n \nu^t {}_tp_x dt$.

Solution.
We have

$$\begin{aligned}\bar{a}_{x:\overline{n}|} =&\frac{1}{\delta}\left[\int_0^n (1-\nu^t){}_tp_x \mu(x+t)dt + (1-\nu^n){}_np_x\right] \\ =&\frac{1}{\delta}\left[{}_np_x - \nu^n {}_np_x + \int_0^n {}_tp_x \mu(x+t)dt - \int_0^n \nu^t {}_tp_x \mu(x+t)dt\right]\end{aligned}$$

But

$$\int_0^n {}_tp_x \mu(x+t)dt = \int_0^n f_T(t)dt = F_T(n) = {}_nq_x = 1 - {}_np_x$$

and using integration by parts we have

$$\begin{aligned}\int_0^n \nu^t {}_tp_x\mu(x+t)dt =& \int_0^n \nu^t \frac{d}{dt}(-{}_tp_x)dt \\ =& -\nu^t {}_tp_x\Big|_0^n - \delta\int_0^n e^{-\delta t} {}_tp_x dt \\ =& 1-\nu^n {}_np_x - \delta\int_0^\infty \nu^t {}_tp_x dt.\end{aligned}$$

By substitution, we find

$$\bar{a}_{x:\overline{n|}} = \int_0^n {}_tE_x dt \blacksquare$$

Example 34.2
You are given the following:
• $T(x)$ is exponential with $\mu = 0.34$.
• A continuous 4-year temporary annuity.
• $\delta = 0.06$.
Calculate $\bar{a}_{x:\overline{4|}}$.

Solution.
We have

$$\bar{a}_{x:\overline{4|}} = \int_0^4 e^{-0.06t}e^{-0.34t}dt = -\frac{e^{-0.4t}}{0.4}\Big|_0^4 = 1.9953 \blacksquare$$

Next, notice that

$$\bar{Y}_{x:\overline{n|}} = \frac{1-\bar{Z}_{x:\overline{n|}}}{\delta}$$

where $\bar{Z}_{x:\overline{n|}}$ is the contingent payment random variable of a $n-$year endowment life insurance. Thus,

$$E(\bar{Y}_{x:\overline{n|}}) = \frac{1-E(\bar{Z}_{x:\overline{n|}})}{\delta} \Longrightarrow \bar{a}_{x:\overline{n|}} = \frac{1-\bar{A}_{x:\overline{n|}}}{\delta}.$$

The variance of the $n-$year temporary annuity random variable $\bar{Y}_{x:\overline{n|}}$ also works like the variance of the continuous whole life annuity. That is,

$$\text{Var}(\bar{Y}_{x:\overline{n|}}) = \frac{1}{\delta^2}\text{Var}(\bar{Z}_{x:\overline{n|}}) = \frac{{}^2\bar{A}_{x:\overline{n|}} - (\bar{A}_{x:\overline{n|}})^2}{\delta^2}.$$

Example 34.3
Find $\text{Var}(\bar{Y}_{x:\overline{20}|})$ if $\delta = 0.06$ and $\mu(x+t) = 0.04$ for all $t \geq 0$.

Solution.
From the formula

$$\bar{A}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} + \nu^n {}_n p_x$$

we find

$$\bar{A}_{x:\overline{20}|} = \frac{1 - e^{-20(0.04+0.06)}}{0.04 + 0.06} + e^{-20(0.06+0.04)} = 0.781982451.$$

Likewise,

$${}^2\bar{A}_{x:\overline{20}|} = \frac{1 - e^{-20(0.04+2(0.06))}}{0.04 + 2(0.06)} + e^{-20(0.04+2(0.06))} = 0.7859984291.$$

Hence,

$$\text{Var}(\bar{Y}_{x:\overline{20}|} = \frac{0.7859984291 - 0.781982451^2}{0.06^2} = 48.4727 \blacksquare$$

Actuarial Accumulated Value
In the theory of interest, the relationship between accumulated value AV and present value PV at time n is given by $PV = \nu^n FV$. The term ν^n is known as the discount factor. In the actuarial context, what is parallel to the discount factor is the term ${}_nE_x = \nu^n {}_np_x$. That is, we have a relation like $PV = {}_nE_x FV$. For example, the actuarial accumulated value at time n of an $n-$year temporary annuity of 1 per year payable continuously while (x) survives is given by

$$\bar{s}_{x:\overline{n}|} = \frac{\bar{a}_{x:\overline{n}|}}{{}_nE_x}.$$

Example 34.4
Show that

$$\bar{s}_{x:\overline{n}|} = \int_0^n \frac{dt}{{}_{n-t}E_{x+t}}.$$

Solution.
We have

$$\begin{aligned}
\bar{s}_{x:\overline{n|}} =& \frac{\bar{a}_{x:\overline{n|}}}{{}_nE_x}\\
=& \frac{\int_0^n \nu^t{}_tp_x dt}{\nu^n{}_np_x}\\
=& \int_0^n \frac{1}{\nu^{n-t}{}_{n-t}p_{x+t}}dt\\
=& \int_0^n \frac{1}{{}_{n-t}E_{x+t}}dt \blacksquare
\end{aligned}$$

Practice Problems

Problem 34.1
Find $\bar{a}_{x:\overline{n}|}$ under a constant force of mortality and constant force of interest.

Problem 34.2
You are given:
• $\mu(x+t) = 0.34$
• $\delta = 0.06$.
Find $\bar{a}_{x:\overline{5}|}$.

Problem 34.3
Show thar $\bar{a}_x = \bar{a}_{x:\overline{n}|} + \nu^n {}_tp_x \bar{a}_{x+n}$.

Problem 34.4
Seven-finned jumping fish can take out continuous whole and temporary life annuities. Each annuity pays $1dt$ in benefits at each time t. A 5-year-old seven-finned jumping fish can get either a one-year temporary life annuity with present value 0.67 or a whole life annuity with present value 5.6. The annual force of interest among seven-finned jumping fish is 0.2, and a 5-year-old seven-finned jumping fish has a probability of 0.77 of surviving to age 6. Find the actuarial present value of a continuous whole life annuity available to a 6-year-old seven-finned jumping fish.

Problem 34.5
Show that $\bar{a}_{x:\overline{m+n}|} = \bar{a}_{x:\overline{m}|} + {}_mE_x \bar{a}_{x+m:\overline{n}|}$.

35 Continuous Deferred Life Annuities

A $n-$**year deferred continuous annuity** guarantees a continuous flow of payments at the rate of 1 per year while the individual is alive starting in n years. The present value random variable of this annuity is defined by

$$ {}_{n|}\bar{Y}_x = \begin{cases} 0, & T < n \\ \bar{a}_{\overline{T-n}|}\nu^n, & T \geq n \end{cases} $$

where T is the remaining lifetime random variable.
The actuarial present value of an $n-$year deferred life annuity of 1 for (x) is denoted by ${}_{n|}\bar{a}_x$. Since an $n-$year deferred life annuity of 1 for (x) is a whole life annuity starting at age $x+n$ discounted back to the date of purchase, we can write

$$ {}_{n|}\bar{a}_x = E({}_{n|}\bar{Y}_x) = \nu^n {}_np_x \bar{a}_{x+n} = {}_nE_x \bar{a}_{x+n}. $$

Example 35.1
Let the remaining lifetime at birth random variable X be uniform on $[0, 100]$. Find ${}_{20|}\bar{a}_{40}$ if $\nu = 0.91$.

Solution.
We have

$$ \begin{aligned} {}_{20}E_{40} =&(0.91)^{20}{}_{20}p_{40} = (0.91)^{20}\frac{40}{60} = 0.1010966087 \\ \bar{A}_{60} =& \int_0^{40} \frac{\nu^t}{40} dt = 0.2589854685 \\ \bar{a}_{60} =& \frac{1-\bar{A}_{60}}{\delta} = \frac{1-0.2589854685}{-\ln 0.91} = 7.857164593 \\ {}_{20|}\bar{a}_{40} =& {}_{20}E_{40}\bar{a}_{60} = (0.1010966087)(7.857164593) = 0.7943 \blacksquare \end{aligned} $$

Example 35.2
Show that

$$ {}_{n|}\bar{a}_x = \int_n^{\infty} \nu^n \bar{a}_{\overline{t-n}|}\ {}_tp_x \mu(x+t) dt. $$

Solution.
We have

$$\begin{aligned}
{}_{n|}\bar{a}_x =& \int_0^\infty {}_{n|}y_x f_T(t)dt \\
=& \int_n^\infty \nu^n \bar{a}_{\overline{t-n|}} f_T(t)dt \\
=& \int_n^\infty \nu^n \bar{a}_{\overline{t-n|}} \ {}_tp_x \mu(x+t)dt \ \blacksquare
\end{aligned}$$

Example 35.3
Show that

$$ {}_{n|}\bar{a}_x = \int_n^\infty \nu^t {}_tp_x dt. $$

Solution.
We have

$$\begin{aligned}
{}_{n|}\bar{a}_x =& \int_n^\infty \nu^n \bar{a}_{\overline{t-n|}} \frac{d}{dt}(-{}_tp_x)dt \\
=& - \nu^n \bar{a}_{\overline{t-n|}} \ {}_tp_x\Big|_n^\infty + \int_n^\infty e^{-\delta n} e^{-\delta(t-n)} {}_tp_x dt \\
=& \int_n^\infty e^{-\delta t} {}_tp_x dt = \int_n^\infty \nu^t {}_tp_x dt \ \blacksquare
\end{aligned}$$

Finally, from the equality

$$ \int_0^\infty \nu^t {}_tp_x dt = \int_0^n \nu^t {}_tp_x dt + \int_n^\infty \nu^t {}_tp_x dt $$

we can write

$$ \bar{a}_x = \bar{a}_{x:\overline{n|}} + {}_{n|}\bar{a}_x. $$

This says that if you purchase an $n-$year temporary annuity to cover the next n years, and an $n-$year deferred annuity to cover the remainder of your lifetime work the same as if you purchase a continuous whole life annuity.

Practice Problems

Problem 35.1
Show that under a constant force of mortality μ and a constant force of interest δ we have

$$ {}_{n|}\bar{a}_x = \frac{e^{-n(\mu+\delta)}}{\mu+\delta}. $$

Problem 35.2
Let the remaining lifetime at birth random variable X be exponential with $\mu = 0.05$. Find ${}_{20|}\bar{a}_{50}$ if $\delta = 0.1$.

Problem 35.3
Show that

$$ {}_{n|}\bar{Y}_x = \frac{Z_{x:\overset{1}{\overline{n|}}} - {}_{n|}\bar{Z}_x}{\delta} $$

where $Z_{x:\overset{1}{\overline{n|}}}$ is the present value of a pure endowment and ${}_{n|}\bar{Z}_x$ that of an $n-$year deferred insurance.

Problem 35.4
Show that

$$ E[({}_{n|}\bar{Y}_x)^2] = 2\nu^{2n}{}_np_x \int_0^\infty \nu^t \bar{a}_{\overline{t|}}\ {}_tp_{x+n}dt. $$

Problem 35.5
Suppose that $\nu = 0.92$, and the force of mortality is $\mu(x+t) = 0.02$, for $t \geq 0$. Find ${}_{20|}\bar{a}_x$ and $\mathrm{Var}({}_{n|}\bar{Y}_x)$.

36 The Certain-and-Life Annuity

A **continuous** $n-$**year certain and life annuity** of a life aged x makes continuous payments at the rate of 1 per year for n years, and if the annuitant survives more than n years, makes contingent payments until his/her death. Under this annuity, the payments up to year n are guaranteed. If the annuitant dies before the completion of n years, payments would be made to his/her estate until n years were up.
The present value random variable of the benefits is defined by

$$\bar{Y}_{\overline{x:\overline{n}|}} = \begin{cases} \bar{a}_{\overline{n}|}, & T(x) < n \\ \bar{a}_{\overline{T}|}, & T(x) \geq n. \end{cases}$$

Example 36.1
Show that

$$\bar{Y}_{\overline{x:\overline{n}|}} = \bar{a}_{\overline{n}|} + {}_{n|}\bar{Y}_x.$$

Solution.
If $T(x) < n$ we have ${}_{n|}\bar{Y}_x = 0$ and $\bar{Y}_{\overline{x:\overline{n}|}} = \bar{a}_{\overline{n}|}$. If $T(x) \geq n$ then

$$\bar{a}_{\overline{n}|} + {}_{n|}\bar{Y}_x = \bar{a}_{\overline{n}|} + \bar{a}_{\overline{T-n}|}\nu^n = \bar{a}_{\overline{T}|} = \bar{Y}_{\overline{x:\overline{n}|}} \blacksquare$$

The actuarial present value of $\bar{Y}_{\overline{x:\overline{n}|}}$ is denoted by $\bar{a}_{\overline{x:\overline{n}|}}$. Thus, we have

$$\bar{a}_{\overline{x:\overline{n}|}} = E(\bar{Y}_{\overline{x:\overline{n}|}}) = E(\bar{a}_{\overline{n}|}) + E({}_{n|}\bar{Y}_x) = \bar{a}_{\overline{n}|} + {}_{n|}\bar{a}_x.$$

This says that the $n-$year certain and life annuity gives you the present value of an $n-$year annuity certain plus the present value of a deferred annuity covering the years after year n.

Example 36.2
Let the remaining lifetime at birth random variable X be uniform $[0, 100]$. Find ${}_{20|}\bar{a}_{40}$ and $\bar{a}_{\overline{40:\overline{20}|}}$ if $\nu = 0.91$.

Solution.
We have

$$\bar{a}_{\overline{20}|} = \frac{1 - (0.91)^{20}}{-\ln 0.91} = 8.9953$$

and by Example 35.1,

$${}_{20|}\bar{a}_{40} = 0.7943.$$

Hence,

$$\bar{Y}_{\overline{40:\overline{20}|}} = 8.9953 + 0.7943 = 9.7896 \blacksquare$$

Practice Problems

Problem 36.1
Let the remaining lifetime at birth random variable X be exponential with $\mu = 0.05$. Find ${}_{20|}\bar{a}_{50}$ and $\bar{a}_{\overline{50:\overline{20}|}}$ if $\delta = 0.1$.

Problem 36.2
Show that
$$\bar{a}_{\overline{x:\overline{n}|}} = \bar{a}_{\overline{n}|}\,{}_nq_x + \int_n^\infty \bar{a}_{\overline{t}|}\ {}_tp_x\mu(x+t)dt.$$

Problem 36.3
Use the previous problem and integration by parts to show that
$$\bar{a}_{\overline{x:\overline{n}|}} = \bar{a}_{\overline{n}|} + \int_n^\infty \nu^t\,{}_tp_xdt.$$

Problem 36.4
Show that
$$\mathrm{Var}(\bar{Y}_{\overline{x:\overline{n}|}}) = \mathrm{Var}({}_{n|}\bar{Y}_x).$$

Problem 36.5
Show that
$$\bar{a}_{\overline{x:\overline{n}|}} = \bar{a}_{\overline{n}|} + {}_nE_x\bar{a}_{x+n}.$$

Problem 36.6
Show that
$$\bar{a}_{\overline{x:\overline{n}|}} = \bar{a}_{\overline{n}|} + (\bar{a}_x - \bar{a}_{x:\overline{n}|}).$$

37 Discrete Life Annuities

In this section we look at discrete annuities where payments are either made at the beginning of the year (annuity due) or at the end of the year (annuity immediate). You will notice that the theory and formulas developed in this section will parallel what has already been developed for the continuous case.

37.1 Whole Life Annuity Due

A **whole life annuity due** is a series of payments made at the beginning of the year while an annuitant is alive. If death occurs in the interval $[x+k, x+k+1)$, then $k+1$ payments have already been made at times $0, 1, \cdots, k$. The present value of a whole life annuity due for (x) with unit payment is denoted by $\ddot{Y}_x$ and is given by

$$\ddot{Y}_x = \ddot{a}_{\overline{K(x)+1}|} = \frac{1-\nu^{K(x)+1}}{d} = \sum_{j=0}^{K(x)} \nu^j.$$

The actuarial present value is denoted by $\ddot{a}_x$.

Example 37.1
You are given the following probability mass function of $K(x)$.

k	0	1	2
$\Pr(K(x)=k)$	0.2	0.3	0.5

Find $\ddot{a}_x$ and ${}^2\ddot{a}_x = E(\ddot{Y}_x^2)$ if $i = 0.05$.

Solution.
We want to find

$$\ddot{a}_x = \ddot{a}_{\overline{1}|}\Pr(K(x)=0) + \ddot{a}_{\overline{2}|}\Pr(K(x)=1) + \ddot{a}_{\overline{3}|}\Pr(K(x)=2)$$

where

$$\begin{aligned}
\ddot{a}_{\overline{1}|} =& \frac{1-\nu}{d} = 1\\
\ddot{a}_{\overline{2}|} =& \frac{1-\nu^2}{d} = 1.9524\\
\ddot{a}_{\overline{3}|} =& \frac{1-\nu^3}{d} = 2.8594.
\end{aligned}$$

Thus,

$$\begin{aligned}\ddot{a}_x =&(1)(0.2) + (1.9524)(0.3) + (2.8594)(0.5) = 2.21542\\ {}^2\ddot{a}_x =&(1)^2(0.2) + (1.9524)^2(0.3) + (2.8594)^2(0.5) = 5.4316 \blacksquare\end{aligned}$$

Next, it is easy to see that

$$\ddot{Y}_x = \frac{1 - Z_x}{d}$$

where Z_x is the discrete whole life insurance paying 1 at the end of year of death.

Thus, if we know A_x we can find $\ddot{a}_x$ as follows:

$$\ddot{a}_x = E(\ddot{Y}_x) = E\left(\frac{1 - Z_x}{d}\right) = \frac{1 - A_x}{d}.$$

A third way for computing $\ddot{a}_x$ is shown in the next example.

Example 37.2
Show that

$$\ddot{a}_x = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|}\,{}_{k|}q_x = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|}\,{}_kp_x q_{x+k}.$$

Solution.
We have

$$\begin{aligned}\ddot{a}_x =&E(\ddot{a}_{\overline{K(x)+1}|}) = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|}\Pr(K(x) = k)\\ =&\sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|}\,{}_kp_x q_{x+k} \blacksquare\end{aligned}$$

The above can have a different form as shown next.

Example 37.3
Show that

$$\ddot{a}_x = \sum_{k=0}^{\infty} \nu^k\,{}_kp_x = \sum_{k=0}^{\infty} {}_kE_x.$$

Solution.
Using Problem 20.14, We have

$$\begin{aligned}\ddot{a}_x=&\sum_{k=0}^{\infty}\ddot{a}_{\overline{k+1}|}\,{}_kp_xq_{x+k}=\frac{1}{d}\left[\sum_{k=0}^{\infty}{}_kp_xq_{x+k}-\sum_{k=0}^{\infty}\nu^{k+1}{}_kp_xq_{x+k}\right]\\ =&\frac{1}{d}\left[\sum_{k=0}^{\infty}\Pr(K(x)=k)-\sum_{k=0}^{\infty}\nu^{k+1}({}_kp_x-{}_{k+1}p_x)\right]\\ =&\frac{1}{d}[1-(\nu-\nu p_x+\nu^2p_x-\nu^2{}_2p_x+\nu^3{}_2p_x-\nu^3{}_3p_x+\cdots)]\\ =&\frac{1}{d}[(1-\nu)+\nu(1-\nu)p_x+\nu^2(1-\nu)_2p_x+\cdots]\\ =&\frac{1}{d}(d+\nu dp_x+\nu^2d_2p_x+\nu^3d_3p_x+\cdots)\\ =&1+\nu p_x+\nu^2{}_2p_x+\nu^3{}_3p_x+\cdots=\sum_{k=0}^{\infty}\nu^k{}_kp_x \blacksquare\end{aligned}$$

The above formula is knwon as the **current payment technique formula** for computing life annuities.

Example 37.4
Consider the following extract from a life table.

x	75	76	77	78	79	80
ℓ_x	120	100	70	40	10	0

Assume $i=0.05$, find
(a) the present value of the random variable $\ddot{Y}_{75}$;
(b) the expected value of $\ddot{Y}_{75}$.

Solution.
(a) We have

$$\ddot{Y}_{75}=\ddot{a}_{\overline{K(75)+1}|}.$$

(b) We have

$$\begin{aligned}
\ddot{a}_{75} =& \sum_{k=0}^{\infty}(1.05)^{-k}\frac{\ell_{75+k}}{\ell_{75}}\\
=&1+(1.05)^{-1}\left(\frac{100}{120}\right)+(1.05)^{-2}\left(\frac{70}{120}\right)\\
&+(1.05)^{-3}\left(\frac{40}{120}\right)+(1.05)^{-4}\left(\frac{10}{120}\right)\\
=&2.6793 \blacksquare
\end{aligned}$$

Next, the variance of $\ddot{Y}_x$ can be expressed in terms of the variance of Z_x :

$$\text{Var}(\ddot{Y}_x) = \text{Var}\left(\frac{1-Z_x}{d}\right) = \frac{\text{Var}(Z_x)}{d^2} = \frac{{}^2A_x - (A_x)^2}{d^2}.$$

Example 37.5
You are given the following:
- $i = 0.06$
- $A_x = 0.369131$
- ${}^2A_x = 0.1774113.$

Find $\text{Var}(\ddot{Y}_x)$.

Solution.
We have

$$\text{Var}(\ddot{Y}_x) = \frac{{}^2A_x - (A_x)^2}{d^2} = \frac{0.1774113 - 0.369131^2}{(0.06)^2(1.06)^{-2}} = 12.8445 \blacksquare$$

Example 37.6
Let $\ddot{Y}_x$ denote the present value of a whole life annuity-due for (x). Suppose that $q_{x+k} = 0.11$ for all nonnegative integer k. Find the expected value of $\ddot{Y}_x$ if $i = 0.25$.

Solution.
Using the current payment method formula we can write

$$\begin{aligned}\ddot{a}_x =&E(\ddot{Y}_x) = \sum_{k=0}^{\infty}\nu^k{}_kp_x\\ =&\sum_{k=0}^{\infty}(0.8)^k(p_x)^k\\ =&\sum_{k=0}^{\infty}(0.8)^k(1-0.11)^k\\ =&\sum_{k=0}^{\infty}(0.712)^k\\ =&\frac{1}{1-0.712} = 3.4722 \blacksquare\end{aligned}$$

Example 37.7 ‡
A government creates a fund to pay this year's lottery winners.
You are given:
(i) There are 100 winners each age 40.
(ii) Each winner receives payments of 10 per year for life, payable annually, beginning immediately.
(iii) Mortality follows the Illustrative Life Table.
(iv) The lifetimes are independent.
(v) $i = 0.06$
(vi) The amount of the fund is determined, using the normal approximation, such that the probability that the fund is sufficient to make all payments is 95%.
Calculate the initial amount of the fund.

Solution.
Let Y_i denote the present value random variable for payments on the i^{th} life, where $i = 1, 2, \cdots, 100$. That is,

$$Y_i = 10\frac{1-\nu^{K+1}}{d}.$$

Then

$$\begin{aligned}E[Y_i] =&10\ddot{a}_{40} = 10(14.8166) = 148.166\\ \mathrm{Var}(Y_i) =&100\frac{{}^2A_{40} - A_{40}^2}{d^2} = 100\frac{0.04863 - 0.16132^2}{0.06(1.06)^{-1}} = 705.55.\end{aligned}$$

The present value random variable for all payments is

$$S = Y_1 + Y_2 + \cdots + Y_{100}.$$

Hence,

$$\begin{aligned}E[S] =&100E[Y_i] = 14,816.60\\ \mathrm{Var}(S) =&100\mathrm{Var}(Y_i) = 70,555\\ \sigma_S =&\sqrt{70,555} = 265.62.\end{aligned}$$

We would like to know the amount of fund F that the insurer must have now to assure that the probability of having sufficient funds to pay all claims is 0.95. That is, we want F such that

$$\Pr(S \le F) = 0.95.$$

In probability language, F is the 95^{th} percentile of the random variable S. Now, using normal approximation, we can write

$$\Pr\left(Z_{0.95} \le \frac{F - E[S]}{\sigma_S}\right) = 0.95.$$

But $Z_{0.95} = 1.645$ from which we find

$$F = E[S] + 1.645\sigma_S = 14,816.60 + 1.645(265.62) = 15,254 \blacksquare.$$

Practice Problems

Problem 37.1
You are given the following probability mass function of $K(x)$.

k	0	1	2
$\Pr(K(x)=k)$	0.2	0.3	0.5

Find $\mathrm{Var}(\ddot{Y}_x)$ if $i=0.05$.

Problem 37.2
You are given $A_x=0.22$ and $i=0.06$. Calculate $\ddot{a}_x$.

Problem 37.3
Show that $\ddot{a}_x=1+\nu p_x\ddot{a}_{x+1}$.

Problem 37.4 ‡
You are given:
(i) $\ddot{a}_x=8$ for all integral values x.
(ii) $i=0.08$.
Calculate ${}_8q_{30}$ for all integral values x.

Problem 37.5
You are given

x	q_x	$\ddot{a}_x$
75	0.03814	7.4927
76	0.04196	7.2226

Calculate the interest rate i.

Problem 37.6 ‡
Your company currently offers a whole life annuity product that pays the annuitant 12,000 at the beginning of each year. A member of your product development team suggests enhancing the product by adding a death benefit that will be paid at the end of the year of death.
Using a discount rate, d, of 8%, calculate the death benefit that minimizes the variance of the present value random variable of the new product.

Problem 37.7 ‡
Your company is competing to sell a life annuity-due with an actuarial present value of 500,000 to a 50-year old individual.
Based on your company's experience, typical 50-year old annuitants have a complete life expectancy of 25 years. However, this individual is not as healthy as your company's typical annuitant, and your medical experts estimate that his complete life expectancy is only 15 years.
You decide to price the benefit using the issue age that produces a complete life expectancy of 15 years. You also assume:
(i) For typical annuitants of all ages, mortality follows De Moivre's Law with the same limiting age, ω
(ii) $i = 0.06$
Calculate the annual benefit that your company can offer to this individual.

Problem 37.8 ‡
For a pension plan portfolio, you are given:
(i) 80 individuals with mutually independent future lifetimes are each to receive a whole life annuity-due.
(ii) i = 0.06
(iii)

Age	# of Annuitants	Annual Payment	$\ddot{a}_x$	A_x	${}^2A_x$
65	50	2	9.8969	0.43980	0.23603
75	30	1	7.2170	0.59149	0.38681

Using the normal approximation, calculate the 95$^{\text{th}}$ percentile of the distribution of the present value random variable of this portfolio.

37.2 Temporary Life Annuity-Due

An $n-$**year temporary life annuity-due** pays 1 at the beginning of each year so long as the annuitant (x) survives, for up to a total of n years, or n payments otherwise. Thus, for $k < n$ there are $k+1$ payments made at time $0, 1, \cdots, k$ and for $k \geq n$ there are n payments made at time $0, 1, \cdots, n-1$. The present value random variable of this life annuity is given by

$$\ddot{Y}_{x:\overline{n}|} = \begin{cases} \ddot{a}_{\overline{K+1}|}, & K < n \\ \ddot{a}_{\overline{n}|}, & n \leq K. \end{cases}$$

Example 37.8
Show that

$$\ddot{Y}_{x:\overline{n}|} = \frac{1 - Z_{x:\overline{n}|}}{d}$$

where $Z_{x:\overline{n}|}$ is the present value random variable of an $n-$year endowment insurance.

Solution.
We have

$$\ddot{Y}_{x:\overline{n}|} = \ddot{a}_{\overline{\min(K+1,n)}|} = \frac{1 - \nu^{\min(K+1,n)}}{d} = \frac{1 - Z_{x:\overline{n}|}}{d} \blacksquare$$

The actuarial present value of the annuity is

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} \Pr(K(x) = k) + \ddot{a}_{\overline{n}|} \Pr(K(x) \geq n) = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\, {}_kp_x q_{x+k} + \ddot{a}_{\overline{n}|}\, {}_np_x.$$

A more convenient formula is given by the current payment technique formula described in the next example.

Example 37.9
Show that

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \nu^k {}_kp_x = \sum_{k=0}^{n-1} {}_kE_x.$$

Solution.
We have

$$\begin{aligned}
\ddot{a}_{x:\overline{n}|} =& \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\ {}_kp_x q_{x+k} + \ddot{a}_{\overline{n}|}\ {}_np_x \\
=& \frac{1}{d}\left[\sum_{k=0}^{n-1} {}_kp_x q_{x+k} - \sum_{k=0}^{n-1} \nu^{k+1} {}_kp_x q_{x+k} + (1-\nu^n)_np_x\right] \\
=& \frac{1}{d}\left[1 - {}_np_x + (1-\nu^n)_np_x - \sum_{k=0}^{n-1} \nu^{k+1} {}_kp_x q_{x+k}\right].
\end{aligned}$$

But

$$\begin{aligned}
\sum_{k=0}^{n-1} \nu^{k+1} {}_kp_x q_{x+k} =& \sum_{k=0}^{n-1} \nu^{k+1}({}_kp_x - {}_{k+1}p_x) \\
=& \nu - \nu p_x + \nu^2 p_x - \nu^2 {}_2p_x + \nu^3 {}_2p_x - \nu^3 {}_3p_x + \cdots + \nu^n {}_{n-1}p_x - \nu^n {}_np_x \\
=& \nu - \nu(1-\nu)p_x - \nu^2(1-\nu)_2p_x - \cdots - \nu^{n-1}(1-\nu)_{n-1}p_x - \nu^n {}_npx \\
=& \nu - d\nu p_x - d\nu^2 {}_2p_x - \cdots - d\nu^{n-1} {}_{n-1}p_x - \nu^n {}_np_x.
\end{aligned}$$

Thus,

$$\begin{aligned}
\ddot{a}_{x:\overline{n}|} =& \frac{1}{d}\left[1 - {}_np_x + (1-\nu^n)_np_x - \sum_{k=0}^{n-1} \nu^{k+1} {}_kp_x q_{x+k}\right] \\
=& \frac{1}{d}[1 - {}_np_x + (1-\nu^n)_np_x - \nu + \nu d p_x + \nu^2 d_2p_x + \cdots + \nu^{n-1} d_{n-1}p_x + \nu^n {}_np_x] \\
=& 1 + \nu p_x + \nu^2 {}_2p_x + \cdots + \nu^{n-1} {}_{n-1}p_x = \sum_{k=0}^{n-1} \nu^k {}_kp_x \ \blacksquare
\end{aligned}$$

Example 37.10
For a 3-year temporary life annuity-due on (30), you are given:
(i) $s(x) = 1 - \frac{x}{80},\ 0 \le x < 80.$
(ii) $i = 0.05.$
(iii)

$$Y = \begin{cases} \ddot{a}_{\overline{K+1}|}, & K = 0, 1, 2 \\ \ddot{a}_{\overline{3}|}, & K = 3, 4, \cdots. \end{cases}$$

Calculate $\mathrm{Var}(Y)$.

Solution.
We have

Y	$\Pr(K(30)=k)$
$\ddot{a}_{\overline{1}\rceil}=1$	q_{30}
$\ddot{a}_{\overline{2}\rceil}=1.95238$	$p_{30}q_{31}$
$\ddot{a}_{\overline{3}\rceil}=2.85941$	${}_2p_{30}$

Finding the probabilities in the table we find

$$\begin{aligned} q_{30} =&1-p_{30}=1-\frac{s(31)}{s(30)}=\frac{1}{60}\\ p_{30}q_{31} =&\frac{59}{60}\left(1-\frac{s(32)}{s(31)}\right)=\frac{59}{60}\cdot\frac{1}{59}=\frac{1}{60}\\ {}_2p_{30} =&p_{30}p_{31}=\frac{59}{60}\cdot\frac{58}{59}=\frac{58}{60}. \end{aligned}$$

Thus,

$$E(Y)=\frac{1}{60}+(1.95238)\times\frac{1}{60}+(2.85941)\times\frac{58}{60}=2.81330$$

and

$$E(Y^2)=\frac{1}{60}+(1.95238)^2\times\frac{1}{60}+(2.85941)^2\times\frac{58}{60}=7.98388.$$

The final answer is

$$\mathrm{Var}(Y)=7.98388-2.81330^2=0.069223\ \blacksquare$$

In the case $A_{x:\overline{n}\rceil}$ is known then we can find $\ddot{a}_{x:\overline{n}\rceil}$ using the formula

$$\ddot{a}_{x:\overline{n}\rceil}=E(\ddot{Y}_{x:\overline{n}\rceil})=\frac{1-A_{x:\overline{n}\rceil}}{d}.$$

We can also find the variance of $\ddot{Y}_{x:\overline{n}\rceil}$. Indeed, we have

$$\mathrm{Var}(\ddot{Y}_{x:\overline{n}\rceil})=\frac{1}{d^2}\mathrm{Var}(Z_{x:\overline{n}\rceil})=\frac{{}^2A_{x:\overline{n}\rceil}-(A_{x:\overline{n}\rceil})^2}{d^2}.$$

Finally, the actuarial accumulated value at the end of the term of an $n-$year temporary life annuity-due is

$$\ddot{s}_{x:\overline{n}\rceil}=\frac{\ddot{a}_{x:\overline{n}\rceil}}{{}_nE_x}.$$

Example 37.11
You are given that $\mu = 0.02$ and $\delta = 0.05$. Let Y be the present value random variable for an 20-year temporary life annuity due of 1 per year issued to (75). Calculate $\ddot{s}_{75:\overline{20}|}$.

Solution.
We have

$$\begin{aligned}\ddot{a}_{75:\overline{20}|} =&1 + \nu p_{75} + \nu^2{}_2p_{75} + \cdots + \nu^{19}{}_{19}p_{75}\\ =&1 + e^{-0.05}e^{-0.02} + e^{-2(0.05)}e^{-2(0.02)} + \cdots + e^{-19(0.05)}e^{-19(0.02)}\\ =&1 + e^{-0.07} + e^{-2(0.07)} + \cdots + e^{-19(0.07)}\\ =&\frac{1 - e^{-20(0.07)}}{1 - e^{-0.07}}.\end{aligned}$$

Also, we have

$${}_{20}E_{75} = e^{-20(0.05)}e^{-20(0.02)} = e^{-20(0.07)}.$$

Hence,

$$\ddot{s}_{75:\overline{20}|} = \frac{e^{20(0.07)}(1 - e^{-20(0.07)})}{1 - e^{-0.07}} = 45.19113 \blacksquare$$

Example 37.12
Show that

$$\ddot{s}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \frac{1}{{}_{n-k}E_{x+k}}.$$

Solution.
We have

$$\begin{aligned}\ddot{s}_{x:\overline{n}|} =&\frac{\ddot{a}_{x:\overline{n}|}}{{}_nE_x}\\ =&\sum_{k=0}^{n-1} \frac{\nu^k{}_kp_x}{\nu^n{}_np_x}\\ =&\sum_{k=0}^{n-1} \frac{1}{\nu^{n-k}{}_{n-k}p_{x+k}}\\ =&\sum_{k=0}^{n-1} \frac{1}{{}_{n-k}E_{x+k}} \blacksquare\end{aligned}$$

Practice Problems

Problem 37.9
Show that

$$\ddot{a}_{x:\overline{n}|} = 1 + \nu p_x \ddot{a}_{x+1:\overline{n-1}|}.$$

Problem 37.10 ‡
For a three-year temporary life annuity due of 100 on (75), you are given:
(i) $i = 0.11$
(ii) $\int_0^x \mu(t)dt = 0.01x^{1.2},\ x > 0.$
Calculate the actuarial present value of this annuity.

Problem 37.11 ‡
For a special 3-year temporary life annuity-due on (x), you are given:
(i)

k	Annuity Payment	p_{x+k}
0	15	0.95
1	20	0.90
2	25	0.85

(ii) $i = 0.06$.
Calculate the variance of the present value random variable for this annuity.

Problem 37.12 ‡
Y is the present-value random variable for a special 3-year temporary life annuity-due on (x). You are given:
(i) ${}_tp_x = 0.9^t, t \geq 0$;
(ii) K is the curtate-future-lifetime random variable for (x).
(iii)

$$Y = \begin{cases} 1.00 & K = 0 \\ 1.87 & K = 1 \\ 2.72 & K = 2, 3, \cdots \end{cases}$$

Calculate Var(Y).

Problem 37.13
John is currently age 50. His survival pattern follows DeMoivre's law with $\omega = 100$. He purchases a three-year temporary life annuity that pays a benefit of 100 at the beginning of each year. Compute the actuarial present value of his benefits if $i = 0.05$.

Problem 37.14
You are given that mortality follows DeMoivre's law with $\omega = 125$ and $\delta = 0.05$. Y is the present value random variable for an 20-year certain and life annuity due of 1 per year issued to (75). Calculate $\ddot{a}_{\overline{75:\overline{20|}}}$.

Problem 37.15
You are given the following mortality table:

x	ℓ_x
90	1000
91	900
92	720
93	432
94	216
95	0

Suppose $i = 0.04$. Calculate $\ddot{s}_{91:\overline{3|}}$.

Problem 37.16
Show that

$$\ddot{a}_x = \ddot{a}_{x:\overline{n|}} + {}_nE_x\ddot{a}_{x+n}.$$

37.3 Discrete Deferred Life Annuity-Due

An $n-$**year deferred whole life annuity-due** of 1 payable at the beginning of each year while (x) survives from age $x+n$ onward is described by the following random variable

$$
{}_{n|}\ddot{Y}_x = \begin{cases} 0, & K \leq n-1 \\ \nu^n \ddot{a}_{\overline{K+1-n}|}, & K \geq n. \end{cases}
$$

Example 37.13
Show that

$$
{}_{n|}\ddot{Y}_x = \ddot{Y}_x - \ddot{Y}_{x:\overline{n}|}.
$$

Solution.
Recall that $\ddot{Y}_x = \ddot{a}_{\overline{K(x)+1}|}$ and

$$
\ddot{Y}_{x:\overline{n}|} = \begin{cases} \ddot{a}_{\overline{K+1}|}, & K < n \\ \ddot{a}_{\overline{n}|}, & n \leq K. \end{cases}
$$

Thus, if $K<n$ then ${}_{n|}\ddot{Y}_x = 0,\ \ddot{Y}_x - \ddot{Y}_{x:\overline{n}|} = 0$ so that the equality holds. If $K \geq n$ then ${}_{n|}\ddot{Y}_x = \nu^n \ddot{a}_{\overline{k+n-1}|},\ \ddot{Y}_x - \ddot{Y}_{x:\overline{n}|} = \ddot{a}_{\overline{K(x)+1}|} - \ddot{a}_{\overline{n}|} = \nu^n \ddot{a}_{\overline{k+n-1}|}$ ■

The actuarial present value of the deferred life annuity-due is denoted by ${}_{n|}\ddot{a}_x$. From the previous example, we can write

$$
\begin{aligned}
{}_{n|}\ddot{a}_x =& E({}_{n|}\ddot{Y}_x) = E(\ddot{Y}_x) - E(\ddot{Y}_{x:\overline{n}|}) = \ddot{a}_x - \ddot{a}_{x:\overline{n}|} \\
=& \sum_{k=0}^{\infty} \nu^k {}_kp_x - \sum_{k=0}^{n-1} \nu^k {}_kp_x \\
=& \sum_{k=n}^{\infty} \nu^k {}_kp_x = {}_nE_x \ddot{a}_{x+n}.
\end{aligned}
$$

Example 37.14
For a 5-year deferred whole life annuity due of 1 on (x), you are given:
(i) $\mu(x+t) = 0.01$
(ii) $i = 0.04$
(iii) $\ddot{a}_{x:\overline{5}|} = 4.542$.
(iv) The random variable S denotes the sum of the annuity payments.
(a) Calculate ${}_{5|}\ddot{a}_x$.
(b) Calculate $\Pr(S > {}_{5|}\ddot{a}_x)$.

Solution.
(a) We will use the formula

$$_{n|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n}|}.$$

For that we need to find

$$\ddot{a}_x = \frac{1 - A_x}{d}.$$

But

$$\begin{aligned}
A_x =& \sum_{k=0}^{\infty} e^{-\ln 1.04(k+1)} e^{-0.01k}(1 - e^{-0.01}) \\
=& (1.04)^{-1}(1 - e^{-0.01}) \sum_{k=0}^{\infty} e^{-k(0.01 + \ln 1.04)} \\
=& \frac{(1.04)^{-1}(1 - e^{-0.01})}{1 - e^{-(\ln 1.04 + 001)}} = 0.199202.
\end{aligned}$$

Thus,

$$\ddot{a}_x = \frac{1 - 0.199202}{0.04(1.04)^{-1}} = 20.821$$

and

$$_{5|}\ddot{a}_x = 20.821 - 4.542 = 16.279.$$

(b) Let K be the curtate future lifetime of (x). Then the sum of the payments is 0 if $K \le 4$ and is $K - 4$ if $K \ge 20$. For $S > 16.279$, we make 17 or more payments with the first payment at time $t = 5$ and the 17$^{\text{th}}$ payment at time $t = 21$. Thus

$$\Pr(S \ge 16.279) = \Pr(K - 4 \ge 17) = \Pr(T \ge 21) = {}_{21}p_x = e^{-21(0.01)} = 0.81 \blacksquare$$

Practice Problems

Problem 37.17
Show that

$$_{n|}\ddot{Y}_x = \frac{Z_{x:\overset{\,1}{\overline{n}|}} - {}_{n|}Z_x}{d}.$$

Problem 37.18
Show that

$$_{n|}\ddot{a}_x = \frac{A_{x:\overset{\,1}{\overline{n}|}} - {}_{n|}A_x}{d}.$$

Problem 37.19
Find $_{n|}\ddot{a}_x$ under a constant force of mortality μ.

Problem 37.20
Suppose that $\nu = 0.91$ and the force of mortality is $\mu = 0.05$. Find $_{20|}\ddot{a}_x$.

Problem 37.21 ‡
For a 20-year deferred whole life annuity-due of 1 per year on (45), you are given:
(i) Mortality follows De Moivre's Law with $\omega = 105$.
(ii) $i = 0$.
Calculate the probability that the sum of the annuity payments actually made will exceed the actuarial present value at issue of the annuity.

37.4 Discrete Certain and Life Annuity-Due

A **discrete** $n-$**year certain and life annuity** of a life aged x makes unit payments at the beginning of the year for n years, and if the annuitant survives more than n years, makes contingent payments until his death. Under this annuity, the payments up to year n are guaranteed. If the annuitant dies before the completion of n years, payments would be made to his/her estate until n years were up.
The present value random variable of the benefits is defined by

$$\ddot{Y}_{\overline{x:\overline{n|}}} = \begin{cases} \ddot{a}_{\overline{n|}}, & K < n \\ \ddot{a}_{\overline{K+1|}}, & K \geq n. \end{cases}$$

The actuarial present value of this annuity is denoted by $\ddot{a}_{\overline{x:\overline{n|}}}$. Thus, we have

$$\begin{aligned} \ddot{a}_{\overline{x:\overline{n|}}} =& E(Z_{\overline{x:\overline{n|}}}) = \sum_{k=0}^{n-1} \ddot{a}_{\overline{n|}\,n}p_x q_{x+k} + \sum_{k=n}^{\infty} \ddot{a}_{\overline{K+1|}\,k}p_x q_{x+k} \\ =& \ddot{a}_{\overline{n|}\,n}q_x + \sum_{k=n}^{\infty} \ddot{a}_{\overline{K+1|}\,k}p_x q_{x+k}. \end{aligned}$$

Example 37.15
Show that

$$\ddot{a}_{\overline{x:\overline{n|}}} = \ddot{a}_{\overline{n|}} + \sum_{k=n}^{\infty} \nu^k {}_kp_x = \ddot{a}_{\overline{n|}} + \ddot{a}_x - \ddot{a}_{x:\overline{n|}}.$$

Solution.
We have

$$\begin{aligned} \ddot{a}_{\overline{x:\overline{n|}}} =& \ddot{a}_{\overline{n|}\,n}q_x + \sum_{k=n}^{\infty} \ddot{a}_{\overline{K+1|}\,k}p_x q_{x+k} \\ =& \ddot{a}_{\overline{n|}} - \frac{1}{d}{}_np_x + \frac{\nu^n}{d}{}_np_x + \frac{1}{d}[{}_np_x - \nu^{n+1}{}_np_x + \nu^{n+1}{}_{n+1}p_x - \cdots] \\ =& \ddot{a}_{\overline{n|}} + \nu^n{}_np_x + \nu^{n+1}{}_{n+1}p_x + \cdots \\ =& \ddot{a}_{\overline{n|}} + \sum_{k=n}^{\infty} \nu^k {}_kp_x \\ =& \ddot{a}_{\overline{n|}} + {}_{n|}\ddot{a}_x \\ =& \ddot{a}_{\overline{n|}} + \ddot{a}_x - \ddot{a}_{x:\overline{n|}} \; \blacksquare \end{aligned}$$

Example 37.16
Show that $\text{Var}(\ddot{Y}_{\overline{x:\overline{n}|}}) = \text{Var}({}_{n|}\ddot{Y}_x)$.

Solution.
We have

$$\ddot{Y}_{\overline{x:\overline{n}|}} = \begin{cases} \ddot{a}_{\overline{n}|}, & K < n \\ \ddot{a}_{\overline{K+1}|}, & K \geq n. \end{cases} = \ddot{a}_{\overline{n}|} + \begin{cases} 0, & K < n \\ \ddot{a}_{\overline{K+1}|} - \ddot{a}_{\overline{n}|}, & K \geq n. \end{cases}$$

Since $\ddot{a}_{\overline{K+1}|} - \ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{K+1-n}|}\nu^n$, we have

$$\ddot{Y}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} + {}_{n|}\ddot{Y}_x.$$

Taking the variance of both sides, we obtain

$$\text{Var}(\ddot{Y}_{\overline{x:\overline{n}|}}) = \text{Var}({}_{n|}\ddot{Y}_x)$$

since $\ddot{a}_{\overline{n}|}$ is a constant ■

Example 37.17
Consider a 5-year certain-and-life annuity due for (60) that pays \$1000 guaranteed at the beginning of the year for five years and continuing thereafter for life. You are given the following:
(i) $i = 0.06$
(ii) $A_{65} = 0.43980$
(iii) $\ell_{60} = 8188$ and $\ell_{65} = 7534$.
Calculate the actuarial present value of this life annuity.

Solution.
From the given information in the problem, we have

$$\begin{aligned} \ddot{a}_{\overline{5}|} =& \frac{1 - 1.06^{-5}}{0.06(1.06)^{-1}} = 4.4651 \\ \ddot{a}_{65} =& \frac{1 - 0.43980}{0.06(1.06)^{-1}} = 9.8969 \\ {}_5E_{60} =& (1.06)^{-5}\left(\frac{7534}{8188}\right) = 0.6876 \\ {}_5\ddot{a}_{60} =& (0.6876)(9.8969) = 6.8051. \end{aligned}$$

Hence, the APV of the given life annuity is

$$1000(\ddot{a}_5 + {}_5\ddot{a}_{60}) = 1000(4.4651 + 6.8051) = 11,270.20 ■$$

Practice Problems

Problem 37.22
You are given that $\mu = 0.02$ and $\delta = 0.05$. Y is the present value random variable for an 20-year certain and life annuity due of 1 per year issued to (75). Calculate $E(Y)$.

Problem 37.23
You are given that mortality follows DeMoivre's law with $\omega = 125$ and $\delta = 0.05$. Y is the present value random variable for an 20-year certain and life annuity due of 1 per year issued to (75). Calculate $\ddot{a}_{\overline{75:\overline{20}|}}$.

Problem 37.24 ‡
A person age 40 wins 10,000 in the actuarial lottery. Rather than receiving the money at once, the winner is offered the actuarially equivalent option of receiving an annual payment of K (at the beginning of each year) guaranteed for 10 years and continuing thereafter for life.
You are given:
(i) $i = 0.04$
(ii) $A_{40} = 0.30$
(iii) $A_{50} = 0.35$
(iv) $A^{\ 1}_{40:\overline{10}|} = 0.09$.
Calculate K.

Problem 37.25
You are given the following:
(i) $s(x) = e^{-0.34x},\ x \geq 0$.
(ii) $\delta = 0.06$
Calculate $\ddot{a}_{\overline{5:\overline{2}|}}$.

Problem 37.26 ‡
At interest rate i, you are given:
(i) $\ddot{a}_x = 5.6$
(ii) The actuarial present value of a 2-year certain and life annuity-due of 1 on (x) is $\ddot{a}_{\overline{x:\overline{2}|}} = 5.6459$.
(iii) $e_x = 8.83$
(iv) $e_{x+1} = 8.29$.
(a) Show that $e_x = p_x(1 + e_{x+1})$.
(b) Calculate i.

37.5 Life Annuity-Immediate

Immediate Life annuities are annuities where payments are made at the end of the year. Immediate annuities can be handled fairly simply in terms of the results we have already obtained for annuities due

Whole Life Discrete Annuity Immediate

A **whole life discrete immediate annuity** is a series of payments made at the end of the year as long as the annuitant is alive. The present value of a whole life immediate annuity for (x) with unit payment is the random variable Y_x. Clearly, $Y_x = \ddot{Y}_x - 1$ where $\ddot{Y}_x$ is the present value of the whole life annuity due.

The actuarial present value of a whole life immediate annuity for (x) with unit payment is denoted by a_x. Clearly, $a_x = \ddot{a}_x - 1$.

All the following can be easily checked by the reader

$$\begin{aligned}
Y_x =& \ddot{a}_{\overline{K}|} \\
Y_x =& \frac{\nu - Z_x}{d} \\
a_x =& \frac{\nu - A_x}{d} \\
\text{Var}(Y_x) =& \frac{{}^2A_x - (A_x)^2}{d^2} \\
a_x =& \sum_{k=1}^{\infty} \nu^k {}_kp_x.
\end{aligned}$$

Example 37.18

Suppose mortality follows De Moivre's Law with $\omega = 100$. Find a_{30} if $i = 0.06$.

Solution.

We have

$$\begin{aligned}
A_{30} =& \sum_{k=0}^{70} \frac{(1.06)^{-(k+1)}}{70} = \frac{1}{70}(1.06)^{-1}\left(\frac{1-(1.06)^{-70}}{1-(1.06)^{-1}}\right) = 0.2341 \\
a_{30} =& \frac{(1.06)^{-1} - 0.2341}{(1.06)^{-1}(0.06)} = 12.531 \blacksquare
\end{aligned}$$

Example 37.19

Suppose that $p_{x+k} = 0.97$, for each integer $k \geq 0$, and $i = 0.065$. Find $\text{Var}(Y_x)$.

Solution.
We have

$$A_x = \sum_{k=0}^{\infty} \nu^{k+1} {}_kp_x q_{x+k} = \nu(1-0.97)\sum_{k=0}^{\infty}(\nu p_x)^k = \frac{(1-0.97)\nu}{1-\nu p_x} = 0.3158$$

$${}^2A_x = \sum_{k=0}^{\infty} \nu^{2(k+1)} {}_kp_x q_{x+k} = \nu^2(1-0.97)\sum_{k=0}^{\infty}(\nu^2 p_x)^k = \frac{(1-0.97)\nu^2}{1-\nu^2 p_x} = 0.1827$$

$$\text{Var}(Y_x) = \frac{0.1827 - 0.3158^2}{(1.065)^{-2}(0.065)^2} = 22.2739 \blacksquare$$

<u>$n-$**year Deferred Annuity Immediate**</u>
An **immediate $n-$year deferred annuity** guarantees payments made at the end of the year while an individual is alive starting n years from now. We denote the present value random variable of this annuity by ${}_{n|}Y_x$ and is given by

$${}_{n|}Y_x = \begin{cases} 0, & K \le n \\ \nu^n a_{\overline{K-n|}}, & K > n. \end{cases}$$

The actuarial present value of an immediate $n-$year deferred annuity for (x) with unit payment is denoted by ${}_{n|}a_x$. The reader can easily verify the following results

$$\begin{aligned} {}_{n|}a_x &= \sum_{k=n+1}^{\infty} \nu^n a_{\overline{k-n|}} {}_kp_x q_{x+k} \\ &= \sum_{k=n+1}^{\infty} \nu^k {}_kp_x = {}_nE_x a_{x+n} \\ &= \nu p_{x n-1|} a_x. \end{aligned}$$

Example 37.20
Find ${}_{n|}a_x$ under a constant force of mortality μ and a constant force of interest δ.

Solution.
We have

$$
\begin{aligned}
{}_{n|}a_x = & \sum_{k=n+1}^{\infty} e^{-k\delta} e^{-\mu k} = \sum_{k=n+1}^{\infty} e^{-k(\mu+\delta)} \\
= & e^{-(n+1)(\mu+\delta)} \sum_{k=0}^{\infty} e^{-k(\mu+\delta)} = \frac{e^{-(n+1)(\mu+\delta)}}{1-e^{-(\mu+\delta)}} \blacksquare
\end{aligned}
$$

Immediate $n-$year Temporary Annuity

An $n-$**year temporary life annuity-due** pays 1 at the end of each year so long as the annuitant (x) survives, for up to a total of n years, or n payments otherwise. Thus, for $k < n$ there are $k+1$ payments made at time $1, \cdots, k+1$ and for $k \geq n$ there are n payments made at time $1, \cdots, n$.
The present value random variable of this life annuity is given by

$$
Y_{x:\overline{n}|} = \begin{cases} \ddot{a}_{\overline{K}|}, & K < n \\ \ddot{a}_{\overline{n}|}, & n \leq K. \end{cases}
$$

The actuarial present value of an $n-$year term life immediate annuity for (x) with unit payment is denoted by $a_{x:\overline{n}|}$. The reader can easily check the following results

$$
\begin{aligned}
Y_{x:\overline{n}|} = & \frac{\nu - Z_{x:\overline{n+1}|}}{d} = \ddot{Y}_{x:\overline{n+1}|} - 1 \\
a_{x:\overline{n}|} = & \sum_{k=1}^{n} a_{\overline{k}|\,k}p_x q_{x+k} + a_{\overline{n}|\,n}p_x = \sum_{k=1}^{n} \nu^k {}_kp_x \\
= & \frac{\nu - A_{x:\overline{n+1}|}}{d} = \ddot{a}_{x:\overline{n+1}|} - 1 \\
\mathrm{Var}(Y_{x:\overline{n}|}) = & \frac{{}^2A_{x:\overline{n+1}|} - (A_{x:\overline{n+1}|})^2}{d^2} = \mathrm{Var}(\ddot{Y}_{x:\overline{n+1}|})
\end{aligned}
$$

Example 37.21
You are given the following:
(i) $p_x = 0.82,\ p_{x+1} = 0.81$, and $p_{x+2} = 0.80$.
(ii) $\nu = 0.78$
(a) Find $a_{x:\overline{3}|}$.
(b) Find $\mathrm{Var}(Y_{x:\overline{3}|})$.

Solution.
(a) We have

$$\begin{aligned}a_{x:\overline{3}|} =&\nu p_x + \nu^2{}_2p_x + \nu^3{}_3p_x\\ =&\nu p_x + \nu^2 p_x p_{x+1} + \nu^3 p_x p_{x+1} p_{x+2}\\ =&(0.78)(0.82) + (0.78)^2(0.82)(0.81)\\ &+(0.78)^3(0.82)(0.81)(0.80) = 1.2959.\end{aligned}$$

(b) We have

$$\begin{aligned}A_{x:\overline{4}|} =&\nu q_x + \nu^2 p_x q_{x+1} + \nu^3{}_2p_x q_{x+2} + \nu^4{}_3p_x\\ =&(0.78)(1-0.82) + (0.78)^2(0.82)(1-0.81)\\ &+(0.78)^3(0.82)(0.81)(1-0.80) + (0.78)^4(0.82)(0.81)(0.80)\\ =&0.4949\\ {}^2A_{x:\overline{4}|} =&\nu^2 q_x + \nu^4 p_x q_{x+1} + \nu^6{}_2p_x q_{x+2} + \nu^8{}_3p_x\\ =&(0.78)^2(1-0.82) + (0.78)^4(0.82)(1-0.81)\\ &+(0.78)^6(0.82)(0.81)(1-0.80) + (0.78)^8(0.82)(0.81)(0.80)\\ =&0.2699\\ \text{Var}(Y_{x:\overline{3}|}) =&\frac{0.2699 - 0.4949^2}{(0.22)^2} = 0.516 \blacksquare\end{aligned}$$

The actuarial accumulated value at time n of an $n-$year term immediate annuity is

$$s_{x:\overline{n}|} = \frac{a_{x:\overline{n}|}}{{}_nE_x}.$$

Example 37.22
Show that

$$s_{x:\overline{n}|} = \sum_{k=1}^{n} \frac{1}{{}_{n-k}E_{x+k}}.$$

Solution.
We have

$$\begin{aligned}s_{x:\overline{n}|} =&\frac{\sum_{k=1}^n \nu^k{}_kp_x}{\nu^n{}_np_x} = \frac{\sum_{k=1}^n {}_kp_x}{\nu^{n-k}{}_kp_x{}_{n-k}p_{x+1}}\\ =&\sum_{k=1}^n \frac{1}{\nu^{n-k}{}_{n-k}p_{x+k}} = \sum_{k=1}^n \frac{1}{{}_{n-k}E_{x+k}} \blacksquare\end{aligned}$$

Certain-and-Life Annuity Immediate

A **discrete** $n-$**year certain and life annuity** of a life aged x makes unit payments at the end of the year for n years, and if the annuitant survives more than n years, makes contingent payments until his death. Under this annuity, the payments up to year n are guaranteed. If the annuitant dies before the completion of n years payments would be made to his/her estate until n years were up.

The present value random variable of the benefits is defined by

$$Y_{\overline{x:\overline{n}|}} = \begin{cases} a_{\overline{n}|}, & K \leq n \\ a_{\overline{K}|}, & K > n. \end{cases}$$

The actuarial present value of this annuity is denoted by $a_{\overline{x:\overline{n}|}}$. The reader can easily prove the following results:

$$\begin{aligned} Y_{\overline{x:\overline{n}|}} =& a_{\overline{n}|} + {}_{n|}Y_x = \ddot{Y}_{\overline{x:\overline{n}|}} - 1 \\ a_{\overline{x:\overline{n}|}} =& \ddot{a}_{\overline{x:\overline{n}|}} - 1 = a_{\overline{n}|\,n}q_x + \sum_{k=n}^{\infty} a_{\overline{k}|\,k}p_x q_{x+k} \\ =& a_{\overline{n}|} + {}_{n|}a_x \ = a_{\overline{n}|} + \sum_{k=n+1}^{\infty} \nu^k {}_kp_x \\ \text{Var}(Y_{\overline{x:\overline{n}|}}) =& \text{Var}({}_{n|}Y_x). \end{aligned}$$

Practice Problems

Problem 37.27
Find a formula for a_x under a constant force of mortality.

Problem 37.28
Show that $a_x = \nu p_x(1 + a_{x+1})$.

Problem 37.29
Suppose that $p_{x+k} = 0.95$, for each integer $k \geq 0$, and $i = 0.075$. Find a_x.

Problem 37.30
Suppose that $p_{x+k} = 0.97$, for each integer $k \geq 0$, and $\nu = 0.91$. Find ${}_{30}a_x$.

Problem 37.31
Find $a_{x:\overline{n}|}$ under a constant force of mortality μ and a constant force of interest δ.

Problem 37.32 ‡
You are given:
(i) $A_x = 0.28$
(ii) $A_{x+20} = 0.40$
(iii) $A_{x:\overset{1}{\overline{20}|}} = {}_{20}E_x = 0.25$
(iv) $i = 0.05$
Calculate $a_{x:\overline{20}|}$.

38 Life Annuities with m^{thly} Payments

In practice, life annuities are often payable on a monthly, quarterly, or semi-annual basis. In this section, we will consider the case life annuities paid m times a year. An m^{thly} **life annuity-due** makes a payment of $\frac{1}{m}$ at the beginning of every m^{thly} period so that in one year the total payment is 1. Here, $m = 12$ for monthly, $m = 4$ for quarterly, and $m = 2$ for semi-annual. We remind the reader of the following concerning a period of length $\frac{1}{m}$:

- The interest factor is given by $(1+i)^{\frac{1}{m}} = 1 + \frac{i^{(m)}}{m}$.
- The effective interest rate is $(1+i)^{\frac{1}{m}} - 1 = \frac{i^{(m)}}{m}$.
- The discount factor is $(1+i)^{-\frac{1}{m}} = (1-d)^{\frac{1}{m}} = 1 - \frac{d^{(m)}}{m}$.
- The effective discount rate is $1 - \nu^{\frac{1}{m}} = \frac{d^{(m)}}{m}$.

Consider a whole life annuity-due with payments made at the beginning $m-$thly time interval while an individual is alive. Let J be the number of complete $m-$ths of a year lived in the year of death. Then $J = \lfloor (T-K)m \rfloor$. See Figure 38.1. For example, for quarterly payments with $T(x) = 36.82$, we have $J = \lfloor (36.82 - 36)(4) \rfloor = 3$.

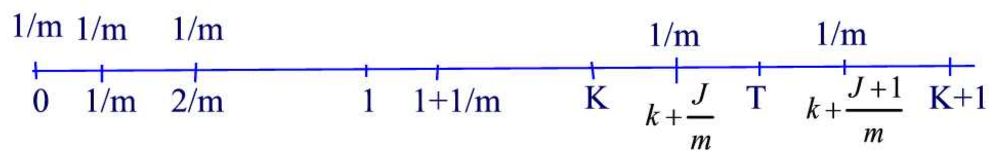


Figure 38.1

Let $\ddot{Y}_x^{(m)}$ denote the present value of a whole life annuity-due for (x) with unit annual payment paid m times a year. Then we can write

$$\ddot{Y}_x^{(m)} = \sum_{j=0}^{mK+J} \frac{1}{m}\nu^{\frac{j}{m}} = \frac{1}{m}\left(\frac{1-\nu^{K+\frac{J+1}{m}}}{1-\nu^{\frac{1}{m}}}\right)$$

$$= \frac{1}{m}\left(\frac{1-\nu^{K+\frac{J+1}{m}}}{\frac{d^{(m)}}{m}}\right) = \frac{1-\nu^{K+\frac{J+1}{m}}}{d^{(m)}} = \ddot{a}^{(m)}_{\overline{K+\frac{J+1}{m}}|}.$$

The actuarial present value of a whole life due annuity for (x) with unit annual payment paid m times a year (with each payment of $\frac{1}{m}$) is denoted by $\ddot{a}_x^{(m)}$. Using the current payment technique formula we can write

$$\ddot{a}_x^{(m)} = \sum_{n=0}^{\infty} \frac{1}{m} \nu^{\frac{n}{m}} {}_{\frac{n}{m}}p_x.$$

Note that, if $Z_x^{(m)}$ denote the present value of a whole life insurance that is paid at the end of the m^{thly} time interval in which death occurs then

$$\ddot{Y}_x^{(m)} = \frac{1 - Z_x^{(m)}}{d^{(m)}}.$$

Taking expectation of both sides we obtain the formula

$$\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}}.$$

Also, it is easy to see that

$$\text{Var}(\ddot{Y}_x^{(m)}) = \frac{{}^2A_x^{(m)} - (A_x^{(m)})^2}{(d^{(m)})^2}.$$

Now, if deaths are assumed to have uniform distribution in each year of age then from Section 30 we can write

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x.$$

Example 38.1
Under UDD assumption, find $\ddot{a}_{60}^{(12)}$ using the Illustrative Life Table.

Solution.
From the Illustrative Life Table we have $i = 0.06$ and $A_{60} = 0.36913$. Now,

$$\frac{i}{i^{(12)}} = \frac{0.06}{12[(1.06)^{\frac{1}{12}} - 1]} = 1.02721$$

and

$$d^{(12)} = m[1 - \nu^{\frac{1}{12}}] = 12[1 - (1.06)^{-\frac{1}{12}}] = 0.05813.$$

Hence,

$$\ddot{a}_{60}^{(12)} = \frac{1 - (1.02721)(0.36913)}{0.05813} = 10.68 \blacksquare$$

Example 38.2
Show that, under UDD assumption, we have

$$\ddot{a}_x^{(m)} = \alpha(m)\ddot{a}_x - \beta(m)$$

for some functions $\alpha(m)$ and $\beta(m)$ to be determined. The values of these functions are given in the Illustrative Life Tables.

Solution.
Recall from Section 37.1, the formula

$$A_x = 1 - d\ddot{a}_x.$$

Thus,

$$\begin{aligned}\ddot{a}_x^{(m)} =&\frac{1 - \frac{i}{i^{(m)}}(1 - d\ddot{a}_x)}{d^{(m)}}\\ =&\frac{i^{(m)} - i + id\ddot{a}_x}{i^{(m)}d^{(m)}}\\ =&\frac{i}{i^{(m)}}\frac{d}{d^{(m)}}\ddot{a}_x - \frac{i - i^{(m)}}{i^{(m)}d^{(m)}}\\ =&\alpha(m)\ddot{a}_x - \beta(m)\end{aligned}$$

where

$$\alpha(m) = \tfrac{i}{i^{(m)}}\tfrac{d}{d^{(m)}} \text{ and } \beta(m) = \tfrac{i-i^{(m)}}{i^{(m)}d^{(m)}} \ \blacksquare$$

We next try to find an approximation of $\ddot{a}_x^{(m)}$. The approximation is based on the fact that ${}_{k+\frac{j}{m}}E_x$ is a linear function j where $j = 0, 1, \cdots, m-1$. Thus, we assume that

$${}_{k+\frac{j}{m}}E_x \approx {}_kE_x + \frac{j}{m}({}_{k+1}E_x - {}_kE_x).$$

With this assumption, we have

$$\begin{aligned}
\ddot{a}_x^{(m)} =& \frac{1}{m}\sum_{k=0}^{\infty}\nu^{\frac{k}{m}}{}_{\frac{k}{m}}p_x\\
=& \frac{1}{m}\sum_{k=0}^{\infty}\sum_{j=0}^{m-1}\nu^{k+\frac{j}{m}}{}_{k+\frac{j}{m}}p_x\\
\approx& \frac{1}{m}\sum_{k=0}^{\infty}\sum_{j=0}^{m-1}\left({}_kE_x+\frac{j}{m}({}_{k+1}E_x-{}_kE_x)\right)\\
=& \frac{1}{m}\sum_{k=0}^{\infty}\left(m{}_kE_x+\frac{m-1}{2}({}_{k+1}E_x-{}_kE_x)\right)\\
=& \frac{m+1}{2m}\sum_{k=0}^{\infty}{}_kE_x+\frac{m-1}{2m}\sum_{k=0}^{\infty}{}_{k+1}E_x\\
=& \frac{m+1}{2m}\sum_{k=0}^{\infty}{}_kE_x+\frac{m-1}{2m}\sum_{k=1}^{\infty}{}_kE_x\\
=& \frac{m+1}{2m}\sum_{k=0}^{\infty}{}_kE_x+\frac{m-1}{2m}\sum_{k=0}^{\infty}{}_kE_x-\frac{m-1}{2m}\\
=& \sum_{k=0}^{\infty}{}_kE_x-\frac{m-1}{2m}\\
=& \ddot{a}_x-\frac{m-1}{2m}.
\end{aligned}$$

Example 38.3
Use the above approximation to estimate $\ddot{a}_{60}^{(12)}$ from Example 38.1.

Solution.
From the Illustrative Life Table, we have $\ddot{a}_{60} = 11.1454$. Thus,

$$\ddot{a}_{60}^{(12)} \approx 11.1454 - \frac{11}{24} = 10.6871 \blacksquare$$

Now, if the payments are made at the end of each $\left(\frac{1}{m}\right)^{\text{th}}$ of a year then the actuarial present value of this annuity is

$$a_x^{(m)} = \frac{1}{m}\sum_{k=1}^{\infty}\nu^{\frac{k}{m}}{}_{\frac{k}{m}}p_x = \ddot{a}_x^{(m)} - \frac{1}{m}.$$

Now, for an $n-$year temporary annuity-due we have

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1}{m}\sum_{k=0}^{mn-1}\nu^{\frac{k}{m}}\,{}_{\frac{k}{m}}p_x$$

and for the annuity-immediate case we have

$$a^{(m)}_{x:\overline{n}|} = \frac{1}{m}\sum_{k=1}^{mn}\nu^{\frac{k}{m}}\,{}_{\frac{k}{m}}p_x.$$

For an $n-$year deferred whole life annuity-due we have

$${}_{n|}\ddot{a}^{(m)}_x = \frac{1}{m}\sum_{k=mn}^{\infty}\nu^{\frac{k}{m}}\,{}_{\frac{k}{m}}p_x$$

and for the annuity-immediate case we have

$${}_{n|}a^{(m)}_x = \frac{1}{m}\sum_{k=mn+1}^{\infty}\nu^{\frac{k}{m}}\,{}_{\frac{k}{m}}p_x.$$

An analogous set of identities to those developed in the annual payment case exist in the m^{th} payment case as well, namley

$$\begin{aligned}
{}_{n|}a^{(m)}_x =& {}_nE_x a^{(m)}_{x+n}\\
{}_{n|}\ddot{a}^{(m)}_x =& {}_nE_x \ddot{a}^{(m)}_{x+n}\\
a^{(m)}_x =& a^{(m)}_{x:\overline{n}|} + {}_{n|}a^{(m)}_x\\
\ddot{a}^{(m)}_x =& \ddot{a}^{(m)}_{x:\overline{n}|} + {}_{n|}\ddot{a}^{(m)}_x
\end{aligned}$$

Example 38.4
(a) Show that ${}_{n|}\ddot{a}^{(m)}_x = {}_{n|}a^{(m)}_x + \frac{1}{m}\cdot {}_nE_x$.
(b) Show that under UDD assumption we have: ${}_{n|}a^{(m)}_x \approx \frac{i}{i^{(m)}}\frac{d}{d^{(m)}}{}_{n|}\ddot{a}_x - \frac{i-i^{(m)}}{i^{(m)}d^{(m)}}{}_nE_x$.
(c) Show that under UDD assumption we have: ${}_{n|}a^{(m)}_x \approx \frac{i}{i^{(m)}}\frac{d}{d^{(m)}}{}_{n|}a_x + \frac{d^{(m)}-d}{i^{(m)}d^{(m)}}{}_nE_x$.
(d) Show that ${}_{n|}a^{(m)}_x \approx {}_{n|}\ddot{a}_x - \frac{m-1}{2m}{}_nE_x$.

Solution.
(a) We have

$$\begin{aligned}{}_{n|}\ddot{a}_x^{(m)} =& \frac{1}{m}\sum_{k=mn}^{\infty}\nu^{\frac{k}{m}}{}_{\frac{k}{m}}p_x\\ =&{}_{n|}a_x^{(m)} + \frac{1}{m}\nu^n{}_np_x = {}_{n|}a_x^{(m)} + \frac{1}{m}\cdot{}_nE_x\end{aligned}$$

(b) As in the case of a m^{thly} whole life insurance, we have

$${}_{n|}a_x^{(m)} = \frac{1-{}_{n|}A_x^{(m)}}{d^{(m)}}.$$

But with the UDD assumption, we have

$${}_{n|}A_x^{(m)} \approx \frac{i}{i^{(m)}}{}_{n|}A_x.$$

Now, recall from Problem 37.15

$${}_{n|}A_x = {}_nE_x - d{}_{n|}\ddot{a}_x.$$

Thus,

$$\begin{aligned}{}_{n|}a_x^{(m)} =& \frac{1-{}_{n|}A_x^{(m)}}{d^{(m)}}\\ \approx& \frac{1-\frac{i}{i^{(m)}}{}_{n|}A_x}{d^{(m)}}\\ =& \frac{1-\frac{i}{i^{(m)}}({}_nE_x - d{}_{n|}\ddot{a}_x)}{d^{(m)}}\\ =& \frac{i}{i^{(m)}}\frac{d}{d^{(m)}}{}_{n|}\ddot{a}_x - \frac{i-i^{(m)}}{i^{(m)}d^{(m)}}{}_nE_x.\end{aligned}$$

(c) We have

$$\begin{aligned}{}_{n|}a_x^{(m)} =& {}_{n|}\ddot{a}_x^{(m)} - \frac{1}{m}{}_nE_x\\ \approx& \frac{i}{i^{(m)}}\frac{d}{d^{(m)}}{}_{n|}\ddot{a}_x - \frac{i-i^{(m)}}{i^{(m)}d^{(m)}}{}_nE_x - \frac{1}{m}{}_nE_x\\ =& \frac{i}{i^{(m)}}\frac{d}{d^{(m)}}({}_{n|}a_x + {}_nE_x) - \frac{i-i^{(m)}}{i^{(m)}d^{(m)}}{}_nE_x - \frac{1}{m}{}_nE_x\\ =& \frac{i}{i^{(m)}}\frac{d}{d^{(m)}}{}_{n|}a_x + \frac{d^{(m)}-d}{i^{(m)}d^{(m)}}{}_nE_x.\end{aligned}$$

(d) From the approximation $\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m}$ we can write that

$$\begin{aligned}
{}_{n|}\ddot{a}_x^{(m)} =& {}_nE_x\ddot{a}_{x+n}^{(m)} \\
\approx& {}_nE_x(\ddot{a}_{x+n} - \frac{m-1}{2m}) \\
=& {}_nE_x\ddot{a}_{x+n} - \frac{m-1}{2m}{}_nE_x \\
=& {}_{n|}\ddot{a}_x - \frac{m-1}{2m}{}_nE_x \blacksquare
\end{aligned}$$

Practice Problems

Problem 38.1
Under UDD assumption, find $\ddot{a}_{50}^{(4)}$ using the Illustrative Life Table.

Problem 38.2
Using the previous problem, find $a_{50}^{(4)}$.

Problem 38.3
You are given the following:
(i) $i = 0.065$
(ii) $\ddot{a}_{80}^{(12)} = 2.5437$
(iii) Uniform distribution of deaths between integral ages.
Find A_{80} and $\bar{a}_{80}$.

Problem 38.4
You are given:
(i) $\ddot{a}_{\infty}^{(4)} = \lim_{n\to\infty} \frac{1-\nu^n}{d^{(4)}} = \frac{1}{d^{(4)}} = 12.287.$
(ii) $A_x = 0.1025.$
(iii) Deaths are uniformly distributed between integral ages.
Calculate $\ddot{a}_x^{(4)}$.

Problem 38.5 ‡
You are given:
• deaths are uniformly distributed between integral ages;
• $i = 0.06$;
• $q_{69} = 0.03$; and
• $\bar{A}_{70} = 0.53$.
Calculate $\ddot{a}_{69}^{(2)}$.

Problem 38.6
Show that $\ddot{a}_{x:\overline{n}|}^{(m)} = a_{x:\overline{n}|}^{(m)} + \frac{1}{m}(1 - {}_nE_x)$.

Problem 38.7
Show that under UDD assumption we have the following approximation:

$$\ddot{a}_{x:\overline{n}|}^{(m)} \approx \frac{i}{i^{(m)}}\frac{d}{d^{(m)}}\ddot{a}_{x:\overline{n}|} - \frac{i - i^{(m)}}{i^{(m)}d^{(m)}}(1 - {}_nE_x).$$

Problem 38.8
Show that under UDD assumption we have the following approximations:
(a) $a_x^{(m)} \approx \frac{i}{i^{(m)}}\frac{d}{d^{(m)}}a_x + \frac{d^{(m)}-d}{i^{(m)}d^{(m)}}$
(b) $a_{x:\overline{n}|}^{(m)} \approx \frac{i}{i^{(m)}}\frac{d}{d^{(m)}}a_{x:\overline{n}|} + \frac{d^{(m)}-d}{i^{(m)}d^{(m)}}(1 - {}_nE_x)$.

Problem 38.9
Using the approximation

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m}$$

derive the following approximations:
(a) $\ddot{a}_{x:\overline{n}|}^{(m)} \approx \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x)$.
(b) $a_x^{(m)} \approx a_x + \frac{m-1}{2m}$.
(c) ${}_{n|}a_x^{(m)} \approx {}_{n|}a_x + \frac{m-1}{2m}{}_nE_x$.
(d) $a_{x:\overline{n}|}^{(m)} \approx a_{x:\overline{n}|} + \frac{m-1}{2m}(1 - {}_nE_x)$.

Problem 38.10
The approximation

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m}$$

is also knwon as the **2-term Woolhouse formula.** A **3-term Woolhouse formula** is given by

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m}(\mu(x)+\delta).$$

Using the 3-term Woolhouse formula, show the following:
(a)

$$\ddot{a}_{x:\overline{n}|}^{(m)} \approx \ddot{a}_{x:\overline{n}|} - \left(\frac{m-1}{m}\right)(1 - {}_nE_x) - \frac{m^2-1}{12m}(\delta + \mu(x) - {}_nE_x(\delta + \mu(x+n))).$$

(b) $\bar{a}_x \approx \ddot{a}_x - \frac{1}{2} - \frac{1}{12}(\delta + \mu(x))$.
(c) $\bar{a}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|} - \frac{1}{2}(1 - {}_nE_x) - \frac{1}{12}(\delta + \mu(x) - {}_nE_x(\delta + \mu(x+n)))$.

39 Non-Level Payments Annuities

Up to this point, we have considered annuities with level payments. In this section, we consider annuities with non-level payments. Our approach for calculating the actuarial present value of such annuities consists of summing over all the payment dates the product with factors the amount of the payment, the probability that the annuitant survives to the payment date, and the appropriate discount factor.

39.1 The Discrete Case

Consider a non-level annuity-due with payment r_k at time k. A time line showing the payment, probability associated with the payment while (x) is alive, and the discount factor is shown in Figure 39.1.

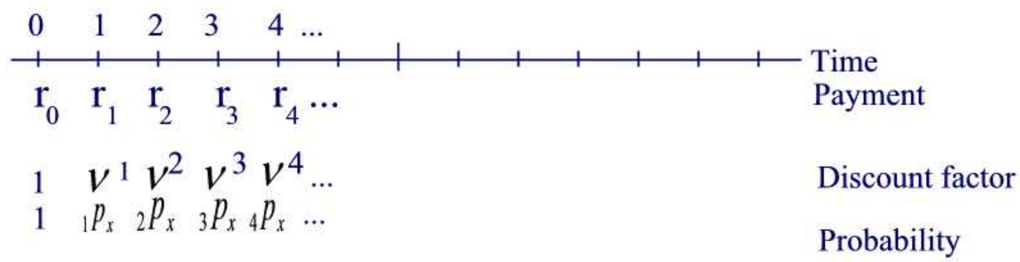


Figure 39.1

From this time diagram, the actuarial present value of this annuity is given by

$$\text{APV} = \sum_{k=0}^{\infty} r_k \nu^k {}_kp_x.$$

As an example, consider an increasing whole life annuity-due with a payment of $r_k = k + 1$ at time k provided that (x) is alive at time k where $k = 0, 1, 2, \cdots$. The actuarial present value of this annuity is given by

$$(I\ddot{a})_x = \sum_{k=0}^{\infty} (k+1)\nu^k {}_kp_x.$$

If the above annuity is restricted to n payments, then the actuarial present value is given by

$$(I\ddot{a})_{x:\overline{n}|}=\sum_{k=0}^{n-1}(k+1)\nu^k{}_kp_x.$$

Example 39.1
Consider a special 3-year annuity due with payments $r_{k-1}=9,000+1,000k$ for $k=1,2,3$. You are given:
(i) Effective interest rate is 6% for the first year and 6.5% for the second year.
(ii) $p_{x+k}=0.98-0.03k$ for $k=0,1$.
Calculate $(I\ddot{a})_{x:\overline{3}|}$.

Solution.
We have

$$(I\ddot{a})_{x:\overline{3}|}=10,000+11,000(1.06)^{-1}(0.98)+12,000(1.065)^{-2}(0.98)(0.95)=30,019.71 \blacksquare$$

Likewise, for a unit arithmetically decreasing $n-$year temporary annuity-due contingent, the actuarial present value is given by

$$(D\ddot{a})_{x:\overline{n}|}=\sum_{k=0}^{n-1}(n-k)\nu^k{}_kp_x.$$

If the annuity-due is replaced by an annutiy-immediate, the actuarial present value is given by

$$(Ia)_x=\sum_{k=1}^{\infty}k\nu^k{}_kp_x$$

for a whole life annuity and

$$(Ia)_{x:\overline{n}|}=\sum_{k=1}^{n}k\nu^k{}_kp_x$$

for an $n-$year temporary annuity.
The unit decreasing $n-$year temporary annuity immediate has the actuarial present value

$$(Da)_{x:\overline{n}|}=\sum_{k=1}^{n}(n+1-k)\nu^k{}_kp_x.$$

Example 39.2
You are given:
(i) $\mu(x+t) = 0.02$
(ii) $\delta = 0.05$.
Find $(Ia)_x$.

Solution.
First, recall from Calculus the following result about infinite series:

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{d}{dx}\left(\sum_{k=0}^{\infty} x^k\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$$

where $|x| < 1$.
Now, we have

$$\begin{aligned}(Ia)_x =& \sum_{k=1}^{\infty} k\nu^k {}_kp_x = \sum_{k=1}^{\infty} ke^{-0.05k}e^{-0.02k}\\ =& \sum_{k=1}^{\infty} ke^{-0.07k} = e^{-0.07}\sum_{k=1}^{\infty} ke^{-0.07(k-1)}\\ =& \frac{e^{-0.07}}{(1-e^{-0.07})^2} \blacksquare\end{aligned}$$

Practice Problems

Problem 39.1
You are given:
(i) $\mu(x+t) = 0.02$
(ii) $\delta = 0.05$.
Find $(I\ddot{a})_x$.

Problem 39.2
Using Life Illustrative Table, calculate the value of an arithmetically increasing term annuity payable in advance for a term of 4 years issued to an individual aged 64, assuming that $i = 0.06$.

Problem 39.3
You are given:
(i) $\mu(x+t) = 0.02$
(ii) $\delta = 0.05$.
Find $(Ia)_{x:\overline{3}|}$.

Problem 39.4
You are given:
(i) $\mu(x+t) = 0.02$
(ii) $\delta = 0.05$.
Find $(D\ddot{a})_{x:\overline{3}|}$.

Problem 39.5
You are given:
(i) $\mu(x+t) = 0.02$
(ii) $\delta = 0.05$.
Find $(Da)_{x:\overline{3}|}$.

39.2 The Continuous Case

Consider a continuous whole life annuity with continuous rate r_t at time t. Then the present value function of the payment at time t is

$$\int_0^t r_s \nu^s ds.$$

Thus, the actuarial present value of this annuity is

$$\begin{aligned}
\text{APV} =& \int_0^\infty \left(\int_0^t r_s \nu^s ds \right) f_{T(x)}(t) dt \\
=& \int_0^\infty \int_0^t r_s \nu^s ds {}_tp_x \mu(x+t) dt \\
=& \int_0^\infty \int_s^\infty r_s \nu^s {}_tp_x \mu(x+t) dt ds \\
=& \int_0^\infty r_s \nu^s \left(\int_s^\infty {}_tp_x \mu(x+t) dt \right) ds \\
=& -\int_0^\infty r_s \nu^s \left(\int_s^\infty \frac{d}{dt} ({}_tp_x) dt \right) ds \\
=& \int_0^\infty r_s \nu^s {}_sp_x ds.
\end{aligned}$$

For an $n-$year temporary annuity, we have the formula

$$\text{APV} = \int_0^n r_t \nu^t {}_tp_x dt.$$

In particular, if $r_t = t$ we have a continuously increasing continuous whole life annuity with actuarial present value

$$(\bar{I}\bar{a})_x = \int_0^\infty t \nu^t {}_tp_x dt.$$

For a continuously increasing continuous $n$year term life annuity with rate of payments t, the actuarial present value is

$$(\bar{I}\bar{a})_{x:\overline{n}|} = \int_0^n t \nu^t {}_tp_x dt.$$

Finally, for a continuously decreasing continuous $n-$year temporary life annuity with rate of payments $n-t$, the actuarial present value is

$$(\bar{D}\bar{a})_{x:\overline{n}|} = \int_0^n (n-t)\nu^t {}_tp_x dt.$$

Example 39.3
You are given:
(i) $\mu(x+t) = 0.02$
(ii) $\delta = 0.05$.
Find $(\bar{D}\bar{a})_{x:\overline{3}|}$.

Solution.
We have, using integration by parts,

$$\begin{aligned}(\bar{D}\bar{a})_{x:\overline{3}|} =& \int_0^3 (3-t)e^{-0.05t}e^{-0.02t}dt \\ =& \int_0^3 3e^{-0.07t}dt - \frac{1}{(0.07)^2}\int_0^{3(0.07)} te^{-t}dt \\ =& -\frac{3}{0.07}e^{-0.07t}\Big|_0^3 - \frac{1}{(0.07)^2}\left[-(1+t)e^{-t}\right]_0^{3(0.07)} \\ =& 4.2 \blacksquare\end{aligned}$$

Example 39.4
You are given:
(i) Continuous payment rate $r(t) = 5000$ for $t \geq 0$
(ii) $\mu = 0.01$
(iii) The force of interest

$$\delta_t = \begin{cases} 0.08 & 0 \leq t < 10 \\ 0.06 & 10 \leq t < \infty. \end{cases}$$

Calculate the actuarial present value of this annuity.

Solution.
We have

$$\nu^t = e^{-\int_0^t \delta_s ds} = \begin{cases} e^{-0.08t} & 0 \leq t < 10 \\ e^{-0.08(10)}e^{-0.06(t-10)} & 10 \leq t < \infty. \end{cases}$$

We have

$$\begin{aligned}
\mathrm{APV} = & \int_0^\infty r_t \nu^t {}_tp_x dt = \int_0^\infty 50,000 e^{-\delta_t} e^{-0.01t} dt \\
= & \int_0^{10} 50,000 e^{-0.09t} dt + \int_{10}^\infty 50,000 e^{0.2-0.07t} dt \\
= & \frac{50,000(1 - e^{0.09(10)})}{0.09} + \frac{50,000 e^{-0.09}}{0.07} = 620,090.40 \blacksquare
\end{aligned}$$

Example 39.5 ‡
For a special increasing whole life annuity-due on (40), you are given:
(i) Y is the present-value random variable.
(ii) Payments are made once every 30 years, beginning immediately.
(iii) The payment in year 1 is 10, and payments increase by 10 every 30 years.
(iv) Mortality follows DeMoivre's Law, with $\omega = 110$.
(v) $i = 0.04$
Calculate $\mathrm{Var}(Y)$.

Solution.
Let $T(40)$ denote the age-at-death of (40). Then $T(40)$ has uniform distribution on $[0, 70]$. Moreover,

$$\Pr(T(40) \leq t) = \frac{t}{70}.$$

We have

Event	Y	Pr(Event)
$T(40) \leq 30$	10	$\frac{30}{70}$
$30 < T(40) \leq 60$	$10 + 20\nu^{30} = 16.1664$	$\frac{30}{70}$
$T(40) \geq 60$	$10 + 20\nu^{30} + 30\nu^{60} = 19.0182$	$\frac{10}{70}$

Hence,

$$\begin{aligned}
E(Y) = & 10\left(\frac{30}{70}\right) + 16.1664\left(\frac{30}{70}\right) + 19.0182\left(\frac{10}{70}\right) = 13.9311 \\
E(Y^2) = & 10^2\left(\frac{30}{70}\right) + 16.1664^2\left(\frac{30}{70}\right) + 19.0182^2\left(\frac{10}{70}\right) = 206.536 \\
\mathrm{Var}(Y) = & E(Y^2) - [E(Y)]^2 = 206.536 - 13.9311^2 = 12.46 \blacksquare
\end{aligned}$$

Practice Problems

Problem 39.6
You are given:
(i) $\mu(x+t) = 0.02$
(ii) $\delta = 0.05$.
Find $(\bar{I}\bar{a})_x$.

Problem 39.7
You are given:
(i) $\mu(x+t) = 0.02$
(ii) $\delta = 0.05$.
Find $(\bar{I}\bar{a})_{x:\overline{3}|}$.

Problem 39.8
Let $(I\bar{a})_x$ be the actuarial present value of an annually increasing continuous whole life annuity with rate of payments $r_t = \lceil t \rceil$. Find a formula for $(I\bar{a})_x$.

Problem 39.9
Let $(I\bar{a})_{x:\overline{n}|}$ be the actuarial present value of an annually increasing continuous $n-$year temporary life annuity with rate of payments $r_t = \lceil t \rceil$. Find a formula for $(I\bar{a})_{x:\overline{n}|}$.

Problem 39.10
Let $(D\bar{a})_{x:\overline{n}|}$ be the actuarial present value of an annually decreasing continuous $n-$year temporary life annuity with rate of payments $r_t = \lceil n-t \rceil$. Find a formula for $(D\bar{a})_{x:\overline{n}|}$.

Calculating Benefit Premiums

Recall that a life insurance involves an insurance company (the issuer) and an insured. When an insurance policy is issued the insurance company assumes its liability. That is, the insurance company will have to make payments to the insured in the future. In this chapter, we discuss the question of how much premium should an insurance company charges the insured for the policy.

If the insured wants a policy with a single payment at the time the policy begins, then the insurer should charge the actuarial present value (also known as the **net single premium**) of the policy at the time of issue.

Usually, the insurance company charges the insured with periodical payments for the policy. There is a loss if the actuarial present value at issue of the contingent benefits is larger than the actuarial present value of the periodic charges. That is, the loss of an insurance contract is the actuarial present value at issue of the net cashflow for this contract. Hence, the loss is a random variable.

The insurance company determines the premiums based on the **equivalence principle** which says that the expected loss of an insurance contract must be zero. This, occurs when the actuarial present value of charges to the insured is equal to the actuarial present value of the benefit payments. In this case, we refer to each charge to the insured as the **benefit premium** or **net premium.** [3] These premiums are usually made annually.

[3] In practice, the insurer includes in the premium expenses and profits. Such premiums are called **contract premiums** to be discussed in Section 70.

40 Fully Continuous Premiums

In this section, we will first work with **fully continuous** premiums, where level annual benefit premiums (i.e., of the same amount per year) are paid on a continuous basis.

40.1 Continuous Whole Life Policies

Consider a continuous whole life policy for a life (x) with benefit of 1 and with an annual premium of $\bar{P}$ which is paid with a continuous annuity while the insured is alive. The loss random variable for this policy is given by

$$\bar{L}_x = \nu^T - \bar{P}\bar{a}_{\overline{T}|} = \bar{Z}_x - \bar{P}\bar{Y}_x.$$

That is, $\bar{L}_x$ is just the present value of the benefit minus the present value of the stream of premiums. Now, applying the equivalence principle, i.e., $E(\bar{L}_x) = 0$, we find

$$0 = E(\bar{L}_x) = E(\nu^T - \bar{P}\bar{a}_{\overline{T}|}) = \bar{A}_x - \bar{P}\bar{a}_x \Longrightarrow \bar{P} = \frac{\bar{A}_x}{\bar{a}_x}.$$

The continuous premium $\bar{P}$ will be denoted by $\bar{P}(\bar{A}_x)$. That is,

$$\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x}.$$

Example 40.1
Show that

$$\bar{a}_x = \frac{1}{\bar{P}(\bar{A}_x) + \delta}.$$

Solution.
Using the definition of $\bar{P}(\bar{A}_x)$ and the relation $\bar{A}_x + \delta\bar{a}_x = 1$ we can write

$$\begin{aligned}
\bar{P}(\bar{A}_x) =& \frac{\bar{A}_x}{\bar{a}_x} = \frac{1-\delta\bar{a}_x}{\bar{a}_x}\\
\bar{P}(\bar{A}_x)\bar{a}_x =& 1 - \delta\bar{a}_x\\
\bar{a}_x(\bar{P}(\bar{A}_x) + \delta) =& 1\\
\bar{a}_x =& \frac{1}{\bar{P}(\bar{A}_x) + \delta} \blacksquare
\end{aligned}$$

Example 40.2
Find the annual benefit premium under a constant force of mortality μ and a constant force of interest δ.

Solution.
We have

$$\begin{aligned}\bar{A}_x =&\frac{\mu}{\mu+\delta}\\ \bar{a}_x =&\frac{1}{\mu+\delta}\\ \bar{P}(\bar{A}_x) =&\frac{\bar{A}_x}{\bar{a}_x} = \mu \blacksquare\end{aligned}$$

The variance of the loss Var(L) under this kind of benefit premium can be found as follows:

$$\begin{aligned}\text{Var}(\bar{L}_x) =&\text{Var}(\nu^T - \bar{P}\bar{a}_{\overline{T}|}) = \text{Var}\left(\nu^T - \bar{P}\frac{1-\nu^T}{\delta}\right)\\ =&\text{Var}\left[\nu^T\left(1+\frac{\bar{P}}{\delta}\right) - \frac{\bar{P}}{\delta}\right]\\ =&\left(1+\frac{\bar{P}}{\delta}\right)^2 \text{Var}(\nu^T)\\ =&[{}^2\bar{A}_x - (\bar{A}_x)^2]\left(1+\frac{\bar{P}}{\delta}\right)^2.\end{aligned}$$

Example 40.3
Show that

$$\text{Var}(\bar{L}_x) = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(\delta\bar{a}_x)^2}.$$

Solution.
We will use the relations $\delta\bar{a}_x + 1 = \bar{A}_x$ and $\bar{P} = \frac{\bar{A}_x}{\bar{a}_x}$. In this case, we have

$$\begin{aligned}\left(1+\frac{\bar{P}}{\delta}\right)^2 =&\left(1+\frac{1-\delta\bar{a}_x}{\bar{a}_x\delta}\right)^2\\ =&\left(1+\frac{1}{\bar{a}_x\delta}-1\right)^2 = \frac{1}{(\delta\bar{a}_x)^2}.\end{aligned}$$

Thus,

$$\mathrm{Var}(\bar{L}_x) == [{}^2\bar{A}_x - (\bar{A}_x)^2]\left(1+\frac{\bar{P}}{\delta}\right)^2 = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(\delta\bar{a}_x)^2} \blacksquare$$

Example 40.4
You are given the following information regarding the type of policy discussed in this section:
(i) $\mu = 0.02$
(ii) $\delta = 0.06$.
Find $\bar{P}(\bar{A}_x)$ and $\mathrm{Var}(\bar{L}_x)$.

Solution.
We have

$$\begin{aligned}
\bar{P}(\bar{A}_x) =&\mu = 0.02\\
\bar{A}_x =&\frac{\mu}{\mu+\delta} = \frac{0.02}{0.02+0.06} = \frac{1}{4}\\
{}^2\bar{A}_x =&\frac{\mu}{\mu+2\delta} = \frac{0.02}{0.02+0.12} = \frac{1}{7}\\
\bar{a}_x =&\frac{1}{\mu+\delta} = \frac{1}{0.02+0.06} = \frac{25}{2}\\
\mathrm{Var}(\bar{L}_x) =&\left(\frac{1}{7} - \left(\frac{1}{4}\right)^2\right)\frac{1}{[(0.06)(12.5)]^2} = \frac{1}{7} \blacksquare
\end{aligned}$$

Example 40.5
You are given:
(i) Mortality follows De Moivre's Law with $\omega = 100$.
(ii) $\delta = 0.03$.
(iii) $\bar{L}_x$ is the loss-at-issue random variable for a fully continuous whole life insurance on (60) with premiums determined based on the equivalence principle.
Calculate $\mathrm{Var}(\bar{L}_x)$.

Solution.
We have

$$\begin{aligned}\bar{A}_{60} =& \int_0^{40} \frac{e^{-0.03t}}{40} dt = 0.58234\\ {}^2A_{60} =& \int_0^{40} \frac{e^{-0.06t}}{40} dt = 0.37887\\ \bar{a}_{60} =& \frac{1-\bar{A}_{60}}{\delta} = 13.922\\ \text{Var}(\bar{L}_x) =& \frac{0.37887 - 0.58234^2}{(0.03\times 13.922)^2} = 0.22787 \blacksquare\end{aligned}$$

Now, suppose that the premiums of the above policy are paid at an annual rate of $\bar{P}$ with a $t-$year temporary annuity. In other words, payments are made for t years in exchange for whole life coverage. In this case, the loss random variable is

$${}_t\bar{L}_x = \nu^T - \bar{P}\bar{a}_{\overline{\min(T,t)}|} = \bar{Z}_x - \bar{P}\bar{Y}_{x:\overline{t}|}.$$

The benefit premium which satisfies the equivalence principle is

$${}_t\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_{x:\overline{t}|}}.$$

Example 40.6
You are given $\mu = 0.02$ and $\delta = 0.05$. Find ${}_{10}\bar{P}(\bar{A}_{75})$.

Solution.
We have

$$\begin{aligned}{}_{10}\bar{P}(\bar{A}_{75}) =& \frac{\bar{A}_{75}}{\bar{a}_{75:\overline{10}|}}\\ =& \frac{\delta\bar{A}_{75}}{1-\bar{A}^{\,1}_{75:\overline{10}|} - {}_{10}E_{75}}\\ =& \frac{\frac{\delta\mu}{\mu+\delta}}{1-\frac{\mu}{\mu+\delta}\left(1-e^{-10(\mu+\delta)}\right) - e^{-10(\mu+\delta)}}\\ =& \frac{\frac{0.05(0.02)}{0.02+0.05}}{1-\frac{0.02}{0.02+0.05}\left(1-e^{-10(0.02+0.05)}\right) - e^{-10(0.02+0.05)}}\\ =& 0.03973 \blacksquare\end{aligned}$$

In the preceding examples the insurance's benefits and the payments annuity are level with amounts of 1. In the next example, we consider a situation where the benefits are variable.

Example 40.7 ‡
For a special fully continuous whole life insurance on (x) :
(i) The level premium is determined using the equivalence principle.
(ii) Death benefits are given $b_t = (1+i)^t$ where i is the interest rate.
(iii) L is the loss random variable at $t = 0$ for the insurance.
(iv) T is the future lifetime random variable of (x).
Which of the following expressions is equal to L?

$$\begin{array}{ll}(A) & \dfrac{\nu^T - \bar{A}_x}{1-\bar{A}_x} \\ (B) & (\nu^T - \bar{A}_x)(1+\bar{A}_x) \\ (C) & \dfrac{\nu^T - \bar{A}_x}{1+\bar{A}_x} \\ (D) & (\nu^T - \bar{A}_x)(1-\bar{A}_x) \\ (E) & \dfrac{\nu^T + \bar{A}_x}{1+\bar{A}_x}\end{array}$$

Solution.
The loss random variable of this type of policy is

$$\bar{L}'_x = b_T\nu^T - \bar{P}\bar{a}_{\overline{T|}} = 1 - \bar{P}\bar{a}_{\overline{T|}}.$$

Under the equivalence principle, we have

$$0 = E(\bar{L}'_x) = 1 - \bar{P}\bar{a}_x \Longrightarrow \bar{P} = \frac{1}{\bar{a}_x}.$$

Thus,

$$\begin{aligned}\bar{L}'_x =& 1 - \frac{\bar{a}_{\overline{T|}}}{\bar{a}_{\overline{x|}}} = 1 - \frac{1}{\frac{1-\bar{A}_x}{\delta}} \cdot \frac{1-\nu^T}{\delta} \\ =& 1 - \frac{\delta}{1-\bar{A}_x} \cdot \frac{1-\nu^T}{\delta} = 1 - \frac{1-\nu^T}{1-\bar{A}_x} \\ =& \frac{\nu^T - \bar{A}_x}{1-\bar{A}_x}.\end{aligned}$$

So the answer is (A) ■

Practice Problems

Problem 40.1 ‡
For a fully continuous whole life insurance of 1:
(i) $\mu = 0.04$
(ii) $\delta = 0.08$
(iii) L is the loss-at-issue random variable based on the benefit premium.
Calculate $\text{Var}(L)$.

Problem 40.2
You are given that Mortality follows De Moivre's Law with $\omega = 125$ and $\delta = 0.05$. Calculate: $\bar{P}(\bar{A}_{75})$ and $\text{Var}(\bar{L}_x)$, where $\bar{L}_x$ is the loss-at-issue random variable based on the benefit premium.

Problem 40.3 ‡
For a fully continuous whole life insurance of 1 on (x), you are given:
(i) The forces of mortality and interest are constant
(ii) ${}^2\bar{A}_x = 0.20$
(iii) $\bar{P}(\bar{A}_x) = 0.03$
(iv) L is the loss-at-issue random variable based on the benefit premium.
Calculate $\text{Var}(L)$.

Problem 40.4 ‡
For a fully continuous whole life insurance of 1 on (x) :
(i) π is the benefit premium.
(ii) L is the loss-at-issue random variable with the premium equal to π.
(iii) L^* is the loss-at-issue random variable with the premium equal to 1.25π.
(iv) $\bar{a}_x = 5.0$.
(v) $\delta = 0.08$
(vi) $\text{Var}(L) = 0.5625$.
Calculate the sum of the expected value and the standard deviation of L^*.

Problem 40.5 ‡
For a fully continuous whole life insurance of 1 on (x) :
(i) $\bar{A}_x = \frac{1}{3}$
(ii) $\delta = 0.10$
(iii) L is the loss at issue random variable using the premium based on the equivalence principle.
(iv) $\text{Var}(L) = \frac{1}{5}$.

(v) L' is the loss at issue random variable using the premium π.
(vi) $\mathrm{Var}(L') = \frac{16}{45}$.
Calculate π.

Problem 40.6
You are given that Mortality follows De Moivre's Law with $\omega = 125$ and $\delta = 0.05$. Calculate ${}_{10}\bar{P}(\bar{A}_{75})$.

40.2 $n-$year Term Policies

Consider an $n-$year term life policy for a life (x) with benefit of 1 and with an annual premium of $\bar{P}$ which is paid with a continuous $n-$year temporary annuity while the insured is alive. The loss random variable for this policy is given by

$$\bar{L}^1_{x:\overline{n}|} = \nu^T \mathbf{I}(T \leq n) - \bar{P}\bar{a}_{\overline{\min(T,n)}|} = \bar{Z}^1_{x:\overline{n}|} - \bar{P}\bar{Y}_{x:\overline{n}|}.$$

Applying the equivalence principle, i.e., $E(\bar{L}^1_{x:\overline{n}|}) = 0$, we find

$$0 = E(\bar{L}^1_{x:\overline{n}|}) = E(\bar{Z}^1_{x:\overline{n}|} - \bar{P}\bar{Y}_{x:\overline{n}|}) = \bar{A}^1_{x:\overline{n}|} - \bar{P}\bar{a}_{x:\overline{n}|} \Longrightarrow \bar{P} = \frac{\bar{A}^1_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}.$$

The continuous premium $\bar{P}$ will be denoted by $\bar{P}(\bar{A}^1_{x:\overline{n}|})$.
The variance of the loss $\text{Var}(\bar{L}^1_{x:\overline{n}|})$ under this kind of benefit premium can be found as follows:

$$\begin{aligned}
\text{Var}(\bar{L}^1_{x:\overline{n}|}) =& \text{Var}(\bar{Z}^1_{x:\overline{n}|} - \bar{P}\bar{Y}_{x:\overline{n}|}) \\
=& \text{Var}\left(\bar{Z}^1_{x:\overline{n}|} - \bar{P}\frac{1-\bar{Z}^1_{x:\overline{n}|}}{\delta}\right) \\
=& \text{Var}\left[\left(1+\frac{\bar{P}}{\delta}\right)\bar{Z}^1_{x:\overline{n}|} - \frac{\bar{P}}{\delta}\right] \\
=& \left(1+\frac{\bar{P}}{\delta}\right)^2 \text{Var}(\bar{Z}^1_{x:\overline{n}|}) \\
=& [{}^2\bar{A}^1_{x:\overline{n}|} - (\bar{A}^1_{x:\overline{n}|})^2]\left(1+\frac{\bar{P}}{\delta}\right)^2 \\
=& \frac{{}^2\bar{A}^1_{x:\overline{n}|} - (\bar{A}^1_{x:\overline{n}|})^2}{(\delta\bar{a}_{x:\overline{n}|})^2}
\end{aligned}$$

where in the last line we used the relation $\delta\bar{a}_{x:\overline{n}|} + \bar{A}^1_{x:\overline{n}|} = 1$.

Example 40.8
You are given:
(i) Mortality is exponential with parameter μ.
(ii) Constant force of interest δ.
(iii) $\bar{L}^1_{x:\overline{n}|}$ is the loss-at-issue random variable for a fully continuous $n-$year term life insurance on (x) with premiums determined based on the equivalence principle.
Calculate $\bar{P}(\bar{A}^1_{x:\overline{n}|})$ and $\text{Var}(\bar{L}^1_{x:\overline{n}|})$.

Solution.
We have

$$\begin{aligned}
\bar{A}^1_{x:\overline{n}|} =&\frac{\mu}{\mu+\delta}(1-e^{-n(\mu+\delta)})\\
{}^2\bar{A}^1_{x:\overline{n}|} =&\frac{\mu}{\mu+2\delta}(1-e^{-n(\mu+2\delta)})\\
\bar{P}(\bar{A}^1_{x:\overline{n}|}) =&\mu\\
\text{Var}(\bar{L}^1_{x:\overline{n}|}) =& \left(1+\frac{\mu}{\delta}\right)^2\left[\frac{\mu}{\mu+2\delta}(1-e^{-n(\mu+2\delta)})-\left(\frac{\mu}{\mu+\delta}\right)^2(1-e^{-n(\mu+\delta)})^2\right] \blacksquare
\end{aligned}$$

Now, if the premiums are paid with a continuous $t-$year temporary annuity (where $t \leq n$)then the loss random variable is given by

$${}_t\bar{L}^1_{x:\overline{n}|} = \nu^T\mathbf{I}(T\leq n) - \bar{P}\bar{a}_{\overline{\min(T,t)}|} = \bar{Z}^1_{x:\overline{n}|} - \bar{P}\bar{Y}_{x:\overline{t}|}.$$

The benefit premium for this insurance is given by

$${}_t\bar{P}(\bar{A}^1_{x:\overline{n}|}) = \frac{\bar{A}^1_{x:\overline{n}|}}{\bar{a}_{x:\overline{t}|}}.$$

Example 40.9
You are given that $\mu = 0.02$ and $\delta = 0.05$. Calculate ${}_{10}\bar{P}(\bar{A}^1_{75:\overline{20}|})$.

Solution.
We have

$$\begin{aligned}
\bar{A}^1_{75:\overline{20}|} =&\frac{\mu}{\mu+\delta}(1-e^{-n(\mu+\delta)}) = 0.215258\\
\bar{A}^1_{75:\overline{10}|} =&\frac{\mu}{\mu+\delta}(1-e^{-n(\mu+\delta)}) = 0.143833\\
{}_{10}\bar{P}(\bar{A}^1_{75:\overline{20}|}) =&\frac{\delta\bar{A}^1_{75:\overline{20}|}}{1-\bar{A}^1_{75:\overline{10}|}} = 0.012571 \blacksquare
\end{aligned}$$

Practice Problems

Problem 40.7
You are given:
(i) Mortality is exponential with parameter $\mu = 0.02$.
(ii) Constant force of interest $\delta = 0.05$.
(iii) $\bar{L}^{1}_{75:\overline{20|}}$ is the loss-at-issue random variable for a fully continuous 20−year term life insurance on (75) with premiums determined based on the equivalence principle.
Calculate $\bar{P}(\bar{A}^{1}_{75:\overline{20|}})$ and $\mathrm{Var}(\bar{L}^{1}_{75:\overline{20|}})$.

Problem 40.8
You are given:
(i) Mortality follows De Moivre's Law with parameter ω.
(ii) Constant force of interest δ.
(iii) $\bar{L}^{1}_{x:\overline{n|}}$ is the loss-at-issue random variable for a fully continuous $n-$year term life insurance on (x) with premiums determined based on the equivalence principle.
Calculate $\bar{P}(\bar{A}^{1}_{x:\overline{n|}})$ and $\mathrm{Var}(\bar{L}^{1}_{x:\overline{n|}})$.

Problem 40.9
You are given:
(i) Mortality follows De Moivre's Law with parameter $\omega = 125$.
(ii) Constant force of interest $\delta = 0.05$.
(iii) $\bar{L}^{1}_{75:\overline{20|}}$ is the loss-at-issue random variable for a fully continuous 20−year term life insurance on (75) with premiums determined based on the equivalence principle.
Calculate $\bar{P}(\bar{A}^{1}_{75:\overline{20|}})$ and $\mathrm{Var}(\bar{L}^{1}_{75:\overline{20|}})$.

Problem 40.10
You are given the following information:
(i) $\delta = 0.05$
(ii) ${}_{20}E_{75} = 0.2207$
(iii) $\bar{P}(\bar{A}^{1}_{75:\overline{20|}}) = 0.02402$
(iv) $\bar{L}^{1}_{75:\overline{20|}}$ is the loss-at-issue random variable for a fully continuous 20−year term life insurance on (75) with premiums determined based on the equivalence principle.
Calculate $\bar{A}^{1}_{75:\overline{20|}}$.

Problem 40.11
You are given the following information:
(i) $\delta = 0.05$
(ii) ${}_{20}E_{75} = 0.2207$
(iii) $\bar{P}(\bar{A}^{\,1}_{75:\overline{20|}}) = 0.02402$
(iv) $\bar{L}^{\,1}_{75:\overline{20|}}$ is the loss-at-issue random variable for a fully continuous 20−year term life insurance on (75) with premiums determined based on the equivalence principle.
Calculate $\bar{A}_{75:\overline{20|}}$.

40.3 Continuous $n-$year Endowment Insurance

Consider an $n-$year endowment policy for a life (x) with benefit of 1 and with an annual premium of $\bar{P}$ which is paid with a continuous $n-$year temporary annuity. The loss random variable for this policy is given by

$$\bar{L}_{x:\overline{n}|} = \nu^{\min(T,n)} - \bar{P}\bar{a}_{\overline{\min(T,n)}|} = \bar{Z}_{x:\overline{n}|} - \bar{P}\bar{Y}_{x:\overline{n}|}.$$

Applying the equivalence principle, i.e., $E(\bar{L}_{x:\overline{n}|}) = 0$, we find

$$0 = E(\bar{L}_{x:\overline{n}|}) = E(\bar{Z}_{x:\overline{n}|} - \bar{P}\bar{Y}_{x:\overline{n}|}) = \bar{A}_{x:\overline{n}|} - \bar{P}\bar{a}_{x:\overline{n}|} \Longrightarrow \bar{P} = \frac{\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}.$$

The continuous premium $\bar{P}$ will be denoted by $\bar{P}(\bar{A}_{x:\overline{n}|})$.
The variance of the loss $\text{Var}(\bar{L}_{x:\overline{n}|})$ under this kind of benefit premium can be found as follows:

$$\begin{aligned}
\text{Var}(\bar{L}_{x:\overline{n}|}) =& \text{Var}(\bar{Z}_{x:\overline{n}|} - \bar{P}\bar{Y}_{x:\overline{n}|}) \\
=& \text{Var}\left(\bar{Z}_{x:\overline{n}|} - \bar{P}\frac{1-\bar{Z}_{x:\overline{n}|}}{\delta}\right) \\
=& \text{Var}\left[\left(1+\frac{\bar{P}}{\delta}\right)\bar{Z}_{x:\overline{n}|} - \frac{\bar{P}}{\delta}\right] \\
=& \left(1+\frac{\bar{P}}{\delta}\right)^2 \text{Var}(\bar{Z}_{x:\overline{n}|}) \\
=& [{}^2\bar{A}_{x:\overline{n}|} - (\bar{A}_{x:\overline{n}|})^2]\left(1+\frac{\bar{P}}{\delta}\right)^2 \\
=& \frac{{}^2\bar{A}_{x:\overline{n}|} - (\bar{A}_{x:\overline{n}|})^2}{(\delta\bar{a}_{x:\overline{n}|})^2} \\
=& \frac{{}^2\bar{A}_{x:\overline{n}|} - (\bar{A}_{x:\overline{n}|})^2}{(1-\bar{A}_{x:\overline{n}|})^2}
\end{aligned}$$

where in the last line we used the formula $\delta\bar{a}_{x:\overline{n}|} + \bar{A}_{x:\overline{n}|} = 1$.

Example 40.10
You are given:
(i) Mortality is exponential with parameter μ.
(ii) Constant force of interest δ.

(iii) $\bar{L}_{x:\overline{n|}}$ is the loss-at-issue random variable for a fully continuous $n-$year endowment insurance on (x) with premiums determined based on the equivalence principle.
Calculate $\bar{P}(\bar{A}_{x:\overline{n|}})$ and $\text{Var}(\bar{L}_{x:\overline{n|}})$.

Solution.
We have

$$\begin{aligned}
\bar{A}_{x:\overline{n|}} =&\bar{A}^{1}_{x:\overline{n|}} + {}_nE_x\\
=&\frac{\mu}{\mu+\delta}(1-e^{-n(\mu+\delta)}) + e^{-n(\mu+\delta)}\\
{}^2\bar{A}_{x:\overline{n|}} =&\frac{\mu}{\mu+2\delta}(1-e^{-n(\mu+2\delta)}) + e^{-n(\mu+2\delta)}\\
\bar{a}_{x:\overline{n|}} =&\frac{1-e^{-(\mu+\delta)n}}{\mu+\delta}\\
\bar{P}(\bar{A}_{x:\overline{n|}}) =&\mu + \frac{(\mu+\delta)e^{-n(\mu+\delta)}}{1-e^{-(\mu+\delta)n}}\\
\text{Var}(\bar{L}_{x:\overline{n|}}) =&\frac{\frac{\mu}{\mu+2\delta}(1-e^{-n(\mu+2\delta)}) + e^{-n(\mu+2\delta)} - \left(\frac{\mu}{\mu+\delta}(1-e^{-n(\mu+\delta)}) + e^{-n(\mu+\delta)}\right)^2}{\left[\frac{\delta(1-e^{-(\mu+\delta)n})}{\mu+\delta}\right]^2} \blacksquare
\end{aligned}$$

Now, if the premiums are paid with a continuous $t-$year temporary annuity (where $t \le n$) then the loss random variable is given by

$$ {}_t\bar{L}_{x:\overline{n|}} = \nu^{\min{(T,n)}} - \bar{P}\bar{a}_{\overline{\min(T,t)|}} = \bar{Z}_{x:\overline{n|}} - \bar{P}\bar{Y}_{x:\overline{t|}}.$$

The benefit premium for this insurance is given by

$$ {}_t\bar{P}(\bar{A}_{x:\overline{n|}}) = \frac{\bar{A}_{x:\overline{n|}}}{\bar{a}_{x:\overline{t|}}}.$$

Example 40.11
You are given that $\mu = 0.02$ and $\delta = 0.05$. Calculate ${}_{10}\bar{P}(\bar{A}_{75:\overline{20|}})$.

Solution.
We have

$$\begin{aligned}
\bar{A}_{75:\overline{20|}} =& \frac{\mu}{\mu+\delta}(1-e^{-n(\mu+\delta)})+e^{-n(\mu+\delta)}=0.461855\\
\bar{A}_{75:\overline{10|}} =& \frac{\mu}{\mu+\delta}(1-e^{-n(\mu+\delta)})+e^{-n(\mu+\delta)}=0.640418\\
{}_{10}\bar{P}(\bar{A}_{75:\overline{20|}}) =& \frac{\delta\bar{A}_{75:\overline{20|}}}{1-\bar{A}_{75:\overline{10|}}}=0.064221 \blacksquare
\end{aligned}$$

Practice Problems

Problem 40.12
You are given:
(i) Mortality is exponential with parameter $\mu = 0.02$.
(ii) Constant force of interest $\delta = 0.05$.
(iii) $\bar{L}_{75:\overline{20|}}$ is the loss-at-issue random variable for a fully continuous $20-$year endowment insurance on (75) with premiums determined based on the equivalence principle.
Calculate $\bar{P}(\bar{A}_{75:\overline{20|}})$

Problem 40.13
Find $\text{Var}(\bar{L}_{75:\overline{20|}})$ in the previous problem.

Problem 40.14
You are given:
(i) Mortality follows De Moivre's Law with parameter ω.
(ii) Constant force of interest δ.
(iii) $\bar{L}_{x:\overline{n|}}$ is the loss-at-issue random variable for a fully continuous $n-$year endowment insurance on (x) with premiums determined based on the equivalence principle.
Calculate $\bar{P}(\bar{A}_{x:\overline{n|}})$ and $\text{Var}(\bar{L}_{x:\overline{n|}})$.

Problem 40.15
You are given:
(i) Mortality follows De Moivre's Law with parameter $\omega = 125$.
(ii) Constant force of interest $\delta = 0.05$.
(iii) $\bar{L}_{75:\overline{20|}}$ is the loss-at-issue random variable for a fully continuous $20-$year endowment insurance on (75) with premiums determined based on the equivalence principle.
Calculate $\bar{P}(\bar{A}_{75:\overline{20|}})$.

Problem 40.16
Find the variance in the previous problem.

Problem 40.17
You are given that mortality follows De Moivre's Law with $\omega = 125$ and $\delta = 0.05$. Calculate ${}_{10}\bar{P}(\bar{A}_{75:\overline{20|}})$.

40.4 Continuous $n-$year Pure Endowment

For this type of insurance, the insurance benefit is an $n-$year pure endowment and the payments are made as a continuous $n-$year temporary annuity. The loss random variable is given by

$$\bar{L}_{x:\overset{1}{\overline{n}|}} = \nu^n \mathbf{I}(T > n) - \bar{P}\bar{a}_{\overline{\min(T,n)}|} = \bar{Z}_{x:\overset{1}{\overline{n}|}} - \bar{P}\bar{Y}_{x:\overline{n}|}.$$

Applying the equivalence principle, i.e., $E(\bar{L}_{x:\overset{1}{\overline{n}|}}) = 0$, we find

$$0 = E(\bar{L}_{x:\overset{1}{\overline{n}|}}) = E(\bar{Z}_{x:\overset{1}{\overline{n}|}} - \bar{P}\bar{Y}_{x:\overline{n}|}) = A_{x:\overset{1}{\overline{n}|}} - \bar{P}\bar{a}_{x:\overline{n}|} \Longrightarrow \bar{P} = \frac{A_{x:\overset{1}{\overline{n}|}}}{\bar{a}_{x:\overline{n}|}}.$$

The continuous premium $\bar{P}$ will be denoted by $\bar{P}(A_{x:\overset{1}{\overline{n}|}})$.

Example 40.12
You are given:
(i) Mortality is exponential with parameter μ.
(ii) Constant force of interest δ.
(iii) $\bar{L}_{x:\overset{1}{\overline{n}|}}$ is the loss-at-issue random variable for a fully continuous $n-$year pure endowment insurance on (x) with premiums determined based on the equivalence principle.
Calculate $\bar{P}(A_{x:\overset{1}{\overline{n}|}})$.

Solution.
We have

$$\begin{aligned}
A_{x:\overset{1}{\overline{n}|}} =& {}_nE_x = e^{-n(\mu+\delta)}\\
\bar{a}_{x:\overline{n}|} =& \frac{1 - e^{-(\mu+\delta)n}}{\mu+\delta}\\
\bar{P}(A_{x:\overset{1}{\overline{n}|}}) =& \frac{(\mu+\delta)e^{-n(\mu+\delta)}}{1 - e^{-n(\mu+\delta)}} \blacksquare
\end{aligned}$$

Now, if the premiums are paid with a continuous $t-$year temporary annuity, where $t < n$, then the loss random variable is given by

$${}_t\bar{L}_{x:\overset{1}{\overline{n}|}} = \nu^n \mathbf{I}(T > n) - \bar{P}\bar{a}_{\overline{\min(T,t)}|} = \bar{Z}_{x:\overset{1}{\overline{n}|}} - \bar{P}\bar{Y}_{x:\overline{t}|}.$$

The benefit premium for this insurance is given by

$${}_t\bar{P}(A_{x:\overset{1}{\overline{n}|}}) = \frac{A_{x:\overset{1}{\overline{n}|}}}{\bar{a}_{x:\overline{t}|}}.$$

Example 40.13
You are given that $\mu = 0.02$ and $\delta = 0.05$. Calculate ${}_{10}\bar{P}(A_{75:\overset{\,1}{\overline{20|}}})$.

Solution.
We have

$$\begin{aligned}
A_{75:\overset{\,1}{\overline{20|}}} =& {}_{20}E_{75} = e^{-n(\mu+\delta)} = 0.2466\\
\bar{a}_{75:\overline{10|}} =& \frac{1-e^{-(\mu+\delta)n}}{\mu+\delta} = 7.19164\\
{}_{10}\bar{P}(A_{75:\overset{\,1}{\overline{20|}}}) =& \frac{A_{75:\overset{\,1}{\overline{20|}}}}{\bar{a}_{75:\overline{10|}}} = 0.03429 \blacksquare
\end{aligned}$$

Practice Problems

Problem 40.18
You are given that mortality follows De Moivre's Law with $\omega = 125$ and $\delta = 0.05$. Calculate $\bar{P}(A_{75:\overset{\,1}{\overline{20}|}})$.

Problem 40.19
You are given that $\mu = 0.02$ and $\delta = 0.05$. Calculate $\bar{P}(A_{75:\overset{\,1}{\overline{20}|}})$.

Problem 40.20
You are given that mortality follows De Moivre's Law with $\omega = 125$ and $\delta = 0.05$. Calculate ${}_{10}\bar{P}(A_{75:\overset{\,1}{\overline{20}|}})$.

Problem 40.21
Show that: ${}_n\bar{P}(\bar{A}_x) = \bar{P}(\bar{A}^1_{x:\overline{n}|}) + \bar{P}(A_{x:\overset{\,1}{\overline{n}|}})\bar{A}_{x+n}$.

Problem 40.22
Show that: $\bar{P}(\bar{A}_{x:\overline{n}|}) + \bar{P}(\bar{A}^1_{x:\overline{n}|}) = \bar{P}(A_{x:\overset{\,1}{\overline{n}|}})$.

40.5 Continuous $n-$year Deferred Insurance

For this insurance, the benefit is a $n-$year deferred whole life insurance and the payments are made as a continuous whole life annuity. The loss random variable is

$$ {}_{n|}\bar{L}_x = \nu^T\mathbf{I}(T > n) - \bar{P}\bar{a}_{\overline{T}|} = {}_{n|}\bar{Z}_x - \bar{P}\bar{Y}_x. $$

The benefit premium is

$$ \bar{P}({}_{n|}\bar{A}_x) = \frac{{}_{n|}\bar{A}_x}{\bar{a}_x}. $$

Example 40.14
You are given that $\mu = 0.02$ and $\delta = 0.05$. Calculate $\bar{P}({}_{10|}\bar{A}_{75})$

Solution. We have

$$ \begin{aligned} {}_{10|}\bar{A}_{75} =& \frac{\mu}{\mu+\delta}e^{-n(\mu+\delta)} = \frac{0.02}{0.07}e^{-10(0.07)} = 0.14188 \\ \bar{a}_{75} =& \frac{1}{\mu+\delta} = \frac{1}{0.07} = 14.28571 \\ \bar{P}({}_{10|}\bar{A}_{75}) =& \frac{0.14188}{14.28571} = 0.0099 \blacksquare \end{aligned} $$

Now, if the premiums are paid with a continuous $t-$year temporary annuity, where $t < n$, then the loss random variable is given by

$$ {}_t({}_{n|}\bar{L}_x) = \nu^T\mathbf{I}(T > n) - \bar{P}\bar{a}_{\overline{\min(T,t)}|} = {}_{n|}\bar{Z}_x - \bar{P}\bar{Y}_{x:\overline{t}|}. $$

The benefit premium for this insurance is given by

$$ {}_t\bar{P}({}_{n|}\bar{A}_x) = \frac{{}_{n|}\bar{A}_x}{\bar{a}_{x:\overline{t}|}}. $$

Practice Problems

Problem 40.23
You are given that mortality follows De Moivre's Law with $\omega = 125$ and $\delta = 0.05$. Calculate $\bar{P}({}_{10|}\bar{A}_{75})$.

Problem 40.24
You are given that $\mu = 0.02$ and $\delta = 0.05$. Calculate ${}_5\bar{P}({}_{10|}\bar{A}_{75})$

Problem 40.25
You are given that mortality follows De Moivre's Law with $\omega = 125$ and $\delta = 0.05$. Calculate ${}_5\bar{P}({}_{10|}\bar{A}_{75})$.

40.6 Continuous $n-$year Deferred Whole Life Annuity

A continuous $n-$year deferred whole life contingent annuity contract with net single premium ${}_{n|}\bar{a}_x$ is often funded by annual premiums paid over the $n-$year deferred period, subject to the continued survival of (x). The loss random variable for this contract is

$$\bar{L}({}_{n|}\bar{a}_x) = \bar{a}_{\overline{T-n}|}\nu^n\mathbf{I}(T(x) > n) - \bar{P}\bar{a}_{\overline{\min\{T,n)\}}|} = {}_{n|}\bar{Y}_x - \bar{P}\bar{Y}_{x:\overline{n}|}.$$

Applying the equivalence principle, i.e., $E[{}_{n|}\bar{a}_x] = 0$, we find

$$0 = E[{}_{n|}\bar{a}_x] = {}_{n|}\bar{a}_x - \bar{P}\bar{a}_{x:\overline{n}|} \Longrightarrow \bar{P} = \frac{{}_{n|}\bar{a}_x}{\bar{a}_{x:\overline{n}|}}.$$

The continuous benefit premium will be denoted by $\bar{P}({}_{n|}\bar{a}_x)$. Thus,

$$\bar{P}({}_{n|}\bar{a}_x) = \frac{{}_{n|}\bar{a}_x}{\bar{a}_{x:\overline{n}|}} = \frac{{}_nE_x\bar{a}_{x+n}}{\bar{a}_{x:\overline{n}|}}.$$

Example 40.15
You are given that $\mu = 0.02$ and $\delta = 0.05$. Calculate $\bar{P}({}_{10|}\bar{a}_{75})$

Solution.
We have

$$\begin{aligned}
\bar{P}({}_{10|}\bar{a}_{75}) =& \frac{{}_{10|}\bar{a}_{75}}{\bar{a}_{75:\overline{10}|}} = \frac{{}_{10}E_{75}\bar{a}_{85}}{\frac{1-\bar{A}_{75:\overline{10}|}}{\delta}}\\
=& \frac{e^{-10(0.02+0.05)}\bar{a}_{85}}{\frac{1-[\bar{A}^{\,1}_{75:\overline{10}|}+{}_{10}E_{75}]}{\delta}}\\
=& \frac{e^{-10(0.07)}(0.02+0.05)^{-1}}{\frac{1-[\frac{0.02}{0.07}(1-e^{-10(0.07)})+e^{-10(0.07)}]}{0.05}}\\
=&0.98643 \blacksquare
\end{aligned}$$

Now, if the premiums are paid with a continuous $t-$year temporary annuity, where $t < n$, then the loss random variable is given by

$${}_t\bar{L}(\bar{a}_x) = \bar{a}_{\overline{T-n}|}\nu^n\mathbf{I}(T(x) > n) - \bar{P}\bar{a}_{\overline{\min\{T,t)\}}|} = {}_{n|}\bar{Y}_x - \bar{P}\bar{Y}_{x:\overline{t}|}.$$

The benefit premium for this insurance is given by

$${}_t\bar{P}({}_{n|}\bar{a}_x) = \frac{{}_{n|}\bar{a}_x}{\bar{a}_{x:\overline{t}|}}.$$

Example 40.16
You are given that $\mu = 0.02$ and $\delta = 0.05$. Calculate ${}_5\bar{P}({}_{10|}\bar{a}_{75})$

Solution.
We have

$$\begin{aligned}
{}_5\bar{P}({}_{10|}\bar{a}_{75}) =& \frac{{}_{10|}\bar{a}_{75}}{\bar{a}_{75:\overline{5}|}} = \frac{{}_{10}E_{75}\bar{a}_{85}}{\frac{1-\bar{A}_{75:\overline{5}|}}{\delta}} \\
=& \frac{e^{-10(0.02+0.05)}\bar{a}_{85}}{\frac{1-[\bar{A}^{\,1}_{75:\overline{5}|}+{}_5E_{75}]}{\delta}} \\
=& \frac{e^{-10(0.07)}(0.02+0.05)^{-1}}{\frac{1-[\frac{0.02}{0.07}(1-e^{-5(0.07)})+e^{-5(0.07)}]}{0.05}} \\
=& 1.6816 \ \blacksquare
\end{aligned}$$

Example 40.17 ‡
Company ABC sets the contract premium for a continuous life annuity of 1 per year on (x) equal to the single benefit premium calculated using:
(i) $\delta = 0.03$
(ii) $\mu(x+t) = 0.02, \ t \geq 0$
However, a revised mortality assumption reflects future mortality improvement and is given by

$$\mu^{Rev}(x+t) = \begin{cases} 0.02 & t \leq 10 \\ 0.01 & t > 10. \end{cases}$$

Calculate the expected loss at issue for ABC (using the revised mortality assumption) as a percentage of the contract premium.

Solution.
The loss at issue is the present value of benefit to be paid minus the present value of premium to be received. Since this is a single premium policy, the premium received is a single payment equal to the contract premium. Let π denote the contract premium. Then

$$\pi = \bar{a}_x = \frac{1}{\delta + \mu} = \frac{1}{0.03 + 0.02} = 20.$$

Let L denote the random variable of loss at issue using the revised mortality assumption. Then $L = \bar{Y}_x - 20$ where $\bar{Y}_x$ is the present value of the continuous life annuity under the revised mortality. Hence,

$$E(L) = \bar{a}_x^{Rev} - 20.$$

We have

$$\begin{aligned}
\bar{a}_x^{Rev} =& \bar{a}_{x:\overline{10}|} + \nu^{10}{}_{10}p_x \bar{a}_{x+10} \\
=& \int_0^{10} e^{-0.03t} e^{-0.02t} dt + e^{-0.3} e^{-0.2} \frac{1}{0.03 + 0.01} \\
=& \frac{(1 - e^{-0.5})(0.02)}{0.05} + \frac{e^{-0.5}}{0.04} = 23.03.
\end{aligned}$$

Thus,

$$E(L) = 23.03 - 20 = 3.03$$

and

$$\frac{E(L)}{\pi} = \frac{3.03}{20} = 15.163\% \blacksquare$$

Practice Problems

Problem 40.26
For a 10-year deferred whole life annuity of 1 on (35) payable continuously:
(i) Mortality follows De Moivre's Law with $\omega = 85$.
(ii) $i = 0$
(iii) Level benefit premiums are payable continuously for 10 years.
Calculate $\bar{P}({}_{10|}\bar{a}_{35})$.

Problem 40.27
For a 10-year deferred whole life annuity of 1 on (35) payable continuously:
(i) Mortality follows De Moivre's Law with $\omega = 85$.
(ii) $\delta = 0.05$
(iii) Level benefit premiums are payable continuously for 10 years.
Calculate $\bar{P}({}_{10|}\bar{a}_{35})$.

Problem 40.28
For a 10-year deferred whole life annuity of 1 on (75) payable continuously:
(i) $\mu = 0.02$
(ii) $\delta = 0.05$
(iii) $\bar{P}({}_{10|}\bar{a}_{75}) = 0.98643$
(iv) Level benefit premiums are payable continuously for 10 years.
Calculate $\bar{L}({}_{10|}\bar{a}_{75})$ given that death occurred at the end of twelfth year.

Problem 40.29
For a 10-year deferred whole life annuity of 1 on (75) payable continuously:
(i) $\mu = 0.02$
(ii) $\delta = 0.05$
(iii) $\bar{P}({}_{10|}\bar{a}_{75}) = 0.98643$
(iv) Level benefit premiums are payable continuously for 10 years.
Calculate $\bar{L}({}_{10|}\bar{a}_{75})$ given that death occurred at the end of the 9^{th} year.

Problem 40.30
For a 10-year deferred whole life annuity of 1 on (75) payable continuously:
(i) ${}_{10|}\bar{a}_{75} = 7.0941$
(ii) $\bar{A}_{75:\overline{10|}} = 0.6404$
(iii) $\bar{P}({}_{10|}\bar{a}_{75}) = 0.98643$
(iv) Level benefit premiums are payable continuously for 10 years.
Calculate δ.

41 Fully Discrete Benefit Premiums

In this section, we study fully discrete policies in which benefit premiums are paid at the beginning of the year and the insurance benefits are paid at the end of the year of death. The benefit premiums are made as far as the individual is alive and the term of the insurance has not expired.

41.1 Fully Discrete Whole Life Insurance

For this type of policy the benefit insurance is a discrete whole life insurance and the payments are made as a discrete whole life annuity-due. For example, suppose that death occurred between the years 2 and 3. Then annual benefit premium P was made at time $t = 0, 1,$ and 2 and the insurance benefit of 1 was paid at time $t = 3$.
The loss random variable for this insurance is given by

$$L_x = \nu^{K+1} - P\ddot{a}_{\overline{K+1}|} = Z_x - P\ddot{Y}_x.$$

Example 41.1
Show that

$$L_x = \left(1 + \frac{P}{d}\right) Z_x - \frac{P}{d}.$$

Solution.
We have

$$\begin{aligned} L_x =& Z_x - P\ddot{Y}_x = Z_x - P\left(\frac{1 - Z_x}{d}\right) \\ =& \left(1 + \frac{P}{d}\right) Z_x - \frac{P}{d} \blacksquare \end{aligned}$$

The actuarial present value of the loss L_x is

$$E(L_x) = \left(1 + \frac{P}{d}\right) A_x - \frac{P}{d}.$$

Applying the equivalence principle we find that

$$P(A_x) = P = \frac{A_x}{\ddot{a}_x} = \frac{dA_x}{1 - A_x}.$$

The variance of L_x under the equivalence principle is found as follows:

$$\begin{aligned}\mathrm{Var}(L_x) =&\mathrm{var}\left[\left(1+\frac{P(A_x)}{d}\right)Z_x - \frac{P(A_x)}{d}\right]\\ =&\left(1+\frac{P(A_x)}{d}\right)^2\mathrm{Var}(Z_x)\\ =&[{}^2A_x-(A_x)^2]\left(1+\frac{P(A_x)}{d}\right)^2\\ =&\frac{{}^2A_x-(A_x)^2}{(d\ddot{a}_x)^2}\\ =&\frac{{}^2A_x-(A_x)^2}{(1-A_x)^2}.\end{aligned}$$

Example 41.2
Consider the following extract from the Illustrative Life Table.

x	ℓ_x	$1000q_x$	$\ddot{a}_x$	$1000A_x$	1000^2A_x
36	9,401,688	2.14	15.2870	134.70	37.26
37	9,381,566	2.28	15.1767	140.94	39.81
38	9,360,184	2.43	15.0616	147.46	42.55
39	9,337,427	2.60	14.9416	154.25	45.48
40	9,313,166	2.78	14.8166	161.32	48.63

Assuming UDD, find $P(A_{36})$ and $\mathrm{Var}(L_{36})$.

Solution.
We have

$$\begin{aligned}P(A_{36}) =&\frac{A_{36}}{\ddot{a}_{36}}=\frac{0.13470}{15.2870}=0.008811\\ \mathrm{Var}(L_{36}) =&\frac{{}^2A_{36}-(A_{36})^2}{(d\ddot{a}_{36})^2}\\ =&\frac{0.03726-0.13470^2}{\left[\left(\frac{0.06}{1.06}\right)(15.2870)\right]^2}=0.02553 \blacksquare\end{aligned}$$

Suppose now that the premiums charged to the insured are paid at the beginning of the year for the next t years while the insured is alive. In this case, the loss random variable is

$${}_tL_x=\nu^{K+1}-P\ddot{a}_{\overline{\min(K+1,t)}|}=Z_x-P\ddot{Y}_{x:\overline{t}|}.$$

The benefit premium for a fully discrete whole life insurance funded for the first t years that satisfies the equivalence principle is

$$ {}_tP(A_x) = \frac{A_x}{\ddot{a}_{x:\overline{t}|}}. $$

Example 41.3
Consider a fully discrete whole life insurance for (30) with a face value of 50000 paid at the end of year of death and with annual premium paid at the beginning of the year for the next 30 years while the insured is alive. Find the annual benefit premium of this policy if mortality is exponential with $\mu = 0.03$ and the force of interest is $\delta = 0.05$.

Solution.
We have

$$ \begin{aligned} A_{30} =& \frac{q_{30}}{q_{30}+i} = \frac{1-e^{-0.03}}{1-e^{-0.03}+e^{0.05}-1} = 0.3657 \\ \ddot{a}_{30:\overline{30}|} =& \frac{1-e^{-n(\mu+\delta)}}{1-e^{-(\mu+\delta)}} = \frac{1-e^{-30(0.08)}}{1-e^{-0.08}} = 11.8267 \\ 50000{}_{30}P(A_{30}) =& 50000\frac{A_{30}}{\ddot{a}_{30:\overline{30}|}} = 50000\frac{0.3657}{11.8267} = 1545.89 \blacksquare \end{aligned} $$

Practice Problems

Problem 41.1
Peter buys a fully discrete whole life insurance with benefit of \$20,000 at the end of year of death. The annual benefit premium is \$50. Assume that $\delta = 0.05$. Find the insurer's loss at the time of the issue of the policy if Peter dies in 8 years and 3 months after the issue of this policy.

Problem 41.2
Peter buys a fully discrete whole life insurance with benefit of \$20,000 at the end of year of death. The annual benefit premium is \$50. Assume that $\delta = 0.05$. Find the insurer's loss at the time of the issue of the policy if Peter dies in 62 years and 9 months after the issue of this policy.

Problem 41.3
Show that $P(A_x) = \nu q_x$ under a constant force of mortality.

Problem 41.4 ‡
For a fully discrete whole life insurance of 150,000 on (x), you are given:
(i) ${}^2A_x = 0.0143$
(ii) $A_x = 0.0653$
(iii) The annual premium is determined using the equivalence principle.
(iv) L is the loss-at-issue random variable.
Calculate the standard deviation of L.

Problem 41.5
For a fully discrete whole life insurance of 1000 on (60) you are given:
- $i = 0.06$
- $1000A_{60} = 369.33$.

Calculate the annual benefit premium.

Problem 41.6
For a fully discrete whole life insurance on (x), you are given the following:
(i) $i = 0.05$
(ii) $P(K(x) = k) = 0.04(0.96)^k,\ k = 0, 1, 2, \cdots$.
Calculate the annual benefit premium for this policy.

Problem 41.7
Show that
$$a_x = \frac{1}{P(A_x) + d}.$$

Problem 41.8 ‡

For a fully discrete whole life insurance of 1000 on (60), you are given:
(i) $i = 0.06$
(ii) Mortality follows the Illustrative Life Table, except that there are extra mortality risks at age 60 such that $q_{60} = 0.015$.
Calculate the annual benefit premium for this insurance.

41.2 Fully Discrete $n-$year Term

For this type of insurance, the benefit insurance is a discrete $n-$year term and the payments are made as a discrete $n-$year temporary annuity-due. The loss random variable is given by

$$L^1_{x:\overline{n}|} = \nu^{K+1}\mathbf{I}(K \le n-1) - P\ddot{a}_{\overline{\min(K+1,n)}|} = Z^1_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{n}|}.$$

The actuarial present value of this policy is given by

$$E(L^1_{x:\overline{n}|}) = A^1_{x:\overline{n}|} - P\ddot{a}_{x:\overline{n}|}.$$

The benefit premium for this policy obtained using the equivalence principle (i.e., $E(L^1_{x:\overline{n}|}) = 0$) is

$$P(A^1_{x:\overline{n}|}) = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}.$$

Example 41.4
You are given that mortality follows the Illustrative Life Table with $i = 6\%$. Calculate $P(A^1_{50:\overline{20}|})$.

Solution.
We have

$$\begin{aligned}
P(A^1_{50:\overline{20}|}) =& \frac{A^1_{50:\overline{20}|}}{\ddot{a}_{50:\overline{20}|}}\\
=& \frac{A_{50} - {}_{20}E_{50}A_{70}}{\ddot{a}_{50} - {}_{20}E_{50}\ddot{a}_{70}}\\
=& \frac{0.24905 - (0.23047)(0.51495)}{13.2668 - (0.23407)(8.5693)} = 0.01155 \blacksquare
\end{aligned}$$

Next, consider a policy where the benefit is a discrete $n-$year term insurance and the payments are made as a discrete $t-$year temporary annuity-due. For such a policy the loss random variable is

$${}_tL^1_{x:\overline{n}|} = \nu^{K+1}\mathbf{I}(K \le n-1) - P\ddot{a}_{\overline{\min(K+1,t)}|} = Z^1_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{t}|}.$$

The actuarial present value of this policy is given by

$$E({}_tL^1_{x:\overline{n}|}) = A^1_{x:\overline{n}|} - P\ddot{a}_{x:\overline{t}|}.$$

The benefit premium for this policy obtained using the equivalence principle (i.e., $E({}_tL^1_{x:\overline{n}|})=0$) is

$${}_tP(A^1_{x:\overline{n}|})=\frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{t}|}}.$$

Example 41.5
For a fully discrete 10−year life insurance of 5000 on (x), you are given:
(i) $\delta=0.10$
(ii) $\mu=0.05$.
(iii) Benefit payment is made at the end of the year of death.
(iv) Annual benefit premium are paid at the beginning of the next 5 years while (x) is alive.
Calculate $5000{}_5P(A^1_{x:\overline{10}|})$.

Solution.
We have

$$\begin{aligned}
A^1_{x:\overline{10}|}=&e^{-\delta}(1-e^{-\mu})\left(\frac{1-e^{-n(\mu+\delta)}}{1-e^{-(\mu+\delta)}}\right)\\
=&\frac{e^{-0.10}(1-e^{-0.05})(1-e^{-1.5})}{1-e^{-0.15}}=0.2461\\
\ddot{a}_{x:\overline{5}|}=&\frac{1-e^{-5(0.05+0.10)}}{1-e^{-(0.10+0.05)}}=3.788\\
5000{}_5P(A^1_{x:\overline{10}|})=&5000\left(\frac{0.2461}{3.788}\right)=324.84\ \blacksquare
\end{aligned}$$

Practice Problems

Problem 41.9
Show that
$$L^1_{x:\overline{n}|} = \left(1+\frac{P}{d}\right)Z^1_{x:\overline{n}|} + \frac{P}{d}Z_{x:\overline{n}|}^{\ \ 1} - \frac{P}{d}.$$

Problem 41.10
You are given the following mortality table.

x	p_x	q_x
90	0.90	0.10
91	0.80	0.20
92	0.60	0.40
93	0.50	0.50

Calculate $P(A^1_{91:\overline{3}|})$ if $i = 0.04$.

Problem 41.11
An insurer offers a four-year life insurance of 90 years old. Mortality is given by the table:

x 90	91	92	93	
q_x	0.10	0.20	0.40	0.50

This life insurance is funded by benefit payments at the beginning of the year. The benefit payment is 10000. Calculate the annual benefit premium using the equivalence principle if $i = 0.04$.

Problem 41.12
You are given:
- ${}_{15}P(A_{45}) = 0.038$
- $A_{60} = 0.0625$
- $\frac{A_{45:\overline{15}|}}{\ddot{a}_{45:\overline{15}|}} = 0.056.$

Calculate $P(A^1_{45:\overline{15}|})$ and interpret this value.

Problem 41.13
For a special fully discrete 10-payment 20−year term life insurance on (x), you are given:
(i) The death benefit of 10000 payable at the end of the year of death.

(ii) The benefit premiums for this insurance are made at the beginning of the next 10 years while (x) is alive.
(iii) $\mu = 0.02$
(iv) $\delta = 0.05$.
Find the annual benefit premium for this policy.

41.3 Fully Discrete $n-$year Pure Endowment

For this policy the benefit is a discrete $n-$year pure endowment insurance and the payments are made as an $n-$year temporary annuity-due. The loss random variable of this type of insurance is

$$L_{x:\overset{1}{\overline{n}|}} = \nu^n \mathbf{I}(K > n-1) - P\ddot{a}_{\overline{\min(K+1,n)}|} = Z_{x:\overset{1}{\overline{n}|}} - P\ddot{Y}_{x:\overline{n}|}.$$

The actuarial present value is

$$E(L_{x:\overset{1}{\overline{n}|}}) = A_{x:\overset{1}{\overline{n}|}} - P\ddot{a}_{x:\overline{n}|}.$$

The benefit premium which satisfies the equivalence principle is

$$P(A_{x:\overset{1}{\overline{n}|}}) = \frac{A_{x:\overset{1}{\overline{n}|}}}{\ddot{a}_{x:\overline{n}|}}.$$

Example 41.6
For a special fully discrete $20-$year pure endowment insurance on (40), you are given:
(i) The death benefit of 50000 payable at the end of the year of death.
(ii) Mortality follows De Moivre's Law with $\omega = 110$
(iii) $\nu = 0.94$.
(iv) Annual benefit premium are paid at the beginning of the next 20 years while (40) is alive.
(a) What is the net single premium?
(b) Find the annual benefit premium of this policy.

Solution.
(a) The net single premium is

$$50000A_{40:\overset{1}{\overline{20}|}} = 50000\nu^{20}{}_{20}p_{40} = 50000(0.94)^{20}\left(\frac{70-20}{70}\right) = 10360.94.$$

(b) We have

$$\begin{aligned}
A_{40:\overline{20}|} =& A^1_{40:\overline{20}|} + A_{40:\overset{1}{\overline{20}|}} \\
=& \frac{a_{\overline{20}|\frac{6}{94}}}{70} + (0.94)^{20}\left(\frac{70-20}{70}\right) = 0.3661 \\
\ddot{a}_{40:\overline{20}|} =& \frac{1 - A_{40:\overline{20}|}}{d} = 10.565.
\end{aligned}$$

Thus, the annual benefit premium is

$$50000P(A_{40:\overline{20|}}) = \frac{10360.94}{10.565} = 980.69 \blacksquare$$

If the payments for an $n-$year pure endowment are made as a discrete $t-$year temporary annuity where $0 \leq t \leq n$, then the loss random variable is

$${}_tL_{x:\overset{1}{\overline{n|}}} = \nu^n \mathbf{I}(K > n-1) - P\ddot{a}_{\overline{\min(K+1,t)|}} = Z_{x:\overset{1}{\overline{n|}}} - P\ddot{Y}_{x:\overline{t|}}.$$

The actuarial present value is

$$E({}_tL_{x:\overset{1}{\overline{n|}}}) = A_{x:\overset{1}{\overline{n|}}} - P\ddot{a}_{x:\overline{t|}}.$$

The benefit premium which satisfies the equivalence principle is

$${}_tP(A_{x:\overset{1}{\overline{n|}}}) = \frac{A_{x:\overset{1}{\overline{n|}}}}{\ddot{a}_{x:\overline{t|}}}.$$

Example 41.7
For a fully discrete 20−year pure endowment of 50000 on (40), you are given:
(i) The death benefit of 50000 payable at the end of the year of death.
(ii) Mortality follows De Moivre's Law with $\omega = 110$
(iii) $\nu = 0.94$.
(iv) Annual benefit premium are paid at the beginning of the next 10 years while (40) is alive.
Find the annual benefit premium of this policy.

Solution.
We have

$$\begin{aligned}
A_{40:\overline{10|}} =& A^{1}_{40:\overline{10|}} + A_{40:\overset{1}{\overline{10|}}}\\
=& \frac{a_{\overline{10|}\frac{6}{94}}}{70} + (0.94)^{10}\left(\frac{70-10}{70}\right) = 0.5649\\
\ddot{a}_{40:\overline{10|}} =& \frac{1 - A_{40:\overline{10|}}}{d} = 7.2517
\end{aligned}$$

Thus, the annual benefit premium is

$$50000P(A_{40:\overline{20|}}) = \frac{10360.94}{7.2517} = 1420.76 \blacksquare$$

Practice Problems

Problem 41.14
For a fully discrete 20−year pure endowment of 50000 on (40), you are given:
(i) The death benefit of 50000 payable at the end of the year of death.
(ii) Mortality follows De Moivre's Law with $\omega = 110$
(iii) $\nu = 0.94$.
(iv) Annual premiums of 1500 are payable at the beginning of each year for 10 years.
(v) L is the loss random variable at the time of issue.
Calculate the minimum value of L.

Problem 41.15
For a special fully discrete 2-year pure endowment of 1000 on (x), you are given:
(i) $\nu = 0.94$
(ii) $p_x = p_{x+1} = 0.70$.
Calculate the benefit premium.

Problem 41.16
You are given:
(i) $d = 0.08$
(ii) $A_{x:\overline{n}|} = 0.91488$
(iii) $A^{1}_{x:\overline{n}|} = 0.70152$.
Calculate $P(A_{x:\overset{1}{\overline{n}|}})$.

Problem 41.17
You are given:
(i) $A^{1}_{x:\overline{n}|} = 0.70152$.
(ii) $A_{x:\overset{1}{\overline{n}|}} = 0.21336$.
(iii) $P(A_{x:\overset{1}{\overline{n}|}}) = 0.2005$.
Calculate i.

Problem 41.18
Define $P(A_{x:\overline{n}|}) = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$. Show that

$$P(A_{x:\overline{n}|}) = P(A^{1}_{x:\overline{n}|}) + P(A_{x:\overset{1}{\overline{n}|}}).$$

41.4 Fully Discrete $n-$year Endowment Insurance

For this policy the benefit is a discrete $n-$year endowment insurance and the payments are made as an $n-$year temporary annuity-due. The loss random variable of this type of insurance is

$$L_{x:\overline{n}|} = \nu^{\min(n,K+1)} - P\ddot{a}_{\overline{\min(K+1,n)}|} = Z_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{n}|}.$$

The actuarial present value is

$$E(L_{x:\overline{n}|}) = A_{x:\overline{n}|} - P\ddot{a}_{x:\overline{n}|}.$$

The benefit premium which satisfies the equivalence principle is

$$P(A_{x:\overline{n}|}) = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}.$$

Example 41.8
Find an expresssion for $\text{Var}(L_{x:\overline{n}|})$.

Solution.
We have

$$\begin{aligned}
\text{Var}(L_{x:\overline{n}|}) =& \text{Var}(Z_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{n}|}) \\
=& \text{Var}\left(Z_{x:\overline{n}|} - P\left(\frac{1 - Z_{x:\overline{n}|}}{d}\right)\right) \\
=& \text{Var}\left[\left(1 + \frac{P}{d}\right)Z_{x:\overline{n}|} - \frac{P}{d}\right] \\
=& \left(1 + \frac{P}{d}\right)^2 \text{Var}(Z_{x:\overline{n}|}) \\
=& \left(1 + \frac{P}{d}\right)^2 [{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2] \blacksquare
\end{aligned}$$

Example 41.9
For a fully discrete $20-$year endowment of 10000 on (x), you are given:
(i) The death benefit of 10000 payable at the end of the year of death.
(ii) The constant force of mortality $\mu = 0.02$.
(iii) $\delta = 0.05$.
(iv) Annual benefit premium are paid at the beginning of the next 20 years while (x) is alive.
(a) Find the annual benefit premium of this policy.
(b) Calculate the variance for the loss at-issue random variable.

Solution.
(a) We have

$$\begin{aligned}
A_{x:\overline{20}|} =&(1-e^{-n(\mu+\delta)})\frac{q_x}{q_x+i}+e^{-n(\mu+\delta)}=0.4565\\
\ddot{a}_{x:\overline{20}|} =&(1-e^{-n(\mu+\delta)})\left(\frac{1}{1-\nu q_x}\right)\\
=&\frac{1-e^{-20(0.02+0.05)}}{1-e^{-(0.02+0.05)}}=11.144\\
10000P(A_{x:\overline{20}|}) =&10000\left(\frac{0.4565}{11.144}\right)=409.6375.
\end{aligned}$$

(b) We have

$$\begin{aligned}
{}^2A_{x:\overline{20}|} =&(1-e^{-n(\mu+2\delta)})\frac{q_x}{q_x+i}+e^{-n(\mu+2\delta)}=0.2348\\
\text{Var}(L_{x:\overline{20}|}) =&[0.2348-0.4565^2]\left(10000+\frac{409.6375}{1-e^{-0.05}}\right)^2\\
=&8939903.78 \blacksquare
\end{aligned}$$

If the payments for an $n-$year endowment are made as a discrete $t-$year temporary annuity where $0 \le t \le n$, then the loss random variable is

$$_tL_{x:\overline{n}|}=\nu^{\min{(n,K+1)}}-P\ddot{a}_{\overline{\min{(K+1,t)}}|}=Z_{x:\overline{n}|}-P\ddot{Y}_{x:\overline{t}|}.$$

The actuarial present value is

$$E(_tL_{x:\overline{n}|})=A_{x:\overline{n}|}-P\ddot{a}_{x:\overline{t}|}.$$

The benefit premium which satisfies the equivalence principle is

$$_tP(A_{x:\overline{n}|})=\frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{t}|}}.$$

Example 41.10
For a fully discrete 20−year endowment of 50000 on (40), you are given:
(i) The death benefit of 50000 payable at the end of the year of death.
(ii) The constant force of mortality $\mu = 0.02$.
(iii) $\delta = 0.05$.
(iv) Annual benefit premium are paid at the beginning of the next 10 years while (40) is alive.
Find the annual benefit premium of this policy.

Solution.
We have

$$\begin{aligned}A_{x:\overline{20}|} =&(1-e^{-n(\mu+\delta)})\frac{q_x}{q_x+i}+e^{-n(\mu+\delta)}=0.4565\\ \ddot{a}_{x:\overline{10}|} =&(1-e^{-n(\mu+\delta)})\left(\frac{1}{1-\nu q_x}\right)\\ =&\frac{1-e^{-10(0.02+0.05)}}{1-e^{-(0.02+0.05)}}=7.4463\end{aligned}$$

Thus, the annual benefit premium is

$$50000_{10}P(A_{40:\overline{20}|})=50000\frac{0.4565}{7.4463}=3065.28 \blacksquare$$

Practice Problems

Problem 41.19
Show that
$$ {}_nP(A_x) = P(A^1_{x:\overline{n}|}) + P(A_{x:\overline{n}|}^{\;\;\,1})A_{x+n}. $$

Problem 41.20
You are given:
(i) $P(A_{x:\overline{n}|}) = 0.00646$
(ii) $P(A_{x:\overline{n}|}^{\;\;\,1}) = 0.00211.$
Calculate $P(A^1_{x:\overline{n}|})$.

Problem 41.21
You are given that mortality follows the Illustrative Life Table with $i = 6\%$.
Calculate $P(A_{50:\overline{20}|})$.

Problem 41.22
You are given that mortality follows the Illustrative Life Table with $i = 6\%$.
Calculate $\mathrm{Var}(L_{50:\overline{20}|})$.

Problem 41.23
You are given the following mortality table.

x	p_x	q_x
90	0.90	0.10
91	0.80	0.20
92	0.60	0.40
93	0.50	0.50
94	0.00	1.00

Calculate $P(A_{90:\overline{3}|})$ if $i = 0.04$.

41.5 Fully Discrete $n-$year Deferred Insurance

For this policy the benefit is a discrete $n-$year deferred whole life insurance and the payments are made as a discrete whole life annuity-due. The loss random variable of this type of insurance is

$$ {}_{n|}L_x = \nu^{K+1}\mathbf{I}(K \geq n) - P\ddot{a}_{K+1} = {}_{n|}Z_x - P\ddot{Y}_x. $$

The actuarial present value is

$$ E({}_{n|}L_x) = {}_{n|}A_x - P\ddot{a}_x. $$

The benefit premium which satisfies the equivalence principle is

$$ P({}_{n|}A_x) = \frac{{}_{n|}A_x}{\ddot{a}_x}. $$

Example 41.11
You are given that mortality follows the Illustrative Life Table with $i = 6\%$. Calculate $P({}_{20|}A_{50})$.

Solution.
We have

$$ \begin{aligned} {}_{20|}A_{50} =& {}_{20}E_{50}A_{70} = (0.23047)(0.51495) = 0.11868 \\ \ddot{a}_{50} =& 13.2668 \\ P({}_{20|}A_{50}) =& \frac{0.11868}{13.2668} = 0.00895 \blacksquare \end{aligned} $$

If the funding scheme is a $t-$year temporary annuity then the loss random variable is

$$ {}_t({}_{n|}L_x) = \nu^{K+1}\mathbf{I}(K \geq n) - P\ddot{a}_{\min\{(K+1,t)\}} = {}_{n|}Z_x - P\ddot{Y}_{x:\overline{t}|}. $$

The actuarial present value is

$$ E({}_t({}_{n|}L_x)) = {}_{n|}A_x - P\ddot{a}_{x:\overline{t}|}. $$

The benefit premium which satisfies the equivalence principle is

$$ {}_tP({}_{n|}A_x) = \frac{{}_{n|}A_x}{\ddot{a}_{x:\overline{t}|}}. $$

Example 41.12
You are given that mortality follows the Illustrative Life Table with $i = 6\%$. Calculate ${}_{10}P({}_{20|}A_{50})$.

Solution.
We have

$$\begin{aligned}
{}_{20|}A_{50} =& {}_{20}E_{50}A_{70} = (0.23047)(0.51495) = 0.11868\\
\ddot{a}_{50:\overline{10|}} =& \ddot{a}_{50} - {}_{10}E_{50}\ddot{a}_{60}\\
=& 13.2668 - (1.06)^{-10}\left(\frac{\ell_{60}}{\ell_{50}}\right)\ddot{a}_{60}\\
=& 13.2668 - (1.06)^{-10}\left(\frac{8,188,074}{8,950,901}\right)(11.1454)\\
=& 7.5749.
\end{aligned}$$

Thus,

$$ {}_{10}P({}_{20|}A_{50}) = \frac{{}_{20|}A_{50}}{\ddot{a}_{50:\overline{10|}}} = \frac{0.11868}{7.5749} = 0.01567 \blacksquare$$

Practice Problems

Problem 41.24
Show that

$$ {}_{n|}L_x = \left(1 + \frac{P}{d}\right) {}_{n|}Z_x + \frac{P}{d} Z^{1}_{x:\overline{n}|} - \frac{P}{d}. $$

Problem 41.25
Show that

$$ E\left[\left({}_{n|}L_x + \frac{P}{d}\right)^2\right] = \left(\frac{P}{d}\right)^2 {}^2A^{1}_{x:\overline{n}|} + \left(1 + \frac{P}{d}\right)^2 {}^2{}_{n|}A_x. $$

Problem 41.26
Connsider an insurance where the benefit is an $n-$year deferred insurance and the payments are made as $t-$year temporary annuity-due. Write an expression for the loss random variable for this policy and then find the actuarial present value.

Problem 41.27
Find the annual benefit premium for a $t-$payments, $n-$year deferred insurance.

Problem 41.28
You are given that mortality follows the Illustrative Life Table with $i = 6\%$. Calculate ${}_{10}P({}_{20|}A_{50})$.

41.6 Fully Discrete $n-$year Deferred Annuity-Due

A fully discrete $n-$year deferred whole life contingent annuity contract with net single premium ${}_{n|}\ddot{a}_x$ is often funded by annual premiums paid over the $n-$year deferred period, subject to the continued survival of (x). The loss random variable for this contract is

$$L({}_{n|}\ddot{a}_x) = \ddot{a}_{\overline{K+1-n}|}\nu^n\mathbf{I}(K \geq n) - P\ddot{a}_{\overline{\min\{K+1,n)\}}|} = {}_{n|}\ddot{Y}_x - P\ddot{Y}_{x:\overline{n}|}.$$

Applying the equivalence principle, i.e., $E[L({}_{n|}\ddot{a}_x)] = 0$, we find

$$0 = E[L({}_{n|}\ddot{a}_x)] = {}_{n|}\ddot{a}_x - P\ddot{a}_{x:\overline{n}|} \Longrightarrow P = \frac{{}_{n|}\ddot{a}_x}{\ddot{a}_{x:\overline{n}|}}.$$

The continuous benefit premium will be denoted by $P({}_{n|}\ddot{a}_x)$. Thus,

$$P({}_{n|}\ddot{a}_x) = \frac{{}_{n|}\ddot{a}_x}{\ddot{a}_{x:\overline{n}|}} = \frac{{}_nE_x\ddot{a}_{x+n}}{\ddot{a}_{x:\overline{n}|}}.$$

Example 41.13

You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Calculate $P({}_{10|}\ddot{a}_{75})$

Solution.

We have

$$\begin{aligned}
P({}_{10|}\ddot{a}_{75}) =&\frac{{}_{10|}\ddot{a}_{75}}{\ddot{a}_{75:\overline{10}|}}\\
=&\frac{{}_{10}E_{75}\ddot{a}_{85}}{\ddot{a}_{75} - {}_{10}E_{75}\ddot{a}_{85}}\\
=&\frac{0.01722(4.6980)}{7.2170 - 0.01722(4.6980)} = 0.01134 \blacksquare
\end{aligned}$$

Now, if the premiums are paid with a discrete $t-$year temporary life annuity-due, where $t < n$, then the loss random variable is given by

$${}_tL({}_{n|}\ddot{a}_x) = \ddot{a}_{\overline{K+1-n}|}\nu^n\mathbf{I}(K \geq n) - P\ddot{a}_{\overline{\min\{K+1,t)\}}|} = {}_{n|}\ddot{Y}_x - P\ddot{Y}_{x:\overline{t}|}.$$

The benefit premium for this insurance is given by

$${}_tP({}_{n|}\ddot{a}_x) = \frac{{}_{n|}\ddot{a}_x}{\ddot{a}_{x:\overline{t}|}}.$$

Example 41.14
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Calculate ${}_5P({}_{10|}\ddot{a}_{75})$

Solution.
We have

$$\begin{aligned}{}_5P({}_{10|}\ddot{a}_{75}) =&\frac{{}_{10|}\ddot{a}_{75}}{\ddot{a}_{75:\overline{5}|}}\\ =&\frac{{}_{10}E_{75}\ddot{a}_{85}}{\ddot{a}_{75}-{}_5E_{75}\ddot{a}_{80}}\\ =&\frac{0.01722(4.6980)}{7.2170-0.54207(5.9050)}=0.02014 \blacksquare\end{aligned}$$

Example 41.15 ‡
For a special 30-year deferred annual whole life annuity-due of 1 on (35):
(i) If death occurs during the deferral period, the single benefit premium is refunded without interest at the end of the year of death.
(ii) $\ddot{a}_{65} = 9.90$
(iii) $A_{35:\overline{30}|} = 0.21$
(iv) $A^{1}_{35:\overline{30}|} = 0.07$
Calculate the single benefit premium for this special deferred annuity.

Solution.
Letπ denote the single benefit premium. Then actuarial present value of the refund of premium without interest if death occurs during the deferral period is $\pi A^{1}_{30:\overline{35}|}$. The single benefit premium for the deferred annuity is ${}_{30|}\ddot{a}_{35}$. Thus, π satisfies the equation

$$\pi = \pi A^{1}_{30:\overline{35}|} + {}_{30|}\ddot{a}_{35} = \pi A^{1}_{30:\overline{35}|} + \nu^{30}{}_{30}p_{35}\ddot{a}_{65}.$$

We have

$$\begin{aligned}\pi =&\pi(0.07) + {}_{30}E_{35}(9.90)\\ =&0.07\pi + (A_{35:\overline{35}|} - A^{1}_{30:\overline{35}|})(9.90)\\ =&0.07\pi + (0.21-0.07)(9.90)\\ \pi =&1.49 \blacksquare\end{aligned}$$

Practice Problems

Problem 41.29
Consider a 20-year deferred annuity-due of 40000 on (45). You are given:
(i) Force of mortality is 0.02
(ii) $\delta = 0.05$
Calculate the annual benefit premium of this policy.

Problem 41.30
For (x), you are given:
(1) The premium for a 20-year endowment of 1 is 0.0349.
(2) The premium for a 20-year pure endowment of 1 is 0.023.
(3) The premium for a 20-year deferred whole life annuity-due of 1 per year is 0.2087.
(4) All premiums are fully discrete net level premiums.
(5) i = 0.05
Calculate the premium for a 20-payment whole life insurance of 1.

42 Benefit Premiums for Semicontinuous Models

A **Semicontinuous** insurance is a policy with a continuous benefit and payments made with a discrete annuity due. That is, the benefit is paid at the moment of death and the premiums are paid at the beginning of the year while the insured is alive.

42.1 Semicontinuous Whole Life Insurance

For this policy, the benefit is a continuous whole life insurance and the payments are made as a discrete whole life annuity-due. The loss random variable of this policy is

$$\overline{SL}_x = \nu^T - P\ddot{a}_{\overline{K+1|}} = \bar{Z}_x - P\ddot{Y}_x.$$

The actuarial present value of the loss is

$$E(\overline{SL}_x) = \bar{A}_x - P\ddot{a}_x.$$

The benefit premium that satisfies the equivalence principle is given by

$$P(\bar{A}_x) = \frac{\bar{A}_x}{\ddot{a}_x}.$$

Example 42.1
You are given that mortality follows De Moivre's Law with $\omega = 125$ and $\delta = 0.05$. Calculate $P(\bar{A}_{75})$.

Solution.
We have

$$\begin{aligned}
\bar{A}_{75} =& \frac{\bar{a}_{\overline{50|}}}{50} = \frac{1 - e^{-0.05(50)}}{0.05(50)} = 0.36717 \\
\ddot{a}_{75} =& 1 + \nu\,{}_1p_{75} + \nu^2\,{}_2p_{75} + \cdots + \nu^{49}\,{}_{49}p_{75} \\
=& \frac{50}{50} + \frac{49}{50}\nu + \frac{48}{50}\nu^2 + \cdots + \frac{1}{50}\nu^{49} \\
=& \frac{1}{50}(D\ddot{a})_{\overline{50|}} = \frac{50 - a_{\overline{50|}}}{50} = 13.16238 \\
P(\bar{A}_{75}) =& \frac{\bar{A}_{75}}{\ddot{a}_{75}} \\
=& \frac{0.36717}{13.16238} = 0.0279 \blacksquare
\end{aligned}$$

Now, for a continuous benefit and payments that are made with a $t-$year temporary annuity-due the loss random variable is

$$ {}_t\overline{SL}_x = \nu^T - P\ddot{a}_{\overline{\min(K+1,t)}|} = \bar{Z}_x - P\ddot{Y}_{x:\overline{t}|}. $$

The actuarial present value of the loss is

$$ E[{}_t\overline{SL}_x] = \bar{A}_x - P\ddot{a}_{x:\overline{t}|} $$

and the benefit premium is

$$ {}_tP(\bar{A}_x) = \frac{\bar{A}_x}{\ddot{a}_{x:\overline{t}|}}. $$

Example 42.2
You are given that mortality follows De Moivre's Law with $\omega = 125$ and $\delta = 0.05$. Calculate ${}_{20}P(\bar{A}_{75})$.

Solution.
We have

$$ \begin{aligned} \ddot{a}_{75:\overline{20}|} =& 1 + \nu {}_1p_{75} + \nu^2 {}_2p_{75} + \cdots + \nu^{19} {}_{19}p_{75} \\ =& \frac{50}{50} + \frac{49}{50}\nu + \frac{48}{50}\nu^2 + \cdots + \frac{31}{50}\nu^{19} \\ =& \frac{30}{50}\ddot{a}_{\overline{20}|} + \frac{1}{50}(D\ddot{a})_{\overline{20}|} \\ =& \frac{30}{50}\left(\frac{1-e^{-20(0.05)}}{e^{0.05}-1}\right)e^{0.05} + \frac{1}{50}\left(\frac{20 - \frac{1-e^{-20(0.05)}}{e^{0.05}-1}}{e^{0.05}-1}\right)e^{0.05} \\ =& 10.92242 \\ {}_{20}P(\bar{A}_{75}) =& \frac{0.36717}{10.92242} = 0.03362 \ \blacksquare \end{aligned} $$

If uniform distribution of deaths is assumed over each year of age then

$$ P(\bar{A}_x) = \frac{\bar{A}_x}{\ddot{a}_x} = \frac{i}{\delta}\frac{A_x}{\ddot{a}_x} = \frac{i}{\delta}P(A_x). $$

Example 42.3
You are given that mortality follows the Illustrative Life Table with $i = 6\%$. Assume that mortality is uniformly distributed between integral ages. Calculate $P(\bar{A}_{50})$.

Solution.
We have

$$
\begin{aligned}
P(\bar{A}_{50}) =& \frac{i}{\delta}\frac{A_{50}}{\ddot{a}_{50}}\\
=& \frac{0.06}{\ln 1.06}\frac{0.24905}{13.2668} = 0.01933 \blacksquare
\end{aligned}
$$

Practice Problems

Problem 42.1
You are given that $\mu = 0.02$ and $\delta = 0.05$. Calculate $P(\bar{A}_{75})$.

Problem 42.2
You are given that $\mu = 0.02$ and $\delta = 0.05$. Calculate ${}_{20}P(\bar{A}_{75})$.

Problem 42.3
You are given:
(i) $i = 0.06$
(ii) $A_{60} = 0.36913$.
Find $P(\bar{A}_{60})$.

Problem 42.4
Interpret the meaning of $10000{}_{20}P(\bar{A}_{40})$.

Problem 42.5
You are given that mortality follows the Illustrative Life Table with $i = 6\%$. Assume that mortality is uniformly distributed between integral ages. Calculate $P(\bar{A}_{65})$.

42.2 Semicontinuous $n-$year Term Insurance

For this type of insurance, the benefit is a continuous $n-$year term insurance and the payments are made as a discrete $n-$year temporary life annuity-due. The loss random variable of this insurance is

$$\overline{SL}^1_{x:\overline{n}|} = \nu^T \mathbf{I}(T \le n) - P\ddot{a}_{\overline{\min(K+1,n)}|} = \bar{Z}^1_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{n}|}.$$

The actuarial present value of the loss is

$$E(\overline{SL}^1_{x:\overline{n}|}) = \bar{A}^1_{x:\overline{n}|} - P\ddot{a}_{x:\overline{n}|}.$$

The benefit premium that satisfies the equivalence principle is given by

$$P(\bar{A}^1_{x:\overline{n}|}) = \frac{\bar{A}^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}.$$

Example 42.4
You are given that mortality follows De Moivre's Law with $\omega = 125$ and $\delta = 0.05$. Calculate $P(\bar{A}^1_{75:\overline{20}|})$.

Solution.
We have

$$\begin{aligned}
\bar{A}^1_{75:\overline{20}|} =& \frac{\bar{a}_{\overline{20}|}}{50} = \frac{1 - e^{-0.05(20)}}{0.05(50)} = 0.25285 \\
\ddot{a}_{75:\overline{20}|} =& 10.92242 \\
P(\bar{A}^1_{75:\overline{20}|}) =& \frac{0.25285}{10.92242} = 0.0231 \blacksquare
\end{aligned}$$

For an $n-$year term insurance with payments made as a discrete $t-$year temporary annuity-due, the loss random variable is given by

$${}_t\overline{SL}^1_{x:\overline{n}|} = \nu^T \mathbf{I}(T \le n) - P\ddot{a}_{\overline{\min(K+1,t)}|} = \bar{Z}^1_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{t}|}.$$

The actuarial present value of the loss is

$$E({}_t\overline{SL}^1_{x:\overline{n}|}) = \bar{A}^1_{x:\overline{n}|} - P\ddot{a}_{x:\overline{t}|}.$$

The benefit premium that satisfies the equivalence principle is given by

$${}_tP(\bar{A}^1_{x:\overline{n}|}) = \frac{\bar{A}^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{t}|}}.$$

Example 42.5
You are given that mortality follows De Moivre's Law with $\omega = 125$ and $\delta = 0.05$. Calculate ${}_{10}P(\bar{A}^{1}_{75:\overline{20}|})$.

Solution.
We have

$$\begin{aligned}
\bar{A}^{1}_{75:\overline{20}|} =& \frac{\bar{a}_{\overline{20}|}}{50} = \frac{1-e^{-0.05(20)}}{0.05(50)} = 0.25285\\
\ddot{a}_{75:\overline{10}|} =& 1 + \nu_1 p_{75} + \nu^2{}_2p_{75} + \cdots + \nu^9{}_9p_{75}\\
=& \frac{50}{50} + \frac{49}{50}\nu + \frac{48}{50}\nu^2 + \cdots + \frac{41}{50}\nu^9\\
=& \frac{40}{50}\ddot{a}_{\overline{10}|} + \frac{1}{50}(D\ddot{a})_{\overline{10}|}\\
=& \frac{40}{50}\left(\frac{1-e^{-10(0.05)}}{e^{0.05}-1}\right)e^{0.05} + \frac{1}{50}\left(\frac{10 - \frac{1-e^{-10(0.05)}}{e^{0.05}-1}}{e^{0.05}-1}\right)e^{0.05}\\
=& 7.4079\\
{}_{10}P(\bar{A}^{1}_{75:\overline{20}|}) =& \frac{0.25285}{7.4079} = 0.0341 \blacksquare
\end{aligned}$$

If uniform distribution of deaths is assumed over each year of age then

$$P(\bar{A}^{1}_{x:\overline{n}|}) = \frac{\bar{A}^{1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{i}{\delta}\frac{A^{1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{i}{\delta}P(A^{1}_{x:\overline{n}|}).$$

Example 42.6
You are given that mortality follows the Illustrative Life Table with $i = 6\%$. Assume that mortality is uniformly distributed between integral ages. Calculate $P(\bar{A}^{1}_{50:\overline{20}|})$.

Solution.
We have

$$\begin{aligned}
P(\bar{A}^{1}_{50:\overline{20}|}) =& \frac{i}{\delta}\frac{A^{1}_{50:\overline{20}|}}{\ddot{a}_{50:\overline{20}|}}\\
=& \frac{i}{\delta}\frac{A_{50} - {}_{20}E_{50}A_{70}}{\ddot{a}_{50} - {}_{20}E_{50}\ddot{a}_{70}}\\
=& \frac{0.06}{\ln 1.06}\cdot\frac{0.24905 - (0.23047)(0.51495)}{13.2668 - (0.23407)(8.5693)}\\
=& 0.0119 \blacksquare
\end{aligned}$$

Practice Problems

Problem 42.6
You are given that $\mu = 0.02$ and $\delta = 0.05$. Calculate $P(\bar{A}^{\,1}_{75:\overline{20}|})$.

Problem 42.7
You are given that $\mu = 0.02$ and $\delta = 0.05$. Calculate ${}_{10}P(\bar{A}^{\,1}_{75:\overline{20}|})$.

Problem 42.8
You are given that mortality follows the Illustrative Life Table with $i = 6\%$. Assume that mortality is uniformly distributed between integral ages. Calculate $P(\bar{A}^{\,1}_{75:\overline{20}|})$.

Problem 42.9
For a semicontinuous 20-year term insurance on (x) you are given:
(i) $\bar{A}_{x:\overline{20}|} = 0.4058$
(ii) $A_{x:\overline{20}|}^{\;\;\;\;1} = 0.3195$
(iii) $\ddot{a}_{x:\overline{20}|} = 12.522$.
Calculate $P(A^{1}_{x:\overline{20}|})$.

Problem 42.10
You are given:
(i) $\bar{A}_x - \bar{A}^{1}_{x:\overline{n}|} = 0.07045$
(ii) ${}_nP(\bar{A}_x) - P(\bar{A}^{1}_{x:\overline{n}|}) = 0.0063$.
Calculate $\ddot{a}_{x:\overline{n}|}$.

Problem 42.11 ‡
For a 10-payment, 20-year term insurance of 100,000 on Pat:
(i) Death benefits are payable at the moment of death.
(ii) Contract premiums of 1600 are payable annually at the beginning of each year for 10 years.
(iii) $i = 0.05$
(iv) L is the loss random variable at the time of issue.
Calculate the minimum value of L as a function of the time of death of Pat.

42.3 Semicontinuous $n-$year Endowment Insurance

For this type of insurance, the benefit is a continuous $n-$year endowment and the payments are made as a discrete $n-$year temporary life annuity-due. The loss random variable of this insurance is

$$\overline{SL}_{x:\overline{n}|} = \nu^{\min(n,T)} - P\ddot{a}_{\overline{\min(K+1,n)}|} = \bar{Z}_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{n}|}.$$

The actuarial present value of the loss is

$$E(\overline{SL}_{x:\overline{n}|}) = \bar{A}_{x:\overline{n}|} - P\ddot{a}_{x:\overline{n}|}.$$

The benefit premium that satisfies the equivalence principle is given by

$$P(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}.$$

Example 42.7
You are given that mortality follows De Moivre's Law with $\omega = 125$ and $\delta = 0.05$. Calculate $P(\bar{A}_{75:\overline{20}|})$.

Solution.
We have

$$\begin{aligned}
\bar{A}_{75:\overline{20}|} =& \bar{A}^{\,1}_{75:\overline{20}|} + A_{75:\overline{20}|}^{\quad\ 1} = 0.25285 + 0.22073 = 0.47358\\
\ddot{a}_{75:\overline{20}|} =& 10.92242\\
P(\bar{A}_{75:\overline{20}|}) =& \frac{0.47358}{10.92242} = 0.0434 \blacksquare
\end{aligned}$$

Now, if the benefit is a continuous $n-$year endowment and the payments are made as a discrete $t-$year temporary life annuity-due then the loss random variable of this insurance is

$${}_t\overline{SL}_{x:\overline{n}|} = \nu^{\min(n,T)} - P\ddot{a}_{\overline{\min(K+1,t)}|} = \bar{Z}_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{t}|}.$$

The actuarial present value of the loss is

$$E({}_t\overline{SL}_{x:\overline{n}|}) = \bar{A}_{x:\overline{n}|} - P\ddot{a}_{x:\overline{t}|}.$$

The benefit premium that satisfies the equivalence principle is given by

$${}_tP(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{t}|}}.$$

Example 42.8
You are given that mortality follows De Moivre's Law with $\omega = 125$ and $\delta = 0.05$. Calculate ${}_{10}P(\bar{A}_{75:\overline{20|}})$.

Solution.
We have

$$\begin{aligned}\bar{A}_{75:\overline{20|}} =&\bar{A}^{1}_{75:\overline{20|}} + A_{75:\overset{\ \ 1}{\overline{20|}}} = 0.25285 + 0.22073 = 0.47358\\ \ddot{a}_{75:\overline{10|}} =&7.4079\\ {}_{10}P(\bar{A}_{75:\overline{20|}}) =&\frac{0.47358}{7.4079} = 0.064 \ \blacksquare\end{aligned}$$

If uniform distribution of deaths is assumed over each year of age then

$$P(\bar{A}_{x:\overline{n|}}) = \frac{i}{\delta}P(A^{1}_{x:\overline{n|}}) + P(A_{x:\overset{\ 1}{\overline{n|}}}).$$

Example 42.9
You are given that mortality follows the Illustrative Life Table with $i = 6\%$. Assume that mortality is uniformly distributed between integral ages. Calculate $P(\bar{A}_{50:\overline{20|}})$.

Solution.
We have

$$\begin{aligned}P(\bar{A}_{50:\overline{20|}}) =&\frac{\frac{i}{\delta}A_{50:\overline{20|}} + {}_{20}E_{50}}{\ddot{a}_{50:\overline{20|}}}\\ =&\frac{\frac{i}{\delta}[A_{50} - {}_{20}E_{50}A_{70}] + {}_{20}E_{50}}{\ddot{a}_{50} - {}_{20}E_{50}\ddot{a}_{70}}\\ =&\frac{(1.02791)[0.24905 - (0.23047)(0.51495)] + 0.23047}{13.2668 - (0.23047)(8.5693)} = 0.03228 \ \blacksquare\end{aligned}$$

Remark 42.1
There is no need for a semicontinuous annual premium $n-$year pure endowment since no death benefit is involved.

Practice Problems

Problem 42.12
You are given that $\mu = 0.02$ and $\delta = 0.05$. Calculate $P(\bar{A}_{75:\overline{20}|})$.

Problem 42.13
You are given that $\mu = 0.02$ and $\delta = 0.05$. Calculate ${}_{10}P(\bar{A}_{75:\overline{20}|})$.

Problem 42.14
You are given that mortality follows the Illustrative Life Table with $i = 6\%$. Assume that mortality is uniformly distributed between integral ages. Calculate $P(\bar{A}_{75:\overline{20}|})$.

Problem 42.15
Simplify $\frac{P(\bar{A}_{x:\overline{n}|}-{}_nP(\bar{A}_x)}{P(A_{x:\overline{n}|}^{\ \ 1})}$.

Problem 42.16
Show that ${}_tP(\bar{A}_{x:\overline{n}|}) - {}_tP(A_{x:\overline{n}|}^{\ \ 1}) = {}_tP(\bar{A}^1_{x:\overline{n}|})$.

42.4 Semicontinuous $n-$year Deferred Insurance

For this type of insurance, the benefit is a continuous $n-$year deferred insurance and the payments are made as a discrete whole life annuity-due. The loss random variable of this insurance is

$$ {}_{n|}\overline{SL}_x = \nu^T \mathbf{I}(T > n) - P\ddot{a}_{\overline{K+1}|} = {}_{n|}\bar{Z}_x - P\ddot{Y}_x. $$

The actuarial present value of the loss is

$$ E({}_{n|}\overline{SL}_x) = {}_{n|}\bar{A}_x - P\ddot{a}_x. $$

The benefit premium that satisfies the equivalence principle is given by

$$ P({}_{n|}\bar{A}_x) = \frac{{}_{n|}\bar{A}_x}{\ddot{a}_x}. $$

Example 42.10
You are given that mortality follows De Moivre's Law with $\omega = 125$ and $\delta = 0.05$. Calculate $P({}_{10|}\bar{A}_{75})$.

Solution.
We have

$$ \begin{aligned} {}_{10|}\bar{A}_{75} =& \nu^{10}{}_{10}p_{75}\bar{A}_{85} = e^{-10\delta}\frac{\bar{a}_{\overline{40}|}}{50} \\ =& e^{-10(0.05)}\frac{1-e^{-40(0.05)}}{50(0.05)} = 0.20978 \\ \ddot{a}_{75} =& 13.16238 \\ P({}_{10|}\bar{A}_{75}) =& \frac{0.20978}{13.16238} = 0.0159 \blacksquare \end{aligned} $$

If uniform distribution of deaths is assumed over each year of age then

$$ P({}_{n|}\bar{A}_x) = \frac{i}{\delta}P({}_{n|}A_x). $$

Now, if the benefit is a continuous $n-$year deferred insurance and the payments are made as a discrete $t-$year temporary annuity-due, then the loss random variable of this insurance is

$$ {}_t({}_{n|}\overline{SL}_x) = \nu^T \mathbf{I}(T > n) - P\ddot{a}_{x:\overline{\min\{(K+1,t)\}}|} = {}_{n|}\bar{Z}_x - P\ddot{Y}_{x:\overline{t}|}. $$

The actuarial present value of the loss is

$$E({}_t({}_{n|}\overline{SL}_x)) = {}_{n|}\bar{A}_x - P\ddot{a}_{x:\overline{t|}}.$$

The benefit premium that satisfies the equivalence principle is given by

$${}_tP({}_{n|}\bar{A}_x) = \frac{{}_{n|}\bar{A}_x}{\ddot{a}_{x:\overline{t|}}}.$$

Practice Problems

Problem 42.17
You are given that $\mu = 0.02$ and $\delta = 0.05$. Calculate $P({}_{10|}\bar{A}_{75})$.

Problem 42.18
You are given that mortality follows the Illustrative Life Table with $i = 6\%$. Assume that mortality is uniformly distributed between integral ages. Calculate $P({}_{20|}\bar{A}_{50})$.

Problem 42.19
Show that

$$P({}_{n|}\bar{A}_x) = \frac{A_{x:\overline{n|}}^{\ \ 1}}{\ddot{a}_{x:\overline{n|}} + {}_nE_x\ddot{a}_{x+n}}\bar{A}_{x+n}.$$

Problem 42.20 ‡
On January 1, 2002, Pat, age 40, purchases a 5-payment, 10-year term insurance of 100,000:
(i) Death benefits are payable at the moment of death.
(ii) Contract premiums of 4000 are payable annually at the beginning of each year for 5 years.
(iii) $i = 0.05$
(iv) L is the loss random variable at time of issue.
Calculate the value of L if Pat dies on June 30, 2004.

43 m^{thly} Benefit Premiums

In this section we consider life insurances where either the benefit is paid at the moment of death or at the end of the year of death and m^{thly} premium payments paid at the beginning of each $\frac{1}{m}^{\text{th}}$ of a period while the individual is alive. Typically, m is 2, 4, or 12.

43.1 m^{thly} Payments with Benefit Paid at Moment of Death

In this subsection, we consider insurance paid at time of death and funded with level payments made at the beginning of each $m-$thly period while the individual is alive. Such premiums are called **true fractional premiums**. The notation $P^{(m)}$ stands for the **true level annual benefit premium** payable in $m-$thly installments at the beginning of each $m-$thly period while the individual is alive so that the $m-$thly payment is $\frac{P^{(m)}}{m}$. We assume the equivalence principle is used.
For a whole life insurance on (x), the loss random variable is

$$\bar{L}_x^{(m)} = \nu^T - P\ddot{Y}_x^{(m)}.$$

The actuarial present value of the loss is

$$E(\bar{L}_x^{(m)}) = \bar{A}_x - P\ddot{a}_x^{(m)}.$$

The true level annual benefit premium is

$$P^{(m)}(\bar{A}_x) = \frac{\bar{A}_x}{\ddot{a}_x^{(m)}}.$$

For a whole life insurance to (x) funded for t years, the loss random variable is

$${}_t\bar{L}_x^{(m)} = \nu^T - P\ddot{Y}_{x:\overline{t|}}^{(m)},$$

where

$$\ddot{Y}_{x:\overline{t|}}^{(m)} = \frac{1 - Z_{x:\overline{n|}}^{(m)}}{d^{(m)}}.$$

The actuarial present value of the loss is

$$E({}_t\bar{L}_x^{(m)}) = \bar{A}_x - P\ddot{a}_{x:\overline{t|}}^{(m)}.$$

The true level annual benefit premium is

$$ {}_tP^{(m)}(\bar{A}_x) = \frac{\bar{A}_x}{\ddot{a}^{(m)}_{x:\overline{t}|}}. $$

For an $n-$year term insurance on (x), the loss random variable is

$$ \bar{L}^{(m)}{}^{1}_{x:\overline{n}|} = \nu^T \mathbf{I}(0 \le T \le n) - P\ddot{Y}^{(m)}_{x:\overline{n}|}. $$

The actuarial present value of the loss is

$$ E(\bar{L}^{(m)}{}^{1}_{x:\overline{n}|}) = \bar{A}^{1}_{x:\overline{n}|} - P\ddot{a}^{(m)}_{x:\overline{n}|}. $$

The true level annual benefit premium is

$$ P^{(m)}(\bar{A}^{1}_{x:\overline{n}|}) = \frac{\bar{A}^{1}_{x:\overline{n}|}}{\ddot{a}^{(m)}_{x:\overline{n}|}}. $$

For an $n-$year term insurance on (x), funded for $t-$years, the loss random variable is

$$ {}_t\bar{L}^{(m)}{}^{1}_{x:\overline{n}|} = \nu^T \mathbf{I}(0 \le T \le n) - P\ddot{Y}^{(m)}_{x:\overline{t}|}. $$

The actuarial present value of the loss is

$$ E({}_t\bar{L}^{(m)}{}^{1}_{x:\overline{n}|}) = \bar{A}^{1}_{x:\overline{n}|} - P\ddot{a}^{(m)}_{x:\overline{t}|}. $$

The true level annual benefit premium is

$$ {}_tP^{(m)}(\bar{A}^{1}_{x:\overline{n}|}) = \frac{\bar{A}^{1}_{x:\overline{n}|}}{\ddot{a}^{(m)}_{x:\overline{t}|}}. $$

For an $n-$year endowment insurance on (x), the loss random variable is

$$ \bar{L}^{(m)}_{x:\overline{n}|} = \nu^{\min(T,n)} - P\ddot{Y}^{(m)}_{x:\overline{n}|}. $$

The actuarial present value of the loss is

$$ E(\bar{L}^{(m)}_{x:\overline{n}|}) = \bar{A}_{x:\overline{n}|} - P\ddot{a}^{(m)}_{x:\overline{n}|}. $$

The true level annual benefit premium is

$$ P^{(m)}(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x:\overline{n}|}}{\ddot{a}^{(m)}_{x:\overline{n}|}}. $$

For an endowment insurance to (x) funded for t years, the loss random variable is

$$ {}_t\bar{L}^{(m)}_{x:\overline{n}|} = \nu^{\min(T,n)} - P\ddot{Y}^{(m)}_{x:\overline{t}|}. $$

The actuarial present value of the loss is

$$ E({}_t\bar{L}^{(m)}_{x:\overline{n}|}) = \bar{A}_{x:\overline{n}|} - P\ddot{a}^{(m)}_{x:\overline{t}|}. $$

The true level annual benefit premium is

$$ {}_tP^{(m)}(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x:\overline{n}|}}{\ddot{a}^{(m)}_{x:\overline{t}|}}. $$

For an $n-$year deferred insurance on (x), the loss random variable is

$$ {}_{n|}\bar{L}^{(m)}_x = \nu^T\mathbf{I}(T > n) - P\ddot{Y}^{(m)}_x. $$

The actuarial present value of the loss is

$$ E({}_{n|}\bar{L}^{(m)}_x) = {}_{n|}\bar{A}_x - P\ddot{a}^{(m)}_x. $$

The true level annual benefit premium is

$$ P^{(m)}({}_{n|}\bar{A}_x) = \frac{{}_{n|}\bar{A}_x}{\ddot{a}^{(m)}_x}. $$

For an $n-$year deferred insurance on (x) funded for t years the loss random variable is

$$ {}_t({}_{n|}\bar{L}^{(m)}_x) = \nu^T\mathbf{I}(T > n) - P\ddot{Y}^{(m)}_{x:\overline{t}|}. $$

The actuarial present value of the loss is

$$ E({}_t({}_{n|}\bar{L}^{(m)}_x)) = {}_{n|}\bar{A}_x - P\ddot{a}^{(m)}_{x:\overline{t}|}. $$

The true level annual benefit premium is

$$ {}_tP^{(m)}({}_{n|}\bar{A}_x) = \frac{{}_{n|}\bar{A}_x}{\ddot{a}^{(m)}_{x:\overline{t}|}}. $$

Example 43.1

Calculate the true level annual benefit premium payable quarterly for a whole life insurance on (50) with benefits of 10,000 paid at the moment of death. Assume a uniform distribution of deaths in each year of age.

Solution.
Under the uniform distribution of deaths assumption we have

$$\begin{aligned}
\bar{A}_{50} =& \frac{i}{\delta}A_{50} = \frac{0.06}{\ln(0.06)}(0.24905) = 0.2564\\
\ddot{a}_{50}^{(4)} =& \frac{id}{i^{(4)}d^{(4)}}\ddot{a}_{50} - \frac{i - i^{(4)}}{i^{(4)}d^{(4)}}\\
=& \frac{(0.06)^2(1.06)^{-1}}{16[(1.06)^{\frac{1}{4}} - 1][1 - (1.06)^{-\frac{1}{4}}]}(13.2668) - \frac{0.06 - 4[(1.06)^{\frac{1}{4}} - 1]}{16[(1.06)^{\frac{1}{4}} - 1][1 - (1.06)^{-\frac{1}{4}}]}\\
=& 12.8861
\end{aligned}$$

The true level annual benefit premium is

$$10000P^{(4)}(A_{50}) = 10000\frac{A_{50}}{\ddot{a}_{50}^{(4)}} = 10000\left(\frac{0.2564}{12.8861}\right) = 198.97 \blacksquare$$

Practice Problems

Problem 43.1
Calculate the true level annual benefit premium payable quarterly for a 20-year payments whole life insurance on (50) with benefits of 10,000 paid at the moment of death. Assume a uniform distribution of deaths in each year of age.

Problem 43.2
Calculate the true level annual benefit premium payable quarterly for a 20-year term life insurance on (50) with benefits of 10,000 paid at the moment of death. Assume a uniform distribution of deaths in each year of age.

Problem 43.3
Calculate the true level annual benefit premium payable quarterly for a 20-year endowment life insurance on (50) with benefits of 10,000 paid at the moment of death. Assume a uniform distribution of deaths in each year of age.

Problem 43.4
Calculate the true level annual benefit premium payable quarterly for a 10-year funded 20-year endowment life insurance on (50) with benefits of 10,000 paid at the moment of death. Assume a uniform distribution of deaths in each year of age.

Problem 43.5
Calculate the true level annual benefit premium payable quarterly for a 20-year deferred life insurance on (50) with benefits of 10,000 paid at the moment of death. Assume a uniform distribution of deaths in each year of age.

43.2 m^{thly} Payments with Benefit Paid at End of Year of Death

In this subsection, we consider insurance paid at the end of year of death and funded with level payments made at the beginning of each $m-$thly period while the individual is alive. We assume the equivalence principle is used.
For a whole life insurance on (x), the loss random variable is

$$L_x^{(m)} = \nu^{K+1} - P\ddot{Y}_x^{(m)}.$$

The actuarial present value of the loss is

$$E(L_x^{(m)}) = A_x - P\ddot{a}_x^{(m)}.$$

The true level annual benefit premium is

$$P^{(m)}(A_x) = \frac{A_x}{\ddot{a}_x^{(m)}}.$$

For a whole life insurance to (x) funded for t years the annual benefit premium is

$${}_tP^{(m)}(A_x) = \frac{A_x}{\ddot{a}_{x:\overline{t}|}^{(m)}}.$$

For an $n-$year term insurance on (x), the loss random variable is

$$L^{(m)}{}^{\,1}_{x:\overline{n}|} = \nu^{K+1}\mathbf{I}(0 \leq K \leq n-1) - P\ddot{Y}_{x:\overline{n}|}^{(m)}.$$

The actuarial present value of the loss is

$$E(L^{(m)}{}^{\,1}_{x:\overline{n}|}) = A^1_{x:\overline{n}|} - P\ddot{a}_{x:\overline{n}|}^{(m)}.$$

The true level annual benefit premium is

$$P^{(m)}(A^1_{x:\overline{n}|}) = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}^{(m)}}.$$

For an $n-$year endowment insurance on (x), the loss random variable is

$$L_{x:\overline{n}|} = \nu^{\min(K+1,n)} - P\ddot{Y}_{x:\overline{n}|}^{(m)}.$$

The actuarial present value of the loss is

$$E(L_{x:\overline{n}|}) = A_{x:\overline{n}|} - P\ddot{a}^{(m)}_{x:\overline{n}|}.$$

The true level annual benefit premium is

$$P^{(m)}(A_{x:\overline{n}|}) = \frac{A_{x:\overline{n}|}}{\ddot{a}^{(m)}_{x:\overline{n}|}}.$$

For an endowment insurance to (x) funded for t years the annual benefit premium is

$${}_tP^{(m)}(A_{x:\overline{n}|}) = \frac{A_{x:\overline{n}|}}{\ddot{a}^{(m)}_{x:\overline{t}|}}.$$

For an $n-$year pure endowment insurance on (x), the loss random variable is

$$L_{x:\overset{1}{\overline{n}|}} = \nu^n \mathbf{I}(K \geq n) - P\ddot{Y}^{(m)}_{x:\overline{n}|}.$$

The actuarial present value of the loss is

$$E(L_{x:\overset{1}{\overline{n}|}}) = A_{x:\overset{1}{\overline{n}|}} - P\ddot{a}^{(m)}_{x:\overline{n}|}.$$

The true level annual benefit premium is

$$P^{(m)}(A_{x:\overset{1}{\overline{n}|}}) = \frac{A_{x:\overset{1}{\overline{n}|}}}{\ddot{a}^{(m)}_{x:\overline{n}|}}.$$

For a pure endowment insurance to (x) funded for t years the annual benefit premium is

$${}_tP^{(m)}(A_{x:\overset{1}{\overline{n}|}}) = \frac{A_{x:\overset{1}{\overline{n}|}}}{\ddot{a}^{(m)}_{x:\overline{t}|}}.$$

Example 43.2
Calculate the true level annual benefit premium payable semi-annually for a whole life insurance on (50) with benefits of 10,000 paid at the end of year of death. Assume a uniform distribution of deaths in each year of age.

Solution.
Under the uniform distribution of deaths assumption we have

$$
\begin{aligned}
A_{50} =& 0.24905\\
\ddot{a}_{50}^{(2)} =& \frac{id}{i^{(2)}d^{(2)}}\ddot{a}_{50} - \frac{i-i^{(2)}}{i^{(2)}d^{(2)}}\\
=& \frac{(0.06)^2(1.06)^{-1}}{4[(1.06)^{\frac{1}{2}}-1][1-(1.06)^{-\frac{1}{2}}]}(13.2668) - \frac{0.06-2[(1.06)^{\frac{1}{2}}-1]}{4[(1.06)^{\frac{1}{2}}-1][1-(1.06)^{-\frac{1}{2}}]}\\
=& 13.0122
\end{aligned}
$$

The true level annual benefit premium is

$$
10000P^{(2)}(\bar{A}_{50}) = 10000\frac{\bar{A}_{50}}{\ddot{a}_{50}^{(2)}} = 10000\left(\frac{0.24905}{13.0122}\right) = 191.40 \blacksquare
$$

Practice Problems

Problem 43.6
Calculate the true level annual benefit premium payable semi-annually for a 20-year payments whole life insurance on (50) with benefits of 10,000 paid at the end of year of death. Assume a uniform distribution of deaths in each year of age.

Problem 43.7
Calculate the true level annual benefit premium payable semi-annually for a 20-year term life insurance on (50) with benefits of 10,000 paid at the end of year of death. Assume a uniform distribution of deaths in each year of age.

Problem 43.8
Calculate the true level annual benefit premium payable semi-annually for a 20-year endowment life insurance on (50) with benefits of 10,000 paid at the end of year of death. Assume a uniform distribution of deaths in each year of age.

Problem 43.9
Calculate the true level annual benefit premium payable semi-annually for a 10-year funded 20-year endowment life insurance on (50) with benefits of 10,000 paid at the end of year of death. Assume a uniform distribution of deaths in each year of age.

44 Non-Level Benefit/Premium Payments and the Equivalence Principle

Some of the problems that appeared on the MLC actuarial exam involve the application of the equivalence principle to variable benefits or premiums. The goal of this section is to address these problems.

Example 44.1 ‡
A fund is established by collecting an amount P from each of 100 independent lives age 70. The fund will pay the following benefits:
• 10 payable at the end of the year of death, for those who die before the age of 72, or
• P, payable at age 72, to those who survive.
You are given:
(i) Mortality follows the Illustrative Life Table.
(ii) $i = 0.08$
Calculate P using the equivalence principle.

Solution.
By the equivalence principle, we must have

$$\text{APV (Payments)} = \text{APV(Benefits)}$$

From the information in the problem, we have that APV(Payments) $= P$. For the actuarial present value of the benefits, recall that for a benefit of b_t at time t, the actuarial present value is

$$b_t\nu^t\text{Pr(Benefit)}.$$

The set of benefits consists of a benefit of 10 if the individual aged 70 dies before reaching age 71, another 10 if the individual dies before reaching age 72, and a benefit of P if the individual is alive at age 72. Hence,

$$\begin{gathered}\text{APV(Benefit)}\\ = 10\nu\Pr(K(70) = 0) + 10\nu^2\Pr(K(70) = 1) + P\nu^2\Pr(K(70) \geq 2)\end{gathered}$$

which can be expressed mathematically as

$$\text{APV(Benefit)} = 10\nu q_{70} + 10\nu^2 p_{70}q_{71} + P\nu^2\,{}_2p_{70}.$$

Using the equivalence principle and the Illustrative Life Table, we have

$$\begin{aligned}P =&10\nu q_{70} + 10\nu^2 p_{70}q_{71} + P{}_2p_{70}\\ =&10\nu q_{70} + 10\nu^2 p_{70}q_{71} + P\nu^2 p_{70}p_{71}\\ P =&\frac{10\nu q_{70} + 10\nu^2 p_{70}q_{71}}{1-\nu^2 p_{70}p_{71}}\\ =&\frac{10(1.08)^{-1}(0.03318) + 10(1.08)^{-2}(1-0.03318)(0.03626)}{1-(1.08)^{-2}(1-0.03318)(1-0.03626)}\\ =&3.02132 \blacksquare\end{aligned}$$

Example 44.2 ‡
Two actuaries use the same mortality table to price a fully discrete 2-year endowment insurance of 1000 on (x).
(i) Kevin calculates non-level benefit premiums of 608 for the first year and 350 for the second year.
(ii) Kira calculates level annual benefit premiums of π.
(iii) $d = 0.05$
Calculate π.

Solution.
We have:
Kevin:

$$\begin{aligned}\text{APVB} =&1000A_{x:\overline{2}|} = 1000(1-d\ddot{a}_{x:\overline{2}|})\\ =&1000[1-0.05(1+\nu p_x)] = 950 - 47.5p_x\\ \text{APVP} =&608 + 350\nu p_x = 608 + 332.50p_x.\end{aligned}$$

Using the equivalence principle, we have

$$608 + 332.50p_x = 950 - 47.5p_x \Longrightarrow p_x = 0.90.$$

Kira:

$$\begin{aligned}\text{APVB} =&950 - 47.5p_x = 907.25\\ \text{APVP} =&\pi + \nu\pi p_x = 1.855\pi.\end{aligned}$$

Using the equivalence principle, we have

$$1.855\pi = 907.25 \Longrightarrow \pi = 489.08 \blacksquare$$

Example 44.3 ‡
For a special 3-year deferred whole life annuity-due on (x) :
(i) $i = 0.04$
(ii) The first annual payment is 1000.
(iii) Payments in the following years increase by 4% per year.
(iv) There is no death benefit during the three year deferral period.
(v) Level benefit premiums are payable at the beginning of each of the first three year.
(vi) $e_x = 11.05$ is the curtate expectation of life for (x).
(vii)

k	1	2	3
${}_kp_x$	0.99	0.98	0.97

Calculate the annual benefit premium.

Solution.
The actuarial present value of benefits is

$$\begin{aligned}\text{APVB} =&0.97\nu^3[1000 + 1000(1.04)\nu p_{x+3} + 1000(1.04)^2\nu^2{}_2p_{x+3}\\&+1000(1.04)^3\nu^3{}_3p_{x+3} + \cdots]\\=&970\nu^3[1 + p_{x+3} + {}_2p_{x+3} + {}_3p_{x+3} + \cdots]\end{aligned}$$
$= 970\nu^3(1 + e_{x+3})$.

Now, using Problem 20.50 repeatedly, we can derive the formula

$$e_x = p_x + {}_2p_x + {}_3p_x + {}_3p_xe_{x+3}.$$

Substituting, we find

$$11.05 = 0.99 + 0.98 + 0.97 + (0.97)e_{x+3} \Longrightarrow e_{x+3} = 8.360825.$$

Hence, the actuarial present value of benefits is

$$\text{APVB} = 970\nu^3(1 + e_{x+3}) = 970(1.04)^3(8.360825) = 8072.09.$$

Now, let π be the annual benefit premium. Than the actuarial present value of premiums is

$$\text{APVP} = \pi\ddot{a}_{x:\overline{3}|} = \pi(1 + \nu p_x + {}_2p_x\nu^2) = 2.857988\pi.$$

By the equivalence principle, we have

$$2.857988\pi = 8072.09 \Longrightarrow \pi = 2824.39 \blacksquare$$

Example 44.4 ‡
For a special fully discrete 3-year term insurance on (x) :
(i)

k	q_{x+k}	b_{k+1}
0	0.200	0
1	0.100	10,000
2	0.097	9,000

(ii) $i = 0.06$.
Calculate the level annual benefit premium for this insurance.

Solution.
Let π denote the level annual benefit premium. We have

$$\begin{aligned}\text{APVB} =&10,000p_xq_{x+1}\nu^2 + 9,000{}_2p_xq_{x+2}\\ =&10,000(0.8)(0.1)(1.06)^{-2} + 9,000(0.8)(0.9)(0.97)(1.06)^{-3}\\ =&1,239.75\\ \ddot{a}_{x:\overline{3}|} =&1 + \nu p_x + \nu^2{}_2p_x\\ =&1 + (0.8)(1.06)^{-1} + (0.8)(0.9)(1.06)^{-2} = 2.3955\\ \pi =&\frac{1,239.75}{2.3955} = 517.53 \blacksquare\end{aligned}$$

Example 44.5 ‡
For a special fully discrete 35-payment whole life insurance on (30):
(i) The death benefit is 1 for the first 20 years and is 5 thereafter.
(ii) The initial benefit premium paid during the each of the first 20 years is one fifth of the benefit premium paid during each of the 15 subsequent years.
(iii) Mortality follows the Illustrative Life Table.
(iv) $i = 0.06$
(v) $A_{30:\overline{20}|} = 0.32307$
(vi) $\ddot{a}_{30:\overline{35}|} = 14.835$
Calculate the initial annual benefit premium.

Solution.
Let π denote the intitial benefit premium. The actuarial present value of benefits is

$$\begin{aligned}\text{APVB} =&5A_{30} - 4A^{\,1}_{30:\overline{20}|} = 5A_{30} - 4(A_{30:\overline{20}|} - {}_{20}E_{30})\\ =&5(0.10248) - 4(0.32307 - 0.29347) = 0.39508.\end{aligned}$$

The actuarial present value of premiums is

$$\begin{aligned}\text{APVP} =&\pi[5\ddot{a}_{30:\overline{35}|} - 4\ddot{a}_{30:\overline{20}|}] = \pi\left[5\ddot{a}_{30:\overline{35}|} - 4\left(\frac{1 - A_{30:\overline{20}|}}{d}\right)\right]\\ =&\pi\left[5(14.835) - 4\left(\frac{1 - 0.32307}{0.06(1.06)^{-1}}\right)\right] = 26.339\pi.\end{aligned}$$

By the equivalence principle, we have

$$26.339\pi = 0.39508 \Longrightarrow \pi = 0.015 \blacksquare$$

Practice Problems

Use the following information to answer Problems 44.1 - 44.5:
For a special fully discrete 10-payment whole life insurance on (30) with level annual benefit premium π :
(i) The death benefit is equal to 1000 plus the refund, without interest, of the benefit premiums paid
(ii) $A_{30} = 0.102$
(iii) ${}_{10|}A_{30} = 0.088.$
(iv) $(IA)^{1}_{30:\overline{10}|} = 0.078$
(v) $\ddot{a}_{30:\overline{10}|} = 7.747.$

Problem 44.1
Write an expression for the actuarial present value of the premiums in terms of π.

Problem 44.2
What is the actuarial present value of the death benefit without the refund?

Problem 44.3
Up to year 10, the return on premiums is a (discrete) increasing 10−year term insurance on (30). What is the actuarial present value of this insurance?

Problem 44.4
Starting in ten years, the return of premium is just a discrete deferred whole life insurance. Find an expression for the actuarial present value of this insurance.

Problem 44.5
Apply the equivalence principle to determine the value of π.

Problem 44.6 ‡
For a special fully discrete whole life insurance of 1000 on (40):
(i) The level benefit premium for each of the first 20 years is π.
(ii) The benefit premium payable thereafter at age x is $1000\nu q_x,\ x = 60, 61, 62, \cdots$
(iii) Mortality follows the Illustrative Life Table.
(iv) $i = 0.06$
Calculate π.

Problem 44.7 ‡
For a fully discrete whole life insurance of 1,000 on (40), you are given:
(i) $i = 0.06$
(ii) Mortality follows the Illustrative Life Table.
(iii) $\ddot{a}_{40:\overline{10}|} = 7.70$
(iv) $\ddot{a}_{50:\overline{10}|} = 7.57$
(v) $1000A^{\,1}_{40:\overline{20}|} = 60$
At the end of the 10th year, the insured elects an option to retain the coverage of 1,000 for life, but pay premiums for the next 10 years only.
Calculate the revised annual benefit premium for the next 10 years.

Problem 44.8 ‡
For a special whole life insurance on (35), you are given:
(i) The annual benefit premium is payable at the beginning of each year.
(ii) The death benefit is equal to 1000 plus the return of all benefit premiums paid in the past without interest.
(iii) The death benefit is paid at the end of the year of death.
(iv) $A_{35} = 0.42898$
(v) $(IA)_{35} = 6.16761$
(vi) $i = 0.05$
Calculate the annual benefit premium for this insurance.

Problem 44.9 ‡
For a special 2-payment whole life insurance on (80):
(i) Premiums of π are paid at the beginning of years 1 and 3.
(ii) The death benefit is paid at the end of the year of death.
(iii) There is a partial refund of premium feature:
If (80) dies in either year 1 or year 3, the death benefit is $1000+\frac{\pi}{2}$. Otherwise, the death benefit is 1000.
(iv) Mortality follows the Illustrative Life Table.
(v) $i = 0.06$
Calculate π, using the equivalence principle.

Problem 44.10 ‡
For a special 3-year term insurance on (30), you are given:
(i) Premiums are payable semiannually.
(ii) Premiums are payable only in the first year.
(iii) Benefits, payable at the end of the year of death, are:

k	b_{k+1}
0	1000
1	500
2	250

(iv) Mortality follows the Illustrative Life Table.
(v) Deaths are uniformly distributed within each year of age.
(vi) $i = 0.06$
Calculate the amount of each semiannual benefit premium for this insurance.

Problem 44.11 ‡
For a special fully discrete 2-year endowment insurance of 1000 on (x), you are given:
(i) The first year benefit premium is 668.
(ii) The second year benefit premium is 258.
(iii) $d = 0.06$
Calculate the level annual premium using the equivalence principle.

Problem 44.12 ‡
For a special fully discrete 5-year deferred whole life insurance of 100,000 on (40), you are given:
(i) The death benefit during the 5-year deferral period is return of benefit premiums paid without interest.
(ii) Annual benefit premiums are payable only during the deferral period.
(iii) Mortality follows the Illustrative Life Table.
(iv) $i = 0.06$
(v) $(IA)^{1}_{40:\overline{5}|} = 0.04042$.
Calculate the annual benefit premiums.

Problem 44.13 ‡
A group of 1000 lives each age 30 sets up a fund to pay 1000 at the end of the first year for each member who dies in the first year, and 500 at the end of the second year for each member who dies in the second year. Each member pays into the fund an amount equal to the single benefit premium for a special 2-year term insurance, with: (i) Benefits:

k	b_{k+1}
0	1000
1	500

(ii) Mortality follows the Illustrative Life Table.
(iii) $i = 0.06$
The actual experience of the fund is as follows:

k	Interest Rate Earned	Number of Deaths
0	0.070	1
1	0.069	1

Calculate the difference, at the end of the second year, between the expected size of the fund as projected at time 0 and the actual fund.

Problem 44.14 ‡
For a special fully discrete 30-payment whole life insurance on (45), you are given:
(i) The death benefit of 1000 is payable at the end of the year of death.
(ii) The benefit premium for this insurance is equal to $1000P(A_{45})$ for the first 15 years followed by an increased level annual premium of π for the remaining 15 years.
(iii) Mortality follows the Illustrative Life Table.
(iv) $i = 0.06$
Calculate π.

Problem 44.15 ‡
For a special fully discrete 20-year term insurance on (30):
(i) The death benefit is 1000 during the first ten years and 2000 during the next ten years.
(ii) The benefit premium, determined by the equivalence principle, is π for each of the first ten years and 2π for each of the next ten years.
(iii) $\ddot{a}_{30:\overline{20}|} = 15.0364$
(iv)

x	$\ddot{a}_{x:\overline{10}\|}$	$1000A^{1}_{x:\overline{10}\|}$
30	8.7201	16.66
40	8.6602	32.61

Calculate π.

Problem 44.16 ‡
For an increasing 10-year term insurance, you are given:

(i) $b_{k+1} = 100,000(1+k),\ \ k = 0, 1, \cdots, 9$
(ii) Benefits are payable at the end of the year of death.
(iii) Mortality follows the Illustrative Life Table.
(iv) $i = 0.06$
(v) The single benefit premium for this insurance on (41) is 16,736.
Calculate the single benefit premium for this insurance on (40).

45 Percentile Premium Principle

Besides the equivalence principle, there are other premium principles that are sometimes considered. In this section, we introduce the percentile premium principle. With the percentile premium principle, premium is assessed to ensure the insurer suffers financial loss with sufficiently low probability.
If we denote the insurer's loss random variable by L, then the percentile premium is the smallest premium π so that $\Pr(L > 0) \leq \alpha$ for some predetermined $\alpha \in (0, 1)$, i.e., the probability of a positive loss or simply loss is at most α. We call π the $100\alpha-$percentile premium of the policy.
We will restrict our discussion to whole life insurance. The other types of insurance can be treated similarly.
Now, for a fully continuous whole life insurance the loss random variable is

$$\bar{L}_x = \nu^T - \pi \bar{a}_{\overline{T|}},\ T \geq 0.$$

If we graph $\bar{L}_x$ versus T, we find a decreasing graph starting from 1 at time $T = 0$ and decreasing to $-\frac{\pi}{\delta} < 0$. Thus, there is a t such that $\bar{L}_x(t) = 0$. Hence, we can write

$$\Pr(\bar{L}_x > 0) = \Pr(T \leq t) \leq \alpha.$$

But this implies that t is the $100\alpha-$th percentile of the remaining lifetime random variable. Since T is continuous, we find t by solving the equation

$$F_{T(x)}(t) = \alpha.$$

Once t is found, the premium π satisfies the equation

$$0 = \bar{L}_x(t) = \nu^t - \pi \bar{a}_{\overline{t|}}$$

Example 45.1
Consider the fully-continuous whole life insurance on (x) with a constant force of mortality μ and a constant force of interest δ. Derive an expression for the premium π using the percentile premium principle with $\alpha = 0.5$. That is, find the 50-th percentile premium of the policy.

Solution.
First, we find t :

$$e^{-\mu t} = 0.5 \Longrightarrow t = -\frac{\ln 0.5}{\mu}.$$

The percentile premium satisfies the equation

$$e^{-\delta\times\frac{-\ln 0.5}{\mu}} - \pi\left(\frac{1-e^{-\delta\times\frac{-\ln 0.5}{\mu}}}{\delta}\right) = 0.$$

Solving this equation, we find

$$\pi = \frac{\delta(0.5)^{\frac{\delta}{\mu}}}{1-(0.5)^{\frac{\delta}{\mu}}} \blacksquare$$

Next, for a fully discrete whole life insurance, the loss random variable is given by

$$L_x = \nu^{K+1} - \pi\ddot{a}_{\overline{K+1}|}.$$

Clearly, L_x is a decreasing function of K and L_x crosses the $K-$axis, say at k. Hence, we can write

$$\Pr(L_x > 0) = \Pr(K \le k) \le \alpha.$$

To determine k, we first note that

$$\Pr(K \le k) = 1 - \Pr(K > k) = 1 - {}_{k+1}p_x = 1 - \frac{\ell_{x+k+1}}{\ell_x}.$$

We want to find the largest k such that

$$\tfrac{\ell_{x+k+1}}{\ell_x} > 1 - \alpha \text{ or } \ell_{x+k+1} > (1-\alpha)\ell_x.$$

This is accomplished by using the Illustrative Life table. Once, k is found, the percentile premium is the solution to the equation

$$0 = L_x(k) = \nu^{k+1} - \pi\ddot{a}_{\overline{k+1}|}.$$

Example 45.2
Using the Illustrative Life Table for (35), find the smallest premium such that the probability of loss of a discrete whole life insurance issued to (35) is less than 0.5.

Solution.
We first find k such that

$$\ell_{35+k+1} > 0.5\ell_{35} = 0.5(9,420,657) = 4,710,328.50.$$

From the Illustrative Life Table, we see that $\ell_{78} = 4,530,360 < 4,710,328.50 < 4,828,182 = \ell_{77}$ so that $k = 42$. Now, we find the premium by solving the equation

$$\nu^{43} - \pi\ddot{a}_{\overline{43}|} = 0 \Longrightarrow \pi = \frac{1}{\ddot{s}_{\overline{43}|}} = 0.005031 \blacksquare$$

Now, consider a fully continuous whole life insurance with annual premium π and benefit payment b. Then the loss random variable is given by

$$\bar{L}_x = b\nu^T - \pi\bar{a}_{\overline{T}|}.$$

The actuarial present value is given by

$$E(\bar{L}_x) = b\bar{A}_x - \pi\bar{a}_x.$$

Note that this expected value is no longer necessarily equal to 0. The variance of this type of policy is found as follows

$$\begin{aligned}
\text{Var}(\bar{L}_x) =& \text{Var}(b\nu^T - \pi\bar{a}_{\overline{T}|}) \\
=& \text{Var}\left(b\nu^T - \pi\frac{1-\nu^T}{\delta}\right) \\
=& \text{Var}\left(\nu^T\left(b + \frac{\pi}{\delta}\right) - \frac{\pi}{\delta}\right) \\
=& \left(b + \frac{\pi}{\delta}\right)^2 \text{Var}(\nu^T) \\
=& \left(b + \frac{\pi}{\delta}\right)^2 [{}^2\bar{A}_x - (\bar{A}_x)^2].
\end{aligned}$$

Example 45.3 ‡

For a fully discrete whole life insurance of 100,000 on each of 10,000 lives age 60, you are given:
(i) The future lifetimes are independent.
(ii) Mortality follows the Illustrative Life Table.
(iii) $i = 0.06$.
(iv) π is the premium for each insurance of 100,000. Using the normal approximation, calculate π, such that the probability of a positive total loss is 1%.

Solution.
The loss at issue for a single policy is

$$\begin{aligned}L =&100,000\nu^{K+1} - \pi\ddot{a}_{\overline{K+1|}} = 100,000\nu^{K+1} - \pi\frac{1-\nu^{K+1}}{0.06(1.06)^{-1}}\\ =&\left(100,000 + \frac{53}{3}\pi\right)\nu^{K+1} - \frac{53\pi}{3}.\end{aligned}$$

Thus,

$$\begin{aligned}E(L) =&\left(100,000 + \frac{53}{3}\pi\right)A_{60} - \frac{53\pi}{3} = \left(100,000 + \frac{53}{3}\pi\right)(0.36913) - \frac{53\pi}{3}\\ =&36913 - 11.14537\pi\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(L) =&\left(100,000 + \frac{53}{3}\pi\right)^2[{}^2A_{60} - A_{60}^2] = \left(100,000 + \frac{53}{3}\pi\right)^2[0.17741 - 0.36913^2]\\ \sigma_L =&20,286.21 + 3.5838976\pi\end{aligned}$$

The total loss on 10,000 policies is $S = L_1 + L_2 + \cdots + L_{10,000}$. Hence, $E(S) = 10,000E(L) = 10,000(36913 - 11.14537\pi)$ and $\sigma_S = \sqrt{10,000}(20,286.21 + 3.5838976\pi)$. Thus, the probability of positive total loss is:

$$\begin{aligned}\Pr(S>0) =&\Pr\left(\frac{S - 10,000(36913 - 11.14537\pi)}{\sqrt{10,000}(20,286.21 + 3.5838976\pi)} > -\frac{10,000(36913 - 11.14537\pi)}{\sqrt{10,000}(20,286.21 + 3.5838976\pi)}\right)\\ =&\Pr\left(Z > -\frac{10,000(36913 - 11.14537\pi)}{\sqrt{10,000}(20,286.21 + 3.5838976\pi)}\right)\\ =&1 - \Pr\left(Z < -\frac{10,000(36913 - 11.14537\pi)}{\sqrt{10,000}(20,286.21 + 3.5838976\pi)}\right)\\ =&0.01.\end{aligned}$$

That is,

$$\Pr\left(Z < -\frac{10,000(36913 - 11.14537\pi)}{\sqrt{10,000}(20,286.21 + 3.5838976\pi)}\right) = 0.99$$

and according to the Standard Normal distribution table, we must have

$$-\frac{10,000(36913 - 11.14537\pi)}{\sqrt{10,000}(20,286.21 + 3.5838976\pi)} = 2.326 \Longrightarrow \pi = 3379.57 \blacksquare$$

Practice Problems

Problem 45.1
Consider a fully continuous whole life insurance on (x) with $\mu = 0.04$ and $\delta = 0.05$. Find the smallest premium π such that the probability of a loss is less than 50%.

Problem 45.2
Given $\delta = 0.06$. $T(20)$ follows De Moivre's Law with $\omega = 100$. Find the 25-th percentile fully continuous premium for a whole life insurance payable at death.

Problem 45.3
Consider a fully continuous 30-year term insurance on (50) with unit payment. Assume that mortality follows De Moivre's Law with $\omega = 120$. $\delta = 0.05$. Find the smallest premium π such that the probability of a loss is less than 20% if $\delta = 0.05$.

Problem 45.4
You are given that mortality follow De Moivre's Law with $\omega = 125$ and $\delta = 0.05$. Calculate the premium that must be charged for a continuous whole life issued to (75) so that the probability of a loss is less than 20

Problem 45.5
You are given that mortality follows the Illustrative Life Table with $i = 6\%$. Calculate the premium for a fully curtate whole life issued to (50) such that the probability of a loss is less than 30%.

Problem 45.6
Consider a fully discrete whole life insurance with benefit payment b and annual premium of π.
(a) Find the actuarial present value of this insurance.
(b) Find the variance of the loss random variable of this policy.

Problem 45.7
For a fully discrete whole life insurance of 1000 issues to (x), you are given the following:
(i) $i = 0.06$
(ii) $A_x = 0.24905$

(iii) ${}^2A_x = 0.09476$
(iv) $\ddot{a}_x = 13.267$
(v) The contract annual premium is $\pi = 0.25$
(vi) Losses are based on the contract premium.
Find the actuarial present value and the variance of this policy.

Problem 45.8 ‡
For a block of fully discrete whole life insurances of 1 on independent lives age x, you are given:
(i) $i = 0.06$
(ii) $A_x = 0.24905$
(iii) ${}^2A_x = 0.09476$
(iv) $\pi = 0.025$, where π is the contract premium for each policy.
(v) Losses are based on the contract premium.
Using the normal approximation, calculate the minimum number of policies the insurer must issue so that the probability of a positive total loss on the policies issued is less than or equal to 0.05.

Problem 45.9 ‡
For a fully discrete 2-year term insurance of 1 on (x) :
(i) 0.95 is the lowest premium such that there is a 0% chance of loss in year 1.
(ii) $p_x = 0.75$
(iii) $p_{x+1} = 0.80$
(iv) Z is the random variable for the present value at issue of future benefits.
Calculate $\mathrm{Var}(Z)$.

Problem 45.10 ‡
For a fully discrete whole life insurance of 10,000 on (30):
(i) π denotes the annual premium and $L(\pi)$ denotes the loss-at-issue random variable for this insurance.
(ii) Mortality follows the Illustrative Life Table.
(iii) $i = 0.06$
Calculate the lowest premium, π', such that the probability is less than 0.5 that the loss $L(\pi')$ is positive.

Benefit Reserves

The "benefit reserve" of an insurance policy stands for the amount of money the insurance company must have saved up to be able to provide for the future benefits of the policy. This chapter covers the calculation of benefit reserves.

The two methods that will be used for calculating reserves are the prospective and retrospective methods. Under the **prospective method**, the reserve at time t is the conditional expectation of the difference between the present value of future benefits and the present value of future benefit premiums, the conditional event being survivorship of the insured to time t. That is, the reserve ${}_tV$ is the conditional expected value of the prospective loss random variable

$${}_tL = \text{PV(Future benefit at t)} - \text{PV(Future premiums at t)}.$$

We write

$${}_tV = E({}_tL|T > t) = \text{APV of future benefits} - \text{APV of future premiums}.$$

Note that the reserve is zero at time 0 (since the level premium is determined by the equivalence principle at issue of the policy).

The use of the word prospective is due to the fact that the calculation of reserve involves future benefits and premiums. In contrast calculation of reserves based only on past activity is known as the **retrospectvie method**. Under this method, the reserve at time t is the conditional expectation of the difference between the accumulated value of past benefits paid and the accumulated value of past premiums received, the conditional event being survivorship of the insured to time t. That is, the reserve is the conditional expectation of the excess random variable

$${}_tE = \text{AV(premiums received)} - \text{AV(benefits paid)}.$$

The sections of this chapter parallel sections of the previous chapter on benefit premiums.

46 Fully Continuous Benefit Reserves

In this section, we will analyze the benefit reserve at time t for various types of continuous contingent contracts.

46.1 Fully Continuous Whole Life

Consider a unit fully continuous whole life insurance with an annual continuous benefit premium rate of $\bar{P}(\bar{A}_x)$.

46.1.1 Reserves by the Prospective Method

The insurer's prospective loss at time t (or at age $x+t$) is:

$$ {}_t\bar{L}(\bar{A}_x) = \nu^{T-t} - \bar{P}(\bar{A}_x)\bar{a}_{\overline{T-t}|} = \bar{Z}_{x+t} - \bar{P}(\bar{A}_x)\bar{Y}_{x+t},\ T > t. $$

Example 46.1
Show that

$$ {}_t\bar{L}(\bar{A}_x) = \bar{Z}_{x+t}\left(1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right) - \frac{\bar{P}(\bar{A}_x)}{\delta}, \quad T > t. $$

Solution.
We have

$$ \begin{aligned} {}_t\bar{L}(\bar{A}_x) =& \bar{Z}_{x+t} - \bar{P}(\bar{A}_x)\bar{Y}_{x+t} \\ =& \bar{Z}_{x+t} - \bar{P}(\bar{A}_x)\left(\frac{1-\bar{Z}_{x+t}}{\delta}\right) \\ =& \bar{Z}_{x+t}\left(1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right) - \frac{\bar{P}(\bar{A}_x)}{\delta} \ \blacksquare \end{aligned} $$

The prospective formula for the whole life reserve at time t is:

$$ {}_t\bar{V}(\bar{A}_x) = E({}_t\bar{L}(\bar{A}_x)|T(x) > t) = \bar{A}_{x+t} - \bar{P}(\bar{A}_x)\bar{a}_{x+t}. $$

Example 46.2
Find ${}_t\bar{V}(\bar{A}_x)$ under a constant force of mortality μ and a constant force of interest δ.

Solution.
We have from Example 40.2,

$$\begin{aligned}{}_t\bar{V}(\bar{A}_x) =& \bar{A}_{x+t} - \bar{P}(\bar{A}_x)\bar{a}_{x+t}\\ =& \frac{\mu}{\mu+\delta} - \mu\left(\frac{1}{\mu+\delta}\right) = 0.\end{aligned}$$

The exponential random variable is memoryless. Thus you might expect that no reserves are required for it, since the initial expected loss was 0 and subsequent times have the same exponential distribution ■

Example 46.3
You are given:
(i) Mortality follows De Moivre's Law with $\omega = 100$.
(ii) $\delta = 0.03$.
(iii) ${}_t\bar{L}(\bar{A}_x)$ is the prospective loss random variable at time t for a fully continuous whole life insurance on (40) with premiums determined based on the equivalence principle.
Calculate ${}_{20}\bar{V}(\bar{A}_{40})$.

Solution.
We have

$$\begin{aligned}\bar{A}_{40} =& \int_0^{60} \frac{e^{-0.03t}}{60}dt = 0.46372 & \bar{a}_{40} =& \frac{1-\bar{A}_{40}}{\delta} = 17.876\\ \bar{A}_{60} =& \int_0^{40} \frac{e^{-0.03t}}{40}dt = 0.58234 & \bar{a}_{60} =& \frac{1-\bar{A}_{60}}{\delta} = 13.922\\ \bar{P}(\bar{A}_{40}) =& \frac{\bar{A}_{40}}{\bar{a}_{40}} = 0.02594\end{aligned}$$

Thus,

$${}_{20}\bar{V}(\bar{A}_{40}) = \bar{A}_{60} - \bar{P}(\bar{A}_{40})\bar{a}_{60} = 0.58234 - 0.02594(13.922) = 0.2212 \ ■$$

The variance of ${}_t\bar{L}(\bar{A}_x)$ is

$$\begin{aligned}
\mathrm{Var}({}_t\bar{L}(\bar{A}_x)) =&\mathrm{Var}\left(\bar{Z}_{x+t}\left(1+\frac{\bar{P}(\bar{A}_x)}{\delta}\right)-\frac{\bar{P}(\bar{A}_x)}{\delta}\right)\\
=&\mathrm{Var}\left(\bar{Z}_{x+t}\left(1+\frac{\bar{P}(\bar{A}_x)}{\delta}\right)\right)\\
=&\left(1+\frac{\bar{P}(\bar{A}_x)}{\delta}\right)^2\mathrm{Var}(\bar{Z}_{x+t})\\
=&\left(1+\frac{\bar{P}(\bar{A}_x)}{\delta}\right)^2[{}^2\bar{A}_{x+t}-(\bar{A}_{x+t})^2].
\end{aligned}$$

Example 46.4
Find $\mathrm{Var}({}_{20}\bar{L}(\bar{A}_{40}))$ for the previous example.

Solution.
We have

$${}^2\bar{A}_{60}=\int_0^{40}\frac{e^{-0.06t}}{40}dt=0.37887.$$

Thus,

$$\begin{aligned}
\mathrm{Var}({}_{20}\bar{L}(\bar{A}_{40}))=&\left(1+\frac{\bar{P}(\bar{A}_{40})}{\delta}\right)^2[{}^2\bar{A}_{60}-(\bar{A}_{60})^2]\\
=&\left(1+\frac{0.02594}{0.03}\right)^2[0.37887-(0.58234)^2]=0.13821 \blacksquare
\end{aligned}$$

Now, in the case of a limited payment funding patterns such as an $h-$payment whole life contract, the loss random variable is

$${}_t^h\bar{L}(\bar{A}_x)=\begin{cases}\nu^{T-t}-{}_h\bar{P}(\bar{A}_x)\bar{a}_{\overline{\min\{(T-t,h-t)\}|}} & t\le h\\ \nu^{T-t}, & t>h.\end{cases}$$

Thus, the prospective benefit reserve is

$${}_t^h\bar{V}(\bar{A}_x)=\bar{A}_{x+t}-{}_t\bar{P}(\bar{A}_x)\bar{a}_{x+t:\overline{h-t}|}$$

for $t\le h$, since the future premium stream continues only to the h^{th} year. For $t>h$ there are no future premiums so that the prospective reserve is simply

$${}_t^h\bar{V}(\bar{A}_x)=\bar{A}_{x+t}.$$

In summary, we have

$$ {}_t^h\bar{V}(\bar{A}_x) = \begin{cases} \bar{A}_{x+t} - {}_t\bar{P}(\bar{A}_x)\bar{a}_{x+t:\overline{h-t}|}, & t \le h \\ \bar{A}_{x+t}, & t > h. \end{cases} $$

Example 46.5

You are given that mortality follows De Moivre's Law with $\omega = 125$ and $\delta = 0.05$. Calculate ${}_5^{10}\bar{V}(\bar{A}_{75})$.

Solution.

We have

$$ \begin{aligned} \bar{A}_{80} =& \frac{\bar{a}_{\overline{45}|}}{45} = \frac{1 - e^{-0.05(45)}}{0.05(45)} = 0.39760 \\ \bar{A}_{75} =& \frac{\bar{a}_{\overline{50}|}}{50} = \frac{1 - e^{-0.05(50)}}{0.05(50)} = 0.36717 \\ \bar{A}^{\,1}_{75:\overline{10}|} =& \frac{\bar{a}_{\overline{10}|}}{50} = \frac{1 - e^{-0.05(10)}}{0.05(50)} = 0.157388 \\ {}_{10}E_{75} =& e^{-0.05(10)}\left(\frac{125 - 75 - 10}{125 - 75}\right) = 0.48522 \\ \bar{A}_{75:\overline{10}|} =& 0.157388 + 0.48522 = 0.642608 \\ \bar{a}_{75:\overline{10}|} =& \frac{1 - \bar{A}_{75:\overline{10}|}}{\delta} = 7.14784 \\ {}_5\bar{P}(\bar{A}_{75}) =& \frac{\bar{A}_{75}}{\bar{a}_{75:\overline{10}|}} \\ =& \frac{0.36717}{7.14784} = 0.05137 \\ \bar{A}_{80:\overline{5}|} =& \frac{\bar{a}_{\overline{5}|}}{45} + {}_5E_{80} = 0.79058 \\ \bar{a}_{80:\overline{5}|} =& \frac{1 - \bar{A}_{80:\overline{5}|}}{\delta} = 4.1884. \end{aligned} $$

Thus,

$$ {}_5^{10}\bar{V}(\bar{A}_{75}) = 0.39760 - 0.05137(4.1884) = 0.1824 \blacksquare $$

Practice Problems

Problem 46.1
Show that

$$_t\bar{V}(\bar{A}_x) = 1 - \frac{\bar{a}_{x+t}}{\bar{a}_x}.$$

Problem 46.2
For a fully continuous whole life policy of \$1 issued to (30), you are given:
(i) $\bar{a}_{50} = 7.5$
(ii) $_{20}\bar{V}(\bar{A}_{30}) = 0.1$.
Calculate $\bar{a}_{30}$.

Problem 46.3
For a fully continuous whole life policy of \$1 issued to (30), you are given:
(i) $\bar{a}_{40} = 8.0$
(ii) $\bar{a}_{50} = 7.5$
(iii) $_{20}\bar{V}(\bar{A}_{30}) = 0.1$.
Calculate $_{10}\bar{V}(\bar{A}_{30})$.

Problem 46.4
For a fully continuous whole life policy of \$1 issued to (40), you are given:
(i) Mortality follows De Moivre's Law with $\omega = 100$.
(ii) The following annuity-certain values: $\bar{a}_{\overline{50}|} = 16.23$, $\bar{a}_{\overline{60}|} = 16.64$.
Calculate $_{10}\bar{V}(\bar{A}_{40})$.

Problem 46.5
For a fully continuous whole life policy of 1 issued to (x), you are given:
(i) $_t\bar{V}(\bar{A}_x) = 0.100$
(ii) $\delta = 0.03$
(iii) $\bar{P}(\bar{A}_x) = 0.105$
Calculate $\bar{a}_{x+t}$.

Problem 46.6
You are given that $\mu = 0.02$ and $\delta = 0.05$. Calculate $\text{var}(_t\bar{L}(\bar{A}_x))$.

Problem 46.7
You are given that mortality follows De Moivre's Law with $\omega = 125$ and $\delta = 0.05$. Calculate $^{10}_{15}\bar{V}(\bar{A}_{75})$.

Problem 46.8 ‡
For a fully continuous whole life insurance of 1 on (x), you are given: (i) The forces of mortality and interest are constant.
(ii) ${}^2\bar{A}_x = 0.20$
(iii) $\bar{P}(\bar{A}_x) = 0.03$
(iv) ${}_0L$ is the loss-at-issue random variable based on the benefit premium.
Calculate $\text{Var}({}_0L)$.

Problem 46.9 ‡
For a fully continuous whole life insurance of 1 on (30), you are given:
(i) The force of mortality is 0.05 in the first 10 years and 0.08 thereafter.
(ii) $\delta = 0.08$
Calculate the benefit reserve at time 10 for this insurance.

46.1.2 Other Special Formulas for the Prospective Reserve

In this section, we derive some equivalent forms of the prospective reserve formula for a continuous whole life contract.

Example 46.6
Show that

$$ {}_t\bar{V}(\bar{A}_x) = 1 - \frac{\bar{a}_{x+t}}{\bar{a}_x}. $$

Solution.
We have

$$ \begin{aligned} {}_t\bar{V}(\bar{A}_x) =& \bar{A}_{x+t} - \bar{P}(\bar{A}_x)\bar{a}_{x+t} \\ =& (1 - \delta\bar{a}_{x+t}) - \left(\frac{1-\delta\bar{a}_x}{\bar{a}_x}\right)\bar{a}_{x+t} \\ =& 1 - \delta\bar{a}_{x+t} - \frac{\bar{a}_{x+t}}{\bar{a}_x} + \delta\bar{a}_{x+t} \\ =& 1 - \frac{\bar{a}_{x+t}}{\bar{a}_x} \blacksquare \end{aligned} $$

This formula for the reserve involves only annuity functions and so is known as the **annuity reserve formula.** Note that $\bar{a}_{x+t} > \bar{a}_x$ so that ${}_t\bar{V}(\bar{A}_x) < 1$.

Example 46.7
Show that

$$ {}_t\bar{V}(\bar{A}_x) = \frac{\bar{A}_{x+t} - \bar{A}_x}{1 - \bar{A}_x}. $$

Solution.
Using the relation $\bar{A}_x + \delta\bar{a}_x = 1$ we can write

$$ \begin{aligned} {}_t\bar{V}(\bar{A}_x) =& 1 - \frac{\bar{a}_{x+t}}{\bar{a}_x} = 1 - \frac{\delta^{-1}(1-\bar{A}_{x+t})}{\delta^{-1}(1-\bar{A}_x)} \\ =& \frac{\bar{A}_{x+t} - \bar{A}_x}{1 - \bar{A}_x} \blacksquare \end{aligned} $$

This formula for the reserve involves only insurance costs. It is known as the **benefit formula.**

Example 46.8
Show that
$$ {}_t\bar{V}(\bar{A}_x) = \bar{A}_{x+t}\left(1 - \frac{\bar{P}(\bar{A}_x)}{\bar{P}(\bar{A}_{x+t})}\right). $$

Solution.
We have
$$ \begin{aligned} {}_t\bar{V}(\bar{A}_x) =& \bar{A}_{x+t} - \bar{P}(\bar{A}_x)\bar{a}_{x+t} \\ =& \bar{A}_{x+t}\left(1 - \bar{P}(\bar{A}_x)\frac{\bar{a}_{x+t}}{\bar{A}_{x+t}}\right) \\ =& \bar{A}_{x+t}\left(1 - \frac{\bar{P}(\bar{A}_x)}{\bar{P}(\bar{A}_{x+t})}\right) \blacksquare \end{aligned} $$

This expression is known as the **paid-up insurance formula.**

Example 46.9
Show that
$$ {}_t\bar{V}(\bar{A}_x) = \frac{\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)}{\bar{P}(\bar{A}_{x+t}) + \delta}. $$

Solution.
We have
$$ \bar{P}(\bar{A}_{x+t}) = \frac{\bar{A}_{x+t}}{\delta^{-1}(1 - \bar{A}_{x+t})} \Longrightarrow \frac{\bar{A}_{x+t}}{\bar{P}(\bar{A}_{x+t})} = \frac{1}{\bar{P}(\bar{A}_{x+t}) + \delta}. $$

Thus, we have
$$ \begin{aligned} {}_t\bar{V}(\bar{A}_x) =& \bar{A}_{x+t}\left(1 - \frac{\bar{P}(\bar{A}_x)}{\bar{P}(\bar{A}_{x+t})}\right) \\ =& \frac{[\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)]\bar{A}_{x+t}}{\bar{P}(\bar{A}_{x+t})} \\ =& \frac{\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)}{\bar{P}(\bar{A}_{x+t}) + \delta} \blacksquare \end{aligned} $$

Practice Problems

Problem 46.10
For a fully continuous whole life policy with unit benefit issued to (x), you are given:
(i) $\bar{a}_x = 12$
(ii) $\bar{a}_{x+t} = 8.4$.
Calculate ${}_t\bar{V}(\bar{A}_x)$.

Problem 46.11
For a fully-continuous whole life insurance of unit benefit issued to (40), you are given:
(i) Mortality follows De Moivre's Law with $\omega = 100$
(ii) $\delta = 0.05$.
Calculate ${}_{15}\bar{V}(\bar{A}_{40})$ using the benefit formula.

Problem 46.12
You are given:
(i) $1000\bar{A}_x = 400$
(ii) $1000\bar{A}_{x+t} = 500$
Calculate ${}_t\bar{V}(\bar{A}_x)$.

Problem 46.13
For a fully-continuous whole life insurance of unit benefit issued to (40), you are given:
(i) Mortality follows De Moivre's Law with $\omega = 100$
(ii) $\delta = 0.05$.
Calculate ${}_{15}\bar{V}(\bar{A}_{40})$ using the paid-up insurance formula.

Problem 46.14
For a fully-continuous whole life insurance of unit benefit issued to (40), you are given:
(i) Mortality follows De Moivre's Law with $\omega = 100$
(ii) $\delta = 0.05$.
Calculate ${}_{15}\bar{V}(\bar{A}_{40})$ using the formula of Example 46.9.

Problem 46.15
Show that

$${}_t\bar{V}(\bar{A}_x) = \bar{a}_{x+t}[\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)].$$

This formula is known as the **premium difference formula**.

Problem 46.16
For a fully-continuous whole life insurance of unit benefit issued to (40), you are given:
(i) Mortality follows De Moivre's Law with $\omega = 100$
(ii) $\delta = 0.05$.
Calculate ${}_{15}\bar{V}(\bar{A}_{40})$ using the difference premium formula.

46.1.3 Retrospective Reserve Formula

The prospective method uses future benefits and premiums to calculate the policy's reserve at a given time. An alternative to finding the reserve at time t is to look back at past activities between the ages of x and $x+t$. We refer to this approach as the **retrospective method.**
Consider the reserves at time t of the fully continuous whole life insurance issued to (x) with benefit 1. Then

$$\begin{aligned}
{}_t\bar{V}(\bar{A}_x) =& \bar{A}_{x+t} - \bar{P}(\bar{A}_x)\bar{a}_{x+t}\\
=& \bar{A}_{x+t} - \bar{P}(\bar{A}_x)\bar{a}_{x+t} + \frac{\bar{P}(\bar{A}_x)\bar{a}_x - \bar{A}_x}{{}_tE_x}\\
=& \bar{P}(\bar{A}_x)\left(\frac{\bar{a}_x - {}_tE_x\bar{a}_{x+t}}{{}_tE_x}\right) - \left(\frac{\bar{A}_x - {}_tE_x\bar{A}_{x+t}}{{}_tE_x}\right)\\
=& \bar{P}(\bar{A}_x)\left(\frac{\bar{a}_{x:\overline{t}|}}{{}_tE_x}\right) - \left(\frac{\bar{A}^{1}_{x:\overline{t}|}}{{}_tE_x}\right)\\
=& \bar{P}(\bar{A}_x)\bar{s}_{x:\overline{t}|} - {}_t\bar{k}_x.
\end{aligned}$$

The first term

$$\bar{P}(\bar{A}_x)\left(\frac{\bar{a}_{x:\overline{t}|}}{{}_tE_x}\right) = \bar{P}(\bar{A}_x)\bar{s}_{x:\overline{t}|}$$

is the (actuarial) accumulated value of the premiums paid during the first t years.
The second term

$${}_t\bar{k}_x = \frac{\bar{A}^{1}_{x:\overline{t}|}}{{}_tE_x}$$

is the actuarial accumulated value of past benefits or the **accumulated cost of insurance**.

Remark 46.1
The prospective and retrospective reserve methods should always yield the same result provided the same assumptions and basis (mortality, interest) are used in the calculations.

Example 46.10
You are given:
(i) Mortality follows De Moivre's Law with $\omega = 100$.

(ii) $\delta = 0.03$.
(iii) ${}_t\bar{E}(\bar{A}_x)$ is the retrospective excess random variable at time t for a fully continuous whole life insurance on (40) with premiums determined based on the equivalence principle.
Calculate ${}_{20}\bar{V}(\bar{A}_{40})$ using the retrospective formula.

Solution.
We have

$$\begin{aligned}
\bar{A}^{\;1}_{40:\overline{20}|} =& \frac{\bar{a}_{\overline{20}|}}{60} = \frac{1-e^{-0.03(20)}}{0.03(60)} = 0.25066\\
\bar{a}_{40:\overline{20}|} =& \frac{1-\bar{A}_{40:\overline{20}|}}{\delta}\\
=& \frac{1-[\bar{A}^{\;1}_{40:\overline{20}|} - \nu^{20}{}_{20}p_{40}]}{\delta}\\
=& \frac{1-\left[0.25066 + e^{-0.03(20)}\left(\frac{100-40-20}{100-40}\right)\right]}{0.03} = 12.7822\\
{}_{20}E_{40} =& e^{-0.03(20)}\left(\frac{100-40-20}{100-40}\right) = 0.36587\\
\bar{A}_{40} =& \int_0^{60}\frac{e^{-0.03t}}{60}dt = 0.46372\\
\bar{a}_{40} =& \frac{1-\bar{A}_{40}}{\delta} = 17.876\\
\bar{P}(\bar{A}_{40}) =& \frac{\bar{A}_{40}}{\bar{a}_{40}} = 0.02594
\end{aligned}$$

Thus,

$${}_{20}\bar{V}(\bar{A}_{40}) = \frac{0.02594(12.7822) - 0.25066}{0.36587} = 0.2211.$$

Compare this answer to the answer in Example 46.3 ■

Practice Problems

Problem 46.17
You are given:
(i) Mortality follows De Moivre's Law with $\omega = 100$.
(ii) $i = 0.06$.
Calculate ${}_{10}\bar{V}(\bar{A}_{40})$ using the retrospective formula.

Problem 46.18
Write prospective and retrospective formulas for ${}_{10}\bar{V}(\bar{A}_{50})$, the reserve at time 10 for a continuous whole life insurance with unit benefit issued to (50).

Problem 46.19
Write the retrospective formula of a continuous whole life insurance entirely in terms of benefit premiums.

Problem 46.20
True or false?
The benefit reserve at duration t is a measure of liability for a policy issued at age x that is still in force at age $x+t$. It is equal to the APV at age $x+t$ of the future benefits less the APV at age $x+t$ of the future premiums.

Problem 46.21
You are given:
(i) ${}_t\bar{V}(\bar{A}_x) = 0.563$
(ii) $\bar{P}(\bar{A}_x) = 0.090$
(iii) $\bar{P}(A_{x:\overset{1}{\overline{n}|}}) = 0.00864$.
Calculate $\bar{P}(\bar{A}^1_{x:\overline{n}|})$.

46.2 Fully Continuous $n-$year Term

For other types of insurances, similar development of prospective and retrospective formulas can be made. The fundamental principles always hold: when developing the reserves, the prospective formula is always the actuarial present value of future benefits minus the actuarial present value of future premiums. For the retrospective formula, the actuarial accumulated value of past premiums minus the actuarial accumulated value of past benefits.

Consider a fully continuous $n-$year term insurance with unit benefit. The insurer's prospective loss at time t (or at age $x+t$) is:

$$\begin{aligned}{}_t\bar{L}(\bar{A}^{1}_{x:\overline{n}|}) =&\nu^{T-t}\mathbf{I}(T\le n) - \bar{P}(\bar{A}^{1}_{x:\overline{n}|})\bar{a}_{\overline{\min\{(T-t,n-t)\}}|}\\ =&Z^{\ 1}_{x+t:\overline{n-t}|} - \bar{P}(\bar{A}^{1}_{x:\overline{n}|})\bar{Y}_{x+t:\overline{n-t}|}, t<n.\end{aligned}$$

If $t=n$ then ${}_n\bar{L}(\bar{A}^{1}_{x:\overline{n}|})=0$. Thus, the prospective formula of the reserve at time t for this contract is

$${}_t\bar{V}(\bar{A}^{1}_{x:\overline{n}|}) = \begin{cases} \bar{A}^{\ 1}_{x+t:\overline{n-t}|} - \bar{P}(\bar{A}^{1}_{x:\overline{n}|})\bar{a}_{x+t:\overline{n-t}|}, & t<n\\ 0, & t=n.\end{cases}$$

Note that the reserve is 0 at time $t=n$ since the contract at that time is expired without value.

Example 46.11

You are given:

(i) Mortality follows De Moivre's Law with $\omega=100$.

(ii) $\delta=0.05$.

(iii) ${}_t\bar{L}(\bar{A}^{1}_{30:\overline{n}|})$ is the prospective loss random variable at time t for a fully continuous $n-$year life insurance on (30) with premiums determined based on the equivalence principle.

Calculate ${}_5\bar{V}(\bar{A}^{\ 1}_{30:\overline{10}|})$.

Solution.

We have

$$\bar{A}^{1}_{35:\overline{5}|} = \frac{\bar{a}_{\overline{5}|}}{65} = \frac{1-e^{-0.05(5)}}{0.05(65)} = 0.0681$$

$$\begin{aligned}\bar{a}_{35:\overline{5}|} =&\frac{1-\bar{A}_{35:\overline{5}|}}{\delta} = \frac{1-[\bar{A}^{\ 1}_{35:\overline{5}|} - \nu^5{}_5p_{35}]}{\delta}\\ =&\frac{1-\left[0.0681 + e^{-0.05(5)}\left(\frac{100-35-5}{100-35}\right)\right]}{0.05} = 4.2601\end{aligned}$$

$$\bar{A}^{\,1}_{30:\overline{10}|} = \frac{\bar{a}_{\overline{10}|}}{70} = \frac{1-e^{-0.05(10)}}{0.05(70)} = 0.1125$$

$$\begin{aligned}
\bar{a}_{30:\overline{10}|} =& \frac{1-\bar{A}_{30:\overline{10}|}}{\delta} = \frac{1-[\bar{A}^{\,1}_{30:\overline{10}|} - \nu^{10}{}_{10}p_{30}]}{\delta}\\
=& \frac{1-\left[0.1125 + e^{-0.05(10)}\left(\frac{100-30-10}{100-30}\right)\right]}{0.05} = 7.3523\\
\bar{P}(\bar{A}^{\,1}_{30:\overline{10}|}) =& \frac{0.1125}{7.3523} = 0.0153.
\end{aligned}$$

Thus,

$${}_5\bar{V}(\bar{A}^{\,1}_{30:\overline{10}|}) = 0.0681 - 0.0153(4.2601) = 0.0029 \blacksquare$$

Example 46.12 ‡
For a 5-year fully continuous term insurance on (x) :
(i) $\delta = 0.10$
(ii) All the graphs below are to the same scale.
(iii) All the graphs show $\mu(x+t)$ on the vertical axis and t on the horizontal axis.
Which of the following mortality assumptions would produce the highest benefit reserve at the end of year 2?

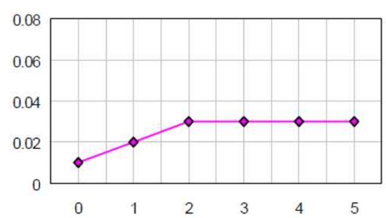
Solution.
The prospective benefit reserve of a 5-year fully continuous term insurance on (x) is

$${}_2\bar{V}(\bar{A}^{\,1}_{x:\overline{5}|}) = \int_0^3 \nu^t{}_tp_{x+2}\mu(x+t+2)dt - \frac{\int_0^5 \nu^t{}_tp_x\mu(x+t)dt}{\int_0^5 \nu^t{}_tp_x dt}\int_0^3 \nu^t{}_tp_{x+2}dt.$$

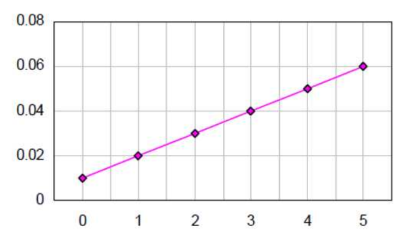
It follows that if $\mu(x+t) = \mu$ for all $0 \le t \le 5$ then ${}_2\bar{V}(\bar{A}^{\,1}_{x:\overline{5}|}) = 0$. That is, the benefit reserve under (E) is zero.
Now, prospectively, the actuarial present value at time $t = 2$ of the future benefits under (B) is larger than the actuarial present value of the future benefits under (C) since people will be dying faster and therefore collecting their benefits. Also, because people are dying faster under (B) than under (C), the actuarial present value of future premiums under (B) is smaller than actuarial present value of future premiums under (C). Hence, the benefit reserve under (B) is larger than the one under (C). In the same token, the

benefit reserve under (B) is larger than the one under (C).
Now, retrospectively, before time $t = 2$, less premiums were received under (D) than under (B). Thus, the actuarial accumulated value at time $t = 2$ of all past premiums under (B) is larger than the actuarial accumulated value of all past premiums under (D). Moreover, more benefits were paid under (D) than under (B) so that the actuarial accumulated value at time $t = 2$ of past benefits paid under (B) is less than the actuarial accumulated value of past benefits paid under (D). Hence, the benefit reserve under (B) is larger than the one under (D).
In conclusion, the mortality given in (B) produce the highest benefit reserve at the end of year 2 ■

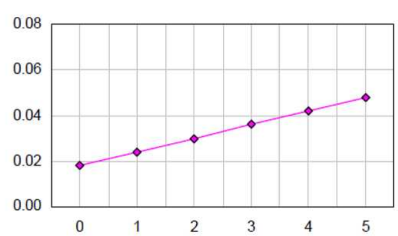
(A)



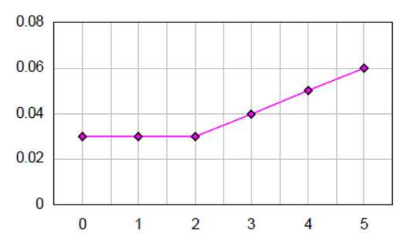
(B)



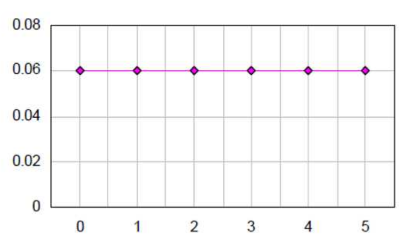
(C)



(D)



(E)



Practice Problems

Problem 46.22
Show that under constant mortality μ and constant force of interest δ, we have ${}_t\bar{V}(\bar{A}^1_{x:\overline{n}|}) = 0$.

Problem 46.23
You are given that $\mu = 0.02$ and $\delta = 0.05$. Calculate ${}_5\bar{V}(\bar{A}^1_{75:\overline{20}|})$.

Problem 46.24
You are given that Mortality follows De Moivre's Law with $\omega = 125$ and $\delta = 0.05$. Calculate ${}_{10}\bar{V}(\bar{A}^1_{75:\overline{20}|})$.

Problem 46.25
Write the premium difference reserve formula for an $n-$year term insurance.

Problem 46.26
Write the paid-up insurance formula for an $n-$year term insurance.

46.3 Fully Continuous $n-$year Endowment Insurance

For this policy, the insurer's prospective loss at time t (or at age $x+t$) is:

$$\begin{aligned}{}_t\bar{L}(\bar{A}_{x:\overline{n}|}) =& \nu^{\min\{(T-t,n-t)\}} - \bar{P}(\bar{A}_{x:\overline{n}|})\bar{a}_{\overline{\min\{(T-t,n-t)\}}|} \\ =& \bar{Z}_{x+t:\overline{n-t}|} - \bar{P}(\bar{A}_{x:\overline{n}|})\bar{Y}_{x+t:\overline{n-t}|},\ t<n\end{aligned}$$

and ${}_n\bar{L}(\bar{A}_{x:\overline{n}|})=1$. The prospective formula of the reserve at time t is

$${}_t\bar{V}(\bar{A}_{x:\overline{n}|}) = \begin{cases} \bar{A}_{x+t:\overline{n-t}|} - \bar{P}(\bar{A}_{x:\overline{n}|})\bar{a}_{x+t:\overline{n-t}|}, & t<n \\ 1, & t=n. \end{cases}$$

Note that the reserve at time $t=n$ is 1 because the contract matures at duration n for the amount of unit endowment benefit.

Example 46.13
You are given:
(i) Mortality follows De Moivre's Law with $\omega = 125$.
(ii) $\delta = 0.05$.
Calculate ${}_{10}\bar{V}(\bar{A}_{75:\overline{20}|})$.

Solution.
We have

$$\begin{aligned}\bar{A}^{1}_{85:\overline{10}|} =& \frac{\bar{a}_{\overline{10}|}}{40} = \frac{1-e^{-0.05(10)}}{0.05(40)} = 0.1967 \\ {}_{10}E_{85} =& e^{-0.05(10)}\left(\frac{125-85-10}{125-85}\right) = 0.4549 \\ \bar{A}_{85:\overline{10}|} =& 0.1967 + 0.4549 = 0.6516 \\ \bar{A}_{75:\overline{20}|} =& \frac{\bar{a}_{\overline{20}|}}{50} + {}_{20}E_{75} = 0.4736 \\ \bar{a}_{75:\overline{20}|} =& \frac{1-\bar{A}_{75:\overline{20}|}}{\delta} = 10.528 \\ \bar{a}_{85:\overline{10}|} =& \frac{1-\bar{A}_{85:\overline{10}|}}{\delta} = 6.968.\end{aligned}$$

Thus.

$${}_{10}\bar{V}(\bar{A}_{75:\overline{20}|}) = 0.6516 - \left(\frac{0.4736}{10.528}\right)(6.968) = 0.3381 \blacksquare$$

Now, in the case of a limited payment funding patterns such as an $h-$year temporary annuity: If $t \leq h < n$, we have

$$ {}_t^h\bar{L}(\bar{A}_{x:\overline{n}|}) = \begin{cases} \nu^{T-t} - {}_h\bar{P}(\bar{A}_{x:\overline{n}|})\bar{a}_{\overline{T-t}|}, & T \leq h \\ \nu^{T-t} - {}_h\bar{P}(\bar{A}_{x:\overline{n}|})\bar{a}_{\overline{h-t}|}, & h < T \leq n \\ \nu^{n-t} - {}_h\bar{P}(\bar{A}_{x:\overline{n}|})\bar{a}_{\overline{h-t}|}, & T > n. \end{cases} $$

If $h < t < n$, we have

$$ {}_t^h\bar{L}(\bar{A}_{x:\overline{n}|}) = \begin{cases} \nu^{T-t}, & T \leq n \\ \nu^{n-t}, & T > n. \end{cases} $$

If $t = n$ then ${}_n^h\bar{L}(\bar{A}_{x:\overline{n}|}) = 1$. Thus, the prospective benefit reserve is

$$ {}_t^h\bar{V}(\bar{A}_{x:\overline{n}|}) = \bar{A}_{x+t:\overline{n-t}|} - {}_h\bar{P}(\bar{A}_{x:\overline{n}|})\bar{a}_{x+t:\overline{h-t}|} $$

for $t \leq h < n$, since the future premium stream continues only to the h^{th} year. For $h < t < n$, there are no future premiums so that the prospective reserve is simply

$$ {}_t^h\bar{V}(\bar{A}_{x:\overline{n}|}) = \bar{A}_{x+t:\overline{n-t}|}. $$

Fot $t = n$, the contract matures at duration n for the amount of unit endowment benefit so that the reserve is 1. In summary, we have

$$ {}_t^h\bar{V}(\bar{A}_{x:\overline{n}|}) = \begin{cases} \bar{A}_{x+t:\overline{n-t}|} - {}_h\bar{P}(\bar{A}_{x:\overline{n}|})\bar{a}_{x+t:\overline{h-t}|}, & 0 < t \leq h < n \\ \bar{A}_{x+t:\overline{n-t}|}, & h < t < n \\ 1, & t = n. \end{cases} $$

Example 46.14
You are given:
(i) Mortality follows De Moivre's Law with $\omega = 125$.
(ii) $\delta = 0.05$.
Calculate ${}_5^{10}\bar{V}(\bar{A}_{75:\overline{20}|})$.

Solution.
We have

$$\begin{aligned}
\bar{A}_{80:\overline{15}|} =& \frac{\bar{a}_{\overline{15}|}}{45} + {}_{15}E_{80} = 0.5494\\
\bar{A}_{80:\overline{5}|} =& \frac{\bar{a}_{\overline{5}|}}{45} + {}_{5}E_{80} = 0.7906\\
\bar{a}_{80:\overline{5}|} =& \frac{1 - \bar{A}_{80:\overline{5}|}}{\delta} = 4.188\\
\bar{A}_{75:\overline{10}|} =& \frac{\bar{a}_{\overline{10}|}}{50} + {}_{10}E_{75} = 0.6426\\
\bar{a}_{75:\overline{10}|} =& \frac{1 - \bar{A}_{75:\overline{10}|}}{\delta} = 7.148\\
\bar{A}_{75:\overline{20}|} =& \frac{\bar{a}_{\overline{20}|}}{50} + {}_{20}E_{75} = 0.4736\\
{}_{10}\bar{P}(\bar{A}_{75:\overline{20}|}) =& \frac{0.4736}{7.148} = 0.0663.
\end{aligned}$$

Thus,

$${}^{10}_{5}\bar{V}(\bar{A}_{75:\overline{20}|}) = 0.5494 - 0.0663(4.188) = 0.2717 \blacksquare$$

Practice Problems

Problem 46.27
You are given:
(i) Mortality follows De Moivre's Law with $\omega = 100$.
(ii) $\delta = 0.05$.
Calculate ${}_5\bar{V}(\bar{A}_{30:\overline{10}|})$.

Problem 46.28
You are given $\mu = 0.02$ and $\delta = 0.05$. Calculate ${}_{10}\bar{V}(\bar{A}_{75:\overline{20}|})$.

Problem 46.29
Show that

$$ {}_t\bar{V}(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x+t:\overline{n-t}|} - \bar{A}_{x:\overline{n}|}}{1 - \bar{A}_{x:\overline{n}|}}. $$

Problem 46.30
You are given:
(i) $\bar{A}_{50:\overline{10}|} = 0.6426$
(ii) $\bar{A}_{40:\overline{10}|} = 0.4559$
Calculate ${}_{10}\bar{V}(\bar{A}_{40:\overline{20}|})$.

Problem 46.31
Find the prospective reserve formula for an $n-$year pure endowment of 1 issued to (x).

Problem 46.32
You are given:
(i) Mortality follows De Moivre's Law with $\omega = 125$.
(ii) $\delta = 0.05$.
Calculate ${}^{10}_{15}\bar{V}(\bar{A}_{75:\overline{20}|})$.

Problem 46.33
You are given:
(i) Mortality follows De Moivre's Law with $\omega = 125$.
(ii) $\delta = 0.05$.
Calculate ${}^{10}_{20}\bar{V}(\bar{A}_{75:\overline{20}|})$.

46.4 Fully Continuous $n-$year Pure Endowment

For a fully continuous $n-$year pure endowment, the insurer's prospective loss at time t (or at age $x+t$) is:

$$ {}_t\bar{L}(A_{x:\overline{n}|}^{\ \ 1}) = \nu^{n-t}\mathbf{I}(T>n) - \bar{P}(A_{x:\overline{n}|}^{\ \ 1})\bar{a}_{\overline{\min\{(T-t,n-t)\}}|}, \ t<n $$

and ${}_n\bar{L}(A_{x:\overline{n}|}^{\ \ 1}) = 1$. The prospective benefit reserve is

$$ {}_t\bar{V}(A_{x:\overline{n}|}^{\ \ 1}) = \begin{cases} A_{x+t:\overline{n-t}|}^{\ \ \ \ \ \ \ 1} - \bar{P}(A_{x:\overline{n}|}^{\ \ 1})\bar{a}_{x+t:\overline{n-t}|} & t<n \\ 1 & t=n. \end{cases} $$

Example 46.15
You are given $\mu = 0.02$ and $\delta = 0.05$. Calculate ${}_5\bar{V}(A_{75:\overline{20}|}^{\ \ \ 1})$.

Solution.
We have

$$ \begin{aligned} A_{75:\overline{20}|}^{\ \ \ 1} =& e^{-20(0.02+0.05)} = 0.24660 \\ \bar{a}_{75:\overline{20}|} =& \frac{1-\bar{A}_{75:\overline{20}|}}{\delta} \\ =& \frac{1-\left[\frac{\mu}{\mu+\delta}(1-e^{-20(\mu+\delta)}) + e^{-20(\mu+\delta)}\right]}{\delta} \\ =& 10.76290 \\ \bar{P}(A_{75:\overline{20}|}^{\ \ \ 1}) =& \frac{0.24660}{10.76290} = 0.0229 \\ A_{80:\overline{15}|}^{\ \ \ 1} =& e^{-15(0.02+0.05)} = 0.3499 \\ \bar{a}_{80:\overline{15}|} =& 9.2866. \end{aligned} $$

Thus,

$$ {}_5\bar{V}(A_{75:\overline{20}|}^{\ \ \ 1}) = 0.3499 - 0.0229(9.2866) = 0.1372 \ \blacksquare $$

Practice Problems

Problem 46.34
You are given:
(i) Mortality follows De Moivre's Law with $\omega = 125$.
(ii) $\delta = 0.05$.
Calculate ${}_{10}\bar{V}(A_{75:\overset{\ \ 1}{\overline{20}|}})$.

Problem 46.35
For a fully continuous 20-year pure endowment of unit benefit on (75), you are given:
(i) ${}_{t}\bar{L}(A_{75:\overset{\ \ 1}{\overline{20}|}})$ is the prospective loss random variable at time t.
(ii) $\delta = 0.05$.
(iii) $\bar{P}(A_{75:\overset{\ \ 1}{\overline{20}|}}) = 0.0229$.
(iv) Premiums are determined by the equivalence principle.
Calculate ${}_{5}\bar{L}(A_{75:\overset{\ \ 1}{\overline{20}|}})$ given that (x) dies in the 25th year after issue.

Problem 46.36
Interpret the meaning of ${}_{10}\bar{V}(A_{75:\overset{\ \ 1}{\overline{20}|}})$.

Problem 46.37
Show that

$$ {}_{t}\bar{V}(\bar{A}_{x:\overset{\ \ 1}{\overline{n}|}}) = {}_{t}\bar{V}(\bar{A}_{x:\overline{n}|}) - {}_{t}\bar{V}(\bar{A}^{1}_{x:\overline{n}|}). $$

Problem 46.38
Show that under constant force of mortality μ and constant δ, we have

$$ {}_{t}\bar{V}(\bar{A}_{x:\overset{\ \ 1}{\overline{n}|}}) = {}_{t}\bar{V}(\bar{A}_{x:\overline{n}|}). $$

46.5 $n-$year Deferred Whole Life Annuity

For an $n-$year deferred contingent annuity-due contract funded continuously over the deferred period, the prospective loss at time t is

$$ {}_t\bar{L}({}_{n|}\bar{a}_x) = \bar{a}_{\overline{T-n}|}\nu^{n-t}\mathbf{I}(T > n) - \bar{P}({}_{n|}\bar{a}_x)\bar{a}_{\overline{\min\{(T-t,n-t)\}}|},\ t \le n $$

and

$$ {}_t\bar{L}({}_{n|}\bar{a}_x) = \bar{a}_{\overline{T-n}|},\ t > n. $$

The prospective benefit reserve for this contract is

$$ {}_t\bar{V}({}_{n|}\bar{a}_x) = \begin{cases} {}_{n-t|}\bar{a}_{x+t} - \bar{P}({}_{n|}\bar{a}_x)\bar{a}_{x+t:\overline{n-t}|} & t \le n \\ \bar{a}_{x+t} & t > n. \end{cases} $$

Example 46.16 ‡
For a 10-year deferred whole life annuity of 1 on (35) payable continuously:
(i) Mortality follows De Moivre's Law with $\omega = 85$.
(ii) $i = 0$
(iii) Level benefit premiums are payable continuously for 10 years.
Calculate the benefit reserve at the end of five years.

Solution.
We want to find

$$ {}_5\bar{V}({}_{10|}\bar{a}_{35}) = {}_{5|}\bar{a}_{40} - \bar{P}({}_{10|}\bar{a}_{35})\bar{a}_{40:\overline{5}|}. $$

We have

$$ \begin{aligned} {}_5E_{40} =& \frac{40}{45} = \frac{8}{9} & \bar{a}_{45} =& \int_0^{40}\left(1 - \frac{t}{40}\right)dt = 20 \\ {}_{5|}\bar{a}_{40} =& \frac{8}{9}(20) = \frac{160}{9} & & \\ {}_{10}E_{35} =& \frac{40}{50} = \frac{4}{5} & \bar{a}_{45} =& \int_0^{40}\left(1 - \frac{t}{40}\right)dt = 20 \\ {}_{10|}\bar{a}_{35} =& \frac{4}{5}(20) = 16 & \bar{a}_{35:\overline{10}|} =& \int_0^{10}\left(1 - \frac{t}{50}\right)dt = 9 \\ \bar{P}({}_{10|}\bar{a}_{35}) =& \frac{16}{9} = 1.778 & & \\ \bar{a}_{40:\overline{5}|} =& \int_0^{5}\left(1 - \frac{t}{45}\right)dt = 4.722 & & \end{aligned} $$

Thus,

$$ {}_5\bar{V}({}_{10|}\bar{a}_{35}) = \frac{160}{9} - 1.778(4.722) = 9.38 \blacksquare $$

Practice Problems

Problem 46.39
For a 10-year deferred whole life annuity of 1 on (35) payable continuously:
(i) Mortality follows De Moivre's Law with $\omega = 85$.
(ii) $i = 0$
(iii) Level benefit premiums are payable continuously for 10 years.
Calculate the benefit reserve at the end of 12 years.

Problem 46.40
For a 10-year deferred whole life annuity of 1 on (35) payable continuously:
(i) Mortality follows De Moivre's Law with $\omega = 85$.
(ii) $\delta = 0.05$
(iii) Level benefit premiums are payable continuously for 10 years.
Calculate the benefit reserve at the end of five years.

Problem 46.41
For a 10-year deferred whole life annuity of 1 on (75) payable continuously:
(i) $\mu = 0.02$
(ii) $\delta = 0.05$
(iii) Level benefit premiums are payable continuously for 10 years.
Calculate the benefit reserve at the end of five years.

Problem 46.42
For a 10-year deferred whole life annuity of 1 on (75) payable continuously:
(i) $\mu = 0.02$
(ii) $\delta = 0.05$
(iii) Level benefit premiums are payable continuously for 10 years.
Calculate the benefit reserve at the end of the twelfth year.

Problem 46.43
For a 10-year deferred whole life annuity of 1 on (75) payable continuously:
(i) $\mu = 0.02$
(ii) $\delta = 0.05$
(iii) Level benefit premiums are payable continuously for 10 years.
Calculate ${}_{12}\bar{L}({}_{10|}\bar{a}_{75})$ given that death occurred at the end of the thirteenth year.

47 Fully Discrete Benefit Reserves

Two important concepts are in order before starting the discussion of discrete benefit reserves. Let ${}_nV$ denote the **terminal benefit reserve** for year n, that is the reserve at the instant year n ends. Let π_n be the premium paid at the beginning of year $n+1$. Then the quantity ${}_nV+\pi_n$ is the **initial benefit reserve** for year $n+1$. See Figure 47.1.

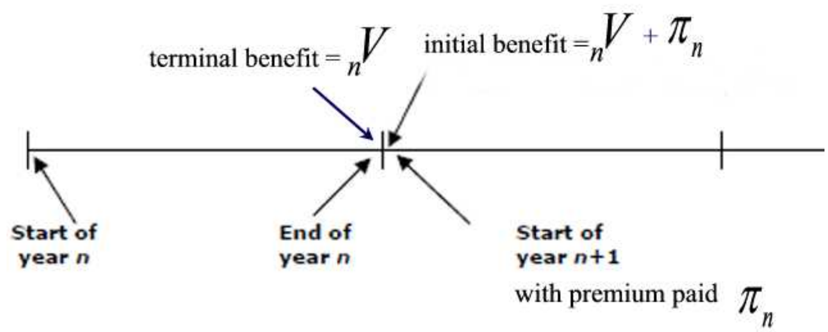


Figure 47.1

The benefit reserves of this section are for discrete insurances with premiums paid at the beginning of the year and benefit paid at the end of the year of death.

47.1 Fully Discrete Whole Life Insurance

Consider a whole life insurance of unit benefit issued to (x) and with benefit premium $P(A_x)$. The loss random variable of (x) surviving k years is defined by

$$ {}_kL(A_x) = \nu^{K(x)-k+1} - P(A_x)\ddot{a}_{\overline{K(x)-k+1}|}. $$

The conditional expectation of the loss function, conditioned on surviving k years, is known as the **prospective** k^{th} **terminal reserve** of the policy and is given by

$$ {}_kV(A_x) = E[{}_kL|K(x) \geq k] = A_{x+k} - P(A_x)\ddot{a}_{x+k}. $$

As before, this formula is the actuarial present value at time k of future benefits from age $x+k$ minus the actuarial present value of future benefit

premiums.
Now, the variance of the loss random variable can be found as follows:

$$\begin{aligned}
\mathrm{Var}({}_kL|K(x)\geq k) =&\mathrm{Var}[\nu^{K(x)-k+1}-P(A_x)\ddot{a}_{\overline{K(x)-k+1}|}|K(x)\geq k]\\
=&\mathrm{Var}\left[\nu^{K(x)-k+1}\left(1+\frac{P(A_x)}{d}\right)-\frac{P(A_x)}{d}|K(x)\geq k\right]\\
=&\mathrm{Var}\left[\nu^{K(x)-k+1}\left(1+\frac{P(A_x)}{d}\right)|K(x)\geq k\right]\\
=&\left(1+\frac{P(A_x)}{d}\right)^2\mathrm{Var}[\nu^{K(x)-k+1}|K(x)\geq k]\\
=&\left(1+\frac{P(A_x)}{d}\right)^2[{}^2A_{x+k}-(A_{x+k})^2].
\end{aligned}$$

Example 47.1
You are given that Mortality follows the Illustrative Life Table with $i=6\%$. Calculate:
(a) ${}_{10}V(A_{60})$.
(b) $\mathrm{Var}({}_{10}L(A_x)|K(x)\geq 10)$.

Solution.
(a) We have

$$\begin{aligned}
A_{60} =&0.36913 & \ddot{a}_{60} =&11.1454\\
P(A_{60}) =&\frac{0.36913}{11.1454}=0.03312\\
A_{70} =&0.51495 & \ddot{a}_{70} =&8.5693.
\end{aligned}$$

Thus, the prospective benefit reserve is

$${}_{10}V(A_{60})=0.51495-0.03312(8.5693)=0.2311.$$

(b) The variance is

$$\begin{aligned}
\mathrm{Var}({}_{10}L(A_x)) =&\left(1+\frac{P(A_{60})}{d}\right)^2[{}^2A_{70}-(A_{70})^2]\\
=&\left(1+\frac{0.03312}{0.06(1.06)^{-1}}\right)^2[0.30642-0.51495^2]=0.10364 \blacksquare
\end{aligned}$$

Now, in the case of a limited payment funding patterns such as an $h-$payment whole life contract, the loss random variable is

$$ {}_k^hL(A_x) = \begin{cases} \nu^{K-k+1} - {}_hP(A_x)\ddot{a}_{\overline{\min\{(K-k+1,h-k)\}}|} & k < h \\ \nu^{K-k+1}, & k \geq h. \end{cases} $$

Thus, the prospective benefit reserve is

$$ {}_k^hV(A_x) = \begin{cases} A_{x+k} - {}_hP(A_x)\ddot{a}_{x+k:\overline{h-k}|}, & k < h \\ A_{x+k}, & k \geq h. \end{cases} $$

Example 47.2
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Calculate:
(a) ${}_5^{10}V(A_{60})$.
(b) ${}_{15}^{10}V(A_{60})$.

Solution.
(a) We have

$$ \begin{aligned} A_{65} =& 0.43980 \\ \ddot{a}_{60} =& 11.1454 \\ {}_{10}E_{60} =& \nu^{10}\frac{\ell_{70}}{\ell_{60}} = (1.06)^{-10}\left(\frac{6,616,155}{8,188,074}\right) = 0.4512 \\ \ddot{a}_{60:\overline{10}|} =& \ddot{a}_{60} - {}_{10}E_{60}\ddot{a}_{70} = 11.1454 - 0.4512(8.5693) = 7.28 \\ {}_{10}P(A_{60}) =& \frac{A_{60}}{\ddot{a}_{60:\overline{10}|}} = \frac{0.36913}{7.28} = 0.0507 \\ \ddot{a}_{65:\overline{5}|} =& \ddot{a}_{65} - {}_5E_{65}\ddot{a}_{70} = 9.8969 - 0.6562(8.5693) = 4.2737. \end{aligned} $$

The prospective benefit reserve is

$$ {}_5^{10}V(A_{60}) = A_{65} - {}_{10}P(A_{65})\ddot{a}_{65:\overline{5}|} = 0.43980 - 0.0612(4.2737) = 0.1782. $$

(b) Since $k = 15 > h = 10$, the prospective benefit reserve is just $A_{75} = 0.59149$ ■

Example 47.3
You are given:
(i) $i = 0.04$
(ii) ${}_{23}^{20}V(A_{15}) = 0.585$
(iii) ${}_{24}^{20}V(A_{15}) = 0.600$.
Calculate p_{38}.

Solution.
Since $k = 23 > h = 20$, the prospective benefit reserve ${}^{20}_{23}V(A_{15})$ is just A_{38}. Likewise, ${}^{20}_{24}V(A_{15}) = A_{39}$. Thus,

$$\begin{aligned}
0.585 =& {}^{20}_{23}V(A_{15}) = A_{38} = \nu q_{38} + \nu p_{38}A_{39} = (1.04)^{-1}[q_{38} + p_{38}(0.600)]\\
0.6084 =& 1 - p_{38} + p_{38}(0.600) = 1 - 0.4p_{38}\\
p_{38} =& \frac{1 - 0.6084}{0.4} = 0.979 \blacksquare
\end{aligned}$$

Example 47.4 ‡
Lottery Life issues a special fully discrete whole life insurance on (25):
(i) At the end of the year of death there is a random drawing. With probability 0.2, the death benefit is 1000. With probability 0.8, the death benefit is 0.
(ii) At the start of each year, including the first, while (25) is alive, there is a random drawing. With probability 0.8, the level premium π is paid. With probability 0.2, no premium is paid.
(iii) The random drawings are independent.
(iv) Mortality follows the Illustrative Life Table.
(v) $i = 0.06$
(vi) π is determined using the equivalence principle.
Calculate the benefit reserve at the end of year 10.

Solution.
By the equivalence principle, we must have

$$\text{APVFB} = \text{APVFP}$$

which in our case is

$$(0.2)(1000)A_{25} = \pi(0.8)\ddot{a}_{25} \Longrightarrow (0.2)(1000)(81.65) = (0.8)(16.2242)\pi \Longrightarrow \pi = 1.258.$$

Hence, the reserve at the end of year 10 is

$$\begin{aligned}
{}_{10}V =& (0.2)(1000)A_{35} - 1.258(0.8)\ddot{a}_{35}\\
=& (0.2)(1000)(0.12872) - 1.258(0.8)(15.3926) = 10.25 \blacksquare
\end{aligned}$$

Practice Problems

Problem 47.1
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Calculate:
(a) ${}_2V(A_{65})$.
(b) $\mathrm{Var}({}_2L(A_{65}))$.

Problem 47.2
Show that

$${}_kV(A_x) = 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x}.$$

This formula is known as the **discrete reserve annuity formula.**

Problem 47.3
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Calculate ${}_2V(A_{65})$ using the discrete reserve annuity formula.

Problem 47.4
Show that

$${}_kV(A_x) = (P(A_{x+k}) - P(A_x))\ddot{a}_{x+k}.$$

This formula is known as the **discrete premium difference formula.**

Problem 47.5
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Calculate ${}_2V(A_{65})$ using the discrete premium difference formula.

Problem 47.6
Show that

$${}_kV(A_x) = A_{x+k}\left(1 - \frac{P(A_x)}{P(A_{x+k})}\right).$$

This formula is known as the **discrete paid-in insurance formula.**

Problem 47.7
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Calculate ${}_2V(A_{65})$ using the discrete paid-in insurance formula.

Problem 47.8
Show that

$$_kV(A_x) = \frac{A_{x+k} - A_x}{1 - A_x}.$$

This formula is known as the **discrete benefit formula.**

Problem 47.9
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Calculate $_2V(A_{65})$ using the discrete benefit formula.

Problem 47.10
Show that

$$_kV(A_x) = \frac{P(A_x)\ddot{a}_{x:\overline{k|}} - A^{1}_{x:\overline{k|}}}{_kE_x}.$$

This formula is known as the **discrete retrospective formula.**

Problem 47.11
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Calculate $_2V(A_{65})$ using the discrete retrospective formula.

Problem 47.12
Show that

$$_kV(A_x) = \frac{P(A_{x+k}) - P(A_x)}{P(A_{x+k}) + d}.$$

Problem 47.13 ‡
For a fully discrete whole life insurance of 1000 on (40), the contract premium is the level annual benefit premium based on the mortality assumption at issue. At time 10, the actuary decides to increase the mortality rates for ages 50 and higher. You are given:
(i) $d = 0.05$
(ii) Mortality assumptions:
• At issue: $_{k|}q_{40} = 0.02, k = 0, 1, 2, \cdots, 49$
• Revised prospectively at time 10: $_{k|}q_{50} = 0.04, k = 0, 1, 2, \cdots, 24$
(iii) $_{10}L$ is the prospective loss random variable at time 10 using the contract premium.
Calculate $E[_{10}L|K(40) \geq 10]$ using the revised mortality assumption.

Problem 47.14 ‡
For a fully discrete whole life insurance of 1000 on (60), the annual benefit premium was calculated using the following:
(i) $i = 0.06$
(ii) $q_{60} = 0.01376$
(iii) $1000A_{60} = 369.33$
(iv) $1000A_{61} = 383.00$
A particular insured is expected to experience a first-year mortality rate ten times the rate used to calculate the annual benefit premium. The expected mortality rates for all other years are the ones originally used.
Calculate the expected loss at issue for this insured, based on the original benefit premium.

Problem 47.15 ‡
You are given:
(i) $P(A_x) = 0.090$
(ii) ${}_nV(A_x) = 0.563$
(iii) $P(A_{x:\overset{1}{\overline{n|}}}) = 0.00864.$
Calculate $P(A^1_{x:\overline{n|}})$. Hint: Problem 47.10.

Problem 47.16 ‡
For a fully discrete whole life insurance of 1000 on (50), you are given:
(i) $1000P(A_{50}) = 25$
(ii) $1000A_{61} = 440$
(iii) $1000q_{60} = 20$
(iv) $i = 0.06$
Calculate $1000{}_{10}V_{50}$.

Problem 47.17 ‡
For a fully discrete whole life insurance of 25,000 on (25), you are given:
(i) $P(A_{25}) = 0.01128$
(ii) $P(A_{25:\overset{1}{\overline{15|}}}) = 0.05107$
(iii) $P(A_{25:\overline{15|}}) = 0.05332.$
Calculate $25,000{}_{15}V_{25}$. Hint: Problem 47.10.

47.2 Fully Discrete $n-$year Term Insurance

For other types of insurances, similar development of prospective formulas can be made. The fundamental principle always hold: when developing the reserves, the prospective formula is always the actuarial present value of future benefits minus the actuarial present value of future premiums.
Consider a fully discrete $n-$year term insurance with unit benefit. The prospective loss at time k (or at age $x+k$) is:

$$_kL(A^1_{x:\overline{n}|}) = \nu^{K-k+1}\mathbf{I}(K<n) - P(A^1_{x:\overline{n}|})\ddot{a}_{\overline{\min\{(K-k+1,n-k)\}}|}$$

for $k<n$. If $k=n$ then $_nL(A^1_{x:\overline{n}|})=0$. The prospective k^{th} terminal reserve for this contract is

$$_kV(A^1_{x:\overline{n}|}) = \begin{cases} A_{x+k:\overline{n-k}|}^{\ 1} - P(A^1_{x:\overline{n}|})\ddot{a}_{x+k:\overline{n-k}|}, & k<n \\ 0, & k=n. \end{cases}$$

Example 47.5
You are given that Mortality follows the Illustrative Life Table with $i=6\%$. Calculate $_5V(A^1_{75:\overline{20}|})$.

Solution.
We have

$$\begin{aligned}
A_{80} =& 0.66575\\
A_{95} =& 0.84214\\
{}_{15}E_{80} =& \nu^{15}\left(\frac{\ell_{95}}{\ell_{80}}\right) = (1.06)^{-15}\left(\frac{297,981}{3,914,365}\right) = 0.0318\\
A^1_{80:\overline{15}|} =& A_{80} - {}_{15}E_{80}A_{95} = 0.66575 - 0.0318(0.84214) = 0.636\\
\ddot{a}_{80:\overline{15}|} =& \ddot{a}_{80} - {}_{15}E_{80}\ddot{a}_{95} = 5.9050 - 0.0318(2.7888) = 5.8163\\
A_{75} =& 0.59149\\
A_{95} =& 0.84214\\
{}_{20}E_{75} =& \nu^{20}\left(\frac{\ell_{95}}{\ell_{75}}\right) = (1.06)^{-20}\left(\frac{297,981}{5,396,081}\right) = 0.0172\\
A^1_{75:\overline{20}|} =& A_{75} - {}_{20}E_{75}A_{95} = 0.59149 - 0.0172(0.84214) = 0.577\\
\ddot{a}_{75:\overline{20}|} =& \ddot{a}_{75} - {}_{20}E_{75}\ddot{a}_{95} = 7.2170 - 0.0172(2.7888) = 7.169\\
P(A^1_{75:\overline{20}|}) =& \frac{0.577}{7.169} = 0.0805
\end{aligned}$$

Thus, the prospective benefit reserve of this contract is

$${}_5V(A^{1}_{75:\overline{20|}}) = A^{1}_{80:\overline{15|}} - P(A^{1}_{75:\overline{20|}})\ddot{a}_{80:\overline{15|}} = 0.636 - 0.0805(5.8163) = 0.1678 \blacksquare$$

Practice Problems

Problem 47.18
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Calculate ${}_2V(A^{1}_{65:\overline{4}|})$.

Problem 47.19
You are given $i = 0.05$ and the following life table

x	95	96	97	98	99	100
ℓ_x	1000	920	550	120	50	0

Calculate ${}_1V(A^{1}_{97:\overline{3}|})$ i.e. the benefit reserve at the end of the first policy year of a fully discrete 3-year term insurance issued to (97).

Problem 47.20
You are given the following:
(i) $A^{\ \ 1}_{101:\overline{2}|} = A^{\ \ 1}_{100:\overline{3}|} = 0$
(ii) $\ddot{a}_{101:\overline{2}|} = 1.2381$ and $\ddot{a}_{100:\overline{3}|} = 1.4717$
(iii) $d = 0.05$.
Calculate ${}_1V(A^{1}_{100:\overline{3}|})$.

Problem 47.21
You are given the force of interest $\delta = 0.04$ and the following life table

x	100	101	102	103
ℓ_x	100	70	30	0

Calculate ${}_1V(A^{1}_{100:\overline{3}|})$ i.e. the benefit reserve at the end of the first policy year of a fully discrete 3-year term insurance issued to (100).

Problem 47.22
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Find the benefit reserve at the end of the 10-th year for a 20-year term contract issued to (35).

47.3 Fully Discrete $n-$year Endowment

For this policy, the insurer's prospective loss at time k (or at age $x+k$) is:

$$ {}_kL(A_{x:\overline{n}|}) = \nu^{\min\{(K-k+1,n-k)\}} - P(A_{x:\overline{n}|})\ddot{a}_{\overline{\min\{(K-k+1,n-k)\}}|}, k < n $$

and ${}_nL(A_{x:\overline{n}|}) = 1$. The prospective formula of the reserve at time k is

$$ {}_kV(A_{x:\overline{n}|}) = \begin{cases} A_{x+k:\overline{n-k}|} - P(A_{x:\overline{n}|})\ddot{a}_{x+k:\overline{n-k}|}, & k < n \\ 1, & k = n. \end{cases} $$

Note that the reserve at time $k = n$ is 1 because the contract matures at duration n for the amount of unit endowment benefit.

Example 47.6
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Calculate ${}_{10}V(A_{60:\overline{20}|})$.

Solution.
We want

$$ {}_{10}V(A_{60:\overline{20}|}) = A_{70:\overline{10}|} - P(A_{60:\overline{20}|})\ddot{a}_{70:\overline{10}|}. $$

We have

$$ \begin{aligned} A^1_{70:\overline{10}|} =& A_{70} - {}_{10}E_{70}A_{80} = 0.51495 - 0.33037(0.66575) = 0.295 \\ A_{70:\overline{10}|} =& A^1_{70:\overline{10}|} + {}_{10}E_{70} = 0.295 + 0.33037 = 0.8254 \\ A^1_{60:\overline{20}|} =& A_{60} - {}_{20}E_{60}A_{80} = 0.36913 - 0.14906(0.66575) = 0.2699 \\ A_{60:\overline{20}|} =& A^1_{60:\overline{20}|} + {}_{20}E_{60} = 0.2699 + 0.14906 = 0.41896 \\ \ddot{a}_{60:\overline{20}|} =& \ddot{a}_{60} - {}_{20}E_{60}\ddot{a}_{80} = 11.1454 - 0.14906(5.9050) = 10.2652 \\ P(A_{60:\overline{20}|}) =& \frac{0.41896}{10.2652} = 0.0408 \\ \ddot{a}_{70:\overline{10}|} =& \ddot{a}_{70} - {}_{10}E_{70}\ddot{a}_{80} = 8.5693 - 0.33037(5.9050) = 6.6185. \end{aligned} $$

Thus,

$$ {}_{10}V(A_{60:\overline{20}|}) = 0.8254 - 0.0408(6.6185) = 0.5554 \blacksquare $$

Now, in the case of a limited payment funding patterns such as an $h-$payment years: If $k \le h < n$, we have

$$ {}^h_kL(A_{x:\overline{n}|}) = \begin{cases} \nu^{K-k+1} - {}_hP(A_{x:\overline{n}|})\ddot{a}_{\overline{K-k+1}|}, & K \le h \\ \nu^{K-k+1} - {}_hP(A_{x:\overline{n}|})\ddot{a}_{\overline{h-k}|}, & h \le K < n \\ \nu^{n-k} - {}_hP(A_{x:\overline{n}|})\ddot{a}_{\overline{h-k}|}, & K \ge n. \end{cases} $$

If $h \le k < n$, we have

$$ {}_k^hL(A_{x:\overline{n}|}) = \begin{cases} \nu^{K-k+1}, & K < n \\ \nu^{n-k}, & K \ge n. \end{cases} $$

If $k = n$ then ${}_n^hL(A_{x:\overline{n}|}) = 1$. Thus, the prospective benefit reserve is

$$ {}_k^hV(A_{x:\overline{n}|}) = A_{x+k:\overline{n-k}|} - {}_hP(A_{x:\overline{n}|})\ddot{a}_{x+k:\overline{h-k}|} $$

for $k \le h < n$, since the future premium stream continues only to the h^{th} year. For $h \le k < n$, there are no future premiums so that the prospective reserve is simply

$$ {}_k^hV(A_{x:\overline{n}|}) = A_{x+k:\overline{n-k}|}. $$

Fot $k = n$, the contract matures at duration n for the amount of unit endowment benefit so that the reserve is 1. In summary, we have

$$ {}_k^hV(A_{x:\overline{n}|}) = \begin{cases} A_{x+k:\overline{n-k}|} - {}_hP(A_{x:\overline{n}|})\ddot{a}_{x+k:\overline{h-k}|}, & k \le h < n \\ A_{x+k:\overline{n-k}|}, & h \le k < n \\ 1, & k = n. \end{cases} $$

Example 47.7
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Calculate ${}_{10}^{15}V(A_{60:\overline{20}|})$.

Solution.
We want

$$ {}_{10}^{15}V(A_{60:\overline{20}|}) = A_{70:\overline{10}|} - {}_{15}P(A_{60:\overline{20}|})\ddot{a}_{70:\overline{5}|}. $$

We have

$$ \begin{aligned} A^1_{70:\overline{10}|} =& A_{70} - {}_{10}E_{70}A_{80} = 0.51495 - 0.33037(0.66575) = 0.295 \\ A_{70:\overline{10}|} =& A^1_{70:\overline{10}|} + {}_{10}E_{70} = 0.295 + 0.33037 = 0.8254 \\ A^1_{60:\overline{20}|} =& A_{60} - {}_{20}E_{60}A_{80} = 0.36913 - 0.14906(0.66575) = 0.2699 \\ A_{60:\overline{20}|} =& A^1_{60:\overline{20}|} + {}_{20}E_{60} = 0.2699 + 0.14906 = 0.41896 \\ \ddot{a}_{60:\overline{15}|} =& \ddot{a}_{60} - {}_{15}E_{60}\ddot{a}_{75} = 11.1454 - 0.275(7.2170) = 9.1607 \\ {}_{15}P(A_{60:\overline{20}|}) =& \frac{0.41896}{9.1607} = 0.04573 \\ \ddot{a}_{70:\overline{5}|} =& \ddot{a}_{70} - {}_5E_{70}\ddot{a}_{75} = 8.5693 - 0.60946(7.2170) = 4.1708. \end{aligned} $$

Thus,

$$ {}_{10}^{15}V(A_{60:\overline{20}|}) = 0.8254 - 0.04573(4.1708) = 0.6347 \blacksquare $$

Example 47.8 ‡
For a fully discrete three-year endowment insurance of 10,000 on (50), you are given:
(i) $i = 0.03$
(ii) $1000q_{50} = 8.32$
(iii) $1000q_{51} = 9.11$
(iv) $10,000{}_1V(A_{50:\overline{3}|}) = 3209$
(v) $10,000{}_2V(A_{50:\overline{3}|}) = 6539$
(vi) ${}_0L$ is the prospective loss random variable at issue, based on the benefit premium.
Calculate the variance of ${}_0L$.

Solution.
We have

$$ {}_0L = \begin{cases} 10,000\nu - 10,000P(A_{50:\overline{3}|})\ddot{a}_{\overline{1}|} & K = 0 \\ 10,000\nu^2 - 10,000P(A_{50:\overline{3}|})\ddot{a}_{\overline{2}|} & K = 1 \\ 10,000\nu^3 - 10,000P(A_{50:\overline{3}|})\ddot{a}_{\overline{3}|} & K > 1. \end{cases} $$

But

$$ P(A_{50:\overline{3}|}) = 10,000\nu - {}_2V(A_{50:\overline{3}|}) = 9708.74 - 6539 = 3169.74. $$

Thus,

$$ {}_0L = \begin{cases} 6539 & K = 0 \\ 3178.80 & K = 1 \\ -83.52 & K > 1. \end{cases} $$

Also,

$$ \begin{aligned} \Pr(K = 0) =& q_{50} = 0.00832 \\ \Pr(K = 1) =& p_{50}q_{51} = (0.99168)(0.00911) = 0.0090342 \\ \Pr(K > 1) =& 1 - 0.00832 - 0.0090432 = 0.98265. \end{aligned} $$

Thus,

$$ \begin{aligned} \mathrm{Var}({}_0L) =& E[{}_0L^2] - [E({}_0L)]^2 = E[{}_0L^2] \\ =& 0.00832 \times 6539^2 + 0.00903 \times 3178.80^2 + 0.98265 \times (-83.52)^2 \\ =& 453,895 \blacksquare \end{aligned} $$

Practice Problems

Problem 47.23 ‡
For a fully discrete 3-year endowment insurance of 1000 on (x), you are given:
(i) ${}_kL$ is the prospective loss random variable at time k.
(ii) $i = 0.10$
(iii) $\ddot{a}_{x:\overline{3}|} = 2.70182$
(iv) Premiums are determined by the equivalence principle.
Calculate ${}_1L$, given that (x) dies in the second year after issue.

Problem 47.24 ‡
For a fully discrete 3-year endowment insurance of 1000 on (x) :
(i) $i = 0.05$
(ii) $p_x = p_{x+1} = 0.7$.
Calculate the second year terminal benefit reserve.

Problem 47.25
Consider a fully discrete $n-$year pure endowment contract.
(a) Find an expression for the insurer's prospectice loss random variable at time k or age $x + k$.
(b) Find the prospective benefit reserve for this policy.

Problem 47.26
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$.
Calculate ${}_{10}V(A_{60:\overline{20}|}^{\ \ \frac{1}{}})$.

Problem 47.27 ‡
For a fully discrete 20-year endowment insurance of 10,000 on (45) that has been in force for 15 years, you are given:
(i) Mortality follows the Illustrative Life Table.
(ii) $i = 0.06$
(iii) At issue, the benefit premium was calculated using the equivalence principle.
(iv) When the insured decides to stop paying premiums after 15 years, the death benefit remains at 10,000 but the pure endowment value is reduced such that the expected prospective loss at age 60 is unchanged.
Calculate the reduced pure endowment value.

47.4 Fully $n-$year Deferred Whole Life Annuity

For an $n-$year deferred contingent annuity-due contract funded by annual premiums over the deferred period, the prospective loss at time k or age $x+k$ is

$$ {}_kL({}_{n|}\ddot{a}_x) = \ddot{a}_{\overline{K-n+1}|}\nu^{n-k}\mathbf{I}(K<n) - P({}_{n|}\ddot{a}_x)\ddot{a}_{\overline{\min\{(K-k+1,n-k)\}}|},\ k<n $$

and

$$ {}_kL({}_{n|}\ddot{a}_x) = \ddot{a}_{\overline{K-n+1}|},\ k \geq n. $$

The prospective benefit reserve for this contract is

$$ {}_kV({}_{n|}\ddot{a}_x) = \begin{cases} {}_{n-k|}\ddot{a}_{x+k} - P({}_{n|}\ddot{a}_x)\ddot{a}_{x+k:\overline{n-k}|} & k<n \\ \ddot{a}_{x+k} & k \geq n. \end{cases} $$

Example 47.9
You are given that Mortality follows the Illustrative Life Table with $i=6\%$. Calculate ${}_{10}V({}_{20|}\ddot{a}_{55})$.

Solution.
We want

$$ {}_{10}V({}_{20|}\ddot{a}_{55}) = {}_{10|}\ddot{a}_{65} - P({}_{20|}\ddot{a}_{55})\ddot{a}_{65:\overline{10}|}. $$

We have

$$ \begin{aligned} {}_{10|}\ddot{a}_{65} =& {}_{10}E_{65}\ddot{a}_{75} = (0.39994)(7.2170) = 2.8864 \\ {}_{20|}\ddot{a}_{55} =& {}_{20}E_{55}\ddot{a}_{75} = (0.19472)(7.2170) = 1.4053 \\ \ddot{a}_{55:\overline{20}|} =& \ddot{a}_{55} - {}_{20}E_{55}\ddot{a}_{75} \\ =& 12.2758 - 0.19472(7.2170) = 10.8705 \\ P({}_{20|}\ddot{a}_{55}) =& \frac{1.4053}{10.8705} = 0.1293 \\ \ddot{a}_{65:\overline{10}|} =& \ddot{a}_{65} - {}_{10}E_{65}\ddot{a}_{75} \\ =& 9.8969 - (0.39994)(7.2170) = 7.0105. \end{aligned} $$

Thus,

$$ {}_{10}V({}_{20|}\ddot{a}_{55}) = 2.8864 - 0.1293(7.0105) = 1.98 \blacksquare $$

Practice Problems

Problem 47.28
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Calculate ${}_{20}V({}_{20|}\ddot{a}_{55})$.

Problem 47.29
Consider a fully discrete 15-year deferred whole life annuity to (65). In this insurance contract, benefits will be paid at the beginning of each year at an annual amount of 1, starting from the end of the 15th year if (65) is still alive at the end of the 15th year, until (65) dies. Premiums are payable during the 15-year period at the beginning of each year at an annual amount of $P({}_{15|}\ddot{a}_{65})$ as long as (65) survives during the 15-year period.
(a) Write the prospective formula for ${}_{3}V({}_{15|}\ddot{a}_{65})$, i.e. the benefitt reserve at the end of the 3rd policy year.
(b) Write the retrospective formula for ${}_{3}V({}_{15|}\ddot{a}_{65})$.

Problem 47.30
Find the retrospective formula of the reserve for an $n-$year deferred annuity due.

Problem 47.31
Find the 10th year terminal reserve on a fully discrete 20-year deferred annuity-due of 1 per year to (40) using retrospective reserve formula.

48 Semicontinuous Reserves

We have had encounter with semicontinuous contracts in Section 42. Recall that the types of semicontinuous contracts we consider here are those with benefits payable at the moment of death and premium payments made at the beginning of the year while the insured is alive.

For a semicontinuous whole life insurance, the prospective loss random variable at time k or age $x+k$ is

$$ {}_kL(\bar{A}_x) = \bar{Z}_{x+k} - P(\bar{A}_x)\ddot{Y}_{x+k}. $$

The k^{th} terminal prospective benefit reserve is given by

$$ {}_kV(\bar{A}_x) = \bar{A}_{x+k} - P(\bar{A}_x)\ddot{a}_{x+k}. $$

If the contract involves a limited funding pattern over the first h years only, then the loss random variable is

$$ {}^h_kL(\bar{A}_x) = \bar{Z}_{x+k} - {}_hP(\bar{A}_x)\ddot{Y}_{x+k:\overline{h-k}|} $$

for $k \le h$ and ${}^h_kL(\bar{A}_x) = \bar{Z}_{x+k}$ for $k > h$. Thus, the k^{th} terminal reserve is given by

$$ {}^h_kV(\bar{A}_x) = \bar{A}_{x+k} - {}_hP(\bar{A}_x)\ddot{a}_{x+k:\overline{h-k}|} $$

for $k \le h$ and ${}^h_kV(\bar{A}_x) = \bar{A}_{x+k}$ for $k > h$.

Now, suppose that we wanted to use the Illustrative Life Table to find either ${}_kV(\bar{A}_x)$ or ${}^h_kV(\bar{A}_x)$. If we assume UDD, then we have

$$ \bar{A}_{x+k} = \frac{i}{\delta}A_{x+k} $$

and

$$ P(\bar{A}_x) = \frac{i}{\delta}P(A_x). $$

Hence, we obtain for example

$$ {}_kV(\bar{A}_x) = \frac{i}{\delta}A_{x+k} - \frac{i}{\delta}P(A_x)\ddot{a}_{x+k}. $$

Other terminal reserve expressions for contracts with immediate payment of claims and premium payments made at the beginning of the year are developed in the exercises.

Example 48.1
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Assume that mortality is uniformly distributed between integral ages. Calculate ${}_{10}V(\bar{A}_{60})$.

Solution.
We want
$$ {}_{10}V(\bar{A}_{60}) = \bar{A}_{70} - P(\bar{A}_{60})\ddot{a}_{70}. $$
Under UDD, we have
$$ \begin{aligned} \bar{A}_{70} =& \frac{i}{\delta}A_{70} = \frac{0.06}{\ln 1.06}(0.51495) = 0.5302 \\ P(\bar{A}_{60}) =& \frac{i}{\delta}P(A_{60}) = \frac{0.06}{\ln 1.06} \times \frac{0.36913}{11.1454} = 0.0341. \end{aligned} $$
Thus,
$$ {}_{10}V(\bar{A}_{60}) = 0.5302 - 0.0341(8.5693) = 0.238 \blacksquare $$

Example 48.2
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Assume that mortality is uniformly distributed between integral ages. Calculate ${}^{20}_{10}V(\bar{A}_{60})$.

Solution.
We want
$$ {}^{20}_{10}V(\bar{A}_{60}) = \bar{A}_{70} - {}_{20}P(\bar{A}_{60})\ddot{a}_{70:\overline{10|}}. $$
We have
$$ \begin{aligned} \bar{A}_{70} =& \frac{i}{\delta}A_{70} = \frac{0.06}{\ln 1.06}(0.51495) = 0.5302 \\ \ddot{a}_{60:\overline{20|}} =& \ddot{a}_{60} - {}_{20}E_{60}\ddot{a}_{80} \\ =& 11.1454 - 0.14906(5.9050) = 10.2652 \\ {}_{20}P(\bar{A}_{60}) =& \frac{i}{\delta}\frac{A_{60}}{\ddot{a}_{60:\overline{20|}}} = \frac{0.06}{\ln 1.06} \times \frac{0.36913}{10.2652} = 0.0370 \\ \ddot{a}_{70:\overline{10|}} =& \ddot{a}_{70} - {}_{10}E_{70}\ddot{a}_{80} \\ =& 8.5693 - 0.33037(5.9050) = 6.6185. \end{aligned} $$
Thus,
$$ {}^{20}_{10}V(\bar{A}_{60}) = 0.5032 - 0.0370(6.6185) = 0.2583 \blacksquare $$

Example 48.3 ‡
A large machine in the ABC Paper Mill is 25 years old when ABC purchases a 5-year term insurance paying a benefit in the event the machine breaks down.
Given:
(i) Annual benefit premiums of 6643 are payable at the beginning of the year.
(ii) A benefit of 500,000 is payable at the moment of breakdown.
(iii) Once a benefit is paid, the insurance contract is terminated.
(iv) Machine breakdowns follow De Moivre's Law with $\ell_x = 100 - x$.
(v) $i = 0.06$
Calculate the benefit reserve for this insurance at the end of the third year.

Solution.
We have

$$\begin{aligned}
{}_3V(\bar{A}^{\,1}_{25:\overline{5}|}) =&500,000\bar{A}^{\,1}_{28:\overline{2}|} - 6643\ddot{a}_{28:\overline{2}|} = 500,000\frac{\bar{a}_{\overline{2}|}}{72} - 6643[1 + \nu p_{28}]\\
=&\frac{500,000}{72}\left(\frac{1 - e^{-2\ln 1.06}}{\ln 1.06}\right) - 6643\left[1 + (1.06)^{-1}\left(\frac{71}{72}\right)\right]\\
=&287.20 \blacksquare
\end{aligned}$$

Example 48.4 ‡
For a whole life insurance of 1 on (x), you are given:
(i) Benefits are payable at the moment of death.
(ii) Level premiums are payable at the beginning of each year.
(iii) Deaths are uniformly distributed over each year of age.
(iv) $i = 0.10$
(v) $\ddot{a}_x = 8$
(vi) $\ddot{a}_{x+10} = 6$
Calculate the 10^{th} year terminal benefit reserve for this insurance.

Solution.
We have

$$\begin{aligned}
A_x =&1-d\ddot{a}_x=1-\frac{0.1}{1.1}(8)=\frac{3}{11}\\
A_{x+10} =&1-d\ddot{a}_{x+10}=1-\frac{0.1}{1.1}(6)=\frac{5}{11}\\
\bar{A}_x =&\frac{i}{\delta}A_x=\frac{3}{11}\frac{0.10}{\ln 1.10}=0.2861\\
\bar{A}_{x+10} =&\frac{i}{\delta}A_{x+10}=\frac{5}{11}\frac{0.10}{\ln 1.10}=0.4769\\
{}_{10}V =&\bar{A}_{x+10}-P(\bar{A}_x)\ddot{a}_{x+10}=0.4769-\left(\frac{0.2861}{8}\right)(6)=0.2623 \blacksquare
\end{aligned}$$

Practice Problems

Problem 48.1
You are given the Illustrative Life Table with $i = 0.06$. Assume UDD, calculate ${}_3V(\bar{A}_{65})$.

Problem 48.2
Consider an $n-$year term insurance contract.
(a) Find an expression for the k^{th} terminal prospective loss random variable.
(b) Find the k^{th} terminal prospective reserve for this contract.

Problem 48.3
Consider an $n-$year term insurance contract with limited funding over the first h years.
(a) Find an expression for the k^{th} terminal prospective loss random variable.
(b) Find the k^{th} terminal prospective reserve for this contract.

Problem 48.4
Consider an $n-$year endowment insurance contract.
(a) Find an expression for the k^{th} terminal prospective loss random variable.
(b) Find the k^{th} terminal prospective reserve for this contract.

Problem 48.5
Consider an $n-$year endowment insurance contract with limited funding over the first h years.
(a) Find an expression for the k^{th} terminal prospective loss random variable.
(b) Find the k^{th} terminal prospective reserve for this contract.

Problem 48.6
Show that, under UDD, we have

$${}^h_kV(\bar{A}_{x:\overline{n}|}) = \frac{i}{\delta}{}^h_kV(A^1_{x:\overline{n}|}) + {}^h_kV(A_{x:\overline{n}|}^{\ \ \ 1}).$$

Problem 48.7
Show that the retrospective reserve formula for a semicontinuous whole life insurance is given by

$${}_kV(\bar{A}_x) = P(\bar{A}_x)\ddot{s}_{x:\overline{k}|} - \frac{\bar{A}^1_{x:\overline{k}|}}{{}_kE_x}.$$

49 Reserves Based on True m^{thly} Premiums

In this section we consider contracts with benefits paid either at the moment of death or at the end of year of death but with premiums paid m^{thly} at the beginning of the period with annual premium denoted by $P^{(m)}$. See Section 43.

Under the whole life model with immediate payment of claims and with annual premium rate of $P^{(m)}(\bar{A}_x)$, the k^{thly} terminal prospective reserve formula for this contract is

$$ {}_kV^{(m)}(\bar{A}_x) = \bar{A}_{x+k} - P^{(m)}(\bar{A}_x)\ddot{a}^{(m)}_{x+k}. $$

If the contract involves a limited funding pattern over the first h years only, then the k^{thly} terminal prospective reserve formula is

$$ {}^h_kV^{(m)}(\bar{A}_x) = \bar{A}_{x+k} - {}_hP^{(m)}(\bar{A}_x)\ddot{a}^{(m)}_{x+k:\overline{h-k}|} $$

for $k \leq h$ and ${}^h_kV^{(m)}(\bar{A}_x) = \bar{A}_{x+k}$ for $k > h$.

For an $n-$year term insurance model, the k^{thly} terminal prospective reserve formula is

$$ {}_kV^{(m)}(\bar{A}^1_{x:\overline{n}|}) = \bar{A}^{\,1}_{x+k:\overline{n-k}|} - P^{(m)}(\bar{A}^1_{x:\overline{n}|})\ddot{a}^{(m)}_{x+k:\overline{n-k}|}, \ k < n. $$

In the case of a limited funding over the first h years, the k^{thly} terminal prospective reserve formula is

$$ {}^h_kV^{(m)}(\bar{A}^1_{x:\overline{n}|}) = \begin{cases} \bar{A}^{\,1}_{x+k:\overline{n-k}|} - {}_hP^{(m)}(\bar{A}^1_{x:\overline{n}|})\ddot{a}^{(m)}_{x+k:\overline{h-k}|} & k < h < n \\ \bar{A}^{\,1}_{x+k:\overline{n-k}|} & h < k < n. \end{cases} $$

For an $n-$year endowment insurance contract, the k^{thly} terminal prospective reserve formula is

$$ {}_kV^{(m)}(\bar{A}_{x:\overline{n}|}) = \bar{A}_{x+k:\overline{n-k}|} - P^{(m)}(\bar{A}_{x:\overline{n}|})\ddot{a}^{(m)}_{x+k:\overline{n-k}|}, \ k < n. $$

In the case of a limited funding over the first h years, the k^{thly} terminal prospective reserve formula is

$$ {}^h_kV^{(m)}(\bar{A}_{x:\overline{n}|}) = \begin{cases} \bar{A}_{x+k:\overline{n-k}|} - {}_hP^{(m)}(\bar{A}_{x:\overline{n}|})\ddot{a}^{(m)}_{x+k:\overline{h-k}|} & k < h < n \\ \bar{A}_{x+k:\overline{n-k}|} & h \leq k < n \\ 1 & k = n. \end{cases} $$

Likewise, we can define prospective reserve formulas for contracts with benefits paid at the end of year of death but with premiums paid m^{thly} at the beginning of the period. For a whole life insurance contract, the prospective reserve is

$${}_kV^{(m)}(A_x) = A_{x+k} - P^{(m)}(A_x)\ddot{a}^{(m)}_{x+k}.$$

If the contract involves a limited funding pattern over the first h years only, then the k^{thly} terminal prospective reserve formula is

$${}^{h}_{k}V^{(m)}(A_x) = A_{x+k} - {}_hP^{(m)}(A_x)\ddot{a}^{(m)}_{x+k:\overline{h-k|}}$$

for $k \le h$ and ${}^{h}_{k}V^{(m)}(A_x) = A_{x+k}$ for $k > h$.
For an $n-$year endowment insurance model, we have

$${}_kV^{(m)}(A^{1}_{x:\overline{n|}}) = A^{\ \ 1}_{x+k:\overline{n-k|}} - P^{(m)}(A^{1}_{x:\overline{n|}})\ddot{a}^{(m)}_{x+k:\overline{n-k|}},\ k < n.$$

In the case of a limited funding over the first h years, the k^{thly} terminal prospective reserve formula is

$${}^{h}_{k}V^{(m)}(A^{1}_{x:\overline{n|}}) = \begin{cases} A^{\ \ 1}_{x+k:\overline{n-k|}} - {}_hP^{(m)}(A^{1}_{x:\overline{n|}})\ddot{a}^{(m)}_{x+k:\overline{h-k|}} & k < h < n \\ A^{\ \ 1}_{x+k:\overline{n-k|}} & h < k < n. \end{cases}$$

For an $n-$year endowment insurance contract, the k^{thly} terminal prospective reserve formula is

$${}_kV^{(m)}(A_{x:\overline{n|}}) = A_{x+k:\overline{n-k|}} - P^{(m)}(A_{x:\overline{n|}})\ddot{a}^{(m)}_{x+k:\overline{n-k|}},\ k < n.$$

In the case of a limited funding over the first h years, the k^{thly} terminal prospective reserve formula is

$${}^{h}_{k}V^{(m)}(A_{x:\overline{n|}}) = \begin{cases} A_{x+k:\overline{n-k|}} - {}_hP^{(m)}(A_{x:\overline{n|}})\ddot{a}^{(m)}_{x+k:\overline{h-k|}} & k < h < n \\ A_{x+k:\overline{n-k|}} & h \le k < n \\ 1 & k = n. \end{cases}$$

For an $n-$year pure endowment insurance model, we have

$${}_kV^{(m)}(A_{x:\overline{n|}}^{\ \ \ 1}) = A_{x+k:\overline{n-k|}}^{\ \ \ \ \ \ 1} - P^{(m)}(A_{x:\overline{n|}}^{\ \ \ 1})\ddot{a}^{(m)}_{x+k:\overline{n-k|}},\ k < n.$$

Example 49.1
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Assume that mortality is uniformly distributed between integral ages. Calculate ${}_{10}V^{(4)}(A_{60})$.

Solution.
We have

$$\begin{aligned}
{}_{10}V^{(4)}(A_{60}) =& A_{70} - P^{(4)}(A_{60})\ddot{a}_{70}^{(4)} = A_{70} - \frac{A_{60}}{\ddot{a}_{60}^{(4)}}\ddot{a}_{70}^{(4)} \\
=& A_{70} - \frac{A_{60}}{\frac{1-\frac{i}{i^{(4)}}A_{60}}{d^{(4)}}}\frac{1-\frac{i}{i^{(4)}}A_{70}}{d^{(4)}} \\
=& A_{70} - \frac{A_{60}(1-\frac{i}{i^{(4)}}A_{70})}{1-\frac{i}{i^{(4)}}A_{60}} \\
=& 0.51495 - \frac{0.36913[1-1.0223(051495)]}{1-1.0223(0.36913)} = 0.23419 \blacksquare
\end{aligned}$$

Example 49.2
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Assume that mortality is uniformly distributed between integral ages. Calculate ${}_{10}V^{(12)}(\bar{A}_{60})$.

Solution.
We have

$$\begin{aligned}
{}_{10}V^{(12)}(\bar{A}_{60}) =& \bar{A}_{70} - P^{(12)}(\bar{A}_{60})\ddot{a}_{70}^{(12)} \\
=& \frac{i}{\delta}A_{70} - \frac{\frac{i}{\delta}A_{60}(1-\frac{i}{i^{(12)}}A_{70})}{1-\frac{i}{i^{(12)}}A_{60}} \\
=& (1.02971)(0.51495) - \frac{(1.02971)(0.36913)(1-1.02721(0.51495))}{1-1.02721(0.36913)} \\
=& 0.24916 \blacksquare
\end{aligned}$$

Practice Problems

Problem 49.1
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Assume that mortality is uniformly distributed between integral ages. Calculate ${}_{20}V^{(12)}(\bar{A}_{30:\overline{35|}})$.

Problem 49.2
You are given that Mortality follows the Illustrative Life Table with $i = 6\%$. Assume that mortality is uniformly distributed between integral ages. Calculate ${}_{3}V^{(12)}(\bar{A}_{65})$.

Problem 49.3
Show that, under UDD, we have
(a) ${}_kV^{(m)}(\bar{A}_x) - {}_kV(\bar{A}_x) = \beta(m)P^{(m)}(\bar{A}_x){}_kV(\bar{A}_x)$.
(b) ${}_kV^{(m)}(A_x) - {}_kV(A_x) = \beta(m)P^{(m)}(A_x){}_kV(A_x)$.

Problem 49.4
Show that, under UDD, we have

$$\frac{{}_kV^{(m)}(A_x) - {}_kV(A_x)}{{}_kV^{(m)}(\bar{A}_x) - {}_kV(\bar{A}_x)} = \frac{\delta}{i}.$$

Problem 49.5
Find the retrospective formula for a whole life contract with benefit payment at the end of the year of death and with premiums paid m^{thly}.

Reserves for Contracts with Nonlevel Benefits and Premiums

Up to this point, we have only discussed reserves for insurances with a level contingent benefit and a level benefit premium. In this chapter, the concept of reserves is extended to general insurances which include contracts with nonlevel benefits and/or premiums.

50 Reserves for Fully Discrete General Insurances

Consider a general fully discrete insurance issued to (x) with the following features:
(i) Let b_j be the death benefit payable at the end of year if death occurs in the $j-$th policy year, where $j = 1, 2, \cdots$.
(ii) Let π_{j-1} be the benefit premium payable at the beginning of the $j-$th policy year, where $j = 1, 2, \cdots$.
This is illustrated in the following diagram.

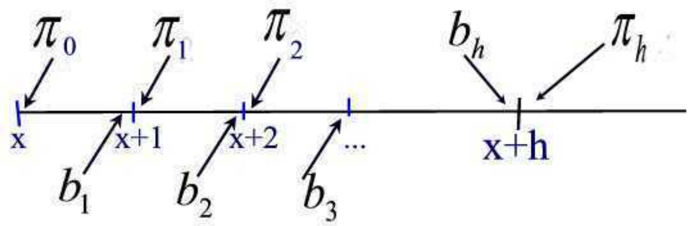


Figure 50.1

From this diagram we see that the actuarial present value of the benefits at issue is

$$\text{APVB} = \sum_{k=0}^{\infty} b_{k+1}\nu^{k+1}\Pr(K(x) = k) = \sum_{k=0}^{\infty} b_{k+1}\nu^{k+1}{}_kp_xq_{x+k}.$$

From Section 39.1, the actuarial present value of the benefit premium stream

$$\text{APVP} = \sum_{k=0}^{\infty} \pi_k\nu^k{}_kp_x.$$

Using the equivalence principle at time 0 we have

$$\sum_{k=0}^{\infty} b_{k+1}\nu^{k+1}{}_kp_xq_{x+k} = \sum_{k=0}^{\infty} \pi_k\nu^k{}_kp_x.$$

Given the benefit payments, we can solve this equation for the benefit premiums. We illustrate this in the next example.

Example 50.1
You are given the following mortality table:

x	90	91	92	93	94	95
ℓ_x	1000	900	720	432	216	0

For a special fully discrete 4 year term issued to (91), you are given:
(i) $i = 4\%$
(ii) The death benefit during the first two years is 1000
(iii) The death benefit during the second two years is 500
(iv) The annual benefit premium for the first two years is twice the annual benefit premium for the last two years.
Calculate the annual premium for this contract.

Solution.
Let π be the premium for each of the third and fourth years. Then we have,

$$\begin{aligned}
\text{APFB} =& 1000\nu q_{91} + 1000\nu^2 p_{91} q_{92} + 500\nu^3 {}_2p_{91} q_{93} + 500\nu^4 {}_3p_{91} \\
=& 1000(1.04)^{-1}\left(\frac{180}{900}\right) + 1000(1.04)^{-2}\left(\frac{720}{900}\right)\left(\frac{288}{720}\right) \\
&+ 500(1.04)^{-3}\left(\frac{432}{900}\right)\left(\frac{216}{432}\right) + 500(1.04)^{-4}\left(\frac{216}{900}\right) \\
=& 697.4217 \\
\text{APFP} =& 2P + 2P\nu p_{91} + P\nu^2 {}_2p_{91} + P\nu^3 {}_3p_{91} \\
=& P\left[2 + 2(1.04)^{-1}\left(\frac{720}{900}\right) + (1.04)^{-2}\left(\frac{432}{900}\right) + (1.04)^{-3}\left(\frac{216}{900}\right)\right] \\
=& 4.1956P.
\end{aligned}$$

By the equivalence principle, we must have

$$4.1956P = 697.4217 \Longrightarrow P = 166.2269 \blacksquare$$

Now, the prospective formula for the h^{th} terminal reserve for this contract at integral duration h is the present value of all future benefits minus the present value of all future premiums, given that the contract has not yet failed, is given by

$$ {}_hV = \sum_{k=0}^{\infty} b_{k+1+h}\nu^{k+1} {}_kp_{x+h} q_{x+h+k} - \sum_{k=0}^{\infty} \pi_{h+k}\nu^k {}_kp_{x+h}.$$

Example 50.2
In Example 50.1, calculate ${}_2V$, the reserve at the end of the second year for this special term insurance.

Solution.
The reserve at the end of the second year for this special term insurance is

$$\begin{aligned}{}_2V =&(b_3\nu q_{93} + b_4\nu^2 p_{93}q_{94}) - (\pi_2 + \pi_3\nu p_{93})\\ =&500(1.04)^{-1}\left(\frac{216}{432}\right) + 500(1.04)^{-2}\left(\frac{216}{432}\right) - 166.2269\left(1 + (1.04)^{-1}\left(\frac{216}{432}\right)\right)\\ =&225.38 \blacksquare\end{aligned}$$

The retrospective formula for the above contract is derived in the next example.

Example 50.3
Write the general retrospective formula corresponding to the prospective formula given above.

Solution.
Recall the retrospective method:

$$\begin{aligned}{}_hV =& \text{ Accumulated Value of the profit from 0 to } h\text{, given that } K(x) < h\\ =& \tfrac{1}{{}_hE_x}[\text{APV at time 0 of premiums over } [0,h] \text{ minus the APV at time 0}\\ & \text{of benefits over } [0,h]].\end{aligned}$$

It follows that

$${}_hV = \frac{1}{{}_hE_x}\left[\sum_{j=0}^{h-1}\pi_j\nu^j{}_jp_x - \sum_{j=0}^{h-1}b_{j+1}\nu^{j+1}{}_jp_xq_{x+j}\right] \blacksquare$$

Example 50.4 ‡
For a fully discrete 5-payment 10-year decreasing term insurance on (60), you are given:
(i) , $b_{k+1} = 1000(10-k),\ k = 0, 1, \cdots, 9$
(ii) Level benefit premiums are payable for five years and equal 218.15 each
(iii) $q_{60+k} = 0.02 + 0.001k,\ k = 0, 1, \cdots, 9$
(iv) $i = 0.06$.
Calculate ${}_2V$, the benefit reserve at the end of year 2.

Solution.
Applying the retrospective formula, we find

$$\begin{aligned}
{}_2V =& \frac{1}{{}_2E_{60}}[218.15 + 218.15\nu p_{60} - (10000\nu q_{60} + 9000\nu^2 p_{60} q_{61}]\\
=& \frac{1}{\nu^2 p_{60} p_{61}}[218.15 + 218.15\nu p_{60} - (10000\nu q_{60} + 9000\nu^2 p_{60} q_{61}]\\
=& \frac{1}{(1.06)^{-2}(0.98)(0.979)}[218.15 + 218.25(1.06)^{-1}(0.98)\\
& -(10000(1.06)^{-1}(0.02) + 9000(1.06)^{-2}(0.98)(0.021))] = 77.66 \blacksquare
\end{aligned}$$

Practice Problems

Problem 50.1
Write down the formula for the insurer's prospective loss random variable for the insurance given at the beginning of this section.

Problem 50.2
A special fully discrete 2-year endowment insurance with a maturity value of 1000 is issued to (x). You are given:
(i) The death benefit in each year is 1000 plus the benefit reserve at then end of the year
(ii) π is the net level annual premium.
(iii) $i = 0.05$
(iv) $q_{x+k} = 0.10(1.10)^k,\ k = 0, 1.$
Calculate π.

Problem 50.3 ‡
For a special fully discrete 2-year endowment insurance on (x) :
(i) The maturity value is 2000.
(ii) The death benefit for year k is $(1000k)$ plus the benefit reserve at the end of year $k,\ k = 1, 2.$
(iii) π is the level annual benefit premium.
(iv) $i = 0.08$
(v) $p_{x+k-1} = 0.9,\ k = 1, 2.$
Calculate π.

Problem 50.4
For a special 20-year endowment policy issued to age 45, you are given:
(i) Death benefit is payable at the end of the year of death with benefit amount equal to:
- \$10 if death is within the first 10 years,
- \$20 if death is within the next 10 years, and
- \$50 if alive at the end of 20 years.

(ii) Mortality follows the Illustrative Life table with $i = 6\%$.
(iii) The level annual benefit premium is payable at the beginning of each year and is determined according to the actuarial equivalence principle.
Using a prospective formula, calculate the 15^{th} year benefit reserve.

Problem 50.5
You are given the following for a special fully discrete whole life issued to (55):
(i) Mortality follows the Illustrative Life Table.
(ii) $i = 0.06$.
(iii) The death benefit is a level 1000 for all years.
(iv) The annual premiums are reduced by 50% at age 65.
Calculate:
(a) The benefit premium during the first 10 years.
(b) The reserve at the end of year 5.
(c) The reserve at the end of year 15.

Problem 50.6 ‡
For a special fully discrete whole life insurance on (40):
(i) The death benefit is 1000 for the first 20 years; 5000 for the next 5 years; 1000 thereafter.
(ii) The annual benefit premium is $1000P_{40}$ for the first 20 years; $5000P_{40}$ for the next 5 years; π thereafter.
(iii) Mortality follows the Illustrative Life Table.
(iv) $i = 0.06$.
Calculate ${}_{21}V$, the benefit reserve at the end of year 21 for this insurance. Hint: Example 50.3.

Problem 50.7 ‡
For a special fully discrete whole life insurance of 1000 on (42):
(i) The contract premium for the first 4 years is equal to the level benefit premium for a fully discrete whole life insurance of 1000 on (40).
(ii) The contract premium after the fourth year is equal to the level benefit premium for a fully discrete whole life insurance of 1000 on (42).
(iii) Mortality follows the Illustrative Life Table.
(iv) $i = 0.06$
(v) ${}_3L$ is the prospective loss random variable at time 3, based on the contract premium.
(vi) $K(42)$ is the curtate future lifetime of (42) .
Calculate $E[{}_3L|K(42) \geq 3]$.

Problem 50.8 ‡
For a special fully discrete 20-year endowment insurance on (40):

(i) The death benefit is 1000 for the first 10 years and 2000 thereafter. The pure endowment benefit is 2000.
(ii) The annual benefit premium, determined using the equivalence principle, is 40 for each of the first 10 years and 100 for each year thereafter.
(iii) $q_{40+k} = 0.001k + 0.001, \;\; k = 8, 9, \cdots, 13$
(iv) $i = 0.05$
(v) $\ddot{a}_{51:\overline{9|}} = 7.1$
Calculate the 10^{th} year terminal reserve using the benefit premiums.

51 Reserves for Fully Continuous General Insurances

Consider a general fully continuous insurance issued to (x) with the following features:
(i) Let b_t be the death benefit payable at the moment of death t.
(ii) Let π_t be the annual rate of benefit premiums at time t and payable continuously.
The actuarial present value of benefits at issue is

$$\text{APVB} = \int_0^\infty b_t \nu^t {}_tp_x \mu(x+t)dt.$$

The actuarial present value of benefit premiums at issue (See Section 39.2) is

$$\text{APVP} = \int_0^\infty \pi_t \nu^t {}_tp_x dt.$$

Using the equivalence principle at time 0 we have

$$\int_0^\infty b_t \nu^t {}_tp_x \mu(x+t)dt = \int_0^\infty \pi_t \nu^t {}_tp_x dt.$$

Given the benefit payments, we can solve this equation for the benefit premiums. We illustrate this in the next example.

Example 51.1 ‡
For a special fully continuous whole life on (65) :
(i) The death benefit at time t is $b_t = 1000e^{0.04t},\ t \geq 0$
(ii) Level benefit premiums are payable for life.
(iii) $\mu(t+65) = 0.02,\ t \geq 0$
(iv) $\delta = 0.04$.
Calculate the level premium π.

Solution.
We have

$$\begin{aligned}
\text{APVB at issue} =& \int_0^\infty b_u \nu^u {}_up_x \mu(x+u)du \\
=& \int_0^\infty 1000e^{-0.04t}e^{-0.02t}(0.02)dt = 1000 \\
\text{APVP at issue} =& \pi \bar{a}_{65} = \frac{\pi}{0.06}.
\end{aligned}$$

By the equivalence principle, we have

$$\frac{\pi}{0.06} = 1000 \Longrightarrow \pi = 60 \blacksquare$$

The prospective formula for the reserve at the end of year t is the present value of all future benefits minus the present value of all future premiums, given that the contract has not yet failed, is given by (See Problem 51.2)

$${}_t\bar{V} = E[{}_t\bar{L}|T(x) > t] = \int_0^\infty b_{u+t}\nu^u {}_up_{x+t}\mu(x+t+u)du - \int_0^\infty \pi_{t+r}\nu^r {}_rp_{x+t}dr.$$

The retrospective formula of benefit reserve of this contract at time t is

$${}_t\bar{V}^R = \frac{1}{{}_tE_x}\left[\int_0^t \pi_r\nu^r {}_rp_xdr - \int_0^t b_r\nu^r {}_rp_x\mu(x+r)dr\right].$$

Example 51.2
For the special insurance of the example above, calculate ${}_2\bar{V}$, the benefit reserve at the end of year 2.

Solution.
We have

$$\begin{aligned}{}_2\bar{V} =& \int_0^\infty 1000e^{-0.04(t+2)}e^{-0.02t}(0.02)dt - 60\bar{a}_{67}\\ =&1000e^{0.08} - \frac{60}{0.06} = 83.29 \blacksquare\end{aligned}$$

Practice Problems

Problem 51.1
Find the formula for the prospective loss random variable ${}_t\bar{L}$ for a general fully continuous insurance issued to (x).

Problem 51.2
Show that under the assumption that the distribution of $T(x)-t|T(x)>t$ is equal to the distribution of $T(x+t)$, we have

$${}_2\bar{V} = \int_0^\infty b_{u+t}\nu^u {}_up_{x+t}\mu(x+t+u)du - \int_0^\infty \pi_{t+r}\nu^r {}_up_{x+t}dr.$$

Problem 51.3 ‡
For a fully continuous whole life insurance on (40), you are given:
(i) The level annual premium is 66, payable for the first 20 years.
(ii) The death benefit is 2000 for the first 20 years and 1000 thereafter.
(iii) $\delta = 0.06$.
(iv) $1000\bar{A}_{50} = 333.33$.
(v) $1000\bar{A}^{\,1}_{50:\overline{10|}} = 197.81$.
(vi) $1000{}_{10}E_{50} = 406.57$.
Calculate ${}_{10}\bar{V}$, the benefit reserve for this insurance at time 10.

Problem 51.4
For a fully continuous whole life insurance issued to (40), you are given:
(i) The death benefit of 100,000 is payable at the time of death.
(ii) Benefit premiums are paid continuously at time t at the annual rate of $\pi_t = \pi e^{0.05t}$.
(iii) Mortality follows De Moivre's Law with $\mu(x) = \frac{1}{100-x}$.
(iv) $\delta = 0.05$.
Calculate π.

Problem 51.5
Show that, for a general fully continuous whole life insurance, the retrospective formula equals the prospective formula.

52 Recursive Formulas for Fully Discrete Benefit Reserves

We derive our basic recursive reserve formula based on general intuition. An elegant discussion is found in Section 8.3 of Bowers [1]. Consider the time interval of the $n+1$st year, that is, $[x+n, x+n+1]$. At the instant year $n+1$ starts, the initial benefit reserve for the policy year $n+1$ is ${}_nV + \pi_n$. (Recall Figure 47.1). This amount is supposed to pay a death benefit if the insured dies within the year plus provide a reserve at the end of the year, i.e., at time $n+1$, if the insured lives. Hence,

$$(1+i)({}_nV + \pi_n) = E[\text{death benefit}] + {}_{n+1}Vp_{x+n} = b_{n+1}q_{x+n} + {}_{n+1}Vp_{x+n}.$$

From this equation, we can find the following backward recursion formula for computing the terminal reserve at the end of policy year n:

$${}_nV = b_{n+1}\nu q_{x+n} + {}_{n+1}V\nu p_{x+n} - \pi_n.$$

Also, a forward recursion formula can be derived for computing the terminal reserve at the end of policy year $n+1$:

$${}_{n+1}V = \frac{(1+i)({}_nV + \pi_n) - b_{n+1}q_{x+n}}{p_{x+n}}.$$

A formula giving the premium at the beginning of policy year $n+1$ can also be found:

$$\pi_n = (b_{n+1} - {}_{n+1}V)\nu q_{x+n} + (\nu_{n+1}V - {}_nV).$$

The expression $b_{n+1} - {}_{n+1}V$ is known as the **net amount at risk.** In insurance terms, ${}_{n+1}V$ is to be available at the end of policy year $n+1$ to offset the death benefit payment b_{n+1}. Therefore, the net amount at risk is the amount of money the insurer will have to produce from sources other than the insured's benefit reserve if the insured dies in policy year $n+1$.

Example 52.1
You are given the following mortality table:

x	90	91	92	93	94	95
ℓ_x	1000	900	720	432	216	0

For a special fully discrete 4 year term issued to (91), you are given:
(i) $i = 4\%$
(ii) The death benefit during the first two years is 1000
(iii) The death benefit during the second two years is 500
(iv) The annual benefit premium for the first two years is twice the annual benefit premium for the last two years.
Calculate ${}_2V$ using
(a) the forward recursion formula of reserves;
(b) the backward recursion formula of reserves.

Solution.
(a) First note that ${}_0V$ is the expected loss at time 0, and the equivalence principle is designed to make that expected loss 0. Now, we have

$$\begin{aligned}
{}_1V =& \frac{(1+i)({}_0V + \pi_0) - b_1 q_{91}}{p_{91}}\\
=& \frac{1.04(2 \times 166.2269) - 1000(180/900)}{720/900} = 182.18994\\
{}_2V =& \frac{(1+i)({}_1V + \pi_1) - b_2 q_{92}}{p_{92}}\\
=& \frac{1.04(182.18994 + 2 \times 166.2266) - 1000(288/720)}{432/720}\\
=& 225.38.
\end{aligned}$$

(b) Note that at age 95 the entire population is deceased and no reserve is necessary. That is, ${}_4V = 0$. We have

$$\begin{aligned}
{}_3V =& b_4 \nu q_{94} + {}_4V \nu p_{94} - \pi_3\\
=& \frac{500}{1.04} - 166.2269 = 314.54233\\
{}_2V =& b_3 \nu q_{93} + {}_3V \nu p_{93} - \pi_2\\
=& 500(1.04)^{-1}\left(\frac{216}{432}\right) + 314.54233(1.04)^{-1}\left(\frac{216}{432}\right) - 166.2269\\
=& 225.38 \blacksquare
\end{aligned}$$

Example 52.2
A discrete 10-pay \$2,000 whole life insurance for (40) is based on the Illustrative Life Table. Find

(a) the premium P,
(b) the reserve ${}_1V$, and
(c) the reserve ${}_{30}V$.

Solution.
(a) We have

$$P = \frac{2000A_{40}}{\ddot{a}_{40:\overline{10}|}}.$$

But

$$\begin{aligned}A_{40} =&0.16132\\ \ddot{a}_{40:\overline{30}|} =&\ddot{a}_{40} - {}_{10}E_{40}\ddot{a}_{50}\\ =&14.8166 - 0.53667(0.24905) = 14.6829.\end{aligned}$$

Hence,

$$P = \frac{2000(0.16132)}{14.6829} = 21.9739.$$

(b) We will use a one year forward recursion.

$$\begin{aligned}{}_1V =&\frac{({}_0V + \pi_0)(1+i) - b_1q_{40}}{p_{40}}\\ =&\frac{21.9739(1.06) - 2000(0.00278)}{1 - 0.00278} = 17.7818.\end{aligned}$$

(c) Since all future premiums are 0, the reserve is the APV of future benefits. That is,

$${}_{30}V = 2000A_{60} = 2000(0.36913) = 738.26 \blacksquare$$

Example 52.3 ‡
For a fully discrete 20-payment whole life insurance of 1000 on (x), you are given:
(i) $i = 0.06$
(ii) $q_{x+19} = 0.01254$
(iii) The level annual benefit premium is 13.72.
(iv) The benefit reserve at the end of year 19 is 342.03.
Calculate $1000P_{x+20}$, the level annual benefit premium for a fully discrete whole life insurance of 1000 on $(x + 20)$.

Solution.
We have

$$\begin{aligned}
1000{}_{20}^{20}V(A_x) =&1000A_{x+20} = \frac{1000({}_{19}^{20}V(A_x) + {}_{20}P(A_x))(1+i) - 1000q_{x+19}}{p_{x+19}}\\
=&\frac{(342.03 + 13.72(1.06) - 1000(0.01254)}{1 - 0.01254} = 369.18\\
\ddot{a}_{x+20} =&\frac{1 - A_{x+20}}{d} = \frac{1 - 0.36918}{0.06(1.06)^{-1}} = 11.1445\\
1000P(A_{x+20}) =&\frac{1000A_{x+20}}{\ddot{a}_{x+20}} = \frac{369.18}{11.1445} = 33.1 \blacksquare
\end{aligned}$$

Example 52.4 ‡
You are given:
(i) ${}_kV^A$ is the benefit reserve at the end of year k for type A insurance, which is a fully discrete 10-payment whole life insurance of 1000 on (x).
(ii) ${}_kV^B$ is the benefit reserve at the end of year k for type B insurance, which is a fully discrete whole life insurance of 1000 on (x).
(iii) $q_{x+10} = 0.004$
(iv) The annual benefit premium for type B is 8.36.
(v) ${}_{10}V^A - {}_{10}V^B = 101.35$
(vi) $i = 0.06$
Calculate ${}_{11}V^A - {}_{11}V^B$.

Solution.
We have

$$\begin{aligned}
{}_{11}V^A =&\frac{({}_{10}V^A + 0)(1+i) - 1000q_{x+10}}{p_{x+10}}\\
{}_{11}V^B =&\frac{({}_{10}V^B + \pi^B)(1+i) - 1000q_{x+10}}{p_{x+10}}\\
{}_{11}V^A - {}_{11}V^B =&\frac{({}_{10}V^A - {}_{10}V^B - \pi^B)(1+i)}{p_{x+10}}\\
=&\frac{(101.35 - 8.36)(1.06)}{1 - 0.004} = 98.97 \blacksquare
\end{aligned}$$

Practice Problems

Problem 52.1
You are given the following mortality table:

x	90	91	92	93	94	95
ℓ_x	1000	900	720	432	216	0

For a special fully discrete 3 year endowment insurance issued to (90), you are given:
(i) $i = 4\%$
(ii) The death benefit and the endowment amount are 2000
(iii) The annual benefit premium for the first year is twice the annual benefit premium for the second year which is twice the annual benefit premium for the third year.
(a) Calculate each of the three premiums.
(b) Calculate the benefit reserves using the recursive formula.

Problem 52.2 ‡
For a fully discrete 3-year endowment insurance of 1000 on (x) :
(i) $q_x = q_{x+1} = 0.20$
(ii) $i = 0.06$
(iii) $1000P(A_{x:\overline{3}|}) = 373.63$.
Calculate $1000({}_2V(A_{x:\overline{3}|}) - {}_1V(A_{x:\overline{3}|}))$ using a recursive formula.

Problem 52.3 ‡
For a fully discrete 5-payment 10-year decreasing term insurance on (60), you are given:
(i) , $b_{k+1} = 1000(10 - k),\ k = 0, 1, \cdots, 9$
(ii) Level benefit premiums are payable for five years and equal 218.15 each
(iii) $q_{60+k} = 0.02 + 0.001k,\ k = 0, 1, \cdots, 9$
(iv) $i = 0.06$.
Calculate ${}_2V$, the benefit reserve at the end of year 2, using a recursive formula.

Problem 52.4
A fully discrete 10-year decreasing term insurance is issued to (40) and pays a death benefit of 10,000 in the first year, 9,000 in the second year, and so

on. The level annual premium is 80. You are given:
(i) ${}_2V = 0$.
(ii) $\nu = 0.95$
(iii) $q_{40} = q_{41}$.
Calculate the benefit reserve at the end of the first policy year.

Problem 52.5
For a fully discrete whole life insurance on (50), you are given:
(i) $i = 0.06$
(ii) $b_{10} = 2500$
(iii) ${}_9V + \pi_9 = {}_{10}V = 500$.
Calculate q_{59}.

Problem 52.6 ‡
For a fully discrete 10-payment whole life insurance of 100,000 on (x), you are given:
(i) $i = 0.05$
(ii) $q_{x+9} = 0.011$
(iii) $q_{x+10} = 0.012$
(iv) $q_{x+11} = 0.014$
(v) The level annual benefit premium is 2078.
(vi) The benefit reserve at the end of year 9 is 32,535.
Calculate $100,000A_{x+11}$.

Problem 52.7 ‡
Michel, age 45, is expected to experience higher than standard mortality only at age 64. For a special fully discrete whole life insurance of 1 on Michel, you are given:
(i) The benefit premiums are not level.
(ii) The benefit premium for year 20, Π_{19}, exceeds P_{45} for a standard risk by 0.010.
(iii) Benefit reserves on his insurance are the same as benefit reserves for a fully discrete whole life insurance of 1 on (45) with standard mortality and level benefit premiums.
(iv) $i = 0.03$ (v) ${}_{20}V_{45} = 0.427$
Calculate the excess of q_{64} for Michel over the standard q_{64}.

Problem 52.8 ‡
For a fully discrete whole life insurance of 1000 on (20), you are given:

(i) $1000P_{20} = 10$
(ii) $1000{}_{20}V_{20} = 490$
(iii) $1000{}_{21}V_{20} = 545$
(iv) $1000{}_{22}V_{20} = 605$
(v) $q_{40} = 0.022$
Calculate q_{41}.

Problem 52.9 ‡
For a special fully discrete 3-year endowment insurance on (75), you are given:
(i) The maturity value is 1000.
(ii) The death benefit is 1000 plus the benefit reserve at the end of the year of death.
(iii) Mortality follows the Illustrative Life Table.
(iv) $i = 0.05$
Calculate the level benefit premium for this insurance.

Problem 52.10 ‡
For a deferred whole life annuity-due on (25) with annual payment of 1 commencing at age 60, you are given:
(i) Level benefit premiums are payable at the beginning of each year during the deferral period.
(ii) During the deferral period, a death benefit equal to the benefit reserve is payable at the end of the year of death.
Show that

$$ {}_{20}V = \left(\frac{\ddot{a}_{\overline{60}|}}{\ddot{s}_{\overline{35}|}}\right)\ddot{s}_{\overline{20}|}. $$

Problem 52.11 ‡
For a special fully discrete 20-year endowment insurance on (55):
(i) Death benefits in year k are given by $b_k = 21 - k,\ k = 1, 2, \cdots, 20$
(ii) The maturity benefit is 1.
(iii) Annual benefit premiums are level.
(iv) ${}_kV$ denotes the benefit reserve at the end of year k, $k = 1, 2, \cdots, 20$.
(v) ${}_{10}V = 5.0$
(vi) ${}_{19}V = 0.6$
(vii) $q_{65} = 0.10$
(viii) $i = 0.08$
Calculate ${}_{11}V$.

Problem 52.12 ‡
For a special fully discrete 3-year term insurance on (x) :
(i) Level benefit premiums are paid at the beginning of each year.
(ii)

k	q_{x+k}	b_{k+1}
0	0.03	200,000
1	0.06	150,000
2	0.09	100,000

(iii) $i = 0.06$.
Calculate the initial benefit reserve for year 2.

Problem 52.13 ‡
For a fully discrete whole life insurance of b on (x), you are given:
(i) $q_{x+9} = 0.02904$
(ii) $i = 0.03$
(iii) The initial benefit reserve for policy year 10 is 343.
(iv) The net amount at risk for policy year 10 is 872.
(v) $\ddot{a}_x = 14.65976$.
Calculate the terminal benefit reserve for policy year 9.

Problem 52.14 ‡
For a special fully discrete 2-year endowment insurance on (x) :
(i) The pure endowment is 2000.
(ii) The death benefit for year k is $1000k$ plus the benefit reserve at the end of year k, $k = 1, 2$.
(iii) π is the level annual benefit premium.
(iv) $i = 0.08$
(v) $p_{x+k-1} = 0.9$, $k = 1, 2$.
Calculate π.

Problem 52.15 ‡
For a fully discrete whole life insurance of 1000 on (45), you are given:

t	$1000{}_tV_{45}$	q_{45+t}
22	235	0.015
23	255	0.020
24	272	0.025

Calculate $1000_{25}V_{45}$.

Problem 52.16 ‡

For a special fully whole life insurance on (x), you are given:
(i) Deaths are distributed according to the Balducci assumption over each year of age.
(ii)

k	Net annual premium at beginning of year k	Death benefit at end of year k	Interest rate used during year k	q_{x+k-1}	${}_kV$
2	—	—	—	—	84
3	18	240	0.07	—	96
4	24	360	0.06	0.101	—

(a) Calculate q_{x+2}.
(b) Calculate ${}_4V$.

53 Miscellaneous Examples

In this section we present several examples to illustrate additional applications of the material presented in this chapter.
We first consider a policy with changes in the first policy year so that the policy is a standard policy (i.e., whole life policy) starting from year 2. We illustrate this idea in the next example.

Example 53.1 ‡
For a special fully discrete whole life insurance on (x) :
(i) The death benefit is 0 in the first year and 5000 thereafter.
(ii) Level benefit premiums are payable for life.
(iii) $q_x = 0.05$
(iv) $\nu = 0.90$
(v) $\ddot{a}_x = 5.00$
(vi) ${}_{10}V_x = 0.20$, where ${}_{10}V_x$ is the benefit reserve for a fully discrete whole life insurance issued to (x).
Calculate ${}_{10}V$, the benefit reserve at the end of year 10 for this special insurance.

Solution.
Let P be tne level benefit premium. For this special insurance, we have

$$\begin{aligned}
{}_{10}V =& \sum_{k=0}^{\infty} b_{k+11}\nu^{k+1}{}_kp_{x+10}q_{x+10+k} - \sum_{k=0}^{\infty} \pi_{k+10}\nu^k{}_kp_{x+10}\\
=&5000\sum_{k=0}^{\infty} \nu^{k+1}{}_kp_{x+10}q_{x+10+k} - P\sum_{k=0}^{\infty} \nu^k{}_kp_{x+10}\\
=&5000A_{x+10} - P\ddot{a}_{x+10}.
\end{aligned}$$

Thus, ${}_{10}V$ is fully determined once we have $A_{x+10}, \ddot{a}_{x+10}$ and P.
Now, using the annuity benefit formula for a fully discrete whole life insurance, we have

$${}_{10}Vx = 1 - \frac{\ddot{a}_{x+10}}{\ddot{a}_x} \Longrightarrow \ddot{a}_{x+10} = (1 - {}_{10}V_x)\ddot{a}_x = 4.$$

Using the equivalence principle at time 0 we have

$$\begin{aligned}
\sum_{k=0}^{\infty} b_{k+1}\nu^{k+1}{}_kp_xq_{x+k} =& \sum_{k=0}^{\infty} \pi_k\nu^k{}_kp_x \\
5000\sum_{k=1}^{\infty} \nu^{k+1}{}_kp_xq_{x+k} =& P\sum_{k=0}^{\infty} \nu^k{}_kp_x \\
5000\nu p_x\sum_{k=0}^{\infty} \nu^{k+1}{}_kp_{x+1}q_{x+1+k} =& P\sum_{k=0}^{\infty} \nu^k{}_kp_x \\
5000\nu p_xA_{x+1} =& P\ddot{a}_x.
\end{aligned}$$

From the recursion formula

$$A_x = \nu q_x + \nu p_xA_{x+1}$$

we obtain $\nu p_xA_{x+1} = A_x - \nu q_x = 1 - d\ddot{a}_x - \nu q_x = 1 - (1 - 0.90)(5.00) - (0.90)(0.05) = 0.455$. Hence,

$$P = \frac{5000(0.455)}{5.00} = 455.$$

Next, we have

$$A_{x+10} = 1 - d\ddot{a}_{x+10} = 1 - 0.10(4) = 0.60.$$

Finally, we have

$${}_{10}V = 5000(0.60) - 455(4) = 1180 \blacksquare$$

Another popular type of problems is one in which the benefit or the payment is doubled or tripled at some point in time.

Example 53.2
You are given the following mortality table:

x	90	91	92	93	94	95
ℓ_x	1000	900	720	432	216	0

For a special fully discrete 3 year endowment insurance issued to (90), you are given:
(i) $i = 4\%$
(ii) The death benefit and the endowment amount are 2000
(iii) The annual benefit premium for the first year is twice the annual benefit premium for the second year which is twice the annual benefit premium for the third year.
(a) Calculate each of the three premiums.
(b) Calculate the benefit reserves using the recursive formula.

Solution.
(a) Let π_2 be the benefit premium for the third year. By the equivalence principle, we have

$$4\pi_2 + 2\pi_2\nu p_{90} + \pi_2\nu^2{}_2p_{90} = 2000\nu q_{90} + 2000\nu^2 p_{90}q_{91} + 2000\nu^3{}_2p_{90}q_{92}.$$

Thus,

$$\pi_2 = 2000(1.04)^{-1}\left(\frac{0.10 + (1.04)^{-1}(0.9)(0.20) + (1.04)^{-2}(0.72)(0.4)}{4 + 2(1.04)^{-1}(0.9) + (1.04)^{-2}(0.72)}\right) = 282.23511.$$

Hence, $\pi_0 = 4(282.23511) = 1128.940$ and $\pi_1 = 2(282.23511) = 564.470$.
(b) We have

$$\begin{aligned}
{}_0V =& 0\\
{}_1V =& \frac{({}_0V + \pi_0)(1+i) - b_1q_{90}}{p_{90}} = 1082.33\\
{}_2V =& \frac{({}_1V + \pi_1)(1+i) - b_2q_{91}}{p_{91}} = 1640.84\\
{}_3V =& \frac{({}_2V + \pi_2)(1+i) - b_3q_{92}}{p_{92}} = 2000 \blacksquare
\end{aligned}$$

Practice Problems

Problem 53.1
A fully discrete whole life insurance with a unit benefit issued to (x) has its first year benefit premium equal to the actuarial present value of the first year's benefit, and the remaining benefit premiums are level and determined by the equivalence principle. Calculate the benefit reserve at the end of the first policy year, i.e., ${}_1V$.

54 Benefit Reserves at Fractional Durations

In this section we seek a formula for ${}_{h+s}V$, where h is a non-negative integer and $0 < s < 1$, for the general insurance introduced in Section 50. See Figure 54.1. We will refer to ${}_{h+s}V$ as the **interim benefit reserve**.

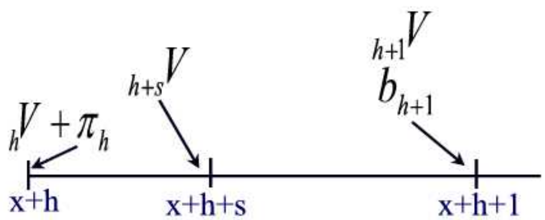


Figure 54.1

Using the above diagram and the basic prospective approach, we can write

$$\begin{aligned}
{}_{h+s}V =&\text{APVB}_{x+h+s} - \text{APVP}_{x+h+s}\\
=&\nu^{1-s}b_{h+1}{}_{1-s}q_{x+h+s} + \nu^{1-s}{}_{1-s}p_{x+h+s}\text{APVB}_{x+h+1} - \nu^{1-s}{}_{1-s}p_{x+h+s}\text{APVP}_{x+h+1}\\
=&\nu^{1-s}b_{h+1}{}_{1-s}q_{x+h+s} + \nu^{1-s}{}_{1-s}p_{x+h+s}({}_{h+1}V).
\end{aligned}$$

Now, if we multiply both sides of the above equation by $\nu^s{}_sp_{x+h}$ we obtain

$$\nu^s{}_sp_{x+h}{}_{h+s}V = \nu b_{h+1}{}_{s|1-s}q_{x+h} + \nu{}_{s|1-s}p_{x+h}({}_{h+1}V). \tag{54.-3}$$

From Section 52, we have

$${}_hV + \pi_h = b_{h+1}\nu q_{x+h} + {}_{h+1}V\nu p_{x+h}$$

so that

$$b_{h+1}\nu q_{x+h} = {}_hV + \pi_h - {}_{h+1}V\nu p_{x+h}.$$

Substituting this into equation () and rearranging, we find

$$\nu^s{}_sp_{x+h}{}_{h+s}V = ({}_hV + \pi_h)\left(\frac{{}_{s|1-s}q_{x+h}}{q_{x+h}}\right) + {}_{h+1}V\nu p_{x+h}\left(1 - \frac{{}_{s|1-s}q_{x+h}}{q_{x+h}}\right). \tag{54.-3}$$

This expression shows that the expected present value at duration h of the interim benefit reserve is the weighted average of the initial reserve ${}_hV + \pi_h$ and the expected present value of the terminal reserve with weights r and $1 - r$ respectively, where

$$r = \frac{{}_{s|1-s}q_{x+h}}{q_{x+h}}.$$

Now, under the assumption of uniform distribution of deaths over the age interval $(x+h, x+h+1)$, (see Section 24.1), we have

$$_{s|1-s}q_{x+h} = q_{x+h} - {}_sq_{x+h} = q_{x+h} - sq_{x+h} = (1-s)q_{x+h}.$$

It follows that $r = 1-s$ and using this in Equation () we obtain

$$\nu^s(1-sq_{x+h})_{h+s}V = ({}_hV + \pi_h)(1-s) + {}_{h+1}V\nu p_{x+h}s.$$

Furthermore, if we let $i = q_{x+h} = 0$ in the interval $(x+h, x+h+1)$, the previous equation becomes

$$_{h+s}V = ({}_hV + \pi_h)(1-s) + {}_{h+1}Vs$$

which can be written as

$$_{h+s}V = {}_hV(1-s) + {}_{h+1}Vs + (1-s)\pi_h.$$

The first component of this equation ${}_hV(1-s) + {}_{h+1}Vs$ is the interpolated reserve and the second component $(1-s)\pi_h$ is the **unearned premium reserve.**

Example 54.1
A fully discrete 20 year endowment insurance of 1 is issued on (50). Mortality follows the Illustrative Life Table with interest at 6%. Benefit premiums are paid annually. Determine ${}_{10.7}V(A_{50:\overline{20}|})$ using both UDD and linear interpolation.

Solution.
We have

$$_{10}V(A_{50:\overline{20}|}) = \frac{A_{60:\overline{10}|} - A_{50:\overline{20}|}}{1 - A_{50:\overline{20}|}}$$

where

$$\begin{aligned}
A_{50:\overline{20}|} =& A_{50} - {}_{20}E_{50}A_{70} + {}_{20}E_{50}\\
=&0.24905 - 0.23047(0.51495) + 0.23047 = 0.36084\\
A_{60:\overline{10}|} =& A_{60} - {}_{10}E_{60}A_{70} + {}_{10}E_{60}\\
=&0.36913 - 0.45120(0.51495) + 0.45120 = 0.58798.
\end{aligned}$$

and

$$P(A_{50:\overline{20}|}) = \frac{dA_{50:\overline{20}|}}{1 - A_{50:\overline{20}|}} = \frac{0.36804(0.06)(1.06)^{-1}}{1 - 0.36804} = 0.031956.$$

Thus,

$$\begin{aligned}
{}_{10}V(A_{50:\overline{20}|}) =& \frac{0.58798 - 0.36084}{1 - 0.36084} = 0.35537 \\
{}_{11}V(A_{50:\overline{20}|}) =& \frac{({}_{10}V + \pi)(1+i) - b_{11}(q_{x+10})}{p_{x+10}} \\
=& \frac{(0.35537 + 0.031956)(1.06) - (1)(0.01376)}{1 - 0.01376} = 0.40234.
\end{aligned}$$

Using linear interpolation of reserve, we find

$$\begin{aligned}
{}_{10.7}V(A_{50:\overline{20}|}) =& (1-0.7)_{10}V + 0.7_{11}V + (1-0.7)\pi \\
=& 0.3(0.35537) + 0.7(0.40234) + 0.3(0.031956) = 0.39784.
\end{aligned}$$

Using UDD, we have

$$\begin{aligned}
{}_{10.7}V(A_{50:\overline{20}|}) =& \frac{({}_{10}V + \pi)(1-0.7) + {}_{11}V\nu p_{x+10}(0.7)}{\nu^{0.7}{}_{0.7}p_{x+10}} \\
=& \frac{(0.35537 + 0.031956)(0.3) + 0.40234(1.06)^{-1}(0.98624)(0.7)}{(1.06)^{-0.7}(0.3)(0.01376)} \\
=& 0.39782 \blacksquare
\end{aligned}$$

Practice Problems

Problem 54.1
For a fully discrete whole life insurance on (65), you are given
(i) ${}_{10}V_{65} = 0.23114$
(ii) ${}_{11}V_{65} = 0.25540$
(iii) $\pi_{65} = 0.03300$.
Calculate ${}_{10\frac{7}{12}}V65$, under the assumptions of a uniform distribution of deaths between integral ages and $i = 0$ and $q_{75} = 0$.

Problem 54.2
Consider a fully discrete whole insurance on (60). You are given:
(i) $b_h = 5000 + 90h$ for $h = 1, 2, \cdots$.
(ii) $delta = 0.06$
(ii) $\mu = 0.02$
(iv) ${}_4V = 240$.
Calculate the benefit reserve at the end of the 39^{th} month.

Problem 54.3
Consider a special fully discrete whole life policy issued to (97). You are given:
(i) Death benefit of \$1000 in the first year, increasing by \$1000 in each subsequent year.
(ii) Level annual benefit premium of \$839.
(iii) Mortality follows De Moivre's Law with $\omega = 100$.
(iv) $\nu = 0.90$.
Calculate the benefit reserve at age 98.5 using linear interpolation.

Problem 54.4
A fully discrete whole life insurance with face amount 100,000 and level premiums is issued at age 80. Mortality follows the Illustrative Life Table with interest 6%.
Let ${}_{10.5}V^{\text{UDD}}$ denote the benefit reserve at time 10.5 assuming UDD over the age interval. Let ${}_{10.5}V^{\text{LIN}}$ denote the benefit reserve at time 10.5 using linear interpolation (i.e., $i = q_{90} = 0.$).
Calculate ${}_{10.5}V^{\text{UDD}} - {}_{10.5}V^{\text{LIN}}$ (the difference between the two approximations).

Problem 54.5

Consider a fully discrete whole life policy issued to (x) on January 1, 2007. You are given:

(i) $b_h = 1000$ for $h = 1, 2, \cdots$

(ii) Initial benefit reserve on January 1, 2009 is 500.

Assuming $i = 6\%$, $q_{x+2} = 0.1$ and uniform distribution of deaths over each year of age, find ${}_{2.25}V$.

Problem 54.6 ‡

For a special fully whole life insurance on (x), you are given:

(i) Deaths are distributed according to the Balducci assumption over each year of age.

(ii)

k	Net annual premium at beginning of year k	Death benefit at end of year k	Interest rate used during year k	q_{x+k-1}	${}_kV$
2	—	—	—	—	84
3	18	240	0.07	—	96
4	24	360	0.06	0.101	101.05

(a) Calculate ${}_{0.5}q_{x+3.5}$.

(b)Calculate ${}_{3.5}V$.

55 Calculation of Variances of Loss Random Variables: The Hattendorf's Theorem

In this section we consider the question of calculating $\text{Var}({}_hL|K(x) \geq h)$ where ${}_hL$ is the loss random variable of the fully discrete general insurance of Section 50. By Problem 50.1, we have

$$ {}_hL = \begin{cases} 0 & K(x) < h \\ b_{K(x)+1+h}\nu^{K(x)+1-h} - \sum_{j=h}^{K(x)} \pi_j \nu^{j-h} & K(x) \geq h. \end{cases} $$

We first look at an example where the mean and variance of ${}_hL$ are found directly from its distribution function.

Example 55.1
You are given the following mortality table:

x	90	91	92	93	94
ℓ_x	100	75	50	25	0

Consider a fully discrete 4-year term policy issued to (90) with unit benefit, level premium and interest $i = 0.06$. Mortality follows De Moivre's Law. Find the expected value and variance of ${}_1L$ given that the life is alive at time 1.

Solution.
The formula for ${}_1L$ is

$$ {}_1L = \nu^{K(90)} - P\ddot{a}_{\overline{K(90)}|}. $$

Let's find the level premium of this policy. We have

$$ \begin{aligned} \text{APFB} =& \nu q_{90} + \nu^2 p_{90} q_{91} + \nu^3 {}_2p_{90} q_{92} + \nu^4 {}_3p_{90} \\ =& (1.06)^{-1}\left(\frac{25}{100}\right) + (1.06)^{-2}\left(\frac{75}{100}\right)\left(\frac{25}{75}\right) \\ &+ (1.06)^{-3}\left(\frac{50}{100}\right)\left(\frac{25}{50}\right) + (1.06)^{-4}\left(\frac{25}{100}\right) \\ =& 0.86628 \\ \text{APFP} =& P + P\nu p_{90} + P\nu^2 {}_2p_{90} + P\nu^3 {}_3p_{90} \\ =& P[1 + (1.06)^{-1}(0.75) + (1.06)^{-2}(0.5) + (1.06)^{-3}(0.25)] \\ =& 2.36245P. \end{aligned} $$

By the equivalence principle, we must have

$$2.36245P = 0.86628 \Longrightarrow P = 0.36669.$$

Now, let's find the distribution function of ${}_1L$. We have

$$\begin{aligned}
K(90) =1 \quad &\Longrightarrow {}_1L = \nu - 0.36669\ddot{a}_{\overline{1|}} = (1.06)^{-1} - 0.36669(1) = 0.57671\\
K(90) =2 \quad &\Longrightarrow {}_1L = \nu^2 - 0.36669\ddot{a}_{\overline{2|}} = (1.06)^{-2} - 0.36669(1.9434) = 0.17737\\
K(90) =3 \quad &\Longrightarrow {}_1L = \nu^3 - 0.36669\ddot{a}_{\overline{3|}} = (1.06)^{-3} - 0.36669(2.83339) = -0.19936
\end{aligned}$$

Thus, the distribution function of ${}_1L$ is summarized in the table below.

$K(90)$	Probability	${}_1L$
1	$\frac{1}{3}$	0.57671
2	$\frac{1}{3}$	0.17737
3	$\frac{1}{3}$	-0.19936

From this distribution function, we can find

$$\begin{aligned}
E({}_1L|K(90) \geq 1) =&\frac{1}{3}(0.57671 + 0.17737 - 0.19936) = 0.18491\\
E(({}_1L)^2|K(90) \geq 1) =&\frac{1}{3}(0.57671^2 + 0.17737^2 + 0.19936^2) = 0.13460\\
\text{Var}({}_1L|K(90) \geq 1) =&0.13460 - 0.18491^2 = 0.10041 \blacksquare
\end{aligned}$$

An alternative way for finding the variance of ${}_hL$ is by means of a result known as **Hattendorf theorem** which we discuss next.
Let h be a non-negative integer. For the year interval $(h, h+1)$, we let Λ_h be the random variable representing the present value at time h of the accrued loss[4] for the $(h+1)-$policy year. Then for Λ_h is given explicitly by

$$\Lambda_h = \begin{cases} 0 & K(x) < h\\ \nu b_{h+1} - \pi_h - {}_hV & K(x) = h\\ \nu_{h+1}V - \pi_h - {}_hV & K(x) \geq h+1.\end{cases}$$

It follows that the distribution of the random variable $\{\Lambda_h | K(x) \geq h\}$ is a two-point distribution with probability distribution

[4]With cashflow, an inflow will be assigned a negative sign while an outflow will be assigned a positive flow.

t	$\Pr(\Lambda_h = t \vert K(x) \geq h)$
$\nu b_{h+1} - \pi_h - {}_hV$	q_{x+h}
$\nu_{h+1}V - \pi_h - {}_hV$	p_{x+h}

Thus,

$$\begin{aligned} E[\Lambda_h|K(x) \geq h) =& (\nu b_{h+1} - \pi_h - {}_hV)q_{x+h} + (\nu_{h+1}V - \pi_h - {}_hV)p_{x+h} \\ =& \nu b_{h+1}q_{x+h} + \nu_{h+1}Vp_{x+h} - (\pi_h + {}_hV). \end{aligned}$$

But, from the forward recursion formula of reserves, the right-hand side is 0. Now, recall the following formula from Section 50,

$$\pi_h + {}_hV = \nu b_{h+1}q_{x+h} + \nu_{h+1}Vp_{x+h}.$$

Using this formula twice, we can write

$$\begin{aligned} \nu b_{h+1} - \pi_h - {}_hV =& \nu(b_{h+1} - {}_{h+1}V)p_{x+h} \\ \nu_{h+1}V - \pi_h - {}_hV =& \nu(b_{h+1} - {}_{h+1}V)q_{x+h}. \end{aligned}$$

Hence, the second moment of the random variable $\{\Lambda_h|K(x) \geq h\}$ is

$$\begin{aligned} \mathrm{Var}(\Lambda_h|K(x) \geq h) =& (\nu b_{h+1} - \pi_h - {}_hV)^2q_{x+h} + (\nu_{h+1}V - \pi_h - {}_hV)^2p_{x+h} \\ =& [\nu(b_{h+1} - {}_{h+1}V)]^2p_{x+h}^2q_{x+h} + [\nu(b_{h+1} - {}_{h+1}V)]^2p_{x+h}q_{x+h}^2 \\ =& [\nu(b_{h+1} - {}_{h+1}V)]^2p_{x+h}q_{x+h}(p_{x+h} + q_{x+h}) \\ =& [\nu(b_{h+1} - {}_{h+1}V)]^2p_{x+h}q_{x+h}. \end{aligned}$$

Example 55.2
Let Λ_{11} denote the accrued cost random variable in the 11^{th} year for a discrete whole life insurance of amount 1000 issued to (40). Calculate the value of $\mathrm{Var}(\Lambda_{11}|K(x) > 10)$, given the following values:
(i) $i = 0.06$
(ii) $\ddot{a}_{40} = 14.8166$
(iii) $\ddot{a}_{50} = 13.2669$
(iv) $\ddot{a}_{51} = 13.0803$.

Solution.
We are asked to find

$$\mathrm{Var}(\Lambda_{11}|K(x) > 10) = \mathrm{Var}(\Lambda_{11}|K(x) \geq 11) = \nu^2(1000 - {}_{11}V)^2p_{50}q_{50}$$

where

$$_{11}V = 1000_{11}V_{40} = 1000\left(1 - \frac{\ddot{a}_{51}}{\ddot{a}_{50}}\right) = 1000\left(1 - \frac{13.0803}{13.2669}\right) = 117.19.$$

Using the relation $\ddot{a}_{50} = 1 + \nu p_{50}\ddot{a}_{51}$ we can write

$$p_{50} = \frac{\ddot{a}_{50} - 1}{\nu \ddot{a}_{51}} = 0.99408.$$

Hence,

$$\text{Var}(\Lambda_{11}|K(x) > 10) = 1.06^{-2}(1000 - 117.19)^2(0.99408)(1 - 0.99408) = 4081.93 \blacksquare$$

Next, the loss random variable $_hL$ can be expressed in terms of the Λ_j's. Indeed, it can be shown that

$$_hL = \sum_{j=h}^{\infty} \nu^{j-h}\Lambda_j + {}_hV$$

which is valid for $K(x) \geq h$. Taking the variance of both sides, we find

$$\begin{aligned}
\text{Var}({}_hL|K(x) \geq h) =& \sum_{j=h}^{\infty} \nu^{2(j-h)}\text{Var}[\Lambda_j|K(x) \geq h]\\
=& \sum_{j=h}^{\infty} \nu^{2(j-h)}{}_{j-h}p_{x+h}\text{Var}[\Lambda_j|K(x) \geq j]\\
=&\text{Var}[\Lambda_h|K(x) \geq h] + \sum_{j=h+1}^{\infty} \nu^{2(j-h)}{}_{j-h}p_{x+h}\text{Var}[\Lambda_j|K(x) \geq j]\\
=&[\nu(b_{h+1} - {}_{h+1}V)]^2 p_{x+h}q_{x+h} + \nu^2 p_{x+h}\text{Var}({}_{h+1}L).
\end{aligned}$$

This recursion formula for $\text{Var}({}_hL|K(x) \geq h)$ is part of the Haddentorf Theorem. The formula says that the variance at time h of all future losses is equal to the variance of the losses of the next year discounted to time h plus the variance of all subsequent years discounted to time h.

Example 55.3
Evaluate $\text{Var}({}_1L|K(90) \geq 1)$ in Example 55.1 by means of Haddentorf theorem.

Solution.
We have

$$\begin{aligned}
{}_1V =& \frac{(1+i)({}_0V+\pi_0)-b_1q_{90}}{p_{90}}\\
=& \frac{1.06(0+0.36669)-0.25}{0.75}=0.18492\\
{}_2V =& \frac{(1+i)({}_1V+\pi_1)-b_2q_{91}}{p_{91}}\\
=& \frac{1.06(0.18492+0.36669)-1/3}{2/3}\\
=& 0.37706.
\end{aligned}$$

Thus, using Problem 55.1

$$\begin{aligned}
[\nu(b_{h+1}-{}_{h+1}V)]^2p_{x+h}q_{x+h} =& \nu^2(1-{}_2V)^2p_{91}q_{91}\\
=& (1.06)^{-2}(1-0.37706)^2(2/9)=0.07675\\
\nu^2p_{x+h}\mathrm{Var}({}_{h+1}L) =& \nu^2p_{91}\mathrm{Var}({}_2L)\\
=& (1.06)^{-2}(2/3)(0.03987)=0.02366.
\end{aligned}$$

Hence,

$$\mathrm{Var}({}_1L|K(90)\geq 1)=0.07675+0.02366=0.10041 \blacksquare$$

Practice Problems

Problem 55.1
You are given the following mortality table:

x	90	91	92	93	94
ℓ_x	100	75	50	25	0

Consider a fully discrete 4-year term policy issued to (90) with unit benefit, level premium and interest $i = 0.06$. Mortality follows De Moivre's Law. Find the expected value and variance of ${}_2L$ using the distribution function of ${}_2L$.

Problem 55.2
A fully discrete \$1000 whole life policy is issued to (40). The mean and the standard deviation of the prospective loss random variable after 2 years are \$100 and \$546.31, respectively. The mean and the standard deviation of the prospective loss random variable after 11 years are \$100 and \$100 respectively. You are given:
(i) $q_{41} = 0.40$
(ii) $q_{41+t} = (1.01)^t q_{41}$
(iii) $d = 0.06$.

Calculate: $\frac{\text{Var}({}_{10}L|K(40)\geq 10)}{\text{Var}({}_1L|K(40)\geq 1)}$.

Problem 55.3
Let L denote the insurer's loss for a fully discrete annual premium two-year endowment of \$10 issued to (x). Premiums are determined by the equivalence principle, but are not necessarily level. You are given
(i) $d = 10\%$
(ii) $\text{Var}(L) = 3.24$
(iii) ${}_1V = 5.00$
(iv) $p_x > q_x$.
Calculate the first year premium.

Problem 55.4 ‡
For a fully discrete 2-year term insurance of 1 on (x) :
(i) $q_x = 0.1$ and $q_{x+1} = 0.2$
(ii) $\nu = 0.9$

(iii) ${}_1L$ is the prospective loss random variable at time 1 using the premium determined by the equivalence principle.
Calculate $\mathrm{Var}({}_1L|K(x) > 0)$.

Multiple Life Models

The actuarial mathematics theory that we have developed for the death benefit of a single life can be extended to multiple lives. The models that we discuss in this chapter involve two lives. Unless otherwise stated, we will assume that the two future lifetime random variables are independent.
The survival of the two lives is referred to as the **status of interest** or simply the **status.** There are two common types of status:
(1) The **joint-life status** is one that requires the survival of *both* lives. Accordingly, the status terminates upon the first death of one of the two lives.
(2) The **last-survivor status** is one that ends upon the death of both lives. That is, the status survives as long as at least one of the component members remains alive.

56 The Joint-Life Status Model

A joint-life status of two lives (x) and (y) will be denoted by (xy) as of time 0. At time n, we will use the notation $(x+n:y+n)$. The remaining lifetime random variable of the status will be denoted by $T(xy)$. Since the status terminates upon the first death, we have

$$T(xy) = \min\{T(x), T(y)\}.$$

All the theory of the previous chapters applies to the status (xy).

56.1 The Joint Survival Function of $T(xy)$

For the joint survival function of $T(xy)$, we have

$$\begin{aligned}
S_{T(xy)}(t) =& {}_tp_{xy} = \Pr(T(xy) > t) = \Pr(\min\{T(x), T(y)\} > t) \\
=& \Pr([T(x) > t] \text{ and } [T(y) > t]) = \Pr(T(x) > t)\Pr(T(y) > t) \\
=& {}_tp_x \ {}_tp_y.
\end{aligned}$$

Example 56.1
You are given:
(i) $T(50)$ and $T(75)$ are independent
(ii) Mortality for both lives follows De Moivre's Law with $\omega = 100$.
Calculate: ${}_tp_{50:75}$.

Solution.
We know that

$$\begin{aligned}
{}_tp_{50} =& 1 - \frac{t}{50}, \quad 0 \le t \le 50 \\
{}_tp_{75} =& 1 - \frac{t}{25}, \quad 0 \le t \le 25.
\end{aligned}$$

Thus,

$$\begin{aligned}
{}_tp_{50:75} =& {}_tp_{50}{}_tp_{75} \\
=& \frac{(50-t)(25-t)}{1250} \\
=& 1 - \left(\frac{75t - t^2}{1250}\right), \quad 0 \le t \le 25 \blacksquare
\end{aligned}$$

Example 56.2
You are given that mortality follows the Illustrative Life Table with $i = 0.06$. Assuming that $T(50)$ and $T(60)$ are independent, calculate ${}_{10}p_{50:60}$.

Solution.
We have

$$ {}_{10}p_{50:60} = {}_{10}p_{50}\,{}_{10}p_{60} = \frac{\ell_{60}}{\ell_{50}}\frac{\ell_{70}}{\ell_{60}} = \frac{\ell_{70}}{\ell_{50}} = \frac{6,616,155}{8,950,901} = 0.73916 \blacksquare $$

Example 56.3 ‡
A 30-year term insurance for Janet age 30 and Andre age 40 provides the following benefits:
• A death benefit of \$140,000 if Janet dies before Andre and within 30 years
• A death benefit of \$180,000 if Andre dies before Janet and within 30 years
You are given:
(i) Mortality follows De Moivre's Law with $\omega = 100$
(ii) $i = 0$
(iii) The death benefit is Payable at the moment of the first death.
(iv) Premiums, $\bar{P}$, are paid continuously while both are alive for a maximum of 20 years.
Calculate the probability that at least one of Janet or Andre will die within 10 years.

Solution.
The probability that both will survive 10 years is ${}_{10}p_{30:40}$. Thus, the probability that at least one of Janet or Andre will die within 10 years is

$$ 1 - {}_{10}p_{30:40} = 1 - {}_{10}p_{30}\,{}_{10}p_{40} = 1 - \left(1 - \frac{10}{70}\right)\left(1 - \frac{10}{60}\right) = \frac{2}{7} \blacksquare $$

Practice Problems

Problem 56.1
Define ${}_tq_x = 1 - {}_tp_y$. Show that

$${}_tq_{xy} = {}_tq_x + {}_tq_y - {}_tq_x\ {}_tq_y.$$

Problem 56.2
Suppose that $T(x)$ and $T(y)$ are independent. Show that

$$\Pr[(T(x) > n) \cup (T(y) > n)] = {}_np_x + {}_np_y - np_{xy}.$$

Problem 56.3
You are given:
(i) $T(25)$ and $T(50)$ are independent
(ii) ${}_{50}p_{25} = 0.2$.
Calculate ${}_{25}p_{25:50}$.

Problem 56.4
You are given:
(i) Mortality follows De Moivre's Law with $\omega = 100$
(ii) $T(80)$ and $T(85)$ are independent
Find the probability that the first death occurs after 5 and before 10 years from now.

Problem 56.5
You are given:
(i) ${}_3p_{40} = 0.990$
(ii) ${}_6p_{40} = 0.980$
(iii) ${}_9p_{40} = 0.965$
(iv) ${}_{12}p_{40} = 0.945$
(v) ${}_{15}p_{40} = 0.920$
(vi) ${}_{18}p_{40} = 0.890$.
For two independent lives aged 40, calculate the probability that the first death occurs after 6 years, but before 12 years.

56.2 The Joint CDF/PDF of $T(xy)$

For the joint cumulative distribution function of $T(xy)$, we have

$$F_{T(xy)}(t) = {}_tq_{xy} = 1 - S_{T(xy)}(t) = 1 - {}_tp_x\ {}_tp_y.$$

Example 56.4
You are given:
(i) $T(50)$ and $T(75)$ are independent
(ii) Mortality for both lives follows De Moivre's Law with $\omega = 100$.
Calculate: ${}_tq_{50:75}$.

Solution.
We have

$$ {}_tq_{50:75} = 1 - \left[1 - \left(\frac{75t - t^2}{1250}\right)\right] = \frac{75t - t^2}{1250}, \quad 0 \le t \le 25 \blacksquare$$

Example 56.5
You are given that mortality follows the Illustrative Life Table with $i = 0.06$. Assuming that $T(50)$ and $T(60)$ are independent, calculate ${}_{10}q_{50:60}$.

Solution.
We have

$$ {}_{10}q_{50:60} = 1 - {}_{10}p_{50:60} = 1 - 0.73916 = 0.26084 \blacksquare$$

Example 56.6
The probability that the time of death of the joint status occurs in the interval $(n, n+1]$ will be denoted by ${}_{n|}q_{xy}$. Thus. ${}_{n|}q_{xy}$ is the probability that the first death occurs in the interval $(n, n+1]$. Derive an expresion for this probability.

Solution.
We have

$$\begin{aligned}
{}_{n|}q_{xy} =&\Pr(n < T(xy) \le n+1) = \Pr(n \le T(xy) < n+1)\\
=&S_{T(xy)}(n) - S_{T(xy)}(n+1) = {}_np_{xy} - {}_{n+1}p_{xy}\\
=&{}_np_{xy} - {}_np_{xy}p_{(x+n:y+n)} = {}_np_{xy}(1 - p_{(x+n:y+n)})\\
=&{}_np_{xy}q_{(x+n:y+n)}.
\end{aligned}$$

Note that if $K(xy)$ is the curtate lifetime of the joint-life status (xy) then ${}_{n|}q_{xy} = \Pr(K(xy) = n)$ ■

The joint probability density function of $T(xy)$ is found as follows:

$$\begin{aligned}
f_{T(xy)}(t) =& -\frac{d}{dt}S_{T(xy)}(t) = -\frac{d}{dt}({}_tp_x{}_tp_y)\\
=& -\left[{}_tp_x\frac{d}{dt}({}_tp_y) + \frac{d}{dt}({}_tp_x){}_tp_y\right]\\
=& -\left[{}_tp_x(-{}_tp_y\mu(y+t) + (-{}_tp_x\mu(x+t)){}_tp_y\right]\\
=& {}_tp_x\ {}_tp_y[\mu(x+t) + \mu(y+t)] = {}_tp_{xy}[\mu(x+t) + \mu(y+t)].
\end{aligned}$$

Example 56.7
You are given:
(i) $T(50)$ and $T(75)$ are independent
(ii) Mortality for both lives follows De Moivre's Law with $\omega = 100$.
Calculate: $f_{50:75}(t)$.

Solution.
We know that

$$\mu(x+50) = \tfrac{1}{50-t} \text{ and } \mu(y+75) = \tfrac{1}{25-t}.$$

It follows that

$$f_{50:75}(t) = \left[1 - \left(\frac{75t - t^2}{1250}\right)\right]\left[\frac{1}{50-t} + \frac{1}{25-t}\right],\ 0 \le t \le 25 \ ■$$

Practice Problems

Problem 56.6
Show that: ${}_tq_{xy} = {}_tq_x + {}_tq_y - {}_tq_x\ {}_tq_y$.

Problem 56.7 ‡
For independent lives $T(x)$ and $T(y)$:
(i) $q_x = 0.05$
(ii) $q_y = 0.10$
(iii) Deaths are uniformly distributed over each year of age.
Calculate ${}_{0.75}q_{xy}$.

Problem 56.8
Find an expression for ${}_{n|m}q_{xy}$.

Problem 56.9
You are given that mortality follows the Illustrative Life Table with $i = 0.06$. Assuming that $T(50)$ and $T(60)$ are independent, calculate ${}_{10|}q_{50:60}$.

Problem 56.10
$T(x)$ and $T(y)$ are independent and each is uniformly distributed over each year of age. Show that

$$18\,{}_{\frac{1}{3}}q_{xy} - 12\,{}_{\frac{1}{2}}q_{xy} = q_xq_y.$$

56.3 The Force of Mortality of $T(xy)$

For the force of mortality of $T(xy)$, we have

$$\begin{aligned}\mu_{xy}(t) =&\mu_{x+t:y+t} = \mu_{T(xy)}(t) = \frac{f_{T(xy)}(t)}{S_{T(xy)}(t)}\\ =&\frac{{}_tp_{xy}[\mu(x+t)+\mu(y+t)]}{{}_tp_{xy}} = \mu(x+t)+\mu(y+t).\end{aligned}$$

Example 56.8
You are given:
(i) $T(50)$ and $T(75)$ are independent
(ii) Mortality for both lives follows De Moivre's Law with $\omega = 100$.
Calculate: $\mu_{50:75}(t)$.

Solution.
We know that

$$\mu(x+50) = \tfrac{1}{50-t} \text{ and } \mu(y+75) = \tfrac{1}{25-t}.$$

It follows that

$$\mu_{50:75}(t) = \frac{1}{50-t} + \frac{1}{25-t} = \frac{75-2t}{(50-t)(25-t)}, \quad 0 \le t \le 25 \blacksquare$$

Example 56.9
Suppose that (x) has a constant mortality μ_x and (y) has a constant mortality μ_y. Show that the joint-life status has a constant mortality $\mu_x + \mu_y$.

Solution.
We know that

$${}_tp_x = e^{-\mu_x t} \text{ and } {}_tp_y = e^{-\mu_y t}.$$

It follows that

$${}_tp_{xy} = {}_tp_{x\,t}p_y = e^{-(\mu_x+\mu_y)t} \blacksquare$$

Practice Problems

Problem 56.11
You are given:
(i) $T(x)$ and $T(y)$ are independent
(ii) $\mu(x+t) = 0.03$
(iii) $\mu(y+t) = 0.05$
Calculate: $\mu_{xy}(t)$.

Problem 56.12
You are given $\mu(x) = \frac{1}{100-x}$, $0 \le x \le 100$. Suppose that $T(40)$ and $T(50)$ are independent. Calculate ${}_{10}p_{40:50}$.

Problem 56.13
You are given that mortality follows Gompertz Law

$$\mu(x) = Bc^x$$

where $B = 10^{-3}$ and $c^{10} = 3$. Suppose that $T(40)$ and $T(50)$ are independent. Calculate ${}_{10}p_{40:50}$.

Problem 56.14
Three lives $(40), (50)$, and (60), with indepent future lifetimes, are each subject to a constant force of mortality with

$$\mu(40+t) = 0.01 \quad \mu(50+t) = 0.02 \quad \mu(60+t) = 0.03.$$

Calculate $\mu_{40:50:60}$.

Problem 56.15
You are given:
(i) $T(x)$ and $T(y)$ are independent
(ii) $\int_0^\infty {}_tp_x = 10$
(iii) $T(x)$ and $T(y)$ are exponential with hazard rates μ_x and μ_y respectively.
Find $\mu_{xy}(t)$.

56.4 Mean and Variance of $T(xy)$

The **complete expectation** of the future lifetime of the joint-life status is

$$\mathring{e}_{xy} = E[T(xy)] = \int_0^\infty t f_{T(xy)}(t)dt = \int_0^\infty {}_tp_{xy}dt.$$

The second moment of $T(xy)$ is

$$E[T(xy)^2] = \int_0^\infty t^2 f_{T(xy)}(t)dt = 2\int_0^\infty t{}_tp_{xy}dt.$$

The variance is

$$\text{Var}[T(xy)] = 2\int_0^\infty t{}_tp_{xy}dt - \mathring{e}_{xy}^2.$$

The **curtate expectation of future lifetime** of the joint-life status is

$$e_{xy} = \sum_{k=1}^\infty {}_kp_{xy}$$

and the $n-$**year curtate lifetime** of the joint-life status is

$$e_{xy:\overline{n}|} = \sum_{k=1}^n {}_kp_{xy}.$$

Example 56.10
Suppose that $T(x)$ and $T(y)$ are exponential with hazard rates μ_x and μ_y respectively. Find an expression for $\mathring{e}_{xy}$.

Solution.
We know that ${}_tp_{xy} = e^{-(\mu_x+\mu_y)t}$. Thus,

$$\mathring{e}_{xy} = \int_0^\infty e^{-(\mu_x+\mu_y)t}dt = -\frac{1}{\mu_x+\mu_y}\, e^{-(\mu_x+\mu_y)t}\Big|_0^\infty = \frac{1}{\mu_x+\mu_y} \blacksquare$$

Example 56.11 ‡
In a population, non-smokers have a force of mortality equal to one half that of smokers. For non-smokers,

$$\ell_x = 500(110-x), 0 \le x \le 110.$$

Calculate $\mathring{e}_{20:25}$ for a smoker (20) and a non-smoker (25) with independent future lifetimes.

Solution.
We have

$$\mu^{NS}(x) = \frac{1}{\omega - x} = \frac{1}{110 - x}; \qquad \mu^{S}(x) = \frac{2}{\omega - x} = \frac{2}{110 - x}$$
$$s^{S}(x) = e^{-\int_0^x \frac{2}{110-s}ds} = \left(1 - \frac{x}{110}\right)^2; \quad s^{NS}(x) = e^{-\int_0^x \frac{1}{110-s}ds} = \left(1 - \frac{x}{110}\right)$$

$$\begin{aligned}
{}_tp_{20}^{S} =& \frac{\ell_{20+t}^{S}}{\ell_{20}^{S}} = \frac{s^S(20+t)}{s^S(20)} = \left(1 - \frac{t}{90}\right)^2 \\
{}_tp_{25}^{NS} =& \frac{\ell_{20+t}^{NS}}{\ell_{20}^{S}} = \frac{s^{NS}(25+t)}{s^{NS}(25)} = \left(1 - \frac{t}{85}\right) \\
\mathring{e}_{20:25} =& \int_0^{85} {}_tp_{20:25} dt = \int_0^{85} {}_tp_{20}^{S} {}_tp_{25}^{NS} dt \\
=& \int_0^{85} \left(1 - \frac{t}{90}\right)^2 \left(1 - \frac{t}{85}\right) dt = \frac{1}{688,500} \int_0^{85} (90-t)^2(85-t)dt \\
=& \frac{1}{688,500} \int_0^{85} u(u+5)^2 du = \frac{1}{688,500}\left[\frac{1}{4}u^4 + \frac{10}{3}u^3 + \frac{25}{2}u^2\right]_0^{85} \\
=& 22.059 \blacksquare
\end{aligned}$$

Example 56.12 ‡
You are given:
(i) (30) and (50) are independent lives, each subject to a constant force of mortality $\mu = 0.05$.
(ii) $\delta = 0.03$.
Calculate $\text{Var}(T(30:50))$.

Solution.
We have

$$\begin{aligned}
\text{Var}(T(30:50)) =& 2\int_0^{\infty} te^{-0.1t}dt - [\mathring{e}_{30:50}]^2 \\
=& 2\left(\frac{1}{2\mu}\right)^2 - \left(\frac{1}{2\mu}\right)^2 = \left(\frac{1}{2\mu}\right)^2 \\
=& \frac{1}{0.01} = 100 \blacksquare
\end{aligned}$$

Practice Problems

Problem 56.16
You are given:
(i) $T(50)$ and $T(75)$ are independent
(ii) Mortality for both lives follows De Moivre's Law with $\omega = 100$
Calculate $\mathring{e}_{50:75}$.

Problem 56.17
You are given:
(i) $T(x)$ and $T(y)$ are independent
(ii) $\mu(x+t) = 0.03$
(iii) $\mu(y+t) = 0.05$
Calculate: $\mathring{e}_{xy}$.

Problem 56.18
You are given:
(i) $T(x)$ and $T(y)$ are independent
(ii) $\mu(x+t) = 0.03$
(iii) $\mu(y+t) = 0.05$
Calculate: ${}_3q_{xy}$.

Problem 56.19
You are given:
(i) $T(x)$ and $T(y)$ are independent
(ii) $\mu(x+t) = 0.03$
(iii) $\mu(y+t) = 0.05$
Calculate: ${}_{5|10}q_{xy}$.

Problem 56.20
In a certain population, smokers have a force of mortality twice that of non-smokers at each age x. For non-smokers, $\ell_x = 1000(100 - x), 0 \leq x \leq 100$. If life x is an 80-year old non-smoker and life y is a 90-year old smoker, calculate $\mathring{e}_{xy}$. Assume $T(x)$ and $T(y)$ are independent.

Problem 56.21
You are given:
(i) $T(40)$ and $T(50)$ are independent
(ii) Mortality for both lives follows De Moivre's Law with $\omega = 100$.
Calculate $\text{Var}[T(40:50)]$.

57 The Last-Survivor Status Model

For lives (x) and (y), the last-survivor status continues so long as either of the lives remain alive. Accordingly, the status terminates when both lives die. Thus, the time of failure of the status is the time the second life dies. We denote the last-survivor status by $(\overline{xy})$.
The future lifetime of the status is given by

$$T(\overline{xy}) = \max\{T(x), T(y)\}.$$

The cumulative distribution function of this random variable is

$$\begin{aligned}F_{T(\overline{xy})}(t) =&{}_tq_{\overline{xy}} = \Pr(T(\overline{xy}) \le t) = \Pr([T(x) \le t] \cap [T(y) \le t])\\ =&\Pr([T(x) \le t])\Pr([T(y) \le t])\\ =&{}_tq_x \ {}_tq_y.\end{aligned}$$

The survival function of $T(\overline{xy})$ is

$$S_{T(\overline{xy})}(t) = {}_tp_{\overline{xy}} = 1 - {}_tq_{\overline{xy}}.$$

Example 57.1
You are given:
(i) (50) and (75) are independent
(ii) Mortality for both lives follows De Moivre's Law with $\omega = 100$.
Calculate ${}_tq_{\overline{50:75}}$ and ${}_tp_{\overline{50:75}}$.

Solution.
We have

$${}_tq_{50} = \tfrac{t}{50},\ 0 \le t \le 50 \text{ and } {}_tq_{75} = \begin{cases} \frac{t}{25} & 0 \le t < 25\\ 1 & 25 \le t \le 50.\end{cases}$$

Thus,

$${}_tq_{\overline{50:75}} = \begin{cases} \frac{t^2}{1250} & 0 \le t < 25\\ \frac{t}{25} & 25 \le t \le 50.\end{cases}$$

and

$${}_tp_{\overline{50:75}} = \begin{cases} 1-\frac{t^2}{1250} & 0 \le t < 25\\ 1-\frac{t}{25} & 25 \le t \le 50 \ \blacksquare\end{cases}$$

Example 57.2
Show that

$$ {}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y - {}_tp_x {}_tp_y. $$

Solution.
We have

$$ {}_tp_{\overline{xy}} = 1 - {}_tq_{\overline{xy}} = 1 - {}_tq_x \ {}_tq_y = 1 - (1 - {}_tp_x)(1 - {}_tp_y) = {}_tp_x + {}_tp_y - {}_tp_x {}_tp_y \ \blacksquare $$

The PDF of $T(\overline{xy})$ is found as follows

$$ \begin{aligned} f_{T(\overline{xy})}(t) = & -\frac{d}{dt}(S_{T(\overline{xy})}(t)) = -\frac{d}{dt}[{}_tp_x + {}_tp_y - {}_tp_{xy}] \\ = & {}_tp_x\mu(x+t) + {}_tp_y\mu(y+t) - {}_tp_{xy}[\mu(x+t) + \mu(y+t)]. \end{aligned} $$

The mortality function of $T(\overline{xy})$ is

$$ \mu_{\overline{xy}} = \frac{f_{T(\overline{xy})}(t)}{S_{T(\overline{xy})}(t)} = \frac{{}_tp_x\mu(x+t) + {}_tp_y\mu(y+t) - {}_tp_{xy}[\mu(x+t) + \mu(y+t)]}{{}_tp_x + {}_tp_y - {}_tp_x {}_tp_y}. $$

The conditional probability of death in the $(n+1)^{\text{st}}$ time interval (i.e. in the time interval $(n, n+1]$) is given by

$$ \begin{aligned} {}_{n|}q_{\overline{xy}} = & \Pr(n < T(\overline{xy}) \le n+1) \\ = & S_{T(\overline{xy})}(n) - S_{T(\overline{xy})}(n+1) \\ = & {}_np_x + {}_np_y - {}_np_x {}_np_y - ({}_{n+1}p_x + {}_{n+1}p_y - {}_{n+1}p_x {}_{n+1}p_y) \\ = & {}_np_{\overline{xy}} - {}_{n+1}p_{\overline{xy}} = {}_{n+1}q_{\overline{xy}} - {}_nq_{\overline{xy}} \\ = & {}_{n|}q_x + {}_{n|}q_y - {}_{n|}q_{xy}. \end{aligned} $$

Example 57.3
You are given the following:
(i) ${}_3p_x = 0.92$ and ${}_3p_y = 0.90$
(ii) ${}_4p_x = 0.91$ and ${}_4p_y = 0.85$
(iii) The two lives are independent.
Calculate the probability that the second death occurs in the fourth year.

Solution.
We have

$$ \begin{aligned} {}_{3|}q_{\overline{xy}} = & {}_3p_x + {}_3p_y - {}_3p_x {}_3p_y - ({}_4p_x + {}_4p_y - {}_4p_x {}_4p_y) \\ = & 0.92 + 0.90 - (0.92)(0.90) - (0.91 + 0.85 - (0.91)(0.85)) = 0.0055 \ \blacksquare \end{aligned} $$

The complete expectation of $T(\overline{xy})$ is

$$\mathring{e}_{\overline{xy}} = \int_0^\infty t f_{T(\overline{xy})}(t)dt = \int_0^\infty {}_tp_{\overline{xy}}dt.$$

Example 57.4
Show that $\mathring{e}_{\overline{xy}} = \mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy}$.

Solution.
We have

$$\mathring{e}_{\overline{xy}} = \int_0^\infty {}_tp_{\overline{xy}}dt = \int_0^\infty ({}_tp_x + {}_tp_y - {}_tp_x {}_tp_y)dt = \mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy} \blacksquare$$

The curtate expectation of $T(\overline{xy})$ is

$$e_{\overline{xy}} = \sum_{k=1}^\infty {}_kp_{\overline{xy}}$$

and the temporary curtate expectation is

$$e_{\overline{xy}:\overline{n}|} = \sum_{k=1}^n {}_kp_{\overline{xy}}.$$

Example 57.5
You are given:
(i) (50) and (75) are independent
(ii) Mortality for both lives follows De Moivre's Law with $\omega = 100$.
Calculate $\mathring{e}_{\overline{50:75}}$.

Solution.
We have

$$\begin{aligned}
\mathring{e}_x =& \frac{\omega - x}{2} \\
{}_tp_x =& 1 - \frac{t}{\omega - x} \\
\mathring{e}_{50} =& \frac{100-50}{2} = 25 \qquad \mathring{e}_{75} = \frac{100-75}{2} = 12.5 \\
\mathring{e}_{50:75} =& \int_0^{25} {}_tp_{50:75}dt = \int_0^{25} {}_tp_{50} {}_tp_{75}dt = \int_0^{25} \left(\frac{50-t}{50}\right)\left(\frac{25-t}{25}\right)dt \\
=& \int_0^{25} \frac{1250 - 75t + t^2}{1250}dt = \frac{1}{1250}\left[1250t - \frac{75}{2}t^2 + \frac{t^3}{3}\right]_0^{25} = 10.42.
\end{aligned}$$

Thus,

$$\mathring{e}_{\overline{50:75}} = \mathring{e}_{50} + \mathring{e}_{75} - \mathring{e}_{50:75} = 25 + 12.5 - 10.42 = 27.08 \blacksquare$$

Example 57.6 ‡

XYZ Co. has just purchased two new tools with independent future lifetimes. Each tool has its own distinct De Moivre survival pattern. One tool has a 10-year maximum lifetime and the other a 7-year maximum lifetime.
Calculate the expected time until both tools have failed.

Solution.

Let x represent the tool with the 10-year maximum lifetime and y that of the 7-year. We are asked to find $\mathring{e}_{\overline{xy}}$. For that we are going to use the formula

$$\mathring{e}_{\overline{xy}} = \mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy}.$$

We have

$$\begin{aligned}
\mathring{e}_x =& \int_0^{10} {}_tp_x dt = \int_0^{10} \left(1 - \frac{t}{10}\right) dt = t - \frac{t^2}{20}\Big|_0^{10} = 5 \\
\mathring{e}_y =& \int_0^{10} {}_tp_y dt = \int_0^{7} \left(1 - \frac{t}{7}\right) dt = t - \frac{t^2}{14}\Big|_0^{7} = 3.5 \\
\mathring{e}_{xy} =& \int_0^{7} {}_tp_{xy} dt = \int_0^{7} \left(1 - \frac{t}{10}\right)\left(1 - \frac{t}{7}\right) dt \\
=& t - \frac{t^2}{20} - \frac{t^2}{14} + \frac{t^3}{210}\Big|_0^{7} = 2.683 \\
\mathring{e}_{\overline{xy}} =& 5 + 3.5 - 2.683 = 5.817 \blacksquare
\end{aligned}$$

Example 57.7 ‡

For independent lives (35) and (45):
(i) ${}_5p_{35} = 0.90$
(ii) ${}_5p_{45} = 0.80$
(iii) $q_{40} = 0.03$
(iv) $q_{50} = 0.05$
Calculate the probability that the last death of (35) and (45) occurs in the 6^{th} year.

Solution.
We have

$$\begin{aligned}
{}_{5|}q_{\overline{35:45}} =& {}_{5|}q_{35} + {}_{5|}q_{45} - {}_{5|}q_{35:45} = {}_5p_{35}q_{40} + {}_5p_{45}q_{50} - {}_5p_{35:45}q_{40:50}\\
=& {}_5p_{35}q_{40} + {}_5p_{45}q_{50} - {}_5p_{35}{}_5p_{45}(1 - p_{40:50})\\
=& {}_5p_{35}q_{40} + {}_5p_{45}q_{50} - {}_5p_{35}{}_5p_{45}(1 - p_{40}p_{50})\\
=&(0.9)(0.3) + (0.8)(0.5) - (0.9)(0.8)[1 - (0.97)(0.95)] = 0.01048 \blacksquare
\end{aligned}$$

Example 57.8 ‡
Kevin and Kira are in a history competition:
(i) In each round, every child still in the contest faces one question. A child is out as soon as he or she misses one question. The contest will last at least 5 rounds.
(ii) For each question, Kevin's probability and Kira's probability of answering that question correctly are each 0.8; their answers are independent.
Calculate the conditional probability that both Kevin and Kira are out by the start of round five, given that at least one of them participates in round 3.

Solution.
Let $T(x)$ denote the round where child x is still in the context. We are looking for $\Pr(T_{\overline{xy}} < 4|T_{\overline{xy}} > 2)$. We have

$$\begin{aligned}
\Pr(T_{\overline{xy}} < 4|T_{\overline{xy}} > 2) =& \frac{\Pr(2 < T_{\overline{xy}} < 4)}{\Pr(T_{\overline{xy}} > 2)}\\
=& \frac{{}_{2|2}q_{\overline{xy}}}{{}_2p_{\overline{xy}}} = \frac{{}_4q_{\overline{xy}} - {}_2q_{\overline{xy}}}{1 - {}_2q_{\overline{xy}}}.
\end{aligned}$$

Let ${}_kp_0$ denote the probability that either Kevin or Kira answers the first k questions correctly, in other words, either Kevin or Kira wins round k. Let x stands for Keving and y for Kira. We have

$$\begin{aligned}
{}_4q_{\overline{xy}} =& ({}_4q_0)^2 = (1 - {}_4p_0)^2 = [1 - (0.8)^4]^2 = 0.34857216\\
{}_2q_{\overline{xy}} =& ({}_2q_0)^2 = (1 - {}_2p_0)^2 = [1 - (0.8)^2]^2 = 0.1296.
\end{aligned}$$

Hence,

$$\Pr(T_{\overline{xy}} < 4|T_{\overline{xy}} > 2) = \frac{0.34857216 - 0.1296}{1 - 0.1296} = 0.2516 \blacksquare$$

Practice Problems

Problem 57.1
You are given:
(i) (x) and (y) are independent
(ii) $\mu(x+t) = 0.02$ and $\mu(y+t) = 0.06$.
Calculate $\mathring{e}_{\overline{xy}}$.

Problem 57.2
You are given:
(i) Mortality follows De Moivre's Law with $\omega = 100$
(ii) $T(80)$ and $T(85)$ are independent
Find the probability that the second death occurs more than 5 years from now.

Problem 57.3
You are given:
(i) $T(90)$ and $T(95)$ are independent
(ii) Both lives have mortality that follows De Moivre's Law with $\omega = 100$.
Calculate $\mathring{e}_{\overline{90:95}}$.

Problem 57.4
Given:
(i) $T(x)$ and $T(y)$ are independent
(ii)

k	0	1	2
q_{x+k}	0.08	0.09	0.10
q_{y+k}	0.10	0.15	0.20

Calculate ${}_{2|}q_{\overline{xy}}$.

Problem 57.5
You are given the following mortality rates for two independent lives, (x) and (y) :

k	0	1	2	3
q_{x+k}	0.08	0.09	0.10	0.11
q_{y+k}	0.10	0.15	0.20	0.26

Calculate ${}_2p_{\overline{xy}}$.

Problem 57.6 ‡
For two independent lives now age 30 and 34, you are given:

x	30	31	32	33	34	35	36	37
q_x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8

Calculate the probability that the last death of these two lives will occur during the 3rd year from now, i.e., ${}_{2|}q_{\overline{30:34}}$.

Problem 57.7 ‡
You are given:
(i) (45) and (65) are independent
(ii) Mortality for both lives follows De Moivre's Law with $\omega = 105$.
Calculate $\mathring{e}_{\overline{45:65}}$.

Problem 57.8
You are given:
(i) (x) and (y) are independent
(ii) $\mu(x+t) = 0.03$
(iii) $\mu(y+t) = 0.05$
(iv) $\delta = 0.05$.
Calculate $\mathring{e}_{\overline{xy}}$.

Problem 57.9 ‡
For (80) and (84), whose future lifetimes are independent:

x	80	81	82	83	84	85	86
p_x	0.5	0.4	0.6	0.25	0.20	0.15	0.10

Calculate the change in the value of ${}_{2|}q_{\overline{80:84}}$ if p_{82} is decreased from 0.60 to 0.30.

Problem 57.10
Show that

$$\mu_{\overline{xy}}(t) = -\frac{\frac{d}{dt}{}_tp_{\overline{xy}}}{{}_tp_{\overline{xy}}} = \frac{{}_tp_{x\,t}q_y\mu(x+t) + {}_tp_{y\,t}q_x\mu(y+t)}{1 - {}_tq_{x\,t}q_y}.$$

Problem 57.11 ‡
You are given:
(i) The future lifetimes of (50) and (50) are independent.
(ii) Mortality follows the Illustrative Life Table.
(iii) Deaths are uniformly distributed over each year of age.
Calculate the force of failure at duration 10.5 for the last survivor status of (50) and (50).

Problem 57.12 ‡
You are given:
(i) The survival function for males is $s(x) = 1 - \frac{x}{75},\ 0 < x < 75$.
(ii) Female mortality follows De Moivre's Law.
(iii) At age 60, the female force of mortality is 60% of the male force of mortality.
For two independent lives, a male age 65 and a female age 60, calculate the expected time until the second death.

Problem 57.13 ‡
For independent lives (50) and (60):

$$\mu(x) = \frac{1}{100 - x},\ 0 \le x < 100.$$

Calculate $\mathring{e}_{\overline{50:60}}$.

Problem 57.14 ‡
For a population whose mortality follows DeMoivre's law, you are given:
(i) $\mathring{e}_{\overline{40:40}} = 3\mathring{e}_{\overline{60:60}}$
(ii) $\mathring{e}_{\overline{20:20}} = k\mathring{e}_{\overline{60:60}}$.
Calculate k.

Problem 57.15 ‡
You are given:
(i) $T(x)$ and $T(y)$ are not independent.
(ii) $q_{x+k} = q_{y+k} = 0.05,\ k = 0, 1, 2, \cdots$
(iii) ${}_kp_{xy} = 1.02{}_kp_{xk}p_y$.
Calculate $e_{\overline{xy}}$.

Problem 57.16 ‡
A 30-year term insurance for Janet age 30 and Andre age 40 provides the following benefits:
• A death benefit of $140,000 if Janet dies before Andre and within 30 years
• A death benefit of $180,000 if Andre dies before Janet and within 30 years
You are given:
(i) Mortality follows De Moivre's Law with $\omega = 100$
(ii) $i = 0$
(iii) The death benefit is Payable at the moment of the first death.
(iv) Premiums, $\bar{P}$, are paid continuously while both are alive for a maximum of 20 years.
Calculate the probability that the second death occurs between times $t = 10$ and $t = 20$.

Problem 57.17 ‡
You are given:
(i) (30) and (50) are independent lives, each subject to a constant force of mortality $\mu = 0.05$.
(ii) $\delta = 0.03$.
(a) Calculate ${}_{10}q_{\overline{30:50}}$.
(b) Calculate $\mathring{e}_{\overline{30:50}}$.

Problem 57.18 ‡
You are given:
(i) Mortality follows De Moivre's Law with $\omega = 100$
(ii) $T(80)$and $T(85)$ are independent.
(iii) G is the probability that (80) dies after (85) and before 5 years from now.
(iv) H is the probability that the first death occurs after 5 and before 10 years from now.
Calculate $G + H$.

58 Relationships Between $T(xy)$ and $T(\overline{xy})$

Recall that $T(xy) = \min\{T(x), T(y)\}$ and $T(\overline{xy}) = \max\{T(x), T(y)\}$. Thus, if $T(xy)$ assumes one of the two values $T(x)$ or $T(y)$ then $T(\overline{xy})$ assumes the other value. Hence, we can write

$$T(xy) + T(\overline{xy}) = T(x) + T(y)$$

and

$$T(xy) \cdot T(\overline{xy}) = T(x) \cdot T(y).$$

Example 58.1
Find a formula for $\text{Cov}(T(xy), T(\overline{xy}))$ for independent $T(x)$ and $T(y)$.

Solution.
We have

$$\begin{aligned}
\text{Cov}(T(xy), T(\overline{xy})) =& E[T(xy) \cdot T(\overline{xy})] - E[T(xy)]E[T(\overline{xy})] \\
=& E[T(x)T(y)] - E[T(xy)]E[T(x) + T(y) - T(xy)] \\
=& E[T(x)]E[T(y)] - E[T(xy)](E[T(x)] + E[T(y)] - E[T(xy)]) \\
=& \mathring{e}_x \mathring{e}_y - \mathring{e}_{xy}(\mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy}) \\
=& \mathring{e}_x \mathring{e}_y - \mathring{e}_{xy}\mathring{e}_x - \mathring{e}_{xy}\mathring{e}_y + (\mathring{e}_{xy})^2 \\
=& (\mathring{e}_x - \mathring{e}_{xy})(\mathring{e}_y - \mathring{e}_{xy}) \blacksquare
\end{aligned}$$

Example 58.2
In a certain population, smokers have a force of mortality twice that of non-smokers at each age x. For non-smokers, $\ell_x = 1000(100 - x), 0 \le x \le 100$. If life x is an 80-year old non-smoker and life y is a 90-year old smoker, calculate $\text{Cov}(T(xy), T(\overline{xy}))$. Assume $T(x)$ and $T(y)$ are independent.

Solution.
We know that

$$\mu^N(x) = \frac{1}{100-x} \text{ and } \mu^S = \frac{2}{100-x}.$$

$$_tp_{80}^N = \frac{20-t}{20} \text{ and } _tp_{90}^S = \left(\frac{10-t}{10}\right)^2.$$

Thus,

$$\begin{aligned}
\mathring{e}_{xy} &= \int_0^{10} {}_tp_{xy}dt = \int_0^{10} {}_tp_{xt}p_ydt \\
&= \int_0^{10} \left(\frac{20-t}{20}\right)\left(\frac{10-t}{10}\right)^2 dt \\
&=\frac{1}{2000}\int_0^{10}(20-t)(10-t)^2dt = \frac{1}{2000}\left[-10\frac{(10-t)^3}{3} - \frac{(10-t)^4}{4}\right]_0^{10} \\
&=2.916667.
\end{aligned}$$

Also,

$$\begin{aligned}
\mathring{e}_x &= \frac{\omega - x}{2} = \frac{100-80}{2} = 10 \\
\mathring{e}_y &= \frac{\omega - x}{3} = \frac{100-90}{3} = \frac{10}{3}.
\end{aligned}$$

By substitution, we find

$$\text{Cov}(T(xy), T(\overline{xy})) = (10 - 2.916667)(10/3 - 2.916667) = 2.9514 \blacksquare$$

Example 58.3 ‡
You are given:
(i) (30) and (50) are independent lives, each subject to a constant force of mortality $\mu = 0.05$.
(ii) $\delta = 0.03$.
Calculate $\text{Cov}(T(30:50), T(\overline{30:50}))$.

Solution.
We have

$$\begin{aligned}
\mathring{e}_{30} =&\mathring{e}_{50} = \int_0^\infty e^{-0.05t}dt = 20 \\
\mathring{e}_{30:50} =& \int_0^\infty e^{-0.1t}dt = 10 \\
\text{Cov}(T(30:50), T(\overline{30:50})) =&(\mathring{e}_{30} - \mathring{e}_{30:50})(\mathring{e}_{50} - \mathring{e}_{30:50}) \\
=&(20-10)(20-10) = 100 \blacksquare
\end{aligned}$$

Practice Problems

Problem 58.1
Show that, in general, we have

$$\mathrm{Cov}(T(xy), T(\overline{xy})) = \mathrm{Cov}(T(x), T(y)) + (\mathring{e}_x - \mathring{e}_{xy})(\mathring{e}_y - \mathring{e}_{xy}).$$

Problem 58.2
You are given:
(i) $E[T(x)] = E[T(y)] = 4.0$
(ii) $\mathrm{Cov}(T(x), T(y)) = 0.01$
(iii) $\mathrm{Cov}(T(xy), T(\overline{xy})) = 0.10$.
Calculate $\mathring{e}_{xy}$.

Problem 58.3
You are given:
(i) $E[T(x)] = E[T(y)] = 4.0$
(ii) $\mathrm{Cov}(T(x), T(y)) = 0.01$
(iii) $\mathrm{Cov}(T(xy), T(\overline{xy})) = 0.10$.
Calculate $\mathring{e}_{\overline{xy}}$.

Problem 58.4
You are given:
(i) $T(x)$ and $T(y)$ are independent
(ii) $\mu_((x+t) = 0.02$ and $\mu_((x+t) = 0.03$.
Calculate the covariance between the first and the second death times.

Problem 58.5
Show that, for $T(x)$ and $T(y)$ independent, we have

$$\mathrm{Cov}(T(xy), T(\overline{xy})) = \mathring{e}_x \mathring{e}_y - \mathring{e}_{xy} \mathring{e}_{\overline{xy}}.$$

59 Contingent Probability Functions

Consider two lives aged (x) and (y) respectively. Let B_{xy} be the event that (x) dies before (y). Thus, $t \in B_{xy}$ means that (x) dies at time t while (y) is alive at that time. The probability of such an event is referred to as **contingent probability.** In the case $T(x)$ and $T(y)$ are independent, we can write

$$\begin{aligned}
{}_\infty q^1_{xy} =&\Pr(B_{xy}) = \int_0^\infty \int_t^\infty f_{T(x)T(y)}(t,s)dsdt\\
=&\int_0^\infty \int_t^\infty f_{T(x)}(t)f_{T(y)}(s)dsdt = \int_0^\infty \left(\int_t^\infty f_{T(y)}(s)ds\right) f_{T(x)}(t)dt\\
=&\int_0^\infty f_{T(x)}(t)S_{T(y)}(t)dt\\
=&\int_0^\infty {}_tp_x\mu(x+t){}_tp_ydt = \int_0^\infty {}_tp_{xy}\mu(x+t)dt.
\end{aligned}$$

The pre-subscript ∞ indicates that the event is satisfied within unlimited time. If we require the event to be satisfied within $n-$year period then we will use the notation ${}_nq^1_{xy}$. In this case,

$${}_nq^1_{xy} = \int_0^n {}_tp_{xy}\mu(x+t)dt.$$

Now, the "upper one" in the notation stands for the fact that (x) will die first. In the case (y) will die first, we will adopt the notation ${}_\infty\, q_{x\overset{1}{y}}$. Clearly,

$${}_\infty\, q_{x\overset{1}{y}} = \int_0^\infty {}_tp_{xy}\mu(y+t)dt.$$

Example 59.1
You are given:
(i) $T(x)$ and $T(y)$ are independent
(ii) $\mu(x+t) = 0.02$
(iii) $\mu(y+t) = 0.04$. Calculate ${}_\infty q^1_{xy}$ and ${}_{10}q^1_{xy}$.

Solution.
We have

$${}_\infty q^1_{xy} = \int_0^\infty {}_tp_{xy}\mu(x+t)dt = \int_0^\infty e^{-0.02t}e^{-0.04t}(0.02)dt = \frac{1}{3}.$$

Also,

$$ {}_{10}q^1_{xy} = \int_0^{10} {}_tp_{xy}\mu(x+t)dt = \int_0^{10} e^{-0.02t}e^{-0.04t}(0.02)dt = 0.15040 \blacksquare $$

Now, let A_{xy} denote the event that (x) dies after (y). That is, $t \in A_{xy}$ means that (x) dies at time t after (y) is already dead at that time. We have

$$ \begin{aligned} {}_\infty q^2_{xy} =& \Pr(A_{xy}) = \int_0^\infty \int_0^t f_{T(x)T(y)}(t,s)dsdt \\ =& \int_0^\infty \int_0^t f_{T(x)}(t)f_{T(y)}(s)dsdt = \int_0^\infty \left(\int_0^t f_{T(y)}(s)ds\right) f_{T(x)}(t)dt \\ =& \int_0^\infty f_{T(x)}(t)F_{T(y)}(t)dt \\ =& \int_0^\infty {}_tp_x\mu(x+t){}_tq_ydt = \int_0^\infty {}_tp_x(1-{}_tp_y)\mu(x+t)dt \\ =& \int_0^\infty {}_tp_x\mu(x+t)dt - \int_0^\infty {}_tp_{xy}\mu(x+t)dt = 1 - {}_\infty q^1_{xy}. \end{aligned} $$

The probability that (x) dies after (y) and within n years is

$$ {}_nq^2_{xy} = \int_0^n {}_tp_x(1-{}_tp_y)\mu(x+t)dt = {}_nq_x - {}_nq^1_{xy}. $$

Example 59.2
You are given:
(i) $T(x)$ and $T(y)$ are independent
(ii) $\mu(x+t) = 0.02$
(iii) $\mu(y+t) = 0.04$. Calculate ${}_\infty\, q^{\ \,2}_{xy}$ and ${}_{10}\, q^{\ \,2}_{xy}$.

Solution.
We have

$$ \begin{aligned} {}_\infty\, q^{\ \,2}_{xy} =& \int_0^\infty {}_tp_y{}_tq_x\mu(y+t)dt \\ =& \int_0^\infty e^{-0.04t}(1-e^{-0.02t})(0.04)dt = \frac{1}{3} \\ {}_{10}\, q^{\ \,2}_{xy} & \\ =& \int_0^{10} e^{-0.04t}(1-e^{-0.02t})(0.04)dt = 0.028888 \blacksquare \end{aligned} $$

Example 59.3 ‡
You are given:
(i) The future lifetimes of (40) and (50) are independent.
(ii) The survival function for (40) is based on a constant force of mortality, $\mu = 0.05$.
(iii) The survival function for (50) follows DeMoivre's law with $\omega = 110$.
Calculate the probability that (50) dies within 10 years and dies before (40).

Solution.
We are looking for

$$_{10}q^{\,1}_{50:40} = \int_0^{10} {}_tp_{50:40}\mu(50+t)dt = \int_0^{10} {}_tp_{50\,t}p_{40}\mu(50+t)dt.$$

We have

$$\begin{aligned} {}_tp_{40} =& e^{-\mu t} = e^{-0.05t} \\ {}_tp_{50}\mu(50+t) =& f_{T(50)}(t) = \frac{1}{110-50} = \frac{1}{60}. \end{aligned}$$

Hence,

$$_{10}q^{\,1}_{50:40} = \int_0^{10} \frac{e^{-0.05t}}{60}dt = \frac{1}{60}\left(\frac{1-e^{-0.05(10)}}{0.05}\right) = 0.1311 \blacksquare$$

Example 59.4 ‡
A 30-year term insurance for Janet age 30 and Andre age 40 provides the following benefits:
• A death benefit of \$140,000 if Janet dies before Andre and within 30 years
• A death benefit of \$180,000 if Andre dies before Janet and within 30 years
You are given:
(i) Mortality follows De Moivre's Law with $\omega = 100$
(ii) $i = 0$
(iii) The death benefit is Payable at the moment of the first death.
(iv) Premiums, $\bar{P}$, are paid continuously while both are alive for a maximum of 20 years.
Calculate $_{10}q^{\,2}_{30:40}$.

Solution.
We have

$$\begin{aligned}
{}_{10}q^{2}_{30:40} =& \int_0^{10}(1-{}_tp_{40})\,{}_tp_{30}\mu(30+t)dt\\
=& \left(\frac{1}{70}\right)\left(\frac{1}{60}\right)\left(\frac{t^2}{2}\right)\Big|_0^{10} = 0.012 \blacksquare
\end{aligned}$$

Example 59.5 ‡
You are given:
(i) (x) and (y) are independent lives
(ii) $\mu(x+t)=5t,\ t\geq 0$
(iii) $\mu(y+t)=t,\ t\geq 0.$
Calculate q^{1}_{xy}.

Solution.
We have

$$\begin{aligned}
q^{1}_{xy} =& \int_0^1 {}_tp_x\,{}_tp_y\mu(x+t)dt\\
{}_tp_y =& e^{-\int_0^t sds} = e^{-\frac{t^2}{2}}\\
{}_tp_x =& e^{-\int_0^t 5sds} = e^{-\frac{5t^2}{2}}\\
q^{1}_{xy} =& \int_0^1 e^{-\frac{t^2}{2}}e^{-\frac{5t^2}{2}}(5t)dt\\
=& \int_0^1 e^{-3t^2}(5t)dt\\
=& \frac{5}{6}\int_{-3}^0 e^u du = \frac{5}{6}(1-e^{-3}) = 0.7918 \blacksquare
\end{aligned}$$

Practice Problems

Problem 59.1
You are given:
(i) $T(x)$ and $T(y)$ are independent
(ii) $T(x)$ is uniform on $[0, 40]$
(iii) $T(y)$ is uniform on $[0, 50]$
Calculate $\Pr(T(x) < T(y))$.

Problem 59.2
You are given:
(i) $T(x)$ and $T(y)$ are independent
(ii) $\mu(x+t) = 0.02$
(iii) $\mu(y+t) = 0.04$.
Calculate ${}_\infty q^2_{xy}$ and ${}_{10}q^2_{xy}$.

Problem 59.3
Show that ${}_nq^1_{xy} + {}_nq_{x\overset{1}{y}} = {}_nq_{xy}$.

Problem 59.4
Assuming the uniform distribution of deaths for each life, and given
(i) $q^1_{xy} = 0.039$
(ii) $q_{xy} = 0.049$.
Calculate q_x.

Problem 59.5
You are given:
(i) Mortality follows the Illustrative Life Table.
(ii) All lives are independent.
(iii) Deaths are uniformly distributed over each year of age.
Calculate $q_{60:\overset{2}{65}}$.

Problem 59.6 ‡
You are given:
(i) (x) is a subject to a uniform distribution of deaths over each year.
(ii) (y) is a subject to a constant force of mortality of 0.25.
(iii) $q^1_{xy} = 0.125$
(iv) $T(x)$ and $T(y)$ are independent.
Calculate q_x.

Problem 59.7 ‡

You are given:
(i) $T(x)$ and $T(y)$ are independent.
(ii) (x) follows De Moivre's Law with $\omega = 95$
(iii) (y) is a subject of a constant force of mortality μ.
(iv) $n < 95 - x$.
Calculate the probability that (x) dies within n years and predeceases (y).

Problem 59.8 ‡

You are given:
(i) $T(30)$ and $T(40)$ are independent.
(ii) Deaths are uniformly distributed over each year of age.
(iii) $q_{30} = 0.4$
(iv) $q_{40} = 0.6$.
Calculate ${}_{0.25}\,q^{\;2}_{30.5:40.5}$.

60 Contingent Policies for Multiple Lives

All the previous material presented so far about insurances and annuities carries to multiple lives statuses. Therefore the genenral concept of a single life (x) can now represent the joint-life status (xy) or the last-survivor status $(\overline{xy})$. It is therefore unecessary to duplicate the general theory presented in earlier chapters simply substituting (xy) or $(\overline{xy})$ for (x). Rather we will list only a selection of the relationships we developed earlier, now recast in joint or last-survivor notation. We illustrate this point through a series of examples

Example 60.1
You are given:
(i) (x) and (y) are independent
(ii) $\mu(x+t) = 0.03$
(iii) $\mu(y+t) = 0.05$
(iv) $\delta = 0.05$.
Calculate the following:
(a) $\bar{A}_{xy}$ (b) $\bar{a}_{xy}$ (c)$\bar{P}(\bar{A}_{xy})$.

Solution.
(a) We have

$$\bar{A}_{xy} = \frac{\mu_{xy}(t)}{\mu_{xy}(t)+\delta} = \frac{0.08}{0.08+0.04} = 0.66667.$$

(b) We have

$$\bar{a}_{xy} = \frac{1}{\mu_{xy}(t)+\delta} = \frac{1}{0.08+0.04} = 8.3333.$$

(c) We have

$$\bar{P}(\bar{A}_{xy}) = \frac{\bar{A}_{xy}}{\bar{a}_{xy}} = 0.08 \blacksquare$$

Example 60.2
Show that
(a) $A_{\overline{xy}} = A_x + A_y - A_{xy}$,
(b) $\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$.
(c) $\ddot{a}_{\overline{xy}} = \ddot{a}_x + \ddot{a}_y - \ddot{a}_{xy}$.
(d) $\bar{a}_{\overline{xy}} = \bar{a}_x + \bar{a}_y - \bar{a}_{xy}$.

Solution.
(a) We have

$$\begin{aligned}A_{\overline{xy}} =& \sum_{k=0}^{\infty} \nu^{k+1}{}_{k|}q_{\overline{xy}} \\ =& \sum_{k=0}^{\infty} \nu^{k+1}({}_{k|}q_x + {}_{k|}q_y - {}_{k|}q_{xy}) \\ =& A_x + A_y - A_{xy}.\end{aligned}$$

(b) We have

$$\begin{aligned}\bar{A}_{\overline{xy}} =& \int_0^{\infty} \nu^t f_{T(\overline{xy})}(t)dt \\ =& \int_0^{\infty} \nu^t[{}_tp_x\mu(x+t) + {}_tp_y\mu(y+t) - ({}_tp_{xy})\mu_{xy}(t)]dt \\ =& \bar{A}_x + \bar{A}_y - \bar{A}_{xy}.\end{aligned}$$

(c) We have

$$\begin{aligned}\ddot{a}_{\overline{xy}} =& \sum_{k=0}^{\infty} \nu^k{}_kp_{\overline{xy}} \\ =& \sum_{k=0}^{\infty} \nu^k[{}_tp_x + {}_tp_y - {}_tp_{xy}] \\ =& \ddot{a}_x + \ddot{a}_y - \ddot{a}_{xy}.\end{aligned}$$

(d) We have

$$\begin{aligned}\bar{a}_{\overline{xy}} =& \int_0^{\infty} \nu^t{}_tp_{\overline{xy}}dt \\ =& \int_0^{\infty} \nu^t[{}_tp_x + {}_tp_y - {}_tp_{xy}]dt \\ =& \bar{a}_x + \bar{a}_y - \bar{a}_{xy} \blacksquare\end{aligned}$$

Example 60.3
You are given that mortality follows the Illustrative Life Table with $i = 0.06$. Assuming that (50) and (60) are independent and that deaths are uniformly distributed between integral ages, calculate:
(a) $A_{\overline{60:70}}$ (b) $\bar{A}_{60:70}$ (c) $\ddot{a}_{60:70}$ (d) $P(A_{60:70})$ (e) $P(\bar{A}_{60:70})$

Solution.
(a) We have

$$A_{\overline{60:70}} = A_{60} + A_{70} - A_{60:70} = 0.36913 + 0.51495 - 0.57228 = 0.31180.$$

(b) We have

$$\bar{A}_{60:70} = \frac{i}{\delta}A_{60:70} = \frac{0.06}{\ln 1.06}(0.57228) = 0.58928.$$

(c) We have from the table $\ddot{a}_{60:70} = \ddot{a}_{60:60+10} = 7.5563$.
(d) We have

$$P(A_{60:70}) = \frac{A_{60:70}}{\ddot{a}_{60:70}} = \frac{0.57228}{7.5563} = 0.07574.$$

(e) We have

$$P(\bar{A}_{60:70}) = \frac{\bar{A}_{60:70}}{\ddot{a}_{60:70}} = \frac{0.58928}{7.5563} = 0.07799 \blacksquare$$

Example 60.4
You are given that mortality follows the Illustrative Life Table with $i = 0.06$. Assuming that (50) and (60) are independent. Calculate $\ddot{a}_{\overline{50:60}}$.

Solution.
We have

$$\begin{aligned}\ddot{a}_{\overline{50:60}} =&\ddot{a}_{50} + \ddot{a}_{60} - \ddot{a}_{50:60}\\ =&13.2668 + 11.1454 - 10.1944 = 14.2178 \blacksquare\end{aligned}$$

Example 60.5
You buy a couple of expensive puppets. You decide to buy an insurance that insures the puppets. The death benefit is 1000 payable at the moment of the second death. The puppets has a constant mortality of $\mu_x = \mu_y = 0.2$. You decide to pay for the insurance through a continuous annuity-certain, $\bar{a}_{\overline{n}|}$, the time length in which is based upon the expected time until the second death. You are given that $\delta = 0.05$.
Assuming the future lifetimes are independent, calculate the annual premium that you will pay for this insurance.

Solution.
We are asked to compute

$$\bar{P}(\bar{A}_{\overline{xy}}) = \frac{\bar{A}_{\overline{xy}}}{\bar{a}_{\overline{n}|}}$$

where n is to be determined.
We have

$$\begin{aligned}\bar{A}_x =& \frac{\mu_x}{\mu_x+\delta} = \frac{0.2}{0.2+0.05} = 0.08\\ \bar{A}_y =& \frac{\mu_y}{\mu_y+\delta} = \frac{0.2}{0.2+0.05} = 0.08\end{aligned}$$

and

$$\bar{A}_{xy} = \frac{\mu_{xy}}{\mu_{xy}+\delta} = \frac{0.4}{0.4+0.05} = 0.8888.$$

Thus,

$$\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy} = 0.8 + 0.8 - 0.8888 = 0.71111.$$

Also,

$$\begin{aligned}\mathring{e}_x =& \mathring{e}_y = \frac{1}{0.2} = 5\\ \mathring{e}_{xy} =& \frac{1}{\mu_x+\mu_y} = \frac{1}{0.4} = 2.5\\ \mathring{e}_{\overline{xy}} =& \mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy} = 5+5-2.5 = 7.5\\ \bar{a}_{\overline{7.5}|} =& \frac{1-e^{-0.05(7.5)}}{0.05} = 6.2542\\ 1000\bar{P}(\bar{A}_{\overline{xy}}) =& \frac{1000(0.71111)}{6.2542} = 113.70 \blacksquare\end{aligned}$$

Example 60.6 ‡
For a temporary life annuity-immediate on independent lives (30) and (40):
(i) Mortality follows the Illustrative Life Table.
(ii) $i = 0.06$
Calculate $a_{30:40:\overline{10}|}$.

Solution.
We have

$$\begin{aligned}
a_{30:40:\overline{10}|} =& a_{30:40} - {}_{10}E_{30:40}a_{40:50}\\
=&(\ddot{a}_{30:40} - 1) - [\nu^{10}{}_{10}p_{30:40}](\ddot{a}_{40:50} - 1)\\
=&(\ddot{a}_{30:40} - 1) - [\nu^{10}{}_{10}p_{30}{}_{10}p_{40}](\ddot{a}_{40:50} - 1)\\
=&(\ddot{a}_{30:40} - 1) - (1+i)^{10}[\nu^{10}{}_{10}p_{30}][\nu^{10}{}_{10}p_{40}](\ddot{a}_{40:50} - 1)\\
=&(\ddot{a}_{30:40} - 1) - (1+i)^{10}({}_{10}E_{30})({}_{10}E_{40})(\ddot{a}_{40:50} - 1)\\
=&(14.2068 - 1) - (1.06)^{10}(0.54733)(0.53667)(12.4784 - 1)\\
=&7.1687 \blacksquare
\end{aligned}$$

Example 60.7 ‡
For an insurance on (x) and (y) :
(i) Upon the first death, the survivor receives the single benefit premium for a whole life insurance of 10,000 payable at the moment of death of the survivor.
(ii) $\mu(x+t) = \mu(y+t) = 0.06$ while both are alive.
(iii) $\mu(xy+t) = 0.12$
(iv) After the first death, $\mu(t) = 0.10$ for the survivor.
(v) $\delta = 0.04$
Calculate the actuarial present value of this insurance on (x) and (y).

Solution.
The actuarial present value of the death benefit is

$$10,000\int_0^\infty \nu^t f_T(t)dt = 10,000\frac{\mu}{\mu+\delta} = 10,000\left(\frac{0.10}{0.10+0.04}\right) = 7142.86.$$

The actuarial present value of this insurance is

$$7142.86\bar{A}_{xy} = 7142.86\left(\frac{\mu(xy+t)}{\mu(xy+t)+\delta}\right) = 7142.86\left(\frac{12}{16}\right) = 5357.15 \blacksquare$$

Practice Problems

Problem 60.1
A fully discrete last-survivor insurance of 1000 is issued on two lives (30) and (40), whose mortality follows the Illustrative Life Table with $i = 0.06$. Net annual premiums are reduced by 25% after the first death. Calculate the initial net annual premium.

Problem 60.2
Z is the present value random variable for a special discrete whole life insurance issued to (x) and (y) which pays 1 at the end of the year of the first death and 1 at the end of the year of the second death. You are given:
(i) $a_x = 11$
(ii) $a_y = 15$
(iii) $i = 0.04$.
Calculate $E(Z)$.

Problem 60.3
For a fully continuous last-survivor whole life insurance of 1 issued to (x) and (y), you are given:
(i) $T(x)$ and $T(y)$ are independent
(ii) $\mu(x+t) = \mu(y+t) = \mu$
(iii) $\delta = 0.04$.
(iv) Premiums of 0.072 per year, set using the equivalence principle, are payable until the first death.
Calculate μ.

Problem 60.4
You are given:
(i) (x) is subject to a uniform distribution of deaths over each year of age.
(ii) (y) is subject to a constant force of mortality of 0.25
(iii) $q_{xy}^{1} = 0.125$
(iv) $T(x)$ and $T(y)$ are independent
(v) $i = 0.05$.
Calculate $A_{x:\overline{1}|}^{1}$.

Problem 60.5
You are given:
(i) $\delta = 0.04$

(ii) $\mu(x+t) = 0.01$
(iii) $\mu(y+t) = \frac{1}{100-t},\ 0 \leq t \leq 100$
(iv) $T(x)$ and $T(y)$ are independent
(a) Calculate $\bar{A}_{xy}$.
(b) Calculate $\bar{a}_{\overline{xy}}$.
(c) Calculate $\text{Cov}(\bar{a}_{\overline{T(xy)}|}, \bar{a}_{\overline{T(\overline{xy})}|})$.

Problem 60.6 ‡
For a special fully continuous last survivor insurance of 1 on (x) and (y), you are given:
(i) $T(x)$ and $T(y)$ are independent.
(ii) $\mu(x+t) = 0.08,\ t > 0$
(iii) $\mu(y+t) = 0.04,\ t > 0$
(iv) $\delta = 0.06$
(v) π is the annual benefit premium payable until the first of (x) and (y) dies.
Calculate π.

Problem 60.7 ‡
For a special fully continuous last survivor insurance of 1 on (x) and (y), you are given:
(i) $T(x)$ and $T(y)$ are independent.
(ii) $\mu(x+t) = \mu(y+t) = 0.07,\ t > 0$
(iii) $\delta = 0.05$
(iv) Premiums are payable until the first of (x) and (y) dies.
Calculate the level annual benefit premium π for this insurance.

Problem 60.8 ‡
(x) and (y) are two lives with identical expected mortality.
You are given:
(i) $P(A_x) = P(A_y) = 0.1$
(ii) $P(A_{\overline{xy}} = 0.06$, where $P(A_{\overline{xy}}$ is the annual benefit premium for a fully discrete insurance of 1 on $\overline{x}y$.
(iii) $d = 0.06$
Calculate the premium $P(A_{xy})$, the annual benefit premium for a fully discrete insurance of 1 on (xy).

Problem 60.9 ‡
You are pricing a special 3-year temporary life annuity-due on two lives each

age x, with independent future lifetimes, each following the same mortality table. The annuity pays 10,000 if both persons are alive and 2000 if exactly one person is alive.
You are given:
(i) $q_{xx} = 0.04$
(ii) $q_{x+1:x+1} = 0.01$
(iii) $i = 0.05$
Calculate the actuarial present value of this annuity.

Problem 60.10 ‡
A 30-year term insurance for Janet age 30 and Andre age 40 provides the following benefits:
• A death benefit of \$140,000 if Janet dies before Andre and within 30 years
• A death benefit of \$180,000 if Andre dies before Janet and within 30 years
You are given:
(i) Mortality follows De Moivre's Law with $\omega = 100$
(ii) $i = 0$
(iii) The death benefit is Payable at the moment of the first death.
(iv) Premiums, $\bar{P}$, are paid continuously while both are alive for a maximum of 20 years.
(a) Calculate the actuarial present value at issue of the death benefits.
(b) Calculate the actuarial present value at issue of the premiums in terms of $\bar{P}$.

Problem 60.11 ‡
You are given:
(i) (30) and (50) are independent lives, each subject to a constant force of mortality $\mu = 0.05$.
(ii) $\delta = 0.03$.
Calculate $\bar{A}^{\,1}_{30:50}$.

Problem 60.12 ‡
Z is the present value random variable for an insurance on the lives of Bill and John. This insurance provides the following benefits.
• 500 at the moment of Bill's death if John is alive at that time, and
• 1000 at the moment of Johns death if Bill is dead at that time.
You are given:
(i) Bill's survival function follows De Moivre's Law with $\omega = 85$

(ii) John's survival function follows De Moivre's Law with $\omega = 84$
(iii) Bill and John are both age 80
(iv) Bill and John are independent lives
(v) $i = 0$
Calculate $E[Z]$.

Problem 60.13 ‡
(i) Z is the present value random variable for an insurance on the lives of (x) and (y) where,

$$Z = \begin{cases} \nu^{T(y)} & T(x) \leq T(y) \\ 0 & T(x) > T(y). \end{cases}$$

(ii) (x) is subject to a constant force of mortality $m(x+t) = 0.07$.
(iii) (y) is subject to a constant force of mortality $m(y+t) = 0.09$.
(iv) $\delta = 0.06$
(v) (x) and (y) are independent lives
Calculate $E(Z)$.

61 Special Two-life Annuities: Reversionary Annuities

A **reversionary annuity** is a special type of two-life annuities. This annuity pays benefit only when one of the lives has failed and then for as long as the other continues to survive.

Commonly reversionary annuities are found in the pensions world - a pensioner will have a pension of (say) \$10,000 per year, and upon his/her death, his/her spouse will receive (say) \$5,000 per year for the rest of their life. Note here that the reversionary benefit requires that the pensioner dies before the spouse. If the spouse dies before the pensioner, there is no reversionary benefit payable.

Let $\bar{Z}$ denote the present value of a continuously-payable reversionary annuity, which pays 1 per year to (y) for the rest of (y)'s lifetime beginning immediately on the death of (x) (note (y) must outlive (x) for this benefit to be payable). Then

$$\bar{Z} = \begin{cases} \bar{a}_{\overline{T(y)}|} - \bar{a}_{\overline{T(x)}|} & \text{if } T(y) > T(x) \\ 0 & \text{if } T(y) \le T(x). \end{cases}$$

This can be written as

$$\bar{Z} = \begin{cases} \bar{a}_{\overline{T(y)}|} - \bar{a}_{\overline{T(x)}|} & \text{if } T(y) > T(x) \\ \bar{a}_{\overline{T(y)}|} - \bar{a}_{\overline{T(y)}|} & \text{if } T(y) \le T(x) \end{cases}$$

or

$$\bar{Z} = \bar{a}_{\overline{T(y)}|} - \bar{a}_{\overline{T(xy)}|}.$$

The actuarial present value of this benefit is denoted by

$$\bar{a}_{x|y} = E(\bar{Z}) = \bar{a}_y - \bar{a}_{xy}.$$

Likewise, in the discrete case, we have

$$a_{x|y} = a_y - a_{xy}.$$

Example 61.1 ‡

A continuous two-life annuity pays

100 while both (30) and (40) are alive
70 while (30) is alive but (40) is dead
50 while (40) is alive but (30) is dead.

The actuarial present value of this annuity is 1180. Continuous single life annuities paying 100 per year are available for (30) and (40) with APV's of 1200 and 1000 respectively.
Calculate the APV of a continuous two-life annuity that pays 100 while at least one of them is alive.

Solution.
We want to find

$$100\bar{a}_{\overline{30:40}} = 100(\bar{a}_{30} + \bar{a}_{40} - \bar{a}_{30:40}).$$

We have

$$\begin{aligned}
1180 =&100\bar{a}_{30:40} + 70(\bar{a}_{30} - \bar{a}_{30:40}) + 50(\bar{a}_{40} - \bar{a}_{30:40})\\
=&-20\bar{a}_{30:40} + 70\bar{a}_{30} + 50\bar{a}_{40}\\
=&-20\bar{a}_{30:40} + 70(12) + 50(10)\\
\bar{a}_{30:40} =&\frac{840 + 500 - 1180}{20} = 8.
\end{aligned}$$

Hence,

$$100\bar{a}_{\overline{30:40}} = 100(12 + 10 - 8) = 1400 \blacksquare$$

Example 61.2
For a special fully continuous last-survivor insurance of 1 on two independent lives (x) and (y), you are given:
(i) Deaths benefits are payable at the moment of the second death
(ii) Level benefit premiums,π, are payable only while (x) is alive and (y) is dead. No premiums are payable while both are alive or if (x) dies first.
(iii) $\delta = 0.05$
(iv) $\mu(x+t) = 0.03$ and $\mu(y+t) = 0.04$.
Calculate π.

Solution.
APV of premiums $= \pi(\bar{a}_x - \bar{a}_{xy})$. Under constant force of mortality, we have

$$\begin{aligned}
\bar{a}_x =&\frac{1}{\mu(x+t)+\delta} = \frac{1}{0.03+0.05} = 12.5\\
\bar{a}_{xy} =&\frac{1}{\mu(x+t)+\mu(y+t)+\delta} = \frac{1}{0.03+0.04+0.05} = 8.3333.
\end{aligned}$$

Now, APV of benefits $=\bar{A}_{\overline{xy}}=\bar{A}_x+\bar{A}_y-\bar{A}_{xy}$ where

$$\begin{aligned}\bar{A}_x=&\frac{\mu(x+t)}{\mu(x+t)+\delta}=\frac{0.03}{0.03+0.05}=0.375\\ \bar{A}_y=&\frac{\mu(y+t)}{\mu(y+t)+\delta}=\frac{0.04}{0.04+0.05}=0.444\\ \bar{A}_{xy}=&\frac{\mu(x+t)+\mu(y+t)}{\mu(x+t)+\mu(y+t)+\delta}=\frac{0.03+0.04}{0.03+0.04+0.05}=0.58333.\end{aligned}$$

Thus,

$$\pi(\bar{a}_x-\bar{a}_{xy})=\bar{A}_x+\bar{A}_y-\bar{A}_{xy}\Longrightarrow\pi=\frac{0.375+0.444+0.58333}{12.5-8.3333}=0.566\ \blacksquare$$

Example 61.3 ‡
You are pricing a special 3-year annuity-due on two independent lives, both age 80. The annuity pays 30,000 if both persons are alive and 20,000 if only one person is alive.
You are given:
(i)

k	${}_kp_{80}$
1	0.91
2	0.82
3	0.72

(ii) $i=0.05$
Calculate the actuarial present value of this annuity.

Solution.
We want

$$\text{APV}=30,000\ddot{a}_{80:80:\overline{3}|}+20,000(\ddot{a}_{80:\overline{3}|}-\ddot{a}_{80:80:\overline{3}|})+20,000(\ddot{a}_{80:\overline{3}|}-\ddot{a}_{80:80:\overline{3}|})$$

where

$$\begin{aligned}\ddot{a}_{80:\overline{3}|}=&1+\nu p_{80}+\nu^2{}_2p_{80}=1+0.91(1.05)^{-1}+0.82(1.05)^{-2}=2.61043\\ \ddot{a}_{80:80:\overline{3}|}=&1+\nu p_{80:80}+\nu^2{}_2p_{80:80}=1+\nu(p_{80})^2+\nu^2({}_2p_{80})^2\\ =&1+(0.91)^2(1.05)^{-1}+(0.82)^2(1.05)^{-2}=2.39855.\end{aligned}$$

Hence,

$$\text{APV}=30,000(2.39855)+(2)(20,000)(2.61043-2.39855)=80,431.70\ \blacksquare$$

Practice Problems

Problem 61.1
A special two-life annuity on (50) and (60) pays a benefit of 1 at the beginning of each year if both annuitants are alive. The annuity pays a benefit of $\frac{2}{3}$ at the beginning of each year if one annuitant is alive.
You are given:
(i) Mortality follows the Illustrative Life Table.
(ii) (50) and (60) are independent lives.
(iii) $i = 0.06$.
Calculate the actuarial present value of this annuity

Problem 61.2
An special two-life annuity on (50) and (60) pays a benefit of 1 at the beginning of each year if both annuitants are alive. The annuity pays a benefit of $\frac{2}{3}$ at the beginning of each year if only (50) is alive. The annuity pays a benefit of $\frac{1}{2}$ at the beginning of each year if only (60) is alive.
You are given:
(i) Mortality follows the Illustrative Life Table.
(ii) (50) and (60) are independent lives.
(iii) $i = 0.06$.
Calculate the actuarial present value of this annuity.

Problem 61.3
A reversionary annuity pays a continuous benefit at a rate of 100 per year to (50) upon the death of (60) provided (50) is alive.
You are given:
(i) (50) and (60) are independent
(ii) $\mu(50 + t) = 0.03$
(iii) $\mu(60 + t) = 0.05$
(iv) $\delta = 0.04$.
Calculate the actuarial present value of this reversionary annuity.

Problem 61.4
You are given:
(i) $a_{\overline{n}|} = 12.1$
(ii) $a_x = 10.3$
(iii) $a_{x:\overline{n}|} = 7.4$.
Calculate the reversionary annuity $a_{x|\overline{n}|}$.

Problem 61.5
Show that

$$\bar{a}_{x|y} = \frac{\bar{A}_{xy} - \bar{A}_y}{\delta}.$$

62 Dependent Future Lifetimes Model: The Common Shock

When two lives are exposed to a common hazard factor on a regular basis then their future lifetimes are dependent. We refer to this common hazard as a **commom shock**. For example, a natural disaster such as an earthquake or a hurricane can be considered as a common shock. The presence of the common shock introduces dependence into what would otherwise be independent future lifetimes.

To model a common shock situation, consider a joint-life status involving two lives (x) and (y) with remaining independent lifetimes without common shock $T^*(x)$ and $T^*(y)$. Let Z be the random variable representing the time the common shock occurs. It follows that the time until failure $T(x)$ will occur at either time $T^*(x)$ or time Z, whichever is earlier. That is,

$$T(x) = \min\{T^*(x), Z\}.$$

Likewise, we have

$$T(y) = \min\{T^*(y), Z\}.$$

Note that $T^*(x), T^*(y)$ and Z are independent random variables. However, $T(x)$ and $T(y)$ are not independent since both depend on the future common random variable Z.

Next, we assume that the common shock, that both (x) and (y) are subject to, can be represented by a constant hazard function which we denote by $\lambda = \mu(z+t)$. Then, by independence, we have

$${}_tp_x = s_{T(x)}(t) = s_{T^*(x)Z}(t) = s_{T^*(x)}(t)s_Z(t) = {}_tp_x^*e^{-\lambda t}.$$

Likewise,

$${}_tp_y = s_{T(y)}(t) = s_{T^*(y)Z}(t) = s_{T^*(y)}(t)s_Z(t) = {}_tp_y^*e^{-\lambda t}.$$

Now, if we let $T = \min\{T^*(x), T^*(y), Z\}$ denote the time of first failure, then we can write

$$\begin{aligned} {}_tp_{xy} =& s_T(t) = s_{T^*(x)T^*(y)Z}(t) \\ =& s_{T^*(x)}(t)s_{T^*(y)}(t)s_Z(t) = {}_tp_x^* {}_tp_y^* e^{-\lambda t}. \end{aligned}$$

Clearly,

$${}_tp_{xy} \neq {}_tp_x {}_tp_y$$

which shows that $T(x)$ and $T(y)$ are dependent.

Remark 62.1
The reader is to be alerted that the notation used in this section is a bit confusing. The notation $T^*(x)$ in this section stands for the remaining lifetime **without the shock**-what we would normally call $T(x)$. In this section, $T(x)$ denotes the lifetime **with the shock.**
Thus the notation $T(x)$ is used to represent the actual lifetime whether or not a shock is included.
When $(x), (y)$ and the common shock are exponential, the shock parameter adds to the force of mortality of $(x), (y), (xy)$, and $(\overline{xy})$.

Example 62.1
Two lives, (x) and (y), are subject to a exponential common shock model with $\lambda = 0.01$. You are given:
(i) $(x), (y)$, and the common shock are independent.
(ii) $\mu(x+t) = 0.04$.
(iii) $\mu(y+t) = 0.06$.
(iv) $\mu(x+t)$ and $\mu(y+t)$ do not reflect the mortality from the common shock.
(v) $\delta = 0.03$.
Calculate:

$$
\begin{array}{lllll}
(a) s_{T^*(x)}(t) & (b) s_{T^*(y)}(t) & (c) s_{T(x)}(t) & (d) s_{T(y)}(t) & (e) s_{T(x)T(y)}(t) \\
(f) \mathring{e}_x & (g) \mathring{e}_y & (h) \mathring{e}_{xy} & (i) \mathring{e}_{\overline{xy}} & (j) \bar{A}_{xy} \\
(k) \bar{A}_x & (\ell) \bar{A}_y & (m) \bar{A}_{\overline{xy}} & (n) \bar{a}_x & (o) \bar{a}_y \\
(p) \bar{a}_{xy} & (q) \bar{a}_{\overline{xy}} & & &
\end{array}
$$

Solution.
We have
(a) $s_{T^*(x)}(t) = {}_tp_x^* = e^{-\mu(x+t)t} = e^{-0.04t}$.
(b) $s_{T^*(y)}(t) = {}_tp_y^* = e^{-\mu(y+t)t} = e^{-0.06t}$.
(c) $s_{T(x)}(t) = {}_tp_x^* e^{-\lambda t} = e^{-0.04t} e^{-0.01t} = e^{-0.05t}$.
(d) $s_{T(y)}(t) = {}_tp_y^* e^{-\lambda t} = e^{-0.06t} e^{-0.01t} = e^{-0.07t}$.
(e) $s_{T(x)T(y)} = {}_tp_x^* {}_tp_y^* e^{-\lambda t} = e^{-0.04t} e^{-0.06t} e^{-0.01t} = e^{-0.11t}$
(f) $\mathring{e}_x = \frac{1}{\mu(x+t)+\lambda} = \frac{1}{0.05} = 20$.
(g) $\mathring{e}_y = \frac{1}{\mu(y+t)+\lambda} = \frac{1}{0.07} = 14.28571$.
(h) $\mathring{e}_{xy} = \frac{1}{\mu(x+t)+\mu(y+t)+\lambda} = \frac{1}{0.11} = 9.09091$.
(i) $\mathring{e}_{\overline{xy}} = \mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy} = 20 + 14.28571 - 9.09091 = 25.1948$.
(j) $\bar{A}_{xy} = \frac{\mu_{xy}}{\mu_{xy}+\delta} = \frac{0.11}{0.11+0.03} = 0.78571$.

(k) $\bar{A}_x = \frac{\mu(x+t)+\lambda}{\mu(x+t)+\lambda+\delta} = \frac{0.05}{0.08} = 0.625.$
(ℓ) $\bar{A}_y = \frac{\mu(y+t)+\lambda}{\mu(y+t)+\lambda+\delta} = \frac{0.07}{0.10} = 0.7.$
(m) $\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy} = 0.625 + 0.7 - 0.78571 = 0.53929.$
(n) $\bar{a}_x = \frac{1}{\mu(x+t)+\lambda+\delta} = \frac{1}{0.08} = 12.5.$
(o) $\bar{a}_y = \frac{1}{\mu(y+t)+\lambda+\delta} = \frac{1}{0.10} = 10.$
(p) $\bar{a}_{xy} = \frac{1-\bar{A}_{xy}}{\delta} = 7.143.$
(q) $\bar{a}_{\overline{xy}} = \bar{a}_x + \bar{a}_y - \bar{a}_{xy} = 12.5 + 10 - 7.143 = 15.357$ ■

Example 62.2 ‡
You have calculated the actuarial present value of a last-survivor whole life insurance of 1 on (x) and (y). You assumed:
(i) The death benefit is payable at the moment of death.
(ii) The future lifetimes of (x) and (y) are independent, and each life has a constant force of mortality with $\mu = 0.06$.
(iii) $\delta = 0.05$
Your supervisor points out that these are not independent future lifetimes. Each mortality assumption is correct, but each includes a common shock component with constant force 0.02.
Calculate the increase in the actuarial present value over what you originally calculated.

Solution.
First, we calculate the actuarial present value of a last-survivor policy under independence. We have

$$\begin{aligned}
\mu(xy+t) =& \mu(x+t) + \mu(y+t) = 0.06 + 0.06 = 0.12\\
\bar{A}_x =& \bar{A}_y = \frac{\mu(x+t)}{\mu(x+t)+\delta} = \frac{0.06}{0.06+0.05} = 0.54545\\
\bar{A}_{xy} =& \frac{\mu(xy+t)}{\mu(xy+t)+\delta} = \frac{0.12}{0.12+0.05} = 0.70588\\
\bar{A}_{\overline{xy}} =& \bar{A}_x + \bar{A}_y - \bar{A}_{xy} = 0.38502.
\end{aligned}$$

Next, we calculate the actuarial present value of a last-survivor policy under the common shock model. We have

$$\begin{aligned}
\mu^{T^*(x)}(x+t) =& \mu(x+t) - \mu(z+t) = 0.06 - 0.02 = 0.04 = \mu^{T^*(y)}(y+t)\\
\mu(xy+t) =& \mu^{T^*(x)}(x+t) + \mu^{T^*(y)}(y+t) + \mu(z+t) = 0.04 + 0.04 + 0.02 = 0.10\\
\bar{A}_x =& \bar{A}_y = 0.54545\\
\bar{A}_{xy} =& \frac{0.10}{0.10 + 0.05} = 0.66667\\
\bar{A}_{\overline{xy}} =& 0.54545 + 0.54545 - 0.66667 = 0.42423.
\end{aligned}$$

Hence, the increase in the actuarial present value over what you originally calculated is $0.42423 - 0.38502 = 0.03921$ ■

Practice Problems

Problem 62.1
You are given:
(i) $\mu(x+t) = 0.02t$.
(ii) $\mu(y+t) = (40-t)^{-1}$.
(iii) The lives are subject to an exponential common shock model with $\lambda = 0.015$.
(iv) $\mu(x+t)$ and $\mu(y+t)$ incorporate deaths from the common shock.
Calculate ${}_{3|}q_{xy}$.

Problem 62.2
Two independent lives (x) and (y) are subject to the following mortality rates when common shock is ignored.

t	q_{x+t}	q_{y+t}
0	0.2	0.3
1	0.4	0.3
2	1.0	1.0

The common shock component follows an exponential distribution with parameter $\lambda = -\ln(0.9)$.
Calculate $e_{\overline{xy}}$.

Problem 62.3
For a special fully continuous last-survivor whole life insurance of 1 on (x) and (y), you are given:
(i) The premium is payable until the first death.
(ii) $T^*(x)$ has an exponential distribution with mean 25.
(iii) $T^*(y)$ has an exponential distribution with mean 16.66667.
(iv) Z, the common shock random variable, has an exponential distribution with mean 50.
(v) $\delta = 0.04$.
Calculate the annual benefit premium P.

Problem 62.4
A life insurance is issued to two lives (x) and (y) and pays 2000 at the moment of the first death and 1000 at the moment of the second death. You are given:

(i) $\mu(x+t) = 0.02$.
(ii) $\mu(y+t) = 0.03$.
(iii) $\delta = 0.04$.
(iv) (x) and (y) are subject to a common shock. The time of shock is exponentially distributed with mean 50.
Calculate the actuarial present value at time 0 of the death benefits under the common shock model.

Problem 62.5 ‡
The mortality of (x) and (y) follows a common shock model with components $T^*(x), T^*(y)$ and Z.
(i) $T^*(x), T^*(y)$ and Z are independent and have exponential distributions with respective forces μ_1, μ_2, and λ.
(ii) The probability that x survives year 1 is 0.96.
(iii) The probability that y survives year 1 is 0.97.
(iv) $\lambda = 0.01$.
Calculate the probability that both (x) and (y) survive 5 years.

63 Joint Distributions of Future Lifetimes

Consider two lives (x) and (y) that are not necessarily independent. Let $f_{T(x)T(y}(t_x, t_y)$ denote the joint density function of the future lifetime random variables $T(x)$ and $T(y)$. The marginal density function of $T(x)$ is

$$f_{T(x)}(t_x) = \int_{t_y} f_{T(x)T(y)}(t_x, t_y)dt_y.$$

Likewise, the marginal density function of $T(y)$ is

$$f_{T(y)}(t_y) = \int_{t_x} f_{T(x)T(y)}(t_x, t_y)dt_x.$$

The expected value of the random variable $T(x)T(y)$ is

$$E[T(x)T(y)] = \int_0^\infty \int_0^\infty t_x t_y f_{T(x)T(y)}(t_x, t)y)dt_x dt_y.$$

Example 63.1
For two lives with joint lifetime random variables $T(x)$ and $T(y)$, you are given

$$f_{T(x)T(y)}(t_x, t_y) = \begin{cases} \frac{t_x+t_y}{216}, & 0 < t_x < 6,\ 0 < t_y < 6 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate $f_{T(x)}(t_x)$.

Solution.
We have

$$\begin{aligned} f_{T(x)}(t_x) =& \int_0^6 f_{T(x)T(y)}(t_x, t_y)dt_y = \frac{1}{216}\int_0^6 (t_x + t_y)dt_y \\ =& \frac{1}{216}\left[t_x t_y + \frac{1}{2}t_y^2\right]_0^6 = \frac{6t_x + 18}{216} = \frac{t_x}{36} + \frac{1}{12} \blacksquare \end{aligned}$$

Example 63.2
For the two lives in Example 63.1, find $E[T(x)T(y)]$.

Solution.
We have

$$\begin{aligned} E[T(x)T(y)] =& \frac{1}{216}\int_0^6 \int_0^6 (t_x^2 t_y + t_x t_y^2)dt_x dt_y = \frac{1}{216}\int_0^6 \left[\frac{1}{3}t_x^3 t_y + \frac{1}{2}t_x^2 t_y^2\right]_0^6 dt_y \\ =& \frac{1}{216}\int_0^6 (72t_y + 18t_y^2)dt_y = \frac{1}{216}\left[36t_y^2 + 6t_y^3\right]_0^6 = 12 \blacksquare \end{aligned}$$

Example 63.3
Find the joint cumulative distribution function for the joint distribution in Exampe 63.1.

Solution.
Consider Figure 63.1.

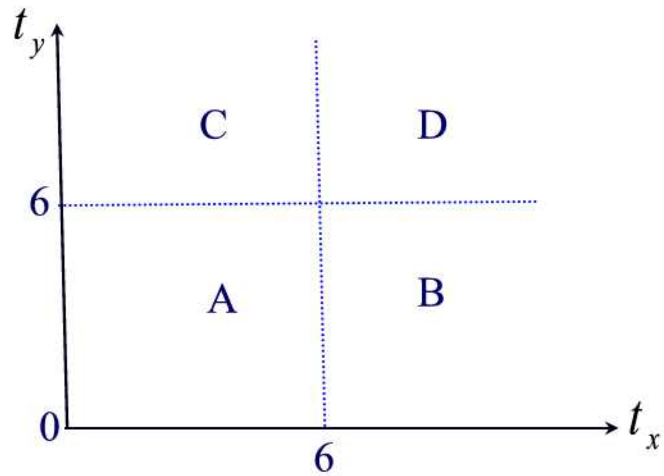


Figure 63.1

For (t_x, t_y) in Region A, we have

$$\begin{aligned}
F_{T(x)T(y)}(t_x, t_y) =&\Pr[(T(x) \le t_x) \cap (T(y) \le t_y)] = \int_0^{t_y}\int_0^{t_x} f_{T(x)T(y)}(r,s)drds\\
=&\frac{1}{216}\int_0^{t_y}\int_0^{t_x}(r+s)drds = \frac{1}{216}\int_0^{t_y}\left[\frac{1}{2}r^2 + rs\right]_0^{t_x} ds\\
=&\frac{1}{216}\int_0^{t_y}\left(\frac{1}{2}t_x^2 + t_x s\right)ds = \frac{1}{216}\left[\frac{1}{2}t_x^2 s + \frac{1}{2}t_x s^2\right]_0^{t_y}\\
=&\frac{t_x^2 t_y + t_x t_y^2}{432}.
\end{aligned}$$

For (t_x, t_y) in Region B, the event $T(x) \le t_x$ is certain to occur since $t_x > 6$. In this case,

$$F_{T(x)T(y)}(t_x, t_y) = \Pr(T(y) \le t_y) = F_{T(y)}(t_y) = \frac{1}{36}.$$

For (t_x, t_y) in Region C, the even $T(y) \leq t_y$ is certain to occur since $t_y > 6$. In this case,

$$F_{T(x)T(y)}(t_x, t_y) = \Pr(T(x) \leq t_x) = F_{T(x)}(t_x) = \frac{1}{36}.$$

For (t_x, t_y) in Region D, both events $T(x) \leq t_x$ and $T(y) \leq t_y$ are certain to occur so that

$$F_{T(x)T(y)}(t_x, t_y) = 1.$$

For (t_x, t_y) in quadrant II, the event $T(x) \leq t_x$ is impossible so that $F_{T(x)T(y)}(t_x, t_y) = 0$. Similar reasoning for (t_x, t_y) in the third or fourth quadrants ■

Now, if we evaluate the joint survival function at a common point (n, n) we find

$$s_{T(x)T(y)}(n, n) = \Pr([T(x) > n] \cap [T(y) > n]) = \Pr(T(xy) > n) = {}_np_{xy}.$$

Therefore, all joint life functions presented in Section 56 can be evaluated from the general joint SDF of $T(x)$ and $T(y)$. Furthermore, the SDF of the joint-life status can be found from the joint SDF of $T(x)$ and $T(y)$.
Likewise, we have

$$F_{T(x)T(y)}(n, n) = \Pr([T(x) \leq n] \cap [T(y) \leq n]) = \Pr(T(\overline{xy}) \leq n) = {}_nq_{\overline{xy}}.$$

Thus, the CDF of the last-survivor status can be found from the joint CDF of $T(x)$ and $T(y)$.

Practice Problems

Problem 63.1
For two lives with joint lifetime random variables $T(x)$ and $T(y)$, you are given

$$f_{T(x)T(y)}(t_x, t_y) = \begin{cases} \frac{t_x+t_y}{216}, & 0 < t_x < 6,\ 0 < t_y < 6 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate $f_{T(y)}(t_y)$.

Problem 63.2
For the two lives in Problem 63.1, find $\text{Var}[T(x)]$.

Problem 63.3
Show that the lives in Problem 63.1 are dependent.

Problem 63.4
For the lives in Problem 63.1, find $\text{Cov}(T(x), T(y))$.

Problem 63.5
For the lives in Problem 63.1, find the coefficient of variation $\rho_{T(x),T(y)}$.

Problem 63.6
For the lives in Problem 63.1, find the joint survival function $s_{T(x)T(y)}(t_x, t_y)$.

Problem 63.7
For the lives in Problem 63.1, find $s_{T(x)T(y)}(n, n)$.

Problem 63.8
For the lives in Problem 63.1, find $p_{2:2}$.

Multiple Decrement Models

So far in this text we have considered a single life status or a multiple life status subject to a single contingency of death (or a **single decrement.**) In this chapter, we consider a single life status subject to multiple contingencies or **multiple decrements.** For example, a pension plan provides benefit for death, disability, employment termination and retirement. Multiple-decrement models also go by the name of **competing risk** models in other contexts.

64 The Continuous Case

Assume that life (x) is a member of a group. Let $T(x) = T$ denote the time of decrement at which life (x) leaves the group. T is a nonnegative continuous random variable. Further, assume that there are m causes of decrement. Let $J(x) = J$ be a discrete random variable and let $(J(x) = j)$ denote cause $j; j = 1, 2, \cdots, m$. Then J takes only m possible values of $1, 2, \cdots, m$.
Let $f_{T,J}(t, j)$ denote the joint probability distribution function of $T(x)$ and $J(x)$. This joint PDF can be used to calculate the probabilities of events defined by $T(x)$ and $J(x)$. For example, the probability of decrement between times a and b due to cause j is

$$\Pr(a < T \le b, J = j) = \int_a^b f_{T,J}(t, j)dt.$$

The probability of decrement between times a and b due to all causes is

$$\Pr(a < T \le b) = \sum_{j=1}^{m} \Pr(a < t \le b, J = j) = \sum_{j=1}^{m} \int_a^b f_{T,J}(t, j)dt.$$

The probability of decrement before time t due to cause j is defined by

$${}_tq_x^{(j)} = \int_0^t f_{T,J}(s, j)ds.$$

The marginal PDF of J is defined by

$$f_J(j) = \Pr(J = j) = \int_0^\infty f_{T,J}(s, j)ds = {}_\infty q_x^{(j)}, \ j = 1, 2, \cdots, m$$

which is the probability of decrement due to cause j at any time in the future. Note that $\sum_{j=1}^m f_J(j) = 1$. Also note the unconventional definition of $f_J(j)$.
The marginal PDF of $T(x)$ is defined by

$$f_T(t) = \sum_{j=1}^{m} f_{T,J}(t, j).$$

The marginal CDF of $T(x)$ is defined as

$$F_T(t) = \int_0^t f_T(s)ds.$$

Thus, the probability of decrement before time t due to all causes of decrement is given by

$$ {}_tq_x^{(\tau)} = \Pr[T(x) \le t] = F_T(t) = \int_0^t f_T(s)ds. $$

It follows that

$$ {}_tq_x^{(\tau)} = \int_0^t f_T(s)ds = \sum_{j=1}^m \int_0^t f_{T,J}(s,j)ds = \sum_{j=1}^m {}_tq_x^{(j)}. $$

Now, the probability that life (x) is still in the group at age $x+t$ is given by

$$ {}_tp_x^{(\tau)} = \Pr[T(x) > t] = 1 - {}_tq_x^{(\tau)} = \int_t^\infty f_T(s)ds. $$

Note that we use τ to indicate that a function refers to all causes.
The expected time of decrement due to all causes is:

$$ E(T) = \int_0^\infty {}_tp_x^{(\tau)}dt. $$

A concept that is parallel to the concept of the force of mortality of a single life contingency is the **total force of mortality** denoted by $\mu^{(\tau)}(x+t)$.
We have

$$ \begin{aligned} \mu^{(\tau)}(x+t) =& \frac{f_T(t)}{1-F_T(t)} = \frac{1}{{}_tp_x^{(\tau)}}\frac{d}{dt}({}_tq_x^{(\tau)}) \\ =& -\frac{1}{{}_tp_x^{(\tau)}}\frac{d}{dt}({}_tp_x^{(\tau)}) = -\frac{d}{dt}[\ln({}_tp_x^{(\tau)})]. \end{aligned} $$

From this, we can write

$$ {}_tp_x^{(\tau)} = e^{-\int_0^t \mu^{(\tau)}(x+s)ds} $$

and

$$ f_T(t) = {}_tp_x^{(\tau)}\mu^{(\tau)}(x+t). $$

We define the **force of decrement due to cause** j by

$$ \mu^{(j)}(x+t) = \frac{f_{T,J}(t,j)}{1-f_T(t)} = \frac{1}{{}_tp_x^{(\tau)}}\frac{d}{dt}({}_tq_x^{(j)}). $$

From this definition, it follows that

$$f_{T,J}(t,j) = \mu^{(j)}(x+t)\,{}_tp_x^{(\tau)}$$

and

$${}_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)}\mu^{(j)}(x+s)ds.$$

Moreover,

$$\begin{aligned}
\mu^{(\tau)}(x+t) =& \frac{1}{{}_tp_x^{(\tau)}}\frac{d}{dt}({}_tq_x^{(\tau)})\\
=& \sum_{j=1}^m \frac{1}{{}_tp_x^{(\tau)}}\frac{d}{dt}({}_tq_x^{(j)})\\
=& \sum_{j=1}^m \mu^{(j)}(x+t).
\end{aligned}$$

Note also the following

$${}_tq_x^{(j)} = \int_0^t f_{T,J}(s,j)ds = \int_0^t {}_sp_x^{(\tau)}\mu^{(j)}(x+s)ds.$$

Example 64.1
For a double-decrement model, you are given:
(i) $\mu^{(1)}(x+t) = \frac{2}{50-t},\ 0 \le t < 50$
(ii) $\mu^{(2)}(x+t) = 0.02,\ t \ge 0.$
Calculate ${}_tp_x^{(\tau)}$ and ${}_{10}q_x^{(1)}$.

Solution.
We have

$${}_tp_x^{(\tau)} = e^{-\int_0^t[\mu^{(1)}(x+s)+\mu^{(2)}(x+s)]ds} = e^{-\int_0^t\left(\frac{2}{50-s}+0.02\right)ds} = \left(\frac{50-t}{50}\right)^2 e^{-0.02t}$$

and

$${}_{10}q_x^{(1)} = \int_0^{10}\left(\frac{50-s}{50}\right)^2 e^{-0.02s}\left(\frac{2}{50-s}\right)ds = 0.3275 \blacksquare$$

Finally, we define the conditional probability that decrement is due to cause j, given decrement at time t, by

$$f_{J|T}(j|t) = \frac{f_{T,J}(t,j)}{f_T(t)} = \frac{{}_tp_x^{(\tau)}\mu^{(j)}(x+t)}{{}_tp_x^{(\tau)}\mu^{(\tau)}(x+t)} = \frac{\mu^{(j)}(x+t)}{\mu^{(\tau)}(x+t)}$$

and the conditional density function of T, given that the cause of decrement is j, by

$$\frac{f_{T,J}(t,j)}{f_J(j)}.$$

Example 64.2
For the double-decrement of the previous example, calculate $f_{J|T}(1|t)$.

Solution.
We have

$$f_{J|T}(1|t) = \frac{\mu^{(1)}(x+t)}{\mu^{(\tau)}(x+t)} = \frac{\frac{2}{50-t}}{\frac{2}{50-t}+0.02} = \frac{1}{1.5-0.01t} \blacksquare$$

Example 64.3
In a triple decrement model you were told that

$$\mu^{(j)}(x+t) = \frac{j}{150},\ t \geq 0, j = 1,2,3.$$

Calculate $E[T|J=3]$.

Solution.
We have

$$f_{T|J}(t|3) = \frac{f_{T,J}(t,3)}{f_J(3)}$$

where

$$f_{T,J}(t,3) = e^{-\int_0^t \frac{6}{150}ds}\left(\frac{3}{150}\right) = \frac{1}{50}e^{-0.04t}$$

and

$$f_J(3) = \int_0^\infty f_{T,J}(t,3)dt = 0.5.$$

Thus,

$$f_{T|J}(t|3) = 0.04e^{-0.04t}.$$

Finally,

$$E[T|J=3] = \int_0^\infty tf_{T|J}(t|3)dt = \int_0^\infty 0.04te^{-0.04t}dt = 25 \blacksquare$$

Example 64.4
Find an expression for the deferred probability ${}_{t|s}q_x^{(j)}$.

Solution.
We have

$$
\begin{aligned}
{}_{t|s}q_x^{(j)} &= \int_t^{t+s} f_{T,J}(r,j)dr = \int_t^{t+s} {}_rp_x^{(\tau)}\mu^{(j)}(x+r)dr \\
&= \int_t^{t+s} {}_tp_x^{(\tau)}{}_{r-t}p_{x+t}^{(\tau)}\mu^{(j)}(x+r)dr = {}_tp_x^{(\tau)} \int_t^{t+s} {}_{r-t}p_{x+t}^{(\tau)}\mu^{(j)}(x+r)dr \\
&= {}_tp_x^{(\tau)} \int_0^s {}_rp_{x+t}^{(\tau)}\mu^{(j)}(x+t+r)dr \\
&= {}_tp_x^{(\tau)}{}_sq_{x+t}^{(j)}.
\end{aligned}
$$

In particular,

$$
{}_{t|}q_x^{(j)} = {}_tp_x^{(\tau)}q_{x+t}^{(j)} \blacksquare
$$

Practice Problems

Problem 64.1
Consider a double-decrement model with $\mu^{(1)}(x+t) = 0.055$ and $\mu^{(2)}(x+t) = 0.005$. Caculate ${}_tp_x^{(\tau)}$.

Problem 64.2
A multiple decrement model with two causes of decrement has forces of decrement given by

$$\mu^{(1)}(x+t) = \frac{1}{100-(x+t)},\ \mu^{(2)}(x+t) = \frac{2}{100-(x+t)},\ 0 \le t \le 100-x.$$

If $x = 50$, obtain expressions (in terms of t) for
(a) $f_{T,J}(t,j)$.
(b) $f_T(t)$.
(c) $f_J(j)$.
(d $f_{J|T}(j|t)$.

Problem 64.3
We consider a three-decrement model: (1) death, (2) disability and (3) withdrawal. Given that
(i) $\mu^{(1)}(x+t) = 0.15\mu^{(\tau)}(x+t),\ 0 \le t \le 20.$
(ii) $\mu^{(2)}(x+t) = 0.25\mu^{(\tau)}(x+t),\ 0 \le t \le 20.$
(iii) ${}_{15}q_x^{(2)} = 0.05.$
Calculate ${}_{15}q_x^{(3)}$.

Problem 64.4
In a triple-decrement model, you are given $\mu^{(j)}(x+t) = 0.015j$ for $j = 1,2,3$. Calculate $E(T)$.

Problem 64.5
A double-decrement model has forces of decrement given by
(i) $\mu^{(1)}(x+t) = 0.05,\ t \ge 0,$
(ii)$\mu^{(2)}(x+t) = \frac{1}{50-t},\ 0 \le t < 50.$
(a) Calculate $f_{T,J}(30,2)$.
(b) Calculate the conditional probability that the decrement is due to cause 2, given decrement at time 10.

Problem 64.6 ‡
For a triple decrement table, you are given:
(i) $\mu^{(1)}(x+t) = 0.3,\ t > 0$
(ii) $\mu^{(2)}(x+t) = 0.5,\ t > 0$
(iii) $\mu^{(3)}(x+t) = 0.7,\ t > 0$
Calculate $q_x^{(2)}$.

Problem 64.7 ‡
(50) is an employee of XYZ Corporation. Future employment with XYZ follows a double decrement model:
(i) Decrement 1 is retirement.
(ii)

$$\mu^{(1)}(50+t) = \begin{cases} 0.00 & 0 \le t < 5 \\ 0.02 & 5 \le t. \end{cases}$$

(iii) Decrement 2 is leaving employment with XYZ for all other causes.
(iv)

$$\mu^{(2)}(50+t) = \begin{cases} 0.05 & 0 \le t < 5 \\ 0.03 & 5 \le t. \end{cases}$$

(v) If (50) leaves employment with XYZ, he will never rejoin XYZ.
Calculate the probability that (50) will retire from XYZ before age 60.

65 Associated Single Decrement Models

For each force of decrement $\mu^{(j)}(x+t)$ in a multiple-decrement model, we define a single-decrement model where survivorship depends only on the cause j without competing with other causes. Thus, $\mu^{(j)}(x+t)$ is the force of mortality of the model. This associated model is called the **associated single decrement model** or the **associated single decrement table.**
In the associated single decrement model we define the functions

$$\begin{aligned} {}_tp_x'^{(j)} =& e^{-\int_0^t \mu^{(j)}(x+s)ds} \\ {}_tq_x'^{(j)} =& 1 - {}_tp_x'^{(j)}. \end{aligned}$$

We call ${}_tq_x'^{(j)}$ the **net probability of decrement, the independent rate of decrement, or the absolute rate of decrement.** The symbol ${}_tq_x'^{(j)}$ is the net probability of decrement due to cause j, without competing with other causes. Thus, the probability represented by $q_x'^{(j)}$ is the same as the one represented by the simpler q_x in the earlier chapters. For this reason, we refer to $q_x'^{(j)}$ as the probability of decrement due to cause j in the associated single-decrement table. In comparison, ${}_tq_x^{(j)}$ is the probability of decrement due to cause j while competing with all other causes.
Now, note the following

$$\begin{aligned} {}_tp_x^{(\tau)} =& e^{-\int_0^t [\mu^{(1)}(x+s)+\mu^{(2)}(x+s)+\cdots+\mu^{(m)}(x+s)]ds} \\ =& e^{-\int_0^t \mu^{(1)}(x+s)ds} e^{-\int_0^t \mu^{(2)}(x+s)ds} \cdots e^{-\int_0^t \mu^{(m)}(x+s)ds} = \prod_{j=1}^m {}_tp_x'^{(j)}. \end{aligned}$$

Thus, for $1 \le j \le m$ we have

$$\frac{{}_tp_x^{(\tau)}}{{}_tp_x'^{(j)}} = e^{-\int_0^t \mu^{(1)}(x+s)ds} \cdots e^{-\int_0^t \mu^{(j-1)}(x+s)ds} e^{-\int_0^t \mu^{(j+1)}(x+s)ds} \cdots e^{-\int_0^t \mu^{(m)}(x+s)ds} \le 1.$$

Hence,

$${}_tp_x'^{(j)} \ge {}_tp_x^{(\tau)}.$$

Now, this last inequality implies

$${}_tp_x'^{(j)} \mu^{(j)}(x+t) \ge {}_tp_x^{(\tau)} \mu^{(j)}(x+t)$$

which upon integration gives

$$q_x'^{(j)} = \int_0^1 {}_tp_x'^{(j)} \mu^{(j)}(x+t)dt \ge \int_0^1 {}_tp_x^{(\tau)} \mu^{(j)}(x+t)dt = q_x^{(j)}.$$

Example 65.1
In a double-increment model you are given the following information:
(i) ${}_tp_x'^{(1)} = \frac{1}{2^t}$. $t \geq 0$.
(ii) ${}_tp_x'^{(2)} = \frac{1}{3^t}$. $t \geq 0$.
Calculate $q_x^{(1)}$.

Solution.
We have

$$\mu^{(1)}(x+t) = -\frac{\frac{d}{dt}\left({}_tp_x'^{(1)}\right)}{{}_tp_x'^{(1)}} = -\frac{-2^{-t}\ln 2}{2^{-t}} = \ln 2$$

and

$${}_tp_x^{(\tau)} = e^{-\int 0^t[\ln 2 + \ln 3]ds} = e^{-t\ln 6}.$$

Thus,

$$q_x^{(1)} = \int_0^1 {}_tp_x^{(\tau)}\mu^{(1)}(x+t)dt = \int_0^1 e^{-t\ln 6}\ln 2 dt = \frac{\ln 2}{\ln 6}[1 - e^{-\ln 6}] = 0.3224 \blacksquare$$

Example 65.2
In a double-decrement model, express $q_x^{(\tau)}$ in terms of $q_x'^{(1)}$ and $q_x'^{(2)}$.

Solution.
We have

$$\begin{aligned} q_x^{(\tau)} =& 1 - p_x^{(\tau)} = 1 - p_x'^{(1)}p_x'^{(2)} \\ =& 1 - (1 - q_x'^{(1)})(1 - q_x'^{(2)}) = q_x'^{(1)} + q_x'^{(2)} - q_x'^{(1)}q_x'^{(2)} \blacksquare \end{aligned}$$

Example 65.3
For a double-decrement table where cause 1 is death and cause 2 is withdrawal, you are given:
(i) Deaths are uniformly distributed over each year of age in the single-decrement table.
(ii) Withdrawals occur only at the beginning of each year of age.
(iii) $\ell_x^{(\tau)} = 1000$
(iv) $q_x^{(2)} = 0.25$
(v) $d_x^{(1)} = 0.04d_x^{(2)}$.
Calculate $q_x'^{(1)}$.

Solution.
Since withdrawals occur at the beginning of the year, the probability of deaths during the year will not affect the probability of withdrawal. That is, $p_x'^{(2)} = p_x^{(2)}$. Now, we have

$$q_x^{(1)} = 0.04q_x^{(2)} = 0.04(0.25) = 0.01$$

and

$$p_x^{(\tau)} = 1 - q_x^{(1)} - q_x^{(2)} = 1 - 0.01 - 0.25 = 0.74.$$

Thus,

$$p_x'^{(1)} = \frac{p_x^{(\tau)}}{p_x'^{(2)}} = \frac{p_x^{(\tau)}}{p_x^{(2)}} = \frac{0.74}{0.75} = \frac{74}{75}$$

and

$$q_x'^{(1)} = 1 - \frac{74}{75} = 0.01333 \blacksquare$$

Example 65.4 ‡
For a double-decrement model:
(i) ${}_tp_{40}^{(1)} = 1 - \frac{t}{60},\ 0 \le t \le 60$
(ii) ${}_tp_{40}^{(2)} = 1 - \frac{t}{40},\ 0 \le t \le 40$
Calculate $\mu^{(\tau)}(40 + t)$.

Solution.
We have

$${}_tp_{40}^{(\tau)} = {}_tp_{40}^{(1)}\,{}_tp_{40}^{(2)} = \left(1 - \frac{t}{60}\right)\left(1 - \frac{t}{40}\right)$$

and

$$\frac{d}{dt}[{}_tp_{40}^{(\tau)}] = -\frac{1}{60}\left(1 - \frac{t}{40}\right) - \frac{1}{40}\left(1 - \frac{t}{60}\right) = -\frac{100}{2400} + \frac{t}{1200}.$$

Hence,

$$\left.\frac{d}{dt}[{}_tp_{40}^{(\tau)}]\right|_{t=20} = -\frac{100}{2400} + \frac{20}{1200} = -0.025.$$

Finally, we have

$$\mu^{(\tau)}(40 + t) = -\frac{\left.\frac{d}{dt}[{}_tp_{40}^{(\tau)}]\right|_{t=20}}{{}_{20}p_{40}^{(\tau)}} = \frac{0.025}{(1/3)} = 0.075 \blacksquare$$

Practice Problems

Problem 65.1
For a double-decrement model, you are given:
(i) $q_x'^{(2)} = 2q_x'^{(1)}$.
(ii) $q_x'^{(1)} + q_x'^{(2)} = q_x^{(\tau)} + 0.18$.
Calculate $q_x'^{(2)}$.

Problem 65.2
In a double-decrement model, you are given:
(i) $2q_{50}'^{(2)} = 3q_{50}'^{(1)}$.
(ii) $p_{50}'^{(1)} + p_{50}'^{(2)} = p_{50}^{(\tau)} + 0.75$.
Calculate the absolute rate of decrement due to cause 1 for age 50.

Problem 65.3
We consider a three-decrement model: (1) death, (2) disability and (3) withdrawal. Given that
(i) $\mu^{(1)}(x+t) = 0.15\mu^{(\tau)}(x+t),\ 0 \le t \le 20$.
(ii) $\mu^{(2)}(x+t) = 0.25\mu^{(\tau)}(x+t),\ 0 \le t \le 20$.
(iii) ${}_{15}q_x^{(2)} = 0.05$.
Calculate ${}_{15}q_x'^{(3)}$.

Problem 65.4
Peter has an old truck, age x, that faces two forces of decrement (as far as Peter is concerned). They are (1) breakdown and (2) sale. Both forces of decrement are constants: $\mu^{(1)}(x+t) = 0.02$ and $\mu^{(2)}(x+t) = 0.03$. Calculate $q_x'^{(1)}$.

Problem 65.5 ‡
Don (50) is an actuarial science professor. His career is subject to two decrements.
(i) Decrement 1 is mortality. The associated single decrement table follows De Moivre's law with $\omega = 100$.
(ii) Decrement 2 is leaving academic employment with $\mu^{(2)}(50+t) = 0.05$.
Calculate the probability that Don remains an actuarial science professor for at least 5 but less than 10 years.

Problem 65.6 ‡
For a double decrement table, you are given:
(i) $\mu^{(1)}(x+t) = 0.2\mu^{(\tau)}(x+t)$
(i) $\mu^{(\tau)}(x+t) = kt^2$
(iii) $q_x'^{(1)} = 0.04$
Calculate ${}_2q_x^{(2)}$.

Problem 65.7 ‡
For a double-decrement table where cause 1 is death and cause 2 is withdrawal, you are given:
(i) Deaths are uniformly distributed over each year of age in the single-decrement table.
(ii) Withdrawals occur only at the end of each year of age.
(iii) $\ell_x^{(\tau)} = 1000$
(iii) $q_x^{(2)} = 0.40$
(iv) $d_x^{(1)} = 0.45d_x^{(2)}$.
Calculate $p_x'^{(2)}$.

66 Discrete Multiple-Decrement Models

In this section we illustrate the discrete multiple-decrement model and introduce new notational concepts.
Consider a group of $\ell_a^{(\tau)}$ lives, each aged a, subject to m causes of decrements. We assume that each member of the group has a joint pdf for time until decrement and cause of decrement given by

$$f_{T,J}(t,j) = {}_tp_a^{(\tau)}\mu^{(i)}(a+t),\ t \geq 0,\ j = 1,2,\cdots,m.$$

Let $\mathscr{L}^{(\tau)}(x)$ be the random variable representing the number of remaining survivors at age $x \geq a$ (out of the $\ell_a^{(\tau)}$ lives.) Then the expected number of individuals remaining in the group at age x is defined by

$$\begin{aligned}
\ell_x^{(\tau)} =& E[\mathscr{L}^{(\tau)}(x)]\\
=&\ell_a^{(\tau)} \times \text{probability that life (a) is still in the group at age x}\\
=&\ell_a^{(\tau)}{}_{x-a}p_a^{(\tau)}.
\end{aligned}$$

Suppose that the $\ell_x^{(\tau)}$ survivors to age x will, at future ages, be fully depleted by the m causes of decrement. Then the group of $\ell_x^{(\tau)}$ survivors can be visualized as consisting of distinct subgroups $\ell_x^{(j)},\ j = 1,2,\cdots,m$, where $\ell_x^{(j)}$ refer to the people in the group who eventually decrement due to cause j. Clearly,

$$\ell_x^{(\tau)} = \sum_{j=1}^m \ell_x^{(j)}.$$

Now, let ${}_n\mathscr{D}_x^{(j)}$ denote the number of individuals exiting the group between ages x and $x+n$ from among the initial $\ell_x^{(\tau)}$ lives due to cause j. Then the expected number of individuals exiting the group between ages x and $x+n$ due to cause j is defined by

$$\begin{aligned}
{}_nd_x^{(j)} =& E[{}_n\mathscr{D}_x^{(j)}]\\
=&\ell_a^{(\tau)} \times \text{probability that an individual exit the group in (x, x + n) due to cause j}\\
=&\ell_a^{(\tau)}\int_{x-a}^{x-a+n} {}_tp_a^{(\tau)}\mu^{(j)}(t+a)dt.
\end{aligned}$$

That is, ${}_nd_x^{(j)}$ is the expected number of individuals exiting the group in the interval $(x, x+n)$ due to cause j.

Next, letting

$$ {}_n\mathscr{D}_x^{(\tau)} = \sum_{j=1}^{m} {}_n\mathscr{D}_x^{(j)} $$

we define

$$ {}_nd_x^{(\tau)} = E[{}_n\mathscr{D}_x^{(\tau)}] = \sum_{j=1}^{m} {}_nd_x^{(j)} = \ell_a^{(\tau)} \int_{x-a}^{x-a+n} {}_tp_a^{(\tau)} \mu^{(\tau)}(t+a)dt. $$

The notation $d_x^{(j)}$ represents the expected number of lives exiting from the population between ages x and $x+1$ due to decrement j. Clearly,

$$ d_x^{(j)} = \ell_x^{(j)} - \ell_{x+1}^{(j)}. $$

Dividing this last formula by $\ell_x^{(\tau)}$ we find

$$ q_x^{(j)} = \frac{d_x^{(j)}}{\ell_x^{(\tau)}} $$

which is the probability that a life (x) will leave the group within one year as a result of decrement j. Also, note that

$$ {}_nd_x^{(j)} = \sum_{t=0}^{n-1} d_{x+t}^{(j)}. $$

The total expected number of exits due to all decrements between the ages of x and $x+1$ is denoted by $d_x^{(\tau)}$. Clearly,

$$ d_x^{(\tau)} = \sum_{j=1}^{m} d_x^{(j)} $$

Note that

$$ d_x^{(\tau)} = \ell_x^{(\tau)} - \ell_{x+1}^{(\tau)} = \ell_x^{(\tau)} q_x^{(\tau)}. $$

From this, we see that the probability that (x) will leave the group within one year (regardless of decrement) is

$$ q_x^{(\tau)} = \frac{d_x^{(\tau)}}{\ell_x^{(\tau)}} = \sum_{j=1}^{m} \frac{d_x^{(j)}}{\ell_x^{(\tau)}} = \sum_{j=1}^{m} q_x^{(j)}. $$

The probability that (x) will remain in the group for at least one year is

$$p_x^{(\tau)} = 1 - q_x^{(\tau)} = \frac{\ell_{x+1}^{(\tau)}}{\ell_x^{(\tau)}} = \frac{\ell_x^{(\tau)} - d_x^{(\tau)}}{\ell_x^{(\tau)}}.$$

The probability of (x) remaining in the group after n years is

$${}_np_x^{(\tau)} = \frac{\ell_{x+n}^{(\tau)}}{\ell_x^{(\tau)}} = p_x^{(\tau)} p_{x+1}^{(\tau)} \cdots p_{x+n-1}^{(\tau)}.$$

The probability that (x) will leave the group within n years (regardless of decrement) is

$${}_nq_x^{(\tau)} = 1 - {}_np_x^{(\tau)} = \sum_{j=1}^{m} {}_nq_x^{(j)}$$

where

$${}_nq_x^{(j)} = \frac{{}_nd_x^{(j)}}{\ell_x^{(\tau)}}$$

is the probability of failure due to decrement j on the interval $(x, x+n]$.
The probability that (x) will leave the group between ages $x+n$ and $x+n+t$ due to decrement j is

$${}_{n|t}q_x^{(j)} = {}_np_x^{(\tau)} {}_tq_{x+n}^{(j)} = \frac{\ell_{x+n}^{(\tau)}}{\ell_x^{(\tau)}} \times \frac{{}_td_{x+n}^{(j)}}{\ell_{x+n}^{(\tau)}} = \frac{{}_td_{x+n}^{(j)}}{\ell_x^{(\tau)}}.$$

The probability that (x) will leave the group between ages $x+n$ and $x+n+1$ regardless to decrement is

$${}_{n|}q_x^{(j)} = {}_np_x^{(\tau)} q_{x+n}^{(j)} = \frac{d_{x+n}^{(j)}}{\ell_x^{(\tau)}}.$$

An important practical problem is that of constructing a **multiple decrement life table.** We illustrate the construction of such a table in the following example.

Example 66.1
Complete the following three-decrement table:

x	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$	$d_x^{(\tau)}$	$\ell_x^{(\tau)}$	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(3)}$	$q_x^{(\tau)}$	$p_x^{(\tau)}$
50	5,168	1,157	4,293		4,832,555					
51	5,363	1,206	5,162							
52	5,618	1,443	5,960							
53	5,929	1,679	6,840							
54	6,277	2,152	7,631							

Solution.
We use the following formulas:

$$\begin{aligned}d_x^{(\tau)} =&d_x^{(1)} + d_x^{(2)} + d_x^{(3)}\\ \ell_{x+1}^{(\tau)} =&\ell_x^{(\tau)} - d_x^{(\tau)}\\ q_x^{(j)} =&\frac{d_x^{(j)}}{\ell_x^{(\tau)}}\\ p_x^{(\tau)} =&1 - q_x^{(\tau)}.\end{aligned}$$

We obtain

x	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$	$d_x^{(\tau)}$	$\ell_x^{(\tau)}$	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(3)}$	$q_x^{(\tau)}$	$p_x^{(\tau)}$
50	5,168	1,157	4,293	10,618	4,832,555	0.00107	0.00024	0.00089	0.00220	0.99780
51	5,363	1,206	5,162	11,731	4,821,937	0.0011	0.00025	0.00107	0.00243	0.99757
52	5,618	1,443	5,960	13,021	4,810,206	0.00117	0.00030	0.00124	0.00271	0.99729
53	5,929	1,679	6,840	14,448	4,797,185	0.00124	0.00035	0.00143	0.00302	0.99698
54	6,277	2,152	7,631	16,060	4,782,737	0.00131	0.00045	0.00160	0.00336	0.99664

Example 66.2
Using the previously given multiple decrement table, compute and interpret the following:
(a) ${}_2d_{51}^{(3)}$ (b) ${}_3p_{50}^{(\tau)}$ (c) ${}_2q_{53}^{(1)}$ (d) ${}_{2|2}q_{50}^{(2)}$ (e) ${}_{2|}q_{51}^{(\tau)}$.

Solution.
(a) ${}_2d_{51}^{(3)}$ is the expected number of 51 year-old exiting the group within the next two years due to decrement 3. We have

$${}_2d_{51}^{(3)} = d_{51}^{(3)} + d_{52}^{(3)} = 5,162 + 5,960 = 11,122.$$

(b) ${}_3p_{50}^{(\tau)}$ is the probability that (50) will survive another 3 years. We have

$${}_3p_{50}^{(\tau)} = \frac{\ell_{53}^{(\tau)}}{\ell_{50}^{(\tau)}} = \frac{4,797,185}{4,832,555} = 0.99268.$$

(c) ${}_2q_{53}^{(1)}$ is the probability that (53) will exit the group whithin two years due to decrement 1. We have

$$ {}_2q_{53}^{(1)} = \frac{d_{53}^{(1)} + d_{54}^{(1)}}{\ell_{53}^{(\tau)}} = \frac{5929 + 6277}{4,797,185} = 0.00254. $$

(d) ${}_{2|2}q_{50}^{(2)}$ is the probability of (50) surviving the next two years and exiting within the following two years due to decrement 2. We have

$$ {}_{2|2}q_{50}^{(2)} = \frac{d_{52}^{(2)} + d_{53}^{(2)}}{\ell_{50}^{(\tau)}} = \frac{1443 + 1679}{4,832,555} = 0.00065. $$

(e) ${}_{2|}q_{51}^{(\tau)}$ is the probability of (52) exiting the group between the ages 53 and 54. We have

$$ {}_{2|}q_{51}^{(\tau)} = \frac{d_{53}^{(1)} + d_{53}^{(2)} + d_{53}^{(3)}}{\ell_{51}^{(\tau)}} = \frac{14448}{4,821,937} = 0.003 \blacksquare $$

Stochastic Analysis of the Model

The discrete multiple-decrement model involves two discrete random variables: The curtate future lifetime $K(x)$ and the random variable $J(x)$ which represents the cause of failure. For example, suppose that the cause of a death is from a heart disease with decrement 1, from an accident with decrement 2, or from other causes with decrement 3. Then $J(x) = 1$ corresponds to decrement by a heart disease, $J(x) = 2$ corresponds to a decrement from an accident, and $J(x) = 3$ corresponds to a decrement by other causes.

Example 66.3

Interpret the meaning of the event $\{K(x) = k, J(x) = j\}$.

Solution.

The event $\{K(x) = k, J(x) = j\}$ denotes the joint event of (x) failing in the interval $(x + k, x + k + 1)$ due to the j^{th} cause. The probability of this joint event will be denoted by ${}_{k|}q_x^{(j)}$ $\blacksquare$

The **joint probability function** of $K(x)$ and $J(x)$ is

$$ p_{K(x),J(x)}(k, j) = \Pr(K(x) = k, J(x) = j) = {}_{k|}q_x^{(j)} = \frac{d_{x+k}^{(j)}}{\ell_x^{(\tau)}} $$

giving the probability that (x) survives all decrements for k years and then fails due to cause j in the $(k+1)^{\mathrm{st}}$ year.

The **marginal probability function** of $K(x)$ is

$$\begin{aligned} p_{K(x)}(k) =& \Pr(K(x)=k) = \sum_{j=1}^{m} p_{K(x),J(x)}(k,j) \\ =& {}_{k|}q_x^{(\tau)} \\ =& \frac{d_{x+k}^{(1)} + d_{x+k}^{(2)} + \cdots + d_{x+k}^{(m)}}{\ell_x^{(\tau)}} \end{aligned}$$

giving the probability that (x) will fail in the $(k+1)^{\mathrm{st}}$ year due to any cause.

The **marginal probability function** of $J(x)$ is

$$p_{J(x)}(j) = \Pr(J(x)=j) = \sum_{k=1}^{\infty} p_{K(x),J(x)}(k,j) = \sum_{k=1}^{\infty} \frac{d_{x+k}^{(j)}}{\ell_x^{(\tau)}}$$

giving the probability that (x) will eventually fail due to cause j without restrictions as to time of failure.

Practice Problems

Problem 66.1
In a double-decrement table you are given

x	$q_x^{(1)}$	$q_x^{(2)}$	$\ell_x^{(\tau)}$
25	0.01	0.15	–
26	0.01	0.10	8400

Calculate the effect on $d_{26}^{(1)}$ if $q_{25}^{(2)}$ changes from 0.15 to 0.25.

Problem 66.2
From a double-decrement table, you are given:
(i) $\ell_{63}^{(\tau)} = 500$
(ii) $q_{63}^{(1)} = 0.050$
(iii) $q_{63}^{(2)} = 0.500$
(iv) ${}_{1|}q_{63}^{(1)} = 0.070$
(v) ${}_{2|}q_{63}^{(1)} = 0.042$
(vi) ${}_{2}q_{63}^{(2)} = 0.600$
(vii) $\ell_{66}^{(\tau)} = 0.$
Calculate $d_{65}^{(2)}$.

Problem 66.3
You are given the following information for a triple-decrement table:
(i) $\ell_x^{(\tau)} = 100,000$
(ii) $\ell_{x+1}^{(\tau)} = 90,000$
(iii) $q_x^{(1)} = 0.02$
(iv) $d_x^{(3)} = 0.6 d_x^{(2)}$.
Calculate $q_x^{(2)}$.

Problem 66.4

x	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(\tau)}$	$\ell_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
30	___	___	0.075	___	___	130
31	0.020	0.05	___	1850	___	___
32	___	___	___	___	54	___

Calculate ${}_3q_{30}^{(1)}$.

Problem 66.5
You are given the following portion of a double-decrement table.

x	$\ell_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
50	___	100	300
51	700	50	___
52	470	40	___
53	320	___	___

Find the probability that (50) will fail first from cause 2 between the ages of 51 and 53.

Problem 66.6 ‡
For a multiple decrement model on (60):
(i) $\mu(60+t)^{(1)},\ t \geq 0$ follows the Illustrative Life Table.
(ii) $\mu(60+t)^{(\tau)} = 2\mu(60+t)^{(1)},\ t \geq 0$
Calcualte ${}_{10|}q_{60}$, the probability that decrement occurs during the 11$^{\text{th}}$ year.

Problem 66.7 ‡
For students entering a college, you are given the following from a multiple decrement model:
(i) 1000 students enter the college at $t = 0$.
(ii) Students leave the college for failure (1) or all other reasons (2).
(iii) $\mu(x+t)^{(1)} = \mu,\ 0 \leq t < 4$.
(iv) $\mu(x+t)^{(2)} = 0.040,\ 0 \leq t < 4$.
(iv) 48 students are expected to leave the college during their first year due to all causes.
Calculate the expected number of students who will leave because of failure during their fourth year.

Problem 66.8 ‡
For a double decrement table with $\ell_{40}^{(\tau)} = 2000$:

x	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{\prime(1)}$	$q_x^{\prime(2)}$
40	0.24	0.10	0.25	y
41	___	___	0.20	$2y$

Calculate $\ell_{42}^{(\tau)}$.

Problem 66.9 ‡
For students entering a three-year law school, you are given:
(i) The following double decrement table:

	For a student at the beginning of that academic year, Probability of		
Academic Year	Academic Failure	Withdrawal for All Other Reasons	Survival Through Academic Year
1	0.40	0.20	—
2	—	0.30	—
3	—	—	0.60

(ii) Ten times as many students survive year 2 as fail during year 3.
(iii) The number of students who fail during year 2 is 40% of the number of students who survive year 2.
Calculate the probability that a student entering the school will withdraw for reasons other than academic failure before graduation.

67 Uniform Distribution of Decrements

With a given multiple-decrement table, one can calculate ${}_tq_x^{(j)}$ and ${}_tq_x^{(\tau)}$ for integral values of t. In this section, we consider the question of nonintegral t. More importantly, we want to establish a relationship between ${}_tq_x^{\prime(j)}$ and ${}_tq_x^{(j)}$.

As in the case of life table of Section 22, some assumptions are needed. One of them is the assumption of uniform distribution of decrements. That is, each decrement is observed to occur uniformly throughout the year of age when other decrements are also present. In a way similar to Section 24.1, this assumption means

$$ {}_tq_x^{(j)} = tq_x^{(j)}, 0 < t \le 1. $$

That is, ${}_tq_x^{(j)}$ is a linear function of t. From this, we can write

$$ tq_x^{(j)} = {}_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)}\mu^{(j)}(x+s)ds. $$

Differentiating both sides with respect to t we obtain

$$ q_x^{(j)} = {}_tp_x^{(\tau)}\mu^{(j)}(x+t) $$

which implies

$$ \mu^{(j)}(x+t) = \frac{q_x^{(j)}}{{}_tp_x^{(\tau)}} = \frac{q_x^{(j)}}{1-tq_x^{(\tau)}}. $$

We also have,

$$ \begin{aligned} {}_tp_x^{\prime(j)} =& e^{-\int_0^t \mu^{(j)}(x+s)ds} \\ =& e^{-q_x^{(j)}\int_0^t \frac{ds}{1-sq_x^{(\tau)}}} = e^{\frac{q_x^{(j)}}{q_x^{(\tau)}}\left[\ln\left(1-sq_x^{(\tau)}\right)\right]_0^t} \\ =& (1-tq_x^{(\tau)})^{\frac{q_x^{(j)}}{q_x^{(\tau)}}}. \end{aligned} $$

This result allows us to compute the absolute rates of decrements $q_x^{\prime(j)}$ given the probabilities of decrements in the multiple decrement model. In particular, when $t=1$, we have

$$ q_x^{\prime(j)} = 1 - (1-q_x^{(\tau)})^{\frac{q_x^{(j)}}{q_x^{(\tau)}}}. $$

Example 67.1
In a double decrement table where cause 1 is death and cause 2 is withdrawal, you are given:
(i) Both deaths and withdrawals are each uniformly distributed over each year of age in the double decrement table.
(ii) $\ell_x^{(\tau)} = 1000$
(iii) $q_x^{(2)} = 0.48$
(iv) $d_x^{(1)} = 0.35 d_x^{(2)}$.
Calculate $q_x'^{(1)}$ and $q_x'^{(2)}$.

Solution.
We have

$$
\begin{aligned}
q_x^{(1)} =& \frac{d_x^{(1)}}{\ell_x^{(\tau)}} = 0.35 q_x^{(2)} = 0.168 \\
q_x^{(\tau)} =& q_x^{(1)} + q_x^{(2)} = 0.168 + 0.48 = 0.648 \\
q_x'^{(1)} =& 1 - (1 - 0.648)^{\frac{0.168}{0.648}} = 0.23715 \\
q_x'^{(2)} =& 1 - (1 - 0.648)^{\frac{0.48}{0.648}} = 0.53857 \blacksquare
\end{aligned}
$$

Next, we explore a more common assumption than the one made earlier in this section. The assumption made above assume that each individual decrement is uniformly distributed in the multiple-decrement context. Now, we make the assumption that the individual decrements are uniformly distributed in the associated single-decrement context. That is,

$$
{}_tq_x'^{(j)} = t q_x'^{(j)}, \ 0 < t \leq 1.
$$

Taking the derivative of both sides with respect to t we find

$$
q_x'^{(j)} = {}_tp_x'^{(j)} \mu^{(j)}(x+t).
$$

Now, using the result

$$
{}_tp_x^{(\tau)} = \prod_{j=1}^{m} {}_tp_x'^{(j)}
$$

we have

$$\begin{aligned}
{}_tq_x^{(j)} =& \int_0^t {}_sp_x^{(\tau)}\mu^{(j)}(x+s)ds \\
=& \int_0^t \prod_{i\neq j} {}_sp_x'^{(i)} {}_sp_x'^{(j)}\mu^{(j)}(x+s)ds \\
=& q_x'^{(j)}\int_0^t \prod_{i\neq j} {}_sp_x'^{(i)}ds \\
=& q_x'^{(j)}\int_0^t \prod_{i\neq j}(1-sq_x'^{(i)})ds.
\end{aligned}$$

Thus, by evaluating the integral, we can find the probabilities of decrement given the absolute rates of decrements. For example, when $m=2$ we have

$${}_tq_x^{(1)} = q_x'^{(1)}\int_0^t (1-sq_x'^{(2)})ds = tq_x'^{(1)}\left(1-\frac{t}{2}q_x'^{(2)}\right).$$

Likewise,

$${}_tq_x^{(2)} = q_x'^{(2)}\int_0^t (1-sq_x'^{(1)})ds = tq_x'^{(2)}\left(1-\frac{t}{2}q_x'^{(1)}\right).$$

Example 67.2
In a triple decrement table where each of the decrement in their associated single decrement tables satisfy the uniform distribution of decrement assumption, you are given:
(i) $q_x'^{(1)} = 0.03$ and $q_x'^{(2)} = 0.06$
(ii) $\ell_x^{(\tau)} = 1,000,000$ and $\ell_{x+1}^{(\tau)} = 902,682$.
Calculate $d_x^{(3)}$.

Solution.
We will use the formula

$$q_x^{(3)} = \frac{d_x^{(3)}}{\ell_x^{(\tau)}}$$

where

$$q_x^{(3)} = q_x'^{(3)}\int_0^1 (1-sq_x'^{(1)})(1-sq_x'^{(2)})ds = q_x'^{(3)}\int_0^1 (1-0.03s)(1-0.06s)ds = 0.9556q_x'^{(3)}.$$

But
$$\frac{\ell_{x+1}^{(\tau)}}{\ell_x^{(\tau)}} = p_x^{(\tau)} = p_x'^{(1)}p_x'^{(2)}p_x'^{(3)} = (1-q_x'^{(1)})(1-q_x'^{(2)})(1-q_x'^{(3)}).$$
Thus,
$$(1-0.03)(1-0.06)(1-q_x'^{(3)}) = \frac{902,682}{1,000,000} \Longrightarrow q_x'^{(3)} = 0.01 \Longrightarrow q_x^{(3)} = 0.009556.$$
Finally,
$$d_x^{(3)} = 1,000,000q_x^{(3)} = 9556 \blacksquare$$

Example 67.3 ‡
For a double decrement model:
(i) In the single decrement table associated with cause (1), ${q'_{40}}^{(1)} = 0.100$ and decrements are uniformly distributed over the year.
(ii) In the single decrement table associated with cause (2), ${q'_{40}}^{(2)} = 0.125$ and all decrements occur at time 0.7.
Calculate ${q_{40}}^{(2)}$.

Solution.
Let ℓ_0 denote the original survivorship group of age 40. In the time interval $[0, 0.7)$, only decrement (1) can reduce the original survivorship group. In that time interval, cause (1) eliminates
$$\ell_{00.7}{q'_{40}}^{(1)} = 0.7\ell_0{q'_{40}}^{(1)} = 0.07\ell_0.$$
Now, at exact time $t = 0.7$, decrement (2) eliminates
$$(\ell_0 - 0.07\ell_0){q'_{40}}^{(2)} = (0.93)(0.125)\ell_0 = 0.11625\ell_0.$$
Hence,
$${q_{40}}^{(2)} = \frac{0.11625\ell_0}{\ell_0} = 0.11625 \blacksquare$$

Example 67.4 ‡
A population of 1000 lives age 60 is subject to 3 decrements, death (1), disability (2), and retirement (3). You are given:
(i) The following absolute rates of decrement:

x	$q_x^{\prime(1)}$	$q_x^{\prime(2)}$	$q_x^{\prime(3)}$
60	0.010	0.030	0.100
61	0.013	0.050	0.200

(ii) Decrements are uniformly distributed over each year of age in the multiple decrement table.
Calculate the expected number of people who will retire before age 62.

Solution.
We are looking for $d_{60}^{(3)} + d_{61}^{(3)}$. We have

$$\begin{aligned}
\ell_{60}^{(\tau)} =&1000\\
p_{60}^{(\tau)} =&p_{60}^{\prime(1)} p_{60}^{\prime(2)} p_{60}^{\prime(3)} = (1-0.010)(1-0.030)(1-0.100) = 0.86427\\
\ell_{61}^{(\tau)} =&\ell_{60}^{(\tau)} p_{60}^{(\tau)} = 1000(0.86427) = 864.27\\
q_{60}^{(3)} =&q_{60}^{(\tau)} \frac{\ln p_{60}^{\prime(3)}}{\ln p_{60}^{(\tau)}} = (1-0.86427)\left(\frac{\ln 0.9}{\ln 0.86427}\right) = 0.09805\\
d_{60}^{(3)} =&\ell_{60}^{(\tau)} q_{60}^{(3)} = 1000(0.09805) = 98.05\\
p_{61}^{(\tau)} =&p_{61}^{\prime(1)} p_{61}^{\prime(2)} p_{61}^{\prime(3)} = (1-0.013)(1-0.050)(1-0.200) = 0.75012\\
q_{61}^{(3)} =&q_{61}^{(\tau)} \frac{\ln p_{61}^{\prime(3)}}{\ln p_{61}^{(\tau)}} = (1-0.75012)\left(\frac{\ln 0.8}{\ln 0.75012}\right) = 0.19393\\
d_{61}^{(3)} =&\ell_{61}^{(\tau)} q_{61}^{(3)} = 864.27(0.19393) = 167.60.
\end{aligned}$$

Hence,

$$d_{60}^{(3)} + d_{61}^{(3)} = 98.05 + 167.60 = 265.65 \blacksquare$$

Practice Problems

Problem 67.1
You are given the following double-decrement table:

x	$\ell_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(3)}$
60	1000	150	50
61	800	160	75
62	565	160	85

Calculate $q_{60}^{\prime(1)}$ assuming each decrement is uniformly distributed over each year of age in the double decrement table.

Problem 67.2
For a double-decrement table, you are given:
(i) $\ell_{60}^{(\tau)} = 5000$ and $\ell_{62}^{(\tau)} = 4050$
(ii) $d_{60}^{(1)} = 210; d_{60}^{(2)} = 235; d_{61}^{(2)} = 306$
(iii) Each decrement is uniformly distributed over each year of age.
Calculate $q_{61}^{\prime(1)}$.

Problem 67.3
In a triple-decrement table you are given:
(i) ${}_{0.8}q_x^{(1)} = 0.016$ and ${}_{0.2}q_x^{(2)} = 0.002$
(ii) $q_x^{(\tau)} = 0.05$.
Assuming that each decrement is uniformly distributed over each year of age, calculate $\mu^{(3)}(x + 0.5)$.

Problem 67.4
Given the following extract from a triple-decrement table:

z	$q_x^{\prime(1)}$	$q_x^{\prime(2)}$	$q_x^{\prime(3)}$
x	0.02	0.02	0.04
$x+1$	0.025	0.02	0.06

Assume that each decrement is uniformly distributed over each year of age, calculate ${}_{1|}q_x^{(1)}$.

Problem 67.5
A group of 500 lives in a suburban community, all age 50, is subject to 3 decrements: death (1), disability (2), and unemployment (3).
You are given:
(i) The following absolute rates of decrement:

x	$q_x'^{(1)}$	$q_x'^{(2)}$	$q_x'^{(3)}$
50	0.008	0.02	0.10
51	0.010	0.04	0.150

(ii) Decrements are uniformly distributed over each year of age in the multiple decrement table.
Calculate the expected number of people who become disabled before age 52.

Problem 67.6 ‡
For a double decrement table, you are given:
(i) $q_x'^{(1)} = 0.2$
(ii) $q_x'^{(2)} = 0.3$
(iii) Each decrement is uniformly distributed over each year of age in the double decrement table.
Calculate ${}_{0.3}q_{x+0.1}^{(1)}$.

Problem 67.7 ‡
You intend to hire 200 employees for a new management-training program. To predict the number who will complete the program, you build a multiple decrement table. You decide that the following associated single decrement assumptions are appropriate:
(i) Of 40 hires, the number who fail to make adequate progress in each of the first three years is 10, 6, and 8, respectively.
(ii) Of 30 hires, the number who resign from the company in each of the first three years is 6, 8, and 2, respectively.
(iii) Of 20 hires, the number who leave the program for other reasons in each of the first three years is 2, 2, and 4, respectively.
(iv) You use the uniform distribution of decrements assumption in each year in the multiple decrement table.
Calculate the expected number who fail to make adequate progress in the third year.

Problem 67.8 ‡
For a multiple decrement table, you are given:
(i) Decrement (1) is death, decrement (2) is disability, and decrement (3) is withdrawal.
(ii) $q_{60}'^{(1)} = 0.010$

(iii) $q_{60}^{\prime(2)} = 0.050$
(iv) $q_{60}^{\prime(3)} = 0.100$
(v) Withdrawals occur only at the end of the year.
(vi) Mortality and disability are uniformly distributed over each year of age in the associated single decrement tables.
Calculate $q_{60}^{(3)}$.

Problem 67.9 ‡
For a double decrement table, you are given:

Age	$\ell_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(1)}$
40	1000	60	55
41	___	___	70
42	750	___	___

Each decrement is uniformly distributed over each year of age in the double decrement table. Calculate $q_{41}^{\prime(1)}$.

Problem 67.10 ‡
For a triple decrement table, you are given:
(i) Each decrement is uniformly distributed over each year of age in its associated single decrement table.
(ii) $q_x^{\prime(1)} = 0.200$
(iii) $q_x^{\prime(2)} = 0.080$
(ii) $q_x^{\prime(3)} = 0.125$
Calculate $q_x^{(1)}$.

68 Valuation of Multiple Decrement Benefits

Multiple Decrement models provide a mathematical framework for the insurance plan with the amount of benefit payment depending on the cause and time of decrement.
Let $B_{x+t}^{(j)}$ denote the value of a benefit at time t if (x) departs due to cause j at time t (or at age $x+t$). The present value of the benefit is

$$Z_x^{(j)} = B_{x+t}^{(j)}\nu^T.$$

The APV of this benefit at age x is denoted as $\bar{A}_x^{(j)}$ and is given by

$$\bar{A}_x^{(j)} = \int_0^\infty B_{x+t}^{(j)}\nu^t {}_tp_x^{(\tau)}\mu^{(j)}(x+t)dt.$$

The APV of the total benefits, denoted as $\bar{A}$ (in a m multiple decrement model) is

$$\bar{A} = \sum_{j=1}^m \bar{A}_x^{(j)} = \sum_{j=1}^m \int_0^\infty B_{x+t}^{(j)}\nu^t {}_tp_x^{(\tau)}\mu^{(j)}(x+t)dt.$$

The corresponding formula in the discrete case is

$$A_x = \sum_{j=1}^m \sum_{k=0}^\infty B_{x+k}^{(j)}\nu^{k+1} {}_kp_x^{(\tau)} q_{x+k}^{(j)}.$$

Example 68.1
You are given the following portion of a double-decrement table:

x	$\ell_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
50	1200	100	300
51	__	200	__
52	300	__	__

A 2-year insurance contract on (50) provides for benefits paid at the end of year of death if this occur within 2 years. The benefit payable is one unit if death is from cause 1 and 2 units if death is from cause 2. Given $i = 50\%$, find the actuarial present value of the benefits.

Solution.
The APV of the benefits is

$$A_x = A_x^{(1)} + A_x^{(2)}$$

where

$$A_x^{(1)} = \nu q_{50}^{(1)} + 2\nu^2 p_{50}^{(\tau)} q_{51}^{(1)}$$

and

$$A_x^{(2)} = \nu q_{50}^{(2)} + 2\nu^2 p_{50}^{(\tau)} q_{51}^{(2)}$$

Now, we have

$$\begin{aligned}
\ell_{51}^{(\tau)} =& \ell_{50}^{(\tau)} - d_{50}^{(1)} - d_{50}^{(2)} = 1200 - 100 - 300 = 800\\
\ell_{52}^{(\tau)} =& \ell_{51}^{(\tau)} - d_{51}^{(1)} - d_{51}^{(2)}\\
300 =& 800 - 200 - d_{51}^{(2)}\\
d_{51}^{(2)} =& 300\\
q_{50}^{(1)} =& \frac{d_{50}^{(1)}}{\ell_{50}^{(\tau)}} = \frac{100}{1200} = \frac{1}{12}\\
q_{50}^{(2)} =& \frac{d_{50}^{(2)}}{\ell_{50}^{(\tau)}} = \frac{300}{1200} = \frac{1}{4}\\
p_{50}^{(\tau)} =& 1 - q_{50}^{(1)} - q_{50}^{(2)} = 1 - \frac{1}{12} - \frac{1}{4} = \frac{2}{3}\\
q_{51}^{(1)} =& \frac{d_{51}^{(1)}}{\ell_{51}^{(\tau)}} = \frac{200}{800} = \frac{1}{4}\\
q_{51}^{(2)} =& \frac{d_{51}^{(2)}}{\ell_{51}^{(\tau)}} = \frac{300}{800} = \frac{3}{8}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\text{APVB} =& A_x^{(1)} + A_x^{(2)} = \frac{1}{1.5}\left[\frac{1}{12} + 2 \times \frac{1}{4}\right] + \frac{1}{1.5^2}\left(\frac{2}{3}\right)\left[\frac{1}{4} + 2 \times \frac{3}{8}\right]\\
=& 0.6852 \blacksquare
\end{aligned}$$

Example 68.2
An insurance policy to (50) will pay \$4 upon death if death is accidental (ad)

and occurs within 25 years. An additional \$1 will be paid regardless of the time or cause of death.
The force of accidental death at all ages is 0.01, and the force of death for all other causes is is 0.05 at all ages.
If $\delta = 0.10$, find the actuarial present value of this policy.

Solution.
We have

$$\begin{aligned}\bar{A}_x^{(\tau)} + 4\bar{A}^{1}_{x:\overline{25|}}{}^{(ad)} =&\frac{\mu^{(\tau)}}{\mu^{(\tau)}+\delta} + 4\frac{\mu^{(ad)}}{\mu^{(\tau)}+\delta}(1-e^{-(\mu^{(\tau)}+\delta)t})\\ =&\frac{0.06}{0.16} + 4\frac{0.01}{0.16}(1-e^{-0.16(25)})\\ =&\frac{5}{8} - \frac{1}{4}e^{-4} \blacksquare\end{aligned}$$

Example 68.3 ‡
A special whole life insurance on (x) pays 10 times salary if the cause of death is an accident and 500,000 for all other causes of death.
You are given:
(i) $\mu(x+t)^{(\tau)} = 0.01,\ t \geq 0$
(ii) $\mu(x+t)^{(\text{Accident})} = 0.001, t \geq 0$
(iii) Benefits are payable at the moment of death.
(iv) $\delta = 0.05$
(v) Salary of (x) at time t is $50,000e^{0.04t}, t \geq 0$.
Calculate the actuarial present value of the benefits at issue.

Solution.
We have

$$\begin{aligned}\bar{A}_x =&\bar{A}_x^{\text{Accident}} + \bar{A}_x^{\text{Non-Accident}}\\ =&10\int_0^\infty 50,000e^{0.04t}e^{-0.05t}e^{-0.01t}(0.001)dt + \int_0^\infty 500,000e^{-0.05t}e^{-0.01t}(0.009)dt\\ =&500\int_0^\infty e^{-0.02t}dt + 4500\int_0^\infty e^{-0.06t}dt\\ =&\left[-25000e^{-0.02t} - 75000e^{-0.06t}\right]_0^\infty = 100,000 \blacksquare\end{aligned}$$

Example 68.4 ‡
For a special whole life insurance of 100,000 on (x), you are given:

(i) $\delta = 0.06$
(ii) The death benefit is payable at the moment of death.
(iii) If death occurs by accident during the first 30 years, the death benefit is doubled.
(iv) $\mu^{(\tau)}(x+t) = 0.008, t \geq 0$
(v) $\mu^{(1)}(x+t) = 0.001, t \geq 0$, where $\mu^{(1)}(x+t)$ is the force of decrement due to death by accident.
Calculate the single benefit premium for this insurance.

Solution.
The actuarial present value of benefits ia

$$\begin{aligned}
\text{APVB} =&100,000\int_0^\infty \nu^t {}_tp_x^{(\tau)}\mu^{(\tau)}(x+t)dt + 100,000\int_0^{30} \nu^t {}_tp_x^{(\tau)}\mu^{(1)}(x+t)dt\\
=&100,000\int_0^\infty e^{-0.06t}e^{-0.008t}(0.008)dt + 100,000\int_0^{30} e^{-0.06t}e^{-0.008t}(0.001)dt\\
=&11765 + 1279 = 13044.
\end{aligned}$$

The death benefit is 100,000 at the time of death, for any reason, at any time (the first integral) plus another 100,000 at the time of death, if death is by accident, during the first 30 years (the second integral) ■

Example 68.5 ‡
A fully discrete 3-year term insurance of 10,000 on (40) is based on a double-decrement model, death and withdrawal:
(i) Decrement 1 is death.
(ii) $\mu^{(1)}(40+t) = 0.02, t > 0$
(iii) Decrement 2 is withdrawal, which occurs at the end of the year.
(iv) $q_{40+k}^{'(2)} = 0.04, k = 0, 1, 2$
(v) $\nu = 0.95$
Calculate the actuarial present value of the death benefits for this insurance.

Solution.
We have

$$\begin{aligned}
\Pr((40)\text{ dies in the first year}) =& q_{40}^{'(1)} = 1 - e^{-0.02} = 0.0198\\
\Pr((40)\text{ dies in the second year}) =& p_{40}^{(\tau)} q_{41}^{'(1)} = p_{40}^{'(1)} p_{40}^{'(2)} q_{41}^{'(1)}\\
=&(1-0.0198)(0.96)(0.0198) = 0.01863\\
\Pr((40)\text{ dies in the third year}) =& p_{40}^{(\tau)} p_{41}^{(\tau)} q_{42}^{'(1)}\\
=& p_{40}^{'(1)} p_{40}^{'(2)} p_{41}^{'(1)} p_{41}^{'(2)} q_{42}^{'(1)}\\
=&(1-0.0198)(0.96)e^{-0.02}(0.96)(1-e^{-0.02}) = 0.01753\\
\text{APVB} =& 10,000(1 + 0.0198\nu + 0.01863\nu^2 + 0.01753\nu^3) = 506.53 \blacksquare
\end{aligned}$$

Practice Problems

Problem 68.1
A 3-year term issued to (16) pays 20,000 at the end of year of death if death results from an accident. Let (a) be the death resulting from an accident and (oc) the death resulting from other causes. You are given:
(i)

x	$\ell_x^{(\tau)}$	$d_x^{(a)}$	$d_x^{(oc)}$
16	20,000	1,300	1,100
17	17,600	1,870	1,210
18	14,520	2,380	1,331

(ii) $i = 0.10$
Calculate the actuarial present value of this policy.

Problem 68.2
An employer provides his employees aged 62 the following one year term benefits, payable at the end of the year of decrement:

$1 if decrement results from cause 1
$2 if decrement results from cause 2
$6 if decrement results from cause 3

In their associated single decrement tables, all three decrements follow De Moivre's Law with $\omega = 65$; only three decrements exist.
If $d = 0.10$, find the actuarial present value at age 62 of the benefits.

Problem 68.3
A multiple-decrement table has two causes of decrement: (1) death by accident and (2) death other than accident. A fully continuous whole life insurance issued to (x) pays 2 if death results by accident and 1 if death results other than by accident. If $\mu^{(1)}(x+t) = \delta$, the force of interest, calculate tha actuarial present value of this policy.

Problem 68.4
Austin who is currently age 35 is a professional race car driver. He buys a three-year term insurance policy which pays $1,000,000 for death from accidents during a race car competition. The policy pays nothing for death due to other causes.

The death benefit is paid at the end of the year of death. Level annual benefit premiums are payable at the beginning of each year and these premiums are determined according to the actuarial equivalence principle.
You are given the following double decrement table:

x	$\ell_x^{(\tau)}$	$d_x^{(a)}$	$d_x^{(oc)}$
35	10,000	16	40
36	9,944	20	60
37	9,864	25	80

where $d_x^{(a)}$ represents deaths from accidents during a race car competition and $d_x^{(oc)}$ represents deaths from other causes.
Calculate the actuarial present value of benefits assuming an interest rate of $i = 0.06$.

Problem 68.5 ‡
For a 3-year fully discrete term insurance of 1000 on (40), subject to a double decrement model:
(i)

x	$\ell_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
40	2000	20	60
41		30	50
42		40	

(ii) Decrement 1 is death. Decrement 2 is withdrawal.
(iii) There are no withdrawal benefits.
(iv) $i = 0.05$.
Calculate the actuarial present value of benefits for this insurance.

Problem 68.6 ‡
A whole life policy provides that upon accidental death as a passenger on an airplane a benefit of 1,000,000 will be paid. If death occurs from other accidental causes, a death benefit of 500,000 will be paid. If death occurs from a cause other than an accident, a death benefit of 250,000 will be paid.
You are given:
(i) Death benefits are payable at the moment of death.
(ii) $\mu^{(1)}(x+t) = \frac{1}{2{,}000{,}000}$, where (1) indicates accidental death as a passenger

on an airplane.
(iii) $\mu^{(2)}(x+t) = \frac{1}{250{,}000}$, where (2) indicates death from other accidental causes.
(iv) $\mu^{(3)}(x+t) = \frac{1}{10{,}000}$, where (3) indicates non-accidental deaths.
(v) $\delta = 0.06$.
Calculate the single benefit premium for this insurance.

Problem 68.7 ‡
XYZ Paper Mill purchases a 5-year special insurance paying a benefit in the event its machine breaks down. If the cause is "minor"(1), only a repair is needed. If the cause is "major" (2), the machine must be replaced.
Given:
(i) The benefit for cause (1) is 2000 payable at the moment of breakdown.
(ii) The benefit for cause (2) is 500,000 payable at the moment of breakdown.
(iii) Once a benefit is paid, the insurance contract is terminated.
(iv) $\mu^{(1)}(x+t) = 0.100$ and $\mu^{(2)}(x+t) = 0.004$, for $t > 0$.
(v) $\delta = 0.04$.
Calculate the actuarial present value of this insurance.

69 Valuation of Multiple Decrement Premiums and Reserves

In multiple decrement context, the premium and reserve are calculated in much the same way as in the single-decrement case, except that care must be taken due to the benefits being made according to the mode of decrement and time of decrement. We will illustrate this point by working out a series of examples.

Example 69.1
A 3-year term issued to (16) pays 20,000 at the end of year of death if death results from an accident. Let (a) be the death resulting from an accident and (oc) the death resulting from other causes. You are given:
(i)

x	$\ell_x^{(\tau)}$	$d_x^{(a)}$	$d_x^{(oc)}$
16	20,000	1,100	1,300
17	17,600	1,210	1,870
18	14,520	1,331	2,380

(ii) $i = 0.10$
Calculate the level annual premium of this policy.

Solution.
We have

$$\begin{aligned}20,000A_{\overset{1}{16}:\overline{3}|}^{(\tau)} =&20,000[\nu q_{16}^{(a)} + \nu^2 p_{16}^{(\tau)} q_{17}^{(a)} + \nu^3 {}_2p_{16}^{(\tau)} q_{18}^{(a)}]\\ =&\frac{20,000}{\ell_{16}^{(\tau))}}[\nu d_{16}^{(a)} + \nu^2 d_{17}^{(a)} + \nu^3 d_{18}^{(a)}]\\ =&\frac{1100}{1.1} + \frac{1210}{1.1^2} + \frac{1331}{1.1^3} = 3000.\end{aligned}$$

Now,

$$\begin{aligned}\ddot{a}_{x:\overline{3}|}^{(\tau)} =&1 + \nu p_{16}^{(\tau)} + \nu^2 {}_2p_{16}^{(\tau)} = 1 + \frac{17,600}{20,000} \cdot \frac{1}{1.10} + \frac{14,520}{20,000} \cdot \frac{1}{1.10^2}\\ =&1 + 0.8 + 0.6 = 2.4.\end{aligned}$$

Hence, the level annual premium is

$$P = \frac{3000}{2.4} = 1250 \blacksquare$$

Example 69.2 ‡
For a fully discrete 4-year term insurance on (40), who is subject to a double-decrement model:
(i) The benefit is 2000 for decrement 1 and 1000 for decrement 2.
(ii) The following is an extract from the double-decrement table for the last 3 years of this insurance:

x	$\ell_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
41	800	8	16
42	__	8	16
43	__	8	16

(iii) $\nu = 0.95$
(iv) The benefit premium, based on the equivalence principle, is 34.
Calculate ${}_2V$, the benefit reserve at the end of year 2.

Solution.
We first complete the table using the formula

$$\ell_{x+1}^{(\tau)} = \ell_x^{(\tau)} - d_x^{(1)} - d_x^{(2)}$$

to obtain

x	$\ell_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
41	800	8	16
42	776	8	16
43	752	8	16

Next, we have

$$\begin{aligned}{}_2V =&\text{APV(Future Benefits)} - \text{APV(Future Premiums)}\\ =&2000[\nu q_{42}^{(1)} + \nu^2 p_{42}^{(\tau)} q_{43}^{(1)}] + 1000[\nu q_{42}^{(2)} + \nu^2 p_{42}^{(\tau)} q_{43}^{(2)}] - 34[1 + \nu p_{42}^{(\tau)}]\\ =&2000\left[0.95\left(\frac{8}{776}\right) + 0.95^2\left(\frac{8}{776}\right)\right] + 1000\left[0.95\left(\frac{16}{776}\right) + 0.95^2\left(\frac{16}{776}\right)\right]\\ &-34\left[1 + 0.95\left(\frac{752}{776}\right)\right] = 11.091 \blacksquare\end{aligned}$$

Example 69.3 ‡
For a special fully discrete 3-year term insurance on (55), whose mortality follows a double decrement model:
(i) Decrement 1 is accidental death; decrement 2 is all other causes of death.
(ii)

x	$q_x^{(1)}$	$q_x^{(2)}$
55	0.002	0.020
56	0.005	0.040
57	0.008	0.060

(iii) $i = 0.06$
(iv) The death benefit is 2000 for accidental deaths and 1000 for deaths from all other causes.
(v) The level annual contract premium is 50.
(vi) ${}_1L$ is the prospective loss random variable at time 1, based on the contract premium.
(vii) $K(55)$ is the curtate future lifetime of (55).
Calculate $E[{}_1L|K(55) > 1]$.

Solution.
We have

$$\begin{aligned}
A^{1\ \ (1)}_{56:\overline{2|}} =& \nu q_{56}^{(1)} + \nu^2 p_{56}^{(\tau)} q_{57}^{(1)} \\
=& (1.06)^{-1}(0.005) + (1.06)^{-2}(0.955)(0.008) = 0.0115165539 \\
A^{1\ \ (2)}_{56:\overline{2|}} =& \nu q_{56}^{(2)} + \nu^2 p_{56}^{(\tau)} q_{57}^{(2)} \\
=& (1.06)^{-1}(0.04) + (1.06)^{-2}(0.955)(0.06) = 0.0887326451 \\
\text{APVFB} =& 2000 A^{1\ \ (1)}_{55:\overline{3|}} + 1000 A^{1\ \ (2)}_{55:\overline{3|}} \\
=& 2000(0.0115165539) + 1000(0.0887326451) = 111.77 \\
\text{APVFP} =& 50\ddot{a}_{56:2} = 50[1 + \nu p_{56}^{(\tau)}] \\
=& 50[1 + (1.06)^{-1}(0.955)] = 95.05 \\
E[{}_1L|K(55) > 1] =& 111.77 - 95.05 = 16.72 \blacksquare
\end{aligned}$$

Example 69.4 ‡
A special whole life insurance of 100,000 payable at the moment of death of (x) includes a double indemnity provision. This provision pays during the first ten years an additional benefit of 100,000 at the moment of death for death by accidental means. You are given:
(i) $\mu^{(\tau)}(x+t) = 0.001,\ t \geq 0$
(ii) $\mu^{(1)}(x+t) = 0.0002,\ t \geq 0$, where $\mu^{(1)}(x+t)$ is the force of decrement due to death by accidental means.
(iii) $\delta = 0.06$
Calculate the single benefit premium for this insurance.

Solution.
The single benefit premium of this insurance is the sum of the actuarial present value of the death benefit due to any cause plus the actuarial present value of the extra benefit due to accidental death.
The actuarial present value of benefit due to any cause is

$$100,000\int_0^\infty e^{-\delta t}{}_tp_x^{(\tau)}\mu^{(\tau)}(x+t)dt = 100,000\int_0^\infty e^{-0.06t}e^{-0.001t}(0.001)dt = 1639.34.$$

The actuarial present value of extra benefit due to an accident is

$$100,000\int_0^{10} e^{-\delta t}{}_tp_x^{(\tau)}\mu^{(1)}(x+t)dt = 100,000\int_0^{10} e^{-0.06t}e^{-0.001t}(0.0002)dt = 149.72.$$

The single benefit premium for this insurance is $1639.34 + 149.72 = 1789.06$ ■

Practice Problems

Problem 69.1
A life (50) is subject to a double decrement model with

x	$\ell_x^{(\tau)}$	$d_x^{(d)}$	$d_x^{(w)}$
50	1,000	15	35
51	___	25	30
52	___	35	___

where decrement (d) is death while decrement (w) is withdrawal.
For a three-year fully discrete term insurance of \$100 on (50), there are no withdrawal benefits and the interest rate is $i = 4\%$.
Compute the amount of the level annual benefit premium for this insurance.

Problem 69.2
A special whole life insurance of 100,000 payable at the moment of death of (40) includes a double indemnity rider. This provision pays during the first ten years an additional benefit of 100,000 at the moment of death for death by accidental means (decrement $J = 1$). You are given:
(i) $\mu^{(1)}(40+t) = 0.0002,\ t \geq 0$
(ii) $\mu^{(\tau)}(40+t) = 0.001,\ t \geq 0$
(iii) $\delta = 0.06$.
For the policy without the rider the level benefit premium π_1 is payable for life. For the policy with the rider an extra benefit premium π_2 in addition to π_1 is charged for the extra benefit for death by accidental means until age 50. Calculate the benefit premiums for the insurance with the rider.

Problem 69.3
The benefit premiums are determined as in the previous problem. Find the benefit reserve for the insurance with the rider at the end of year 2.

Problem 69.4 ‡
For a 3-year fully discrete term insurance of 1000 on (40), subject to a double decrement model:
(i)

x	$\ell_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
40	2000	20	60
41		30	50
42		40	

(ii) Decrement 1 is death. Decrement 2 is withdrawal.
(iii) There are no withdrawal benefits.
(iv) $i = 0.05$.
Calculate the level annual benefit premium for this insurance.

Problem 69.5 ‡
For a special whole life insurance:
(i) The benefit for accidental death is 50,000 in all years.
(ii) The benefit for non-accidental death during the first 2 years is return of the single benefit premium without interest.
(iii) The benefit for non-accidental death after the first 2 years is 50,000.
(iv) Benefits are payable at the moment of death.
(v) Force of mortality for accidental death: $\mu(x+t)^{(1)} = 0.01, t > 0$
(vi) Force of mortality for non-accidental death: $\mu(x+t)^{(2)} = 2.29, t > 0$
(vii) $\delta = 0.10$
Calculate the single benefit premium for this insurance.

Problem 69.6 ‡
Michael, age 45, is a professional motorcycle jumping stuntman who plans to retire in three years. He purchases a three-year term insurance policy. The policy pays 500,000 for death from a stunt accident and nothing for death from other causes. The benefit is paid at the end of the year of death.
You are given:
(i) i = 0.08
(ii)

x	$\ell_x^{(\tau)}$	$d_x^{(-s)}$	$d_x^{(s)}$
45	2500	10	4
46	2486	15	5
47	2466	20	6

where $d_x^{(s)}$ represents deaths from stunt accidents and $d_x^{(-s)}$ represents deaths from other causes.
(iii) Level annual benefit premiums are payable at the beginning of each year.

(iv) Premiums are determined using the equivalence principle.
Calculate the annual benefit premium.

Incorporating Expenses in Insurance Models

Our analysis of benefit premiums in the precedings chapters was based on the equivalence principle. The premium obtained by this principle does not incorporate expenses of operations. In practice, the premiums are set higher than the ones obtained through the equivalence principle in order to generate revenue to pay the expenses as well as the contingent benefit payments.
In this chapter we explore the theory of determining the annual premium that includes operating expenses. We refer to such annual premium as the **gross annual premium.** Synonymous terms are the **contract premium**, the **expense-loaded premium** or the **expense-augmented premium.**
The second topic of this chapter is the insurance **asset shares.**

70 Expense-Augmented Premiums

As pointed out in the introduction, the **expense-augmented premium** is the premium necessary to cover both the policy benefits as well as the related expenses. In practice, expenses are treated as if they are benefits. We will denote the expense-augmented premium by the letter G. An extended equivalence principle is used to find G which states that the actuarial present value of future gross premiums is equal to the actuarial present value of benefits plus the actuarial present value of expenses. This principle is illustrated in the next example for constant expenses.

Example 70.1
Consider a whole life policy issued to (x) with the following characteristics:
(i) benefit is B payable at the end of the year of death.
(ii) premium is payable once at the beginning of each year for h years.
(iii) annual constant expense E payable at the beginning of the year.
Find an expression for the gross premium G.

Solution.
By the extended equivalence principle, we have

$$G\ddot{a}_{x:\overline{h}|} = BA_x + E\ddot{a}_{x:\overline{h}|}.$$

Solving for G we find

$$G = B_hP(A_x) + E.$$

In this case, the annual constant expense is a simple addition to the annual benefit premium ■

In practice, the first year expense tends to be higher than the continuing level expense due to agents' commission from selling the contract and the cost of preparing new policies and records administration. These expenses are usually recovered in later years. Non-first year expense is called **renewal expense** (used for maintaining and continuing the policy).

Example 70.2
Consider again the previous example. Suppose that the first year expense is E_1 and the renewal expenses are E_r. Find an expression of G.

Solution.
By the extended equivalence principle, we have

$$G\ddot{a}_{x:\overline{h}|} = BA_x + E_1 + E_r a_{x:\overline{h-1}|} = BA_x + E_1 + E_r(\ddot{a}_{x:\overline{h}|} - 1) = BA_x + (E_1 - E_r) + E_r\ddot{a}_{x:\overline{h}|}.$$

Thus,

$$G = B_h P(A_x) + E_r + \frac{E_1 - E_r}{\ddot{a}_{x:\overline{h}|}}.$$

In this case, the excess of the first-year to renewal expenses is spread throughout the premium-payment period ■

Example 70.3 ‡
For a fully discrete whole life insurance of 100,000 on (35) you are given:
(i) Percent of premium expenses are 10% per year.
(ii) Per policy expenses are 25 per year.
(iii) Annual maintenance expense of 2.50 for each 1000 of face value.
(iv) All expenses are paid at the beginning of the year.
(v) $1000P_{35} = 8.36$
Calculate the level annual expense-loaded premium using the equivalence principle.

Solution.
By the extended equivalence principle we have

$$\text{APV (Premiums)} = \text{APV(Benefits)} + \text{APV(Expenses)}.$$

That is,

$$G\ddot{a}_{35} = 100,000A_{35} + (250 + 25 + 0.10G)\ddot{a}_{35}.$$

Now,

$$8.36 = 1000P_{35} = 1000\left(\frac{A_{35}}{\ddot{a}_{35}}\right) \Longrightarrow \frac{A_{35}}{\ddot{a}_{35}} = 0.00836.$$

Hence

$$G = 100,000(0.00836) + 275 + 0.10G \Longrightarrow G = 1234.44 \text{ ■}$$

With the gross premiums, it is now a simple matter to include the expense factors, along with the gross premium, to determine **gross premium reserves** or **total reserves.** The general prospective formula for the t^{th} total reserves now reads

$${}_tV^E = \text{APV(future benefits)} + \text{APV(future expenses)} - \text{APV(future gross premiums)}.$$

Example 70.4
For a fully discrete whole life insurance of 1,000 on (45) you are given:
(i) Expenses include 10% per year of gross premium.
(ii) Additional expenses of 3 per year.
(iii) All expenses are paid at the beginning of the year.
(iv) Mortality follows the Illustrative Life Table with interest rate $i = 6\%$.
(a) Calculate the annual benefit premium.
(b) Calculate the annual gross premium.
(c) Calculate the annual expense premium.
(d) Calculate the benefit reserves at the end of year 1.
(e) Calculate the total reserves at the end of year 1.
(f) Calculate the expense reserves at the end of year 1. The expense reserves is defined as the APV of future expenses minus the APV of future expense premiums.

Solution.
(a) First we find the benefit premium:

$$\pi = 1000P(A_{45}) = 1000\frac{A_{45}}{\ddot{a}_{45}} = 1000\left(\frac{0.20120}{14.1121}\right) = 14.2573.$$

(b) Next, we find the gross premium. We have

$$G\ddot{a}_{45} = 1000A_{45} + 0.10G\ddot{a}_{45} + 3\ddot{a}_{45}$$

or

$$14.1121G = 1000(0.20120) + 0.10G(14.1121) + 3(14.1121).$$

Solving for G, we find $G = 19.1747$.
(c) The annual expense premium is $19.1747 - 14.2573 = 4.9174$.
(d) The benefit reserves at the end of year 1 is

$${}_1V = 1000\left(1 - \frac{\ddot{a}_{46}}{\ddot{a}_{45}}\right) = 1000\left(1 - \frac{13.9546}{14.1121}\right) = 11.1606.$$

(e) The total reserve is

$$\begin{aligned}{}_1V^E =&1000A_{46} + 0.10G\ddot{a}_{46} + 3\ddot{a}_{46} - G\ddot{a}_{46}\\ =&1000(0.21012) - 0.9(19.1747)(13.9546) + 3(13.9546) = 11.1661.\end{aligned}$$

(f) The expense reserve is $0.1(19.1747)\ddot{a}_{46} + 3\ddot{a}_{46} - 4.9174\ddot{a}_{46} = 0.00098$ ■

Example 70.5 ‡
For a fully discrete whole life insurance of 1000 on (60), you are given:
(i) The expenses, payable at the beginning of the year, are:

Expense Type	First Year	Renewal Years
% of Premium	20%	6%
Per Policy	8	2

(ii) The level expense-loaded premium is 41.20.
(iii) $i = 0.05$
Calculate the value of the expense augmented loss variable, ${}_0L_e$, if the insured dies in the third policy year.

Solution.
We have

$$\begin{aligned}{}_0L_e =&\text{PVB} + \text{PVE} - \text{PVP}\\ =&1000\nu^3 + (0.20G + 8) + (0.06G + 2)\nu + (0.06G + 2)\nu^2 - G\ddot{a}_{3}\\ =&1000(1.05)^{-3} + [0.20(41.20) + 8] + [0.06(41.20) + 2](1.05)^{-1}\\ &+[0.06(41.20) + 2](1.05)^{-2} - 41.20\frac{1 - 1.05^{-3}}{0.05(1.05)^{-1}} = 770.815 \blacksquare\end{aligned}$$

Practice Problems

Problem 70.1
For a fully discrete whole life insurance of 1,000 on (45) you are given:
(i) Percent of premium expenses are 40% for first year and 10% thereafter.
(ii) Per policy expenses are 5 for first year and 2.5 thereafter.
(iii) Per thousand expenses are 1.00 for first year and 0.5 for thereafter.
(iv) All expenses are paid at the beginning of the year.
(v) Mortality follows the Illustrative Life Table with interest rate $i = 6\%$.
Calculate the expense-loaded annual premium.

Problem 70.2
For a fully discrete whole life insurance of 1,000 on (40) you are given:
(i) Percent of premium expenses are 10% per year.
(ii) Per policy expenses are 5 per year.
(iii) All expenses are paid at the beginning of the year.
(iv) Mortality follows the Illustrative Life Table with interest rate $i = 6\%$.
(a) Calculate the annual benefit premium.
(b) Calculate the annual gross premium.
(c) Calculate the annual expense premium.
(d) Calculate the benefit reserves at the end of year 10.
(e) Calculate the total reserves at the end of year 10.
(f) Calculate the expense reserves at the end of year 10.

Problem 70.3 ‡
For a special fully discrete whole life insurance on (x), you are given:
(i) The net single premium is 450.
(ii) The level annual expense loaded premium determined by the equivalence principle is 85.
(iii) Death is the only decrement.
(iv) Expenses, which occur at the beginning of the policy year, are as follows:

	First Year	renewal Year
% of Premiums	80%	10%
Per policy	25	25

Calculate a_x.

Problem 70.4 ‡
For a fully discrete 15-payment whole life insurance of 100,000 on (x), you are given:
(i) The expense-loaded level annual premium using the equivalence principle is 4669.95.
(ii) $100,000A_xx = 51,481.97$
(iii) $\ddot{a}_{x:\overline{15}|} = 11.35$
(iv) $d = 0.02913$
(v) Expenses are incurred at the beginning of the year.
(vi) Percent of premium expenses are 10% in the first year and 2% thereafter.
(vii) Per policy expenses are K in the first year and 5 in each year thereafter until death.
Calculate K.

Problem 70.5 ‡
For a semicontinuous 20-year endowment insurance of 25,000 on (x), you are given:
(i) The following expenses are payable at the beginning of the year:

	% Premium	Per 1000 insurance	Per Policy
First Year	25%	2.00	15.00
Renewal Years	5%	0.50	3.00

(ii) $\bar{A}_{x:\overline{20}|} = 0.4058$
(iii) $\ddot{a}_{x:\overline{20}|} = 12.522$.
(iv) Premiums are determined using the equivalence principle.
Calculate the level annual expense-loaded premium.

Problem 70.6 ‡
For a fully continuous whole life insurance of 1 on (x), you are given:
(i) $\delta = 0.04$
(ii) $\bar{a}_x = 12$
(iii) $\mathrm{Var}(\nu^T) = 0.10$
(iv) ${}_0L_e = {}_0L + E$ is the expense-augmented loss variable, where
• ${}_0L = \nu^T - \bar{P}(\bar{A}_x)\bar{a}_{\overline{T}|}$
• $E = c_0 + (g - e)\bar{a}_{\overline{T}|}$, where $c_0 =$ initial expenses, $g = 0.0030$, is the annual rate of continuous maintenance expense and $e = 0.0066$, is the annual expense loading in the premium.
Calculate $\mathrm{Var}({}_0L_e)$.

Problem 70.7 ‡
For a fully continuous whole life insurance on (x), you are given:
(i) The benefit is 2000 for death by accidental means (decrement 1).
(ii) The benefit is 1000 for death by other means (decrement 2).
(iii) The initial expense at issue is 50.
(iv) Settlement expenses are 5% of the benefit, payable at the moment of death.
(v) Maintenance expenses are 3 per year, payable continuously.
(vi) The gross or contract premium is 100 per year, payable continuously.
(vii) $\mu^{(1)}(x+t) = 0.004, t > 0$
(viii) $\mu^{(x+t)} = 0.040, t > 0$
(ix) $\delta = 0.05$
Calculate the actuarial present value at issue of the insurer's expense-augmented loss random variable for this insurance.

Problem 70.8 ‡
For a fully discrete 10-year endowment insurance of 1000 on (35), you are given:
(i) Expenses are paid at the beginning of each year.
(ii) Annual per policy renewal expenses are 5.
(iii) Percent of premium renewal expenses are 10% of the expense-loaded premium.
(iv) $1000P_{35:\overline{10}|} = 76.87$
(v) The expense reserve at the end of year 9 is negative 1.67.
(vi) Expense-loaded premiums were calculated using the equivalence principle.
Calculate the expense-loaded premium for this insurance.

Problem 70.9 ‡
For a special single premium 2-year endowment insurance on (x), you are given:
(i) Death benefits, payable at the end of the year of death, are:

$$b_1 = 3000$$
$$b_2 = 2000$$

(ii) The maturity benefit is 1000.
(iii) Expenses, payable at the beginning of the year:
(a) Taxes are 2% of the expense-loaded premium.

(b) Commissions are 3% of the expense-loaded premium.
(c) Other expenses are 15 in the first year and 2 in the second year.
(iv) $i = 0.04$
(v) $p_x = 0.9$
(vi) $p_{x+1} = 0.8$
Calculate the expense-loaded premium G using the equivalence principle.

Problem 70.10 ‡
For a fully discrete 2-payment, 3-year term insurance of 10,000 on (x), you are given:
(i) $i = 0.05$
(ii)

$$q_x = 0.10$$
$$q_{x+1} = 0.15$$
$$q_{x+2} = 0.20$$

(iii) Death is the only decrement.
(iv) Expenses, paid at the beginning of the year, are:

Policy Year	Per policy	Per 1000 of insurance	Fraction of premium
1	25	4.50	0.20
2	10	1.50	0.10
3	10	1.50	0.00

(v) Settlement expenses, paid at the end of the year of death, are 20 per policy plus 1 per 1000 of insurance.
(vi) G is the expense-loaded level annual premium for this insurance.
(vii) The single benefit premium for this insurance is 3499.
Calculate G, using the equivalence principle.

Problem 70.11 ‡
For a fully discrete 20-year endowment insurance of 10,000 on (50), you are given:
(i) Mortality follows the Illustrative Life Table.
(ii) $i = 0.06$
(iii) The annual contract premium is 495.
(iv) Expenses are payable at the beginning of the year.
(v) The expenses are:

	% Premium	Per 1000 insurance	Per Policy
First Year	35%	15.00	20
Renewal Years	5%	1.50	5

Calculate the actuarial present value of amounts available for profit and contingencies.

71 Types of Expenses

In the previous section we included expenses without further classifications of these expenses. In this section, we describe in more detail the types of expenses that insurance companies encounter.
Expenses are broken into investment expenses and insurance-related expenses:
• **Investment-related expenses:**
Costs of analyzing, buying, selling, and servicing the investments used to back insurance company reserves.

• **Insurance-related expenses:**
1. Acquisition (agents' commission, underwriting[4] costs, preparing new records).
2. Maintenance (premium collection, policyholder correspondence).
3. General (research, actuarial, accounting, taxes).
4. Settlement (claim investigation, legal defense, disbursement).

It is common in practice to call that portion of the gross premium that is independent of the benefit amount the **expense policy fee** or simply the **policy fee.** It is important to keep in mind that percent-of-premium charges apply to the policy fee as well as the rest of the gross premium.The following example illustrates the calculation of policy fee.

Example 71.1
A 1,000 fully discrete whole life policy issued to (45) with level annual premiums is priced with the following expense assumptions:

	% of Premium	Per 1,000	Per Policy
First Year	40%	1.0	5.0
Renewal Year	10%	0.5	2.5

Mortality follows the Illustrative Life Table with interest $i = 6\%$. Calculate the expense policy fee for this policy.

Solution.
Let g denote the expense policy fee. Then g satisfies the equation

$$g\ddot{a}_{45} = 0.4g + 5 + (0.10g + 2.5)a_{45}.$$

[4]Process used by insurance companies to assess the eligibility of a customer to receive their products.

That is,

$$g(14.1121) = 0.4g + 5 + (0.10g + 2.5)(13.1121).$$

Solving this equation for g we find $g = 3.05$ ■

Example 71.2

For a fully discrete 10-year endowment insurance on (50), you are given the following:

(i) Percent of premium expenses consist of commissions equal to 50% of the gross premium in the first year followed by 5% of gross premium in the renewal years.

(ii) Expenses include acquisition expense of 20, settlement expenses equal to 10 per 1000 of face amount plus 80, and annual maintenance expenses equal to 5 plus 2 per 1000 of face amount.

(iii) Acquisition expense is due at time of policy issue, settlement expenses are due when the benefit is paid, and annual maintenance expenses are due at the time annual premium is paid.

Find an expression for the expense policy fee.

Solution.

Let g denote the expense policy fee. Then g satisfies the equation

$$g\ddot{a}_{50:\overline{10}|} = 0.50g + 0.05ga_{50:\overline{9}|} + 20 + 80A_{50:\overline{10}|} + 5\ddot{a}_{50:\overline{10}|}$$

or

$$g\ddot{a}_{50:\overline{10}|} = 0.45g + 0.05g\ddot{a}_{50:\overline{10}|} + 20 + 80A_{50:\overline{10}|} + 5\ddot{a}_{50:\overline{10}|}.$$

Thus,

$$g = \frac{80A_{50:\overline{10}|} + 5.05 + \ddot{a}_{50:\overline{10}|} + 20}{0.95\ddot{a}_{50:\overline{10}|} - 0.45} \quad ■$$

Example 71.3 ‡

For a 10-payment 20-year endowment insurance of 1000 on (40), you are given:

(i) The following expenses:

	First Year		Subsequent Year	
	Percent of Premium	Per Policy	Percent of Premium	Per Policy
Taxes	4%	0	4%	0
Sales Commission	25%	0	5%	0
Policy Maintenance	0	10	0	5

(ii) Expenses are paid at the beginning of each policy year.
(iii) Death benefits are payable at the moment of death.
(iv) The expense-loaded premium G is determined using the equivalence principle.
Find an expression for G.

Solution.
The actuarial present value of benefit is $1000\bar{A}_{40:\overline{20}|}$. The actuarial present value of premiums is $G\ddot{a}_{40:\overline{10}|}$. The actuarial present value of expenses is

$$(0.04+0.25)G+10+(0.04+0.05)Ga_{40:\overline{9}|}+5a_{40:\overline{19}|}$$

which simplifies to

$$0.2G+10+0.09G\ddot{a}_{40:\overline{10}|}+5a_{40:\overline{19}|}.$$

By the extended equivalence principle, we have

$$G\ddot{a}_{40:\overline{10}|}=1000\bar{A}_{40:\overline{20}|}+0.2G+10+0.09G\ddot{a}_{40:\overline{10}|}+5a_{40:\overline{19}|}.$$

Solving this equation for G we find

$$G=\frac{1000\bar{A}_{40:\overline{20}|}+10+5a_{40:\overline{19}|}}{0.91\ddot{a}_{40:\overline{10}|}-0.2} \blacksquare$$

Example 71.4 ‡
For a fully discrete whole life insurance of 100,000 on (x), you are given:
(i) Expenses, paid at the beginning of the year, are as follows:

Year	Percentage of Premium Expenses	Per 1000 Expenses	Per Policy Expenses
1	50%	2.0	150
2+	4%	0.5	25

(ii) $i=0.04$
(iii) $\ddot{a}_x=10.8$
(iv) Per policy expenses are matched by a level policy fee to be paid in each year.
Calculate the expense-loaded premium using the equivalence principle.

Solution.
Let g be the expense policy fee. Then g satisfies the equation

$$g\ddot{a}_x = 0.5g + 0.04ga_x + 150 + 25a_x.$$

Solving for g, we find

$$g = \frac{150 + 25a_x}{\ddot{a}_x - 0.04a_x - 0.5} = \frac{150 + 25(9.8)}{10.8 - 0.04(9.8) - 0.5} = 39.87.$$

Let G^* denote the expense-loaded premium excluding policy fee. We have

$$\begin{aligned}
\text{APVB} =& 100,000A_x = 100,000(1 - d\ddot{a}_x) = 100,000\left[1 - \frac{0.04}{1.04}(10.8)\right] = 58461.54\\
\text{APVP} =& G^*\ddot{a}_x = 10.8G^*\\
\text{APVE} =& 0.5G^* + 2.0(100) + 0.04G^*a_x + 0.5(100)a_x = 0.892G^* + 690.
\end{aligned}$$

Using the extended equivalence principle, we find

$$10.8G^* = 58461.54 + 0.892G^* + 690 \Longrightarrow G^* = 5970.08.$$

Let G be the loaded-expense premium. Then $G = 5970.08 + 39.87 = 6009.95$ ■

Practice Problems

Problem 71.1
For a fully discrete whole life insurance of $1,000 on (40), you are given:
(i) Expenses, paid at the beginning of the year, are as follows:

	% of Premium	Per 1,000	Per Policy
First Year	10%	2.5	15
Renewal Year	3%	0.5	5

(ii) $A_{40} = 0.369$.
(iii) $i = 0.03$.
Calculate:
(a) the level annual expense-loaded premium using the equivalence principle, and
(b) the policy fee.

Problem 71.2 ‡
For a fully discrete whole life insurance of 1000 on (50), you are given:
(i) The annual per policy expense is 1.
(ii) There is an additional first year expense of 15.
(iii) The claim settlement expense of 50 is payable when the claim is paid.
(iv) All expenses, except the claim settlement expense, are paid at the beginning of the year.
(v) Mortality follows De Moivre's law with $\omega = 100$.
(vi) $i = 0.05$
Calculate the level expense-loaded premium using the equivalence principle.

Problem 71.3 ‡
For a semicontinuous 20-year endowment insurance of 25,000 on (x), you are given:
(i) The following expenses are payable at the beginning of the year:

	% Premium	Per 1000 insurance	Per Policy
First Year	25%	2.00	15.00
Renewal Years	5%	0.50	3.00

(ii) $\bar{A}_{x:\overline{20}|} = 0.4058$.
(iii) $\ddot{a}_{x:\overline{20}|} = 12.522$.

(iv) Premiums are determined using the equivalence principle.
(a) Calculate the annual expense-loaded premium excluding the per-policy expense.
(b) Calculate the expense-loaded first-year premium including policy fee assuming that per-policy expenses are matched separately by first-year and renewal policy fees.

Problem 71.4 ‡
For a semicontinuous 20-year endowment insurance of 25,000 on (x), you are given:
(i) The following expenses are payable at the beginning of the year:

	% Premium	Per 1000 insurance	Per Policy
First Year	25%	2.00	15.00
Renewal Years	5%	0.50	3.00

(ii) $\bar{A}_{x:\overline{20}|} = 0.4058$.
(iii) $\ddot{a}_{x:\overline{20}|} = 12.522$.
(iv) Premiums are determined using the equivalence principle.
Calculate the expense-loaded renewal years premium including policy fee assuming that per-policy expenses are matched separately by first-year and renewal policy fees.

Problem 71.5 ‡
For a semicontinuous 20-year endowment insurance of 25,000 on (x), you are given:
(i) The following expenses are payable at the beginning of the year:

	% Premium	Per 1000 insurance	Per Policy
First Year	25%	2.00	15.00
Renewal Years	5%	0.50	3.00

(ii) $\ddot{a}_{x:\overline{20}|} = 12.522$.
(iii) Premiums are determined using the equivalence principle.
Calculate the level annual policy fee to be paid each year.

Problem 71.6 ‡
For a fully discrete 5-payment 10-year deferred 20-year term insurance of 1000 on (30), you are given:
(i) The following expenses:

	Year 1		Year 2-10	
	Percent of Premium	Per Policy	Percent of Premium	Per Policy
Taxes	5%	0	5%	0
Sales Commission	25%	0	10%	0
Policy Maintenance	0	20	0	10

(ii) Expenses are paid at the beginning of each policy year.
(iii) The expense-loaded premium is determined using the equivalence principle.
Find an expression for the expense-loaded premium G.

72 The Mathematics of Asset Share

In this section we discuss the mathematical structure of asset share-type calculations. More specifically, we give a recursive relation for determining an asset share.

Asset share is a tool used to project the accumulation of assets backing a block of insurance policies and to assign those assets to each policy. To illustrate, consider an insurance company who sold ℓ_0 policy to a group of individuals aged x. Let

- $\ell_{x+k}^{(\tau)}$ denote the number of original policy holders surviving to age $x+k$.
- G is the annual gross premium received at the beginning of the year.
- c_k denote the fraction of the gross premium paid at time k for expenses.
- e_k denote the per policy expenses at time k. All expenses occur at the beginning of the year.
- b_k denote the benefit paid at time k for a death in the k^{th} policy year.
- ${}_kCV$ is the **cash value** or withdrawal benefit paid to those insureds who cancel their policy in the k^{th} policy year. This amount is paid at the end of the year of withdrawal.
- $d_{x+k}^{(d)}$ denote the number of policy holders dying at age $x+k$.
- $d_{x+k}^{(w)}$ denote the number of policy holders withdrawing at age $x+k$.
- $q_{x+k}^{(d)}$ is the probability of decrement by death before age $x+k+1$ for an insured who is now $x+k$;
- $q_{x+k}^{(w)}$ is the probability of decrement by withdrawal before age $x+k+1$ for an insured who is now $x+k$. It follows that the probability of staying in force in the time interval $(x+k, x+k+1]$ is therefore

$$p_{x+k}^{(\tau)} = 1 - q_{x+k}^{(d)} - q_{x+k}^{(w)}.$$

Denote the asset share at the end of year k by ${}_kAS$ with an initial asset share at time 0 of ${}_0AS$ which may or may not be zero.[5] Standard cash flow analysis gives the fundamental relationship

$${}_{k+1}AS\ell_{x+k+1}^{(\tau)} = ({}_kAS + G - c_kG - e_k)(1+i)\ell_{x+k}^{(\tau)} - b_{k+1}d_{x+k}^{(d)} - {}_{k+1}CVd_{x+k}^{(w)}.$$

Dividing both sides of this equation by $\ell_{x+k}^{(\tau)}$ produces a second useful recursion formula connecting successive asset shares

$${}_{k+1}ASp_{x+k}^{(\tau)} = ({}_kAS + G - c_kG - e_k)(1+i) - b_{k+1}q_{x+k}^{(d)} - {}_{k+1}CVq_{x+k}^{(w)}.$$

[5] For a new policy/contract, we may assume ${}_0AS = 0$.

Example 72.1
For a portfolio of fully discrete whole life insurances of \$1,000 on (30), you are given:
(i) the contract annual premium is \$9.50;
(ii) renewal expenses, payable at the start of the year, are 3% of premium plus a fixed amount of \$2.50;
(iii) ${}_{20}AS = 145$ is the asset share at the end of year 20;
(iv) ${}_{21}CV = 100$ is the cash value payable upon withdrawal at the end of year 21;
(v) interest rate is $i = 7.5\%$ and the applicable decrement table is given below:

x	$q_x^{(d)}$	$q_x^{(w)}$
50	0.0062	0.0415
51	0.0065	0.0400

Calculate the asset share at the end of year 21 (or age 51).

Solution.
We have

$$\begin{aligned}
{}_{21}AS =&\frac{({}_{20}AS + G - c_{20}G - e_{20})(1+i) - b_{21}q_{50}^{(d)} - {}_{21}CVq_{50}^{(w)}}{1 - q_{50}^{(d)} - q_{50}^{(w)}}\\
=&\frac{(145 + 9.50 - 0.03(9.50) - 2.50)(1.075) - 1000(0.0062) - 100(0.0415)}{1 - 0.0062 - 0.0415}\\
=&160.39 \blacksquare
\end{aligned}$$

One natural use of the idea of asset shares is to determine the gross premium G required in order to achieve a certain asset goal at a future time.

Example 72.2 ‡
For a fully discrete whole life insurance of 1000 on (40), you are given:
(i) Death and withdrawal are the only decrements.
(ii) Mortality follows the Illustrative Life Table.
(iii) $i = 0.06$
(iv) The probabilities of withdrawal are $q_{40}^{(w)} = 0.2$ and $q_{40+k}^{(w)} = 0$ for $k > 0$.
(v) Withdrawals occur only at the end of the year. The following expenses are payable at the beginning of the year:

	% of Premium	Per 1000 insurance
All Years	10%	1.50

(vii) ${}_kCV = \frac{1000k}{3}{}_kV_{40}, k = 1, 2, 3.$
(viii) ${}_2AS = 24.$
(ix) The asset share at time 0 is 0.
Calculate the gross premium, G.

Solution.
We use the formula

$$ {}_2ASp_{41}^{(\tau)} = ({}_1AS + G - c_1G - e_1)(1+i) - b_2q_{41}^{(d)} - {}_1CVq_{41}^{(w)}. $$

Thus,

$$ 24(1 - q_{41}^{(d)}) = ({}_1AS + G - 0.10G - 1.5)(1.06) - 1000q_{41}^{(d)} $$

and using ILL we have

$$ 24(1 - 0.00298) = ({}_1AS + 0.90G - 1.5)(1.06) - 1000(0.00298). $$

We find ${}_1AS$ by means of the relation

$$ {}_1ASp_{40}^{(\tau)} = ({}_0AS + G - c_0G - e_0)(1+i) - b_1q_{40}^{(d)} - {}_1CVq_{40}^{(w)} $$

or

$$ {}_1AS(1-0.2-0.00278) = (0+G-0.10G-1.5)(1.06)-1000(0.00278)-{}_1CV(0.2). $$

But

$$ {}_1CV = \frac{1000}{3}{}_1V = \frac{1000}{3}\left(1 - \frac{\ddot{a}_{41}}{\ddot{a}_{40}}\right) = \frac{1000}{3}\left(1 - \frac{14.6864}{14.8166}\right) = 2.9291. $$

Hence

$$ {}_1AS(0.79722) = 1.197G - 6.22. $$

Finally, we have

$$ 24(1-0.00298) = (1.187G-6.22+0.90G-1.5)(1.06)-1000(0.00298) \Longrightarrow G = 15.863 \blacksquare $$

Example 72.3 ‡
For a fully discrete insurance of 1000 on (x), you are given:
(i) ${}_4AS = 396.63$
(ii) ${}_5AS = 694.50$
(iii) $G = 281.77$
(iv) ${}_5CV = 572.12$
(v) $c_4 = 0.05$ is the fraction of the gross premium paid at time 4 for expenses.
(vi) $e_4 = 7.0$ is the amount of per policy expenses paid at time 4.
(vii) $q_{x+4}^{(1)} = 0.09$ is the probability of decrement by death.
(viii) $q_{x+4}^{(2)} = 0.26$ is the probability of decrement by withdrawal.
Calculate i.

Solution.
We have

$$ {}_5AS = \frac{({}_4AS + G(1-c_4) - e_4)(1+i) - 1000q_{x+4}^{(1)} - {}_5CV\, q_{x+4}^{(2)}}{1 - q_{x+4}^{(1)} - q_{x+4}^{(2)}}. $$

Substituting, we find

$$ 694.50 = \frac{(396.63 + 281.77(1-0.05) - 7.0)(1+i) - 1000(0.09) - 572.12(0.26)}{1 - 0.09 - 0.26}. $$

Solving this equation for i, we find $i = 5\%$ ∎

Practice Problems

Problem 72.1
For a fully discrete whole life insurance policy of 20,000 on (40) with level annual premiums, you are given:
(i) ${}_{10}AS = 2000$
(ii) ${}_{11}AS = 1792.80$
(iii) $e_{10} = 10$
(iv) $c_{10} = 5\%$
(v) $i = 0.07$
(vi) $q_{50}^{(d)} = 0.03$ and $q_{(50)}^{(w)} = 0.15$
(vii) the contract premium is one cent for every dollar of insurance.
Calculate the cash value payable upon withdrawal at the end of 11 years.

Problem 72.2 ‡
For a fully discrete 3-year endowment insurance of 1000 on (x), you are given:
(i) $i = 0.10$
(ii) Expenses, which occur at the beginning of the policy year, are as follows:

	% Premium	Per Policy
First Year	20%	8
Renewal Years	6%	2

(iii) The gross annual premium is equal to 314.
(iv) The following double-decrement table:

k	$p_{x+k}^{(\tau)}$	$q_{x+k}^{(d)}$	$q_{x+k}^{(w)}$
0	0.54	0.08	0.38
1	0.62	0.09	0.29
2	0.50	0.50	0.00

(v) The following table of cash values and asset shares:

k	${}_{k+1}CV$	${}_{k}AS$
0	247	0
1	571	173

Calculate ${}_{2}AS$.

Problem 72.3 ‡
For a fully discrete whole life insurance of 1000 on (x) :
(i) Death is the only decrement.
(ii) The annual benefit premium is 80.
(iii) The annual goss premium is 100.
(iv) Expenses in year 1, payable at the start of the year, are 40% of gross premiums.
(v) $i = 0.10$
(vi) $1000_1V_x = 40$
(vii) The asset share at time 0 is 0.
Calculate the asset share at the end of the first year.

Problem 72.4
(a) Assuming that ${}_0AS = 0$. show that

$$\sum_{k=0}^{n-1}[{}_{k+1}AS\nu^{k+1}\ell_{x+k+1}^{(\tau)} - {}_kAS\nu^k\ell_{x+k}^{(\tau)}] = {}_nAS\nu^n\ell_{x+n}^{(\tau)}.$$

(b) Show that

$${}_nAS\nu^n\ell_{x+n}^{(\tau)} = G\sum_{k=0}^{n-1}(1-c_k)\nu^k\ell_{x+k}^{(\tau)} - \sum_{k=0}^{n-1}e_k\nu^k\ell_{x+k}^{(\tau)} - \sum_{k=0}^{n-1}(b_{k+1}d_{x+k}^{(d)} - {}_{k+1}CVd_{x+k}^{(w)})\nu^{k+1}.$$

Problem 72.5
If ${}_{10}AS_1$ is the asset share at the end of 10 years based on premium G_1 and ${}_{10}AS_2$ is the asset share at the end of 10 years based on premium G_2, find a formula for ${}_{10}AS_1 - {}_{10}AS_2$.

Problem 72.6 ‡
For a fully discrete whole life insurance of 10,000 on (x), you are given:
(i) ${}_{10}AS = 1600$
(ii) $G = 200$
(iii) ${}_{11}CV = 1700$
(iv) $c_{10} = 0.04$ is the fraction of gross premium paid at time 10 for expenses.
(v) $e_{10} = 70$ is the amount of per policy expense paid at time 10.
(vi) Death and withdrawal are the only decrements.
(vii) $q_{x+10}^{(d)} = 0.02$
(viii) $q_{x+10}^{(w)} = 0.18$
(ix) $i = 0.05$
Calculate ${}_{11}AS$.

Problem 72.7 ‡
For a fully discrete whole life insurance of 1000 on (x), you are given:
(i) $G = 30$
(ii) $e_k = 5,\ k = 1, 2, 3, \cdots$
(iii) $c_k = 0.02,\ k = 1, 2, 3, \cdots$
(iv) $i = 0.05$
(v) ${}_4CV = 75$
(vi) $q_{x+3}^{(d)} = 0.013$
(vii) $q_{x+3}^{(w)} = 0.05$
(viii) ${}_3AS = 25.22$
If withdrawals and all expenses for year 3 are each 120% of the values shown above, by how much does ${}_4AS$ decrease?

Multiple-State Transition Models

A **multi-state transition model** is defined as a probability model that describes the random movement of a subject among various states.
The main objective of the multi-state transition models is to generalize the formulation of probabilities of contingent events and the valuation of cash flows related to the occurrence of contingent events. The framework of such generalization is based on the discrete-time Markov chain model.

73 Introduction to Markov Chains Process

A **stochastic process** is a collection of random variables $\{X_t : t \in T\}$. The index t is often interpreted as time. X_t is called the **state** of the process at time t. The set T is called the **index set:** If T is countable, the stochastic process is said to be **a discrete-time process.** If T is an interval of the real line, the stochastic process is said to be a **continuous-time process.** The **state space** E is the set of all possible values that the random variables can assume.

An example of a discrete-time stochastic process is the winnings of a gambler at successive games of blackjack; $T = \{1, 2, \cdots, 10\}, E =$ set of integers. An example of a time-continuous stochastic process is the hourly temperature measurement during a specific day: $T = \{12AM, 1AM, \cdots\}$ and $E =$ set of real numbers.

A **discrete time Markov chain** $\{X_0, X_1, X_2, \cdots\}$ is a discrete time stochastic process with values in the countable set $E = \{0, 1, 2, 3 \cdots\}$ and with the properties

(i) $\Pr(X_{n+1} \in E | X_n = i) = 1$.

(ii) $\Pr(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0) = \Pr(X_{n+1} = j | X_n = i)$.

This last property is referred to a the **Markov property**. Intuitively, this property says that the future probabilistic behavior of the sequence depends only on the present value of the sequence and not on the entire history of the sequence. This property is a type of memoryless property.

An example to illustrate the definitions given above, consider a coin that is thrown out repeatedly. let X_n be the number of heads obtained in the first n throws, $n = 0, 1, 2, \cdots$. Then $\{X_n : n = 0, 1, 2, \cdots\}$ is a discrete time Markov chain with state space $E = \{0, 1, 2, \cdots\}$. Moreover, we have

$$\Pr(X_{n+1} = j | X_n = i) = \begin{cases} \frac{1}{2} & \text{if } j = i \\ \frac{1}{2} & \text{if } j = i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

A **finite Markov chain** is one with a finite state space E. The probability of transition from state i to state j at time n will be denoted by $Q_n(i, j)$ and will be referred to as the **one-step transition probability.** The matrix

$$Q_n = \begin{pmatrix} Q_n(0,0) & Q_n(0,1) & \cdots & Q_n(0,r) \\ Q_n(1,0) & Q_n(1,1) & \cdots & Q_n(1,r) \\ \cdots & \cdots & \cdots & \cdots \\ Q_n(r,0) & Q_n(r,1) & \cdots & Q_n(r,r) \end{pmatrix}$$

where $E = \{0, 1, 2, \cdots, r\}$, is called the **transition probability matrix** at time n. Notice that the row in the matrix correspond to the departing state and that of the column corresponds to the arriving state. For example, the entry $Q_n(i, j)$ is the transition probability from state i at time n to state j at time $n + 1$.

Example 73.1
Consider a Markov chain with $E = \{0, 1\}$ and

$$Q_5 = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}$$

(i) Calculate $\Pr(X_6 = j|X_5 = 1), j = 0, 1$.
(ii) Calculate $\Pr(X_6 = 0|X_5 = j), j = 0, 1$.

Solution.
(i) We have

$$\begin{aligned} \Pr(X_6 = 0|X_5 = 1) =& 0.4 \\ \Pr(X_6 = 1|X_5 = 1) =& 0.7 \end{aligned}$$

(ii) We have

$$\begin{aligned} \Pr(X_6 = 0|X_5 = 0) =& 0.6 \\ \Pr(X_6 = 0|X_5 = 1) =& 0.4 \blacksquare \end{aligned}$$

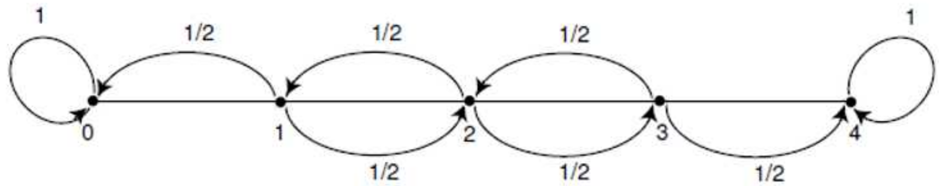
If the transition probability matrix Q_n depends on the time n, it is said to be a **non-homogeneous** Markov chain. Othewise, it is called a **homogeneous** Markov chain, and we shall simply denote the transition probability matrix by Q and the $ij-$th entry by $Q(i, j)$. For a homogeneous Markov chain a probability such as

$$\Pr(X_{n+1} = j|X_n = i)$$

is independent of n.

Example 73.2 *(Drunkard's Walk)*
A man walks along a four-block stretch of Park Avenue. If he is at corner 1, 2, or 3, then he walks to the left or right with equal probability. He continues until he reaches corner 4, which is a bar, or corner 0, which is his home. If he reaches either home or the bar, he stays there. Write the transition matrix of this homogeneous Markov chain.

Solution.
A **transition diagram** of this chain is shown below.



From this diagram, the transition matrix is

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

States 0 and 1 are called **absorbing state** since $Q(0,0) = Q(4,4) = 1$. In general, a state i such that $Q_n(i,i) = 1$ for all n is called an absorbing state ■

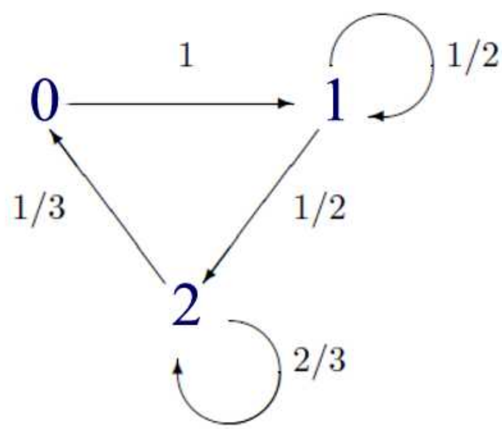
Practice Problems

Problem 73.1
Show that $\sum_{j\in E} Q_n(i,j) = 1$ for each i.

Problem 73.2
Can the following matrix be a legitimate transition matrix?

$$Q_0 = \begin{pmatrix} 0.6 & 0.4 \\ 0.5 & 0.7 \end{pmatrix}$$

Problem 73.3
Find the transition matrix of the Markov chain described by the following transition diagram.



Problem 73.4
A meteorologist studying the weather in a region decides to classify each day as simply sunny or cloudy. After analyzing several years of weather records, he finds:
• the day after a sunny day is sunny 80% of the time, and cloudy 20% of the time; and
• the day after a cloudy day is sunny 60% of the time, and cloudy 40% of the time.
Find the transition matrix of this Markov chain and draw the corresponding transition diagram.

Problem 73.5
Consider a basic survival model where state 0 be that (x) is alive and state

1 be that (x) is dead. Find the transition matrix at time n of this Markov chain

Problem 73.6
A multiple-decrement survival model is a Markov chain with state 0 representing death and state j representing the cause of decrement where $j = 1, 2, \cdots, m$. Find the one-step transition probabilities.

Problem 73.7 ‡
Kevin and Kira are modeling the future lifetime of (60).
(i) Kevin uses a double decrement model:

x	$\ell_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
60	1000	120	80
61	800	160	80
62	560	–	–

(ii) Kira uses a non-homogeneous Markov model:
(a) The states are 0 (alive), 1 (death due to cause 1), 2 (death due to cause 2).
(b) Q_{60} is the transition matrix from age 60 to 61; Q_{61} is the transition matrix from age 61 to 62.
(iii) The two models produce equal probabilities of decrement.
Calculate Q_{61}.

74 Longer Term Transition Probabilities

The transition probabilities $Q_n(i,j)$ and the transition probability matrix Q_n only provide information about the probability distribution of the state one time step in the future. In practice it is often important to know about longer periods of time.
The one-step transition probability is a special case of the k^{th} step transition probability which we define next. The probability of going from state i at time n to state j at time $n+k$ is denoted by

$$_kQ_n(i,j) = \Pr(X_{n+k} = j|X_n = i).$$

We call $_kQ_n(i,j)$ a k^{th} **step transition probability.** The k^{th} step transition matrix is denoted by $_kQ_n$.

Example 74.1
Consider a homogeneous Markov chain with transition matrix Q. Show that $_2Q = QQ$.

Solution.
The $ij-$ entry of $_2Q$ is $\Pr(X_{n+2} = j|X_n = i)$. We have

$$\begin{aligned}
\Pr(X_{n+2} = j|X_n = i) =& \Pr\left([X_{n+2} = j] \cap [\cup_{k=0}^{\infty}[X_{n+1} = k]]\,|X_n = i\right)\\
=& \sum_{k=0}^{\infty} \Pr(X_{n+2} = j, X_{n+1} = k|X_n = i)\\
=& \sum_{k=0}^{\infty} \Pr(X_{n+2} = j|X_{n+1} = k, X_n = i)\Pr(X_{n+1} = k|X_n = i)\\
=& \sum_{k=0}^{\infty} \Pr(X_{n+2} = j|X_{n+1} = k)\Pr(X_{n+1} = k|X_n = i)\\
=& \sum_{k=0}^{\infty} Q_{ik}Q_{kj} \blacksquare
\end{aligned}$$

Using induction on k, one can show

$$_kQ = Q \times Q \times \cdots \times Q = Q^k.$$

The argument used in the above example extends easily to the general case of longer-term probabilities for non-homogeneous Markov chains, resulting in

Theorem 74.1
The k^{th} step transition probability ${}_kQ_n(i,j)$ can b computed as the (i,j) entry of the matrix product $Q_nQ_{n-1}\cdots Q_{n+k-1}$. That is,

$$ {}_kQ_n = Q_nQ_{n-1}\cdots Q_{n+k-1}. \tag{74.-5}$$

Equation (74.1) is a matrix analogue of the survival probability identity

$$ {}_kp_x = p_xp_{x+1}\cdots p_{x+k-1}. $$

Example 74.2
Consider a basic survival model where state 0 be that (x) is alive and state 1 be that (x) is dead. Calculate ${}_2Q_0$.

Solution.
From Problem 73.5, we found

$$ Q_n = \begin{pmatrix} p_{x+n} & q_{x+n} \\ 0 & 1 \end{pmatrix} $$

Thus,

$$ \begin{aligned} {}_2Q_0 =& Q_0Q_1 = \begin{pmatrix} p_x & q_x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{x+1} & q_{x+1} \\ 0 & 1 \end{pmatrix} \\ =& \begin{pmatrix} p_xp_{x+1} & p_xq_{x+1} + q_x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} {}_2p_x & {}_2q_x \\ 0 & 1 \end{pmatrix} \blacksquare \end{aligned} $$

Example 74.3
The status of residents in a Continuing Care Retirement Community (CCRC) is modeled by a non-homogeneous Markov chain with three states: Independent Living (1), Health Center (2), and Gone (3). The transition probability matrices at time $t = 0, 1, 2, 3$ are

$$ Q_0 = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0 & 1 \end{pmatrix} \qquad Q_1 = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0 & 0.4 & 0.6 \\ 0 & 0 & 1 \end{pmatrix} $$

$$ Q_2 = \begin{pmatrix} 0.3 & 0.2 & 0.5 \\ 0 & 0.2 & 0.8 \\ 0 & 0 & 1 \end{pmatrix} \qquad Q_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} $$

Find ${}_2Q_0$ and ${}_4Q_0$.

Solution.
We have

$$
\begin{aligned}
{}_2Q_0 =& Q_0Q_1 = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.1 & 0.4 & 0.6 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.35 & 0.29 & 0.36 \\ 0.05 & 0.27 & 0.68 \\ 0 & 0 & 1 \end{pmatrix}\\
{}_4Q_0 =& Q_0Q_1Q_2Q_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \blacksquare
\end{aligned}
$$

Example 74.4 ‡
For a homogeneous Markov model with three states, Healthy (0), Disabled (1), and Dead (2):
(i) The annual transition matrix is given by

$$
\begin{array}{c c}
 & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0.70 & 0.20 & 0.10 \\ 0.10 & 0.65 & 0.25 \\ 0 & 0 & 1 \end{pmatrix}
\end{array}
$$

(ii) There are 100 lives at the start, all Healthy. Their future states are independent.
(iii) Assume that all lives have the same age at the start.
Calculate the variance of the number of the original 100 lives who die within the first two years.

Solution.
Let N denote the number of the original 100 lives who die within the first two years. Then N is a binomial random variable with parameters 100 and p where $p = {}_2q_x$ is the probability of failure within the the first two years. p is just the $(0,2)-$entry of the transition matrix ${}_2Q = Q^2$. From linear algebra, it is known that the matrix product $\mathbf{e}_1Q^2$ gives the first row of Q^2 where $\mathbf{e}_1 = (1,0,0)$. Hence,

$$
\mathbf{e}_1Q^2 = \begin{bmatrix} 0.51 \\ 0.27 \\ 0.22 \end{bmatrix} \Longrightarrow p = 0.22.
$$

Hence,

$$
\mathrm{Var}(N) = np(1-p) = 100(0.22)(0.78) = 17.16 \blacksquare
$$

Practice Problems

Problem 74.1
Consider a critical illness model with 3 states: healthy (H), critically ill (I) and dead (D).
Suppose you have the homogeneous Markov Chain with transition matrix

$$\begin{pmatrix} 0.92 & 0.05 & 0.03 \\ 0.00 & 0.76 & 0.24 \\ 0.00 & 0.00 & 1.00 \end{pmatrix}$$

What are the probabilities of being in each of the state at times $t = 1, 2, 3$.

Problem 74.2
Suppose that an auto insurer classifies its policyholders according to Preferred (State #1) or Standard (State #2) status, starting at time 0 at the start of the first year when they are first insured, with reclassifications occurring at the start of each new policy year.
You are given the following $t-$th year non-homogeneous transition matrix:

$$Q_t = \begin{pmatrix} 0.65 & 0.35 \\ 0.5 & 0.5 \end{pmatrix} + \frac{1}{t+1}\begin{pmatrix} 0.15 & \text{-}0.15 \\ \text{-}0.20 & 0.20 \end{pmatrix}$$

Given that an insured is Preferred at the start of the second year:
(a) Find the probability that the insured is also Preferred at the start of the third year.
(b) Find the probability that the insured transitions from being Preferred at the start of the third year to being Standard at the start of the fourth year.

Problem 74.3
A bond issue has three possible ratings: A, B and C. Transitions to these various ratings occur only at the end of each year, and are being modeled as a non-homogeneous Markov chain.
The transition matrices for periods 0, 1 and 2 are given as follows:

$$Q_0 = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.5 & 0.3 & 0.2 \\ 0.8 & 0.2 & 0.0 \end{pmatrix}, \; Q_1 = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.5 & 0.3 & 0.2 \\ 0.9 & 0.1 & 0.0 \end{pmatrix}, \; Q_2 = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.5 & 0.3 & 0.2 \\ 1.0 & 0.0 & 0.0 \end{pmatrix}$$

Calculate the probability that a bond issue with an A rating at the start of period 0 will have a rating of B at the end of period 2 and then a rating of C at the end of period 3.

Problem 74.4
For a non-homogeneous Markov model with three states: a, b, and c, you are given the annual transition matrix Q_n as follows:

$$Q_n = \begin{pmatrix} 0.60 & 0.20 & 0.20 \\ 0.00 & 0.00 & 1.00 \\ 0.00 & 0.00 & 1.00 \end{pmatrix}, \ n = 0, 1$$

and

$$Q_n = \begin{pmatrix} 0.10 & 0.20 & 0.70 \\ 0.00 & 0.00 & 1.00 \\ 0.00 & 0.00 & 1.00 \end{pmatrix}, \ n \geq 2,$$

Assuming transitions occur at the end of each year, calculate the probability that an individual who starts in state a will be in state c at the end of three years.

Problem 74.5
An automobile insurance company classifies drivers according to various states:
State 1: Excellent
State 2: Good
State 3: Bad
State 4: Terrible and has to be discontinued
Assume transitions follow a time-homogeneous Markov Chain model with the following transition matrix:

$$Q = \begin{pmatrix} 0.8 & 0.10 & 0.10 & 0.00 \\ 0.20 & 0.50 & 0.20 & 0.10 \\ 0.00 & 0.10 & 0.60 & 0.30 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix}$$

At the start of year 1, ten new drivers are insured and classified as Excellent drivers.
Calculate the probability that during the first 3 years, half of these new drivers become Terrible drivers and therefore have to be discontinued.

Problem 74.6 ‡
A certain species of flower has three states: sustainable, endangered and extinct. Transitions between states are modeled as a non-homogeneous Markov

chain with transition matrices Q_i as follows:

$$Q_0 = \begin{array}{c} \\ \textit{Sustainable} \\ \textit{Endagered} \\ \textit{Extinct} \end{array} \left(\begin{array}{ccc} \text{Sustainable} & \text{Endangered} & \text{Extinct} \\ 0.85 & 0.15 & 0.00 \\ 0.00 & 0.70 & 0.30 \\ 0.00 & 0.00 & 1.00 \end{array} \right)$$

$$Q_1 = \begin{pmatrix} 0.90 & 0.10 & 0.00 \\ 0.10 & 0.70 & 0.20 \\ 0.00 & 0.00 & 1.00 \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} 0.95 & 0.05 & 0.00 \\ 0.20 & 0.70 & 0.10 \\ 0.00 & 0.00 & 1.00 \end{pmatrix}$$

$$Q_i = \begin{pmatrix} 0.95 & 0.05 & 0.00 \\ 0.50 & 0.50 & 0.00 \\ 0.00 & 0.00 & 1.00 \end{pmatrix}, \ i = 3, 4, \cdots$$

Calculate the probability that a species endangered at time 0 will ever become extinct.

Problem 74.7 ‡
A homogeneous Markov model has three states representing the status of the members of a population.

State 1	=	healthy, no benefits
State 2	=	disabled, receiving Home Health Care benefits
State 3	=	disabled, receiving Nursing Home benefits

(i) The annual transition matrix is given by:

$$Q = \begin{pmatrix} 0.80 & 0.15 & 0.05 \\ 0.05 & 0.90 & 0.05 \\ 0.00 & 0.00 & 1.00 \end{pmatrix}$$

(ii) Transitions occur at the end of each year.
(iii) At the start of year 1, there are 50 members, all in state 1, healthy.
Calculate the variance of the number of those 50 members who will be receiving Nursing Home benefits during year 3.

75 Valuation of Cash Flows

The main goal of this section is the generalization of the valuation of insurance and annuity payments, benefit premiums and reserves. That is, we are concerned with cash flows while the subject is in a particular state or upon transition from one state to another in non-homogeneous Markov chains.
Events resulting in payments are classified into two categories: **Cash flows upon transitions** looks for payments that are made upon transition from one state to another. **Cash flows while in states** involves payments made due to being in a certain state for a particular time period.

75.1 Cash Flows Upon Transitions

The underlying principle by which payments are valued in the Markov chain context is the same as the one used for valuing insurance and annuity payments. The principle is described in the following way. Suppose that a payment of amount B_k is to be made k years from now contingent on event E_k occurring. If event E_k occurs then the payment will be made, and if the event does not occur then the payment will not be made. The actuarial present value now of the payment is

$$B_k \times \nu^k \times \Pr(E_k).$$

Take for example, the actuarial present value of a level insurance benefit of 1 for a discrete whole life

$$A_x = \sum_{k=0}^{\infty} C\nu^{k+1}{}_kp_xq_{x+k}.$$

If we think in terms of states, payment was made when there was a transition from state 1 (alive) to state 2 (dead). This was payment that provided a cash flow upon transition from the state 1 to state 2. We see that each term in the above sum is of the form

benefit for transition $(k, k+1)\times$ discount factor $\times$ probability of transition.

Example 75.1
Show that the actuarial present value of a one year discrete term insurance of amount b_1 at age (x) obeys the principle described above.

Solution.
The actuarial present value of a one year discrete term insurance of amount b_1 at age (x)

$$b_1 \cdot \nu \cdot q_x.$$

So the payment will be made at time $k = 1$, the payment amount is b_1, and the event leading to the payment being made is the death of (x) within a year, which has probability q_x ■

Next, we present the general form of the APV of contingent payments to be made for transfer between specific states in the future. We let ${}_{\ell+1}C^{(i,j)}$ denote the payment made at time if the chain made the trnasition from state i at time ℓ to state j at time $\ell+1$. Suppose that at time t the chain is in state s. Suppose that a payment of amount ${}_{t+k+1}C^{(i,j)}$ will be made at time $t+k+1$ ($k+1$ years from now) if the chain transfers from state i at time $t+k$ to state j at time $t+k+1$. See Figure 75.1.

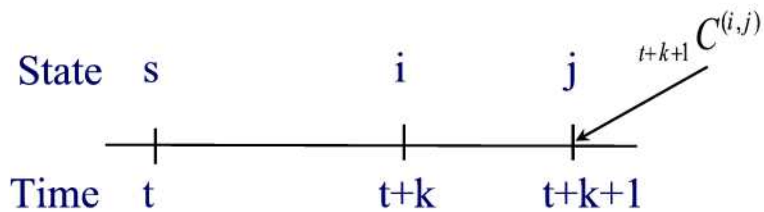


Figure 75.1

The APV of the payment will be

$${}_{t+k+1}C^{(i,j)} \times \nu^{k+1} \times {}_kQ_t(S,i)Q_{t+k}(i,j)$$

where the product ${}_kQ_t(S,i)Q_{t+k}(i,j)$ is the probability that the chain will be in state i at time $t+k$ and then will transfer to state j at time $t+k+1$. The APV of a series of such payments is

$$\text{APV}_{s@t} = \sum_{k=0}^{\infty} {}_{t+k+1}C^{(i,j)} \times \nu^{k+1} \times {}_kQ_t(s,i)Q_{t+k}(i,j).$$

Example 75.2 ‡
An insurance company issues a special 3-year insurance to a high risk individual. You are given the following homogeneous Markov chain model: (i)

$$\begin{array}{lcc} \text{State } 1 & = & \text{Active} \\ \text{State } 2 & = & \text{Disabled} \\ \text{State } 3 & = & \text{Withdrawn} \\ \text{State } 4 & = & \text{Dead} \end{array}$$

Transition probability matrix:

$$Q = \begin{pmatrix} 0.4 & 0.2 & 0.3 & 0.1 \\ 0.2 & 0.5 & 0 & 0.3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(ii) Changes in state occur at the end of the year.
(iii) The death benefit is 1000, payable at the end of the year of death.
(iv) $i = 0.05$
(v) The insured is disabled at the end of year 1.
Calculate the actuarial present value of the prospective death benefits at the beginning of year 2.

Solution.
We have

Possible Transition	Probability	Discounted Death Benefit	APV
$2 \to 4$	0.3	1000ν	$0.3(1000\nu)$
$2 \to 2 \to 4$	$(0.5)(0.3)$	$1000\nu^2$	$(0.5)(0.3)(1000\nu^2)$
$2 \to 1 \to 4$	$(0.2)(0.1)$	$1000\nu^2$	$(0.2)(0.1)(1000\nu^2)$

The actuarial present value of the prospective death benefits at the beginning of year 2 is

$$\text{APV} = 0.3(1000\nu) + 0.17(1000\nu^2) = 300(1.05)^{-1} + 170(1.05)^{-2} = 439.91 \blacksquare$$

75.2 Cash Flows while in State

Recall from Section 37.1 that a whole life annuity due is a series of payments made at the beginning of the year while an annuitant is alive. Its actuarial present value is given by

$$\ddot{a}_x = \sum_{k=0}^{\infty} \nu^k {}_kp_x.$$

If we think in terms of states, the annuity payment is made as long as the annuitant remains in state 1 (alive). This is a payment that provides a cash flow **while in a state.** The probability of remaining in state 1 (i.e., surviving) is given by ${}_kp_x$. We see that each term in the above sum is of the form

Payment at time $k\times$ discount factor $\times$ probability of being in state k.

Next, we present the general formulation for the actuarial present value of a series of payments to be made when the chain is in state i. Let ${}_\ell C^{(i)}$ denote the payment made at time ℓ if the chain is in state i at time ℓ. If the chain is in state s at time t then the actuarial present value at time t of the payment to be made at time $t+k$ is

$${}_{t+k}C^{(i)} \times \nu^k \times {}_kQ_t(s,i)$$

where ${}_kQ_t(s,i)$ is the probability of being in state i at time $t+k$ if in state s at time t. The APV of a series of such payments is

$$\mathrm{APV}_{s@t} = \sum_{k=0}^{\infty} {}_{t+k}C^{(i)} \times \nu^k \times {}_kQ_t(s,i).$$

Example 75.3

The status of residents in a Continuing Care Retirement Community (CCRC) is modeled by a non-homogeneous Markov chain with three states: Independent Living (1), Health Center (2), and Gone (3). The transition probability matrices at time $t = 0, 1, 2, 3$ are

$$Q_0 = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0 & 1 \end{pmatrix} \qquad Q_1 = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0 & 0.4 & 0.6 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} 0.3 & 0.2 & 0.5 \\ 0 & 0.2 & 0.8 \\ 0 & 0 & 1 \end{pmatrix} \qquad Q_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

A CCRC resident begins in independent living at time $t = 0$. He will pay a premium of P at the beginning of each year that he is in independent living, but no premium otherwise. Find the APV of these premiums. Use the interest rate $i = 25\%$.

Solution.
We have

Time	Possible Transition	Probability	Discounted DB	APV
0	$IL \to IL$	1	P	P
1	$IL \to IL$	0.7	$P\nu$	$0.7P\nu$
2	$IL \to IL \to IL$	(0.7)(0.5)	$P\nu^2$	$0.35\nu^2$
2	$IL \to HC \to IL$	(0.2)(0)	$P\nu^2$	0
3	$IL \to IL \to IL \to IL$	(0.7)(0.5)(0.3)	$P\nu^3$	$0.105P\nu^3$
3	$IL \to HC \to IL \to IL$	(0.2)(0)(0.3)	$P\nu^3$	0
3	$IL \to IL \to HC \to IL$	(0.7)(0.3)(0)	$P\nu^3$	0
3	$IL \to HC \to HC \to IL$	(0.3)(0.6)(0)	$P\nu^3$	0

The APV of the premiums is

$$P[1 + 0.7\nu + 0.35\nu^2 + 0.105\nu^3] = 1.8378P \blacksquare$$

75.3 Benefit Premiums and Reserves

Benefit premiums and reserves can be found by applying the Equivalence Principle to any contingent payment situation.

Example 75.4
A four-state homogeneous Markov model represents the joint mortality of a married couple: a husband and a wife. The states are: 1 = husband alive, wife alive; 2 = husband dead, wife alive; 3 = husband alive, wife dead, and 4 = both husband and wife dead.
The one-year transition probabilities are:

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0.95 & 0.02 & 0.02 & 0.01 \\ 0.00 & 0.90 & 0.0 & 0.10 \\ 0.00 & 0.00 & 0.85 & 0.15 \\ 0.00 & 0.00 & 0.00 & 1.00 \end{pmatrix} \end{array}$$

A life insurer sells a two-year term insurance contract to a married couple who are both age 60. The death benefit of 100 is payable at the end of the year in which the second life dies, if both die within 2 years.
Premiums are payable so long as at least one of them is alive and annually in advance. Interest rate $i = 5\%$.
Calculate the annual benefit premium

Solution.
Let P denote the annual benefit premium. We have

Year	Possible Transition	Probability	APV
0	–	1	P
1	$1 \to 1$	0.95	$0.95P\nu$
1	$1 \to 2$	0.02	$0.02P\nu$
1	$1 \to 3$	0.02	$0.02P\nu$.

Thus, APVP $= P + 0.99\nu P = 1.943P$.
For the benefits, we have

Year	Possible Transition	Probability	APV
1	$1 \to 4$	0.01	$100(0.01)\nu$
2	$1 \to 1 \to 4$	(0.95)(0.01)	$100(0.0095)\nu^2$
2	$1 \to 2 \to 4$	(0.02)(0.10)	$100(0.0002)\nu^2$
2	$1 \to 3 \to 4$	(0.02)(0.15)	$100(0.003)\nu^2$.

Thus,

$$\text{APVB} = 100(0.01)\nu + (100)(0.0145)\nu^2 = 2.270.$$

By the equivalence principle, we have

$$1.943P = 2.270 \Longrightarrow P = \frac{2.270}{1.943} = 1.1683 \blacksquare$$

Example 75.5 ‡
The CAS Insurance Company classifies its auto drivers as Preferred (State #1) or Standard (State #2) starting at time 0 at the start of the first year when they are first insured, with reclassifications occurring at the start of each new policy year. The transition-probability matrices Q_n from the state at time n at the start of year $n+1$ to the state at time $n+1$ are

$$Q_n = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} + \frac{1}{n+1}\begin{pmatrix} 0.1 & -0.1 \\ -0.2 & 0.2 \end{pmatrix}$$

Driver F is Standard now, at the start of the fourth year. For $k = 0, 1$, there is a cost of $10(1.1)^k$ at the end of year $4+k$ for a transition from Standard at the start of that year to Preferred at the start of the next year. These costs will be funded by allocations ("premiums") P paid at time 3 if Driver F is Standard at time 3 and paid at time 4 if Driver F is Standard at time 4. The allocation is determined to be $P = 3.1879$ by the Equivalence Principle, using 15% interest. Suppose that Driver F is Standard at the start of the fifth year; find the benefit reserve.

Solution.
We have

$$\begin{aligned} {}_4V =& \text{APVFB} - \text{APVFP} \\ =& Q_4^{(2,1)}\nu(11) - 3.1879 \\ =& 0.36(1.15)^{-1}(11) - 3.1879 = 0.25558 \blacksquare \end{aligned}$$

Practice Problems

Problem 75.1 ‡

For a perpetuity-immediate with annual payments of 1:
(i) The sequence of annual discount factors follows a Markov chain with the following three states:

State Number	0	1	2
Annual Discount Factor, ν	0.95	0.94	0.93

(ii) The transition matrix for the annual discount factors is:

$$\left[\begin{array}{ccc} 0.0 & 1.0 & 0.0 \\ 0.9 & 0.0 & 0.1 \\ 0.0 & 1.0 & 0.0 \end{array}\right]$$

Y is the present value of the perpetuity payments when the initial state is 1. Calculate $E(Y)$.

Problem 75.2 ‡

A machine is in one of four states (F, G, H, I) and migrates annually among them according to a Markov process with transition matrix:

	F	G	H	I
F	0.20	0.80	0.00	0.00
G	0.50	0.00	0.50	0.00
H	0.75	0.00	0.00	0.25
I	1.00	0.00	0.00	0.00

At time 0, the machine is in State F. A salvage company will pay 500 at the end of 3 years if the machine is in State F.
Assuming $\nu = 0.90$, calculate the actuarial present value at time 0 of this payment. Hint: Recall from linear algebra the fact that $\mathbf{e}_i Q^n$ is the i^{th} row of Q^n where $\mathbf{e}_i$ is the unit vector with 1 at the ith component and 0 elsewhere.

Problem 75.3 ‡

For a Markov model for an insured population: (i) Annual transition probabilities between health states of individuals are as follows:

	Healthy	Sick	Terminated
Healthy	0.7	0.1	0.2
Sick	0.3	0.6	0.1
Terminated	0.0	0.0	1.0

(ii) The mean annual healthcare claim each year for each health state is:

	Mean
Healthy	500
Sick	3000
Terminated	0

(iii) Transitions occur at the end of the year.
(iv) $i = 0$
A contract premium of 800 is paid each year by an insured not in the terminated state.
Calculate the expected value of contract premiums.

Problem 75.4
The status of residents in a Continuing Care Retirement Community (CCRC) is modeled by a non-homogeneous Markov chain with three states: Independent Living (1), Health Center (2), and Gone (3).
(i) The transition probability matrices at time $t = 0, 1, 2, 3$ are

$$Q_0 = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0 & 1 \end{pmatrix} \qquad Q_1 = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0 & 0.4 & 0.6 \\ 0 & 0 & 1 \end{pmatrix}$$
$$Q_2 = \begin{pmatrix} 0.3 & 0.2 & 0.5 \\ 0 & 0.2 & 0.8 \\ 0 & 0 & 1 \end{pmatrix} \qquad Q_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(ii) Transitions oocur at the end of each year.
(iii) The CCRC incurs a cost of 1000 at the end of year k for a transition from Independent Living at the start of that year to Health Center at the start of the next year, for all k.
(iv) The CCRC wishes to charge a fee P at the start of each of the first three years for each resident then in Independent Living.
(v) Nathan enters Independent Living at time 0.
(vi) $i = 0.25$
Find P using the equivalence principle.

Problem 75.5 ‡
A non-homogenous Markov model has:

(i) Three states: 0, 1, and 2
(ii) Annual transition matrix Q_n as follows:

$$Q_n = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ n = 0, 1$$

$$Q_n = \begin{pmatrix} 0 & 0.3 & 0.7 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ n = 2, 3, \cdots$$

An individual starts out in state 0 and transitions occur mid-year.
An insurance is provided whereby: • A premium of 1 is paid at the beginning of each year that an individual is in state 0 or 1.
• A benefit of 4 is paid at the end of any year that the individual is in state 1 at the end of the year.
• $i = 0.1$
Calculate the actuarial present value of premiums minus the actuarial present value of benefits at the start of this insurance.

Problem 75.6 ‡
For a special 3-year term insurance:
(i) Insureds may be in one of three states at the beginning of each year: active, disabled, or dead. All insureds are initially active. The annual transition probabilities are as follows:

	Active	Disabled	Dead
Active	0.8	0.1	0.1
Disabled	0.1	0.7	0.2
Dead	0.0	0.0	1.0

(ii) A 100,000 benefit is payable at the end of the year of death whether the insured was active or disabled.
(iii) Premiums are paid at the beginning of each year when active. Insureds do not pay any annual premiums when they are disabled.
(iv) $d = 0.10$
Calculate the level annual benefit premium for this insurance.

Problem 75.7 ‡
For a Markov model for an insured population: (i) Annual transition probabilities between health states of individuals are as follows:

	Healthy	Sick	Terminated
Healthy	0.7	0.1	0.2
Sick	0.3	0.6	0.1
Terminated	0.0	0.0	1.0

(ii) The mean annual healthcare claim each year for each health state is:

	Mean
Healthy	500
Sick	3000
Terminated	0

(iii) Transitions occur at the end of the year.
(iv) $i = 0$
A contract premium of 800 is paid each year by an insured not in the terminated state.
Calculate the expected value of contract premiums less healthcare costs over the first 3 years for a new healthy insured.

Probability Models: Poisson Processes

Poisson process is a sepcial case of a continuous Markov chains. It plays an important role in insurance application: The total insurance claims consists usually of a sum of individual claim amounts. The number of claims is usually assumed to occur according to a Poisson process.
Simply put, the Poisson process is a counting process for the number of events that have occurred up to a particular time. The purpose of this chapter is to cover the material on Poisson processes needed for Exams MLC/3L.

76 The Poisson Process

A stochastic process $\{N(t) : t \geq 0\}$ is called a **counting process** if N(t) represents the total number of events that have occurred up to time t.
A counting process $N(t)$ must satisfy:
(i) $N(t) \geq 0$.
(ii) $N(t)$ is integer valued.
(iii) If $s < t$, then $N(s) \leq N(t)$, and $N(t) - N(s)$ is the number of events that have occurred from (after) time s up to (and including) time t.
A counting process $\{N(t) : t \geq 0\}$ is said to possess the **independent increments property** if for each $s < t < u$ the increments $N(t) - N(s)$ and $N(u) - N(t)$ are independent random variables. That is, the numbers of events occurring in disjoint intervals of time are independent of one another.
A counting process has **stationary increments** if the number of events in an interval depend only on the length of the interval. In other words, for any r and s and $t \geq 0$, the distribution of $N(t+r) - N(r)$ is the same as the distribution of $N(t+s) - N(s)$ (both intervals have time length t).
A counting process $\{N(t) : t \geq 0\}$ is said to be a **Poisson process** with **rate** λ if the following conditions are satisfied:
(i) $N(0) = 0$
(ii) $\{N(t) : t \geq 0\}$ satisfies the independent increments property.
(iii) The number of events occurring in a time interval of length t has a Poisson distribution with mean λt. In other words, for any $s \geq 0$ and $t \geq 0$ the random variable $N(t+s) - N(s)$ is a Poisson distribution with mean λt so that

$$\Pr(N(t+s) - N(s) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!},\ k = 0, 1, 2, \cdots$$

From this, the expected number (and variance) of events occurring in a time interval of length t is

$$\mathrm{Var}(N(t+s) - N(s)) = E[N(t+s) - N(s)] = \lambda t.$$

Notice also that in the right hand side of the formula in (iii) , s doesn't show up at all. This tells us that no matter when we start a period of time of length t, the distribution for the number of occurrences is the same. So a Poisson Process has stationary increments.

Example 76.1
Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 2$. Calculate $\Pr(N(5) = 4)$.

Solution.
The answer is

$$\Pr(N(5) = 4) = e^{-10}\frac{10^4}{4!} = 0.0189 \blacksquare$$

Example 76.2
Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate per unit time $\lambda = 2$. Calculate
(a) $E[2N(3) - 4N(5)]$
(b) $\mathrm{Var}(2N(3) - 4N(5))$.

Solution.
(a) We have

$$E[2N(3) - 4N(5)] = 2E[N(3)] - 4E[N(5)] = 2(2 \times 3) - 4(2 \times 5) = -28.$$

(b) We have

$$\begin{aligned}\mathrm{Var}(2N(3) - 4N(5)) =& \mathrm{Var}[-2(N(3) - N(0)) - 4(N(5) - N(3)]\\ =&(-2)^2\mathrm{Var}(N(3) - N(0)) + (-4)^2\mathrm{Var}(N(5) - N(3))\\ =&4(2 \times 3) + 16(2 \times 2) = 88 \blacksquare\end{aligned}$$

When finding probabilities involving numbers of events or event times in a Poisson process it is usually convenient to take advantage of the independent increments property, if possible. We illustrate this point in the next example.

Example 76.3
Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate per unit time $\lambda = 1$. Find $\Pr(N(2) = 1|N(1) = 0)$.

Solution.
We have

$$\begin{aligned}\Pr(N(2) = 1|N(1) = 0) =& \frac{\Pr([N(1) = 0] \cap [N(2) = 1])}{\Pr(N(1) = 0)}\\ =&\frac{\Pr([N(1) = 0] \cap [N(2) - N(1) = 1])}{\Pr(N(1) = 0)}\\ =&\Pr(N(2) - N(1) = 1) = e^{-1} \blacksquare\end{aligned}$$

Theorem 76.1
Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate per unit time λ. Let $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ and $0 \leq k_1 \leq k_2 \leq \cdots \leq k_n$. Then

$$\begin{aligned}\Pr(N(t_1) = k_1, N(t_2) = k_2, \cdots, N(t_n) = k_n) =& \frac{e^{-\lambda t_1}(\lambda t_1)^{k_1}}{k_1!} \times \frac{e^{-\lambda(t_2-t_1)}(\lambda(t_2-t_1))^{k_2}}{k_2!} \\ & \times \cdots \times \frac{e^{-\lambda(t_n-t_{n-1})}(\lambda(t_n-t_{n-1}))^{k_n}}{k_n!}.\end{aligned}$$

Proof.
First, note that the events

$$[N(t_1) = k_1], [N(t_2) - N(t_1) = k_2 - k_1], \cdots, [N(t_n) - N(t_{n-1}) = k_n - k_{n-1}]$$

are indepedent. Thus,

$$\begin{aligned}\Pr(N(t_1) = k_1, N(t_2) = k_2, \cdots, N(t_n) = k_n) =& \\ \Pr(N(t_1) = k_1, N(t_2) - N(t_1) = k_2 - k_1, \cdots, N(t_n) - N(t_{n-1}) = k_n - k_{n-1}) =& \\ \Pr(N(t_1) = k_1)\Pr(N(t_2) - N(t_1) = k_2 - k_1) \cdots \Pr(N(t_n) - N(t_{n-1}) = k_n - k_{n-1}) =& \\ \frac{e^{-\lambda t_1}(\lambda t_1)^{k_1}}{k_1!} \times \frac{e^{-\lambda(t_2-t_1)}(\lambda(t_2-t_1))^{k_2}}{k_2!} \times \cdots \times \frac{e^{-\lambda(t_n-t_{n-1})}(\lambda(t_n-t_{n-1}))^{k_n}}{k_n!} & \blacksquare\end{aligned}$$

The following theorem says that a Poisson process is a Markov chain with continuous time and state space $E = \{0, 1, \cdots\}$.

Theorem 76.2
Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate per unit time λ. Let $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < s$ and $0 \leq k_1 \leq k_2 \leq \cdots \leq k_n \leq j$. Then

$$\Pr(N(s) = j | N(t_1) = k_1, \cdots, N(t_n) = k_n) = \Pr(N(s) = j | N(t_n) = k_n).$$

Proof.
Using Bayes formula we have

$$\Pr(N(s) = j | N(t_1) = k_1, \cdots, N(t_n) = k_n) = \frac{\Pr([N(t_1) = k_1, \cdots, N(t_n) = k_n] \cap [N(s) = j])}{\Pr(N(t_1) = k_1, \cdots, N(t_n) = k_n)}.$$

Let

$$A = \Pr(N(t_1) = k_1, N(t_2) - N(t_1) = k_2 - k_1, \cdots, N(t_n) - N(t_{n-1}) = k_n - k_{n-1}).$$

Using the independent increments property, we have

$$\Pr(N(t_1) = k_1, \cdots, N(t_n) = k_n, N(s) = j) = A \times \Pr(N(s) - N(t_n) = j - k_n).$$

Hence,

$$\begin{aligned}\Pr(N(s) = j | N(t_1) = k_1, \cdots, N(t_n) = k_n) =& \Pr(N(s) - N(t_n) = j - k_n)\\ =& \Pr(N(s) - N(t_n) = j - k_n | N(t_n) = k_n)\\ =& \Pr(N(s) = j | N(t_n) = k_n) \blacksquare\end{aligned}$$

Theorem 76.3
Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate per unit time λ. Let $s, t > 0$ and $k, j \geq 0$. Then

$$\Pr(N(s + t) = k | N(s) = j) = \Pr(N(t) = k - j).$$

That is, the distribution of $N(t + s)$ given $N(s) = j$ is $j+$ Poisson (λt). Hence,

$$E[N(t + s) | N(s) = j] = j + \lambda t \text{ and } \mathrm{Var}[N(t + s) | N(s) = j] = \lambda t.$$

Proof.
Since $N(s)$ and $N(t + s) - N(s)$ are independent, we have

$$\Pr[N(s+t) = k | N(s) = j] = \Pr[N(s+t) - N(s) = k - j | N(s) = j] = \Pr[N(s+t) - N(s) = k - j] \blacksquare$$

Example 76.4
Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate per unit time $\lambda = 3$. Compute $E[2N(5) - 3N(7) | N(3) = 2]$.

Solution.
(a) We have

$$\begin{aligned}E[2N(5) - 3N(7) | N(3) = 2] =& 2E[N(5) | N(3) = 2] - 3E[N(7) | N(3) = 2]\\ =& 2[2 + 3(2)] - 3[2 + 3(4)] = -26 \blacksquare\end{aligned}$$

Example 76.5 ‡
For a water reservoir:
(i) The present level is 4999 units.
(ii) 1000 units are used uniformly daily.
(iii) The only source of replenishment is rainfall.
(iv) The number of rainfalls follows a Poisson process with $\lambda = 0.2$ per day.
(v) The distribution of the amount of a rainfall is as follows:

Amount	Probability
8000	0.2
5000	0.8

(vi) The numbers and amounts of rainfalls are independent. Calculate the probability that the reservoir will be empty sometime within the next 10 days.

Solution.
The reservoir has initially 4999 units and 1000 units are used per day so that one way for the reservoir to be empty sometime within the next 10 days is to have norainfall within the next 5 days. The probability of such thing to happen is

$$\Pr(0 \text{ rainfall in 5 days}) = e^{-0.2(5)} = 0.3679.$$

Another way for the reservoir to be empty sometime within the next 10 days is to have one rainfall of 1000 units daily for the next five days and no rainfall in the following five days. The probability of such event to occur is

$$\begin{aligned}
&\Pr([1 \text{ rainfall and 5000 units in 5 days}]and[0 \text{ rainfall in 5 days}]) = \\
&\quad \Pr(1 \text{ rainfall in 5 days})\Pr(5000 \text{ units in 5 days})\Pr(0 \text{ rainfall in 5 days}) \\
&=0.2(5)e^{-0.2(5)}(0.8)e^{-0.2(5)} = 0.1083.
\end{aligned}$$

Hence, the probability that the reservoir will be empty sometime within the next 10 days is $0.3679 + 0.1083 = 0.476$ ■

Practice Problems

Problem 76.1
Show that for each $0 \leq e \leq t$, we have

$$\text{Cov}(N(s), N(t)) = \lambda s.$$

Problem 76.2
Let $\{N(t), t \geq 0\}$ be a Poisson process with rate $\lambda = 2$. Find $\Pr(N(5) = 4, N(6) = 9, N(10) = 15)$.

Problem 76.3
Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate per unit time $\lambda = 3$. Find $\Pr(N(5) = 7|N(3) = 2)$.

Problem 76.4
Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate per unit time $\lambda = 2$. Find $E[2N(3) - 4N(5)]$ and $\text{Var}[2N(3) - 4N(5)]$.

Problem 76.5
Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate per unit time λ. Let $s, t \geq 0$. Show that

$$\Pr(N(t) = k|N(s+t) = n) = \left(\begin{array}{c} n \\ k \end{array}\right)\left(\frac{t}{t+s}\right)^k\left(\frac{s}{t+s}\right)^{n-k}.$$

That is, the distribution of $N(t)$ given $N(s+t) = n$ is a binomial distribution with parameters n and $\frac{t}{t+s}$. Hence,

$$E[N(t)|N(s+t) = n] = \tfrac{nt}{t+s} \text{ and } \text{Var}[N(t+s)|N(s) = j] = n\tfrac{t}{t+s}\tfrac{s}{t+s}.$$

Problem 76.6
Customers arrive at a store according to a Poisson process with a rate 40 customers per hour. Assume that three customers arrived during the first 15 minutes. Calculate the probability that no customer arrived during the first five minutes.

Problem 76.7 ‡
A member of a high school math team is practicing for a contest. Her advisor has given her three practice problems: #1, #2, and #3.

She randomly chooses one of the problems, and works on it until she solves it. Then she randomly chooses one of the remaining unsolved problems, and works on it until solved. Then she works on the last unsolved problem.
She solves problems at a Poisson rate of 1 problem per 5 minutes.
Calculate the probability that she has solved problem #3 within 10 minutes of starting the problems.

Problem 76.8 ‡
For a tyrannosaur with 10,000 calories stored:
(i) The tyrannosaur uses calories uniformly at a rate of 10,000 per day. If his stored calories reach 0, he dies.
(ii) The tyrannosaur eats scientists (10,000 calories each) at a Poisson rate of 1 per day.
(iii) The tyrannosaur eats only scientists.
(iv) The tyrannosaur can store calories without limit until needed.
(a) Calculate the probability that the tyrannosaur dies within the next 2.5 days.
(b) Calculate the expected calories eaten in the next 2.5 days.

77 Interarrival and Waiting Time Distributions

Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ. For each positive integer n, we define

$$S_n = \inf\{t \geq 0 : N(t) = n\}.$$

We call S_n the **waiting time** or the **arrival time** of the n^{th} event. Let $T_n = S_n - S_{n-1}$ be the time elapsed between the $(n-1)^{\text{st}}$ event and the n^{th} event. We call T_n the **interarrival time**. Note that T_n is the arrival time of the n^{th} event from the time of arrival of the $(n-1)^{\text{st}}$ event. A simple induction on n shows that $S_n = \sum_{i=1}^{n} T_i$.
Next, we establish the following important result.

Theorem 77.1
The interarrival times $T_n, n = 1, 2, \cdots$ are identically independent exponential random variables with mean $\frac{1}{\lambda}$.

Proof.
We will show the result for $n = 2$. The general case can be derived by using mathematical induction on n.
We have

$$\Pr(T_1 > t) = \Pr(N(t) = 0) = e^{-\lambda t}.$$

Thus,

$$f_{T_1}(t) = \lambda e^{-\lambda t} \Longrightarrow T_1 \text{ is exponential with parameter } \lambda.$$

If $n = 2$, we have

$$\begin{aligned}\Pr(T_2 > t | T_1 = s) =& \Pr(N(t+s) - N(s) = 0 | N(s) - N(0) = 1)\\ =& \Pr(N(t+s) - N(s) = 0) = e^{-\lambda t}.\end{aligned}$$

Hence

$$f_{T_2}(t) = \lambda e^{-\lambda t} \Longrightarrow T_2 \text{ is exponential with parameter } \lambda \text{ and } T_1, T_2 \text{ are independent} \blacksquare$$

Remark 77.1
This result should not come as a surprise because the assumption of independent and stationay increments means that the process from any moment on is independent of all that occured before and also has the same distribution

as the process started at 0. In other words the process is memoryless, and we know from probability theory that any continuous random variable on $(0, \infty)$ with the memoryless property has to have an exponential distribution.
Also, the previous theorem says that if the rate of events is λ events per unit of time, then the expected waiting time between events is $\frac{1}{\lambda}$.

Theorem 77.2
The waiting time S_n is a Gamma distribution with parameters n and λ.

Proof.
Let $Y = X_1 + X_2 + \cdots + X_n$ where each X_i is an exponential random variable with parameter λ. Then

$$M_Y(t) = \prod_{k=1}^{n} M_{X_k}(t) = \prod_{k=1}^{n} \left(\frac{\lambda}{\lambda - t}\right) = \left(\frac{\lambda}{\lambda - t}\right)^n, \quad t < \lambda.$$

Since this is the mgf of a gamma random variable with parameters n and λ we can conclude that Y is a gamma random variable with parameters n and λ ■
By the previous theorem and Section 14.4, we have $E(S_n) = \frac{n}{\lambda}$ and $\text{Var}(S_n) = \frac{n}{\lambda^2}$. Moreover, S_n has the density function

$$f_{S_n}(t) = e^{-\lambda t} \frac{\lambda^n t^{n-1}}{(n-1)!}, \ t \geq 0.$$

Using Problem 77.1 we have

$$\begin{aligned} F_{S_n}(t) =& \Pr(S_n \leq t) = \Pr(N(t) \geq n) \\ =& \sum_{k=n}^{\infty} \Pr(N(t) = k) = e^{-\lambda t} \sum_{k=n}^{\infty} \frac{(\lambda t)^k}{k!}. \end{aligned}$$

Example 77.1
Up to yesterday a store has 999,856 customers. They are planning to hold a little party when the 1,000,000$^{\text{th}}$ customer comes into the store. From experience they know that a customer arrives about every 4 minutes, and the store is open from 9am to 6pm. What is the probability that they will have the party today?

Solution.
They will have the party if at least 144 customers come into the store today. Let's assume that the customers arrive according to a Poisson process with rate 0.25 customer per minutes then we want the probability $\Pr(S_{144} < 9 \times 60 = 540)$. But S_{144} is the Gamma distribution $\Gamma(144, 0.25)$. Thus,

$$\Pr(S_{144} < 540) = \int_0^{540} 0.25^{144} \frac{t^{143} e^{-0.25t}}{143!} dt \blacksquare$$

Example 77.2
Suppose that people immigrate to a particular territory at a Poisson rate of $\lambda = 1.5$ per day.
(a) What is the expected time until the 100^{th} immigrant arrives?
(b) What is the probability that the elapsed time between the 100^{th} immigrant and the next immigrant's arrival exceeds 2 days?
(c) What is the probability that the 100^{th} immigrant will arrive after one year? Assume there are 365 days in a year.

Solution.
(a) We have $E(S_{100}) = \frac{n}{\lambda} = \frac{100}{1.5} = 66.7$ days.
(b) We have $\Pr(T_{101} > 2) = e^{-1.5(2)} = 0.04979$.
(c) We have

$$\Pr(S_{100} > 365) = 1 - \Pr(S_{100} < 365) = 1 - F_{S_{100}}(365) = 1 - e^{-1.5(365)} \sum_{k=100}^{\infty} \frac{[1.5(365)]^k}{k!} \blacksquare$$

Theorem 77.3
Given that exactly one event of a Poisson process $\{N(t); t \geq 0\}$ has occurred during the interval $[0, t]$, the time of occurrence of this event is uniformly distributed over $[0, t]$.

Proof.
For $0 \leq s \leq t$, we have

$$\Pr(S_1 \leq s | N(t) = 1) = \frac{\Pr([S_1 \leq s)] \cap [N(t) = 1])}{\Pr(N(t) = 1)}.$$

Using the equivalence $\{S_1 \leq s\} \Leftrightarrow \{N(s) = 1\}$ we can write

$$\begin{aligned}
\{S_1 \leq s\} \cap \{N(t) = 1\} =& \{N(s) = 1\} \cap \{N(t) = 1\} \\
=& \{N(s) = 1\} \cap \{N(t) - N(s) = 0\}.
\end{aligned}$$

Applying the independence of increments over non-overlapping intervals. we have

$$\begin{aligned}\Pr(S_1 \le s | N(t) = 1) =& \frac{\Pr(N(s) = 1)\Pr(N(t) - N(s) = 0)}{\Pr(N(t) = 1)}\\ =& \frac{e^{-\lambda s}(\lambda s)e^{-\lambda(t-s)}}{e^{-\lambda t}(\lambda t)} = \frac{s}{t}.\end{aligned}$$

Hence,

$$f_{S_1|N(t)=1}(s) = \frac{1}{t},\ 0 \le s \le t.$$

This, shows that $T_1|N(t) = 1$ is uniformly distributed over $[0, t]$ ■

The above theorem is immediately generalized to n events. For any set of real variables s_j satisfying $0 = s_0 < s_1 < s_2 < \cdots < s_n < t$ and given that n events of a Poisson process $\{N(t) : t \ge 0\}$ have occurred during the interval $[0, t]$, the probability of the successive occurrence times $0 < S_1 < S_2 < \cdots < S_n < t$ of these n Poisson events is

$$\Pr(S_1 \le s_1, \cdots, S_n \le s_n | N(t) = n) = \frac{\Pr([S_1 \le s_1, \cdots, S_n \le s_n] \cap [N(t) = n])}{\Pr(N(t) = n)}.$$

Using a similar argument as in the proof of the above theorem, we have

$$\begin{aligned}p =& \Pr([S_1 \le s_1, \cdots, S_n \le s_n] \cap [N(t) = n]))\\ =& \Pr(N(s_1) - N(s_0) = 1, \cdots, N(s_n) - N(s_{n-1}) = 1, N(t) - N(s_n) = 0)\\ =& \left(\prod_{i=1}^{n} \Pr(N(s_i) - N(s_{i-1}) = 1)\right) \Pr(N(t) - N(s_n) = 0)\\ =& \left(\prod_{i=1}^{n} \lambda(s_i - s_{i-1})e^{-\lambda(s_i - s_{i-1})}\right) e^{-\lambda(t-s_n)}\\ =& \lambda^n \prod_{i=1}^{n}(s_i - s_{i-1})e^{-\lambda[\sum_{i=1}^{n}(s_i - s_{i-1}) + t - s_n]} = \lambda^n e^{-\lambda t} \prod_{i=1}^{n}(s_i - s_{i-1}).\end{aligned}$$

Thus,

$$\Pr(S_1 \le s_1, \cdots, S_n \le s_n | N(t) = n) = \frac{\lambda^n e^{-\lambda t} \prod_{i=1}^{n}(s_i - s_{i-1})}{\frac{(\lambda t)^n}{n!}e^{-\lambda t}} = \frac{n!}{t^n} \prod_{i=1}^{n}(s_i - s_{i-1})$$

from which the density function

$$f_{S_1,S_2,\cdots,S_n|N(t)=n}(s_1,s_2,\cdots,s_n) = \frac{\partial^n}{\partial s_1\cdots\partial s_n}[\Pr(S_1\le s_1,\cdots,S_n\le s_n|N(t)=n)]$$

follows as

$$f_{S_1,S_2,\cdots,S_n|N(t)=n}(s_1,s_2,\cdots,s_n) = \frac{n!}{t^n}.$$

Example 77.3
Show that for $0\le s\le t$ and $0\le k\le n$ we have

$$\Pr(N(s)=k|N(t)=n) = \left(\begin{array}{c} n \\ k \end{array}\right)\frac{s^k}{t^n}(t-s)^{n-k}.$$

Solution.
We have

$$\begin{aligned}
\Pr(N(s)=k|N(t)=n) =&\frac{\Pr([N(s)=k]\cap[N(t)=n])}{\Pr(N(t)=n)}\\
=&\frac{\Pr([N(s)=k]\cap[N(t)-N(s)=n-k])}{\Pr(N(t)=n)}\\
=&\frac{\Pr(N(s)=k)\Pr(N(t)-N(s)=n-k)}{\Pr(N(t)=n)}\\
=&\frac{e^{-\lambda s}\frac{(\lambda s)^k}{k!}e^{-\lambda(t-s)}\frac{[\lambda(t-s)^{n-k}]}{(n-k)!}}{e^{-\lambda t}\frac{(\lambda t)^n}{n!}}\\
=&\left(\begin{array}{c} n \\ k \end{array}\right)\frac{s^k}{t^n}(t-s)^{n-k}\\
=&\left(\begin{array}{c} n \\ k \end{array}\right)p^k(1-p)^{(n-k)}
\end{aligned}$$

where $p=\frac{s}{t}$. Thus, given $N(t)=n$, the number of occurrences in the interval $[0,t]$ has a binomial distribution with n trials and success probability $\frac{s}{t}$ ■

Practice Problems

Problem 77.1
Show that if $S_n \leq t$ then $N(t) \geq n$.

Problem 77.2
Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 3$. Let S_n denote the time of the occurrence of the n^{th} event. Calculate $\Pr(S_3 > 5)$.

Problem 77.3
Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 3$. Let S_n denote the time of the occurrence of the n^{th} event. Calculate the expected value and the variance of S_3.

Problem 77.4
Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 3$. Calculate the density function of T_5.

Problem 77.5
Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 3$. Find the expected value and the variance of T_5.

Problem 77.6 ‡
The time elapsed between the claims processed is modeled such that T_n represents the time elapsed between processing the $(n-1)^{\text{st}}$ and n^{th} claim where T_1 is the time until the first claim is processed, etc.
You are given:
(i) $T_1, T_2, \cdots$ are mutually independent; and
(ii) $f_{T_n}(t) = 0.1e^{-0.1t}$, for $t > 0$, where t is measured in half-hours.
Calculate the probability that at least one claim will be processed in the next 5 hours.

Problem 77.7 ‡
Subway trains arrive at a station at a Poisson rate of 20 per hour. 25% of the trains are express and 75% are local. The type of each train is independent of the types of preceding trains. An express gets you to the stop for work in 16 minutes and a local gets you there in 28 minutes. You always take the first train to arrive. Your co-worker always takes the first express. You both are waiting at the same station.
Compare your expected arrival time to that of your co-worker.

78 Superposition and Decomposition of Poisson Process

In this section, we will consider two further properties of the Poisson process that both have to do with deriving new processes from a given Poisson process.

Theorem 78.1
Let $\{N_1(t)\}, \{N_2(t)\}, \cdots, \{N_n(t)\}$ be independent Poisson processes with rates $\lambda_1, \lambda_2, \cdots, \lambda_n$, respectively. Then $N(t) := N_1(t) + N_2(t) + \ + N_n(t)$ is a Poisson processes with rate $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Proof.
Let's check the three Poisson axioms of the definition stated in Section 76.

(i) $N(0) = 0$.
This is easy to verify since

$$N(0) = \sum_{i=1}^{n} N_i(0) = 0$$

since each process $N_i(t)$ is Poisson.
(ii) Independent Increments: Consider times $0 < s < t < u$ and consider the increments

$$N(t) - N(s) \text{ and } N(u) - N(t).$$

Using the fact that $N(t)$ is defined as $\sum_{i=1}^{n} N_i(t)$ these become

$$\textstyle\sum_{i=1}^{n}[N_i(t) - N_i(s)] \text{ and } \sum_{i=1}^{n}[N_i(u) - N_i(t)].$$

Now, each term in the first sum is independent of all terms in the second sum with a different subscript since the processes $\{N_i(t)\}$ and $\{N_j(t)\}$ are, for $i \neq j$, independent by assumption. Also, the term $N_i(t) - N_i(s)$ is independent of $N_i(u) - N_i(t)$ for all $i = 1, 2, \cdots, n$ since $\{N_i(t)\}$ is a Poisson process. Therefore,

$$\textstyle\sum_{i=1}^{n}[N_i(t) - N_i(s)] \text{ and } \sum_{i=1}^{n}[N_i(u) - N_i(t)]$$

are independent.
(iii) $N(s+t) - N(s)$ is a Poisson random variable with parameter λt. See Section 16.2.
Since all three Poisson axioms are satisfied, we have that $\{N(t) : t \geq 0\}$ is a Poisson process with rate $\sum_{i=1}^{n} \lambda_i$ ■
We call $\{N(t) : t \geq 0\}$ the **superposition** of $\{N_1(t) : t \geq 0\}, \{N_2(t) : t \geq 0\}, \cdots, \{N_n(t) : t \geq 0\}$.

Example 78.1
A building can be accessed from two different entrances: The west entrance and the east entrance. The flows of people arriving to the building from these two entrances are independent Poisson processes with rates $\lambda_W = 0.5$ per minute and $\lambda_E = 1.5$ per minute, respectively.
Estimate the probability that more than 200 people entered the building during a fixed 30-minute time interval?

Solution.
Let $N_E(t)$ and $N_W(t)$ denote the number of people entering the building in the time interval $[0, t]$ from the East and West entrances respectively. Then $N(t) = N_E(t) + N_W(t)$ represents tne number of people entering the building in that interval. $\{N(t) : t \geq 0\}$ is a Poisson process with rate $\lambda = 2$ people per minute. Moreover, $E(N(30)) = 30(2) = 60$ and $\mathrm{Var}(N(30)) = 30(2) = 60$. Hence,

$$\Pr(N(30) > 200) \approx \Pr\left(Z > \frac{200 - 60}{\sqrt{60}}\right) = \Pr(Z > 18.07) \approx 0 \ ■$$

Theorem 78.2
Let $\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ be two independent Poisson processes with respective rates λ_1 and λ_2. Let $N(t) = N_1(t) + N_2(t), t \geq 0$. Let $\lambda = \lambda_1 + \lambda_2$. Then, the conditional distribution of $N_1(t)$ given $N(t) = n$ is binomial with parameters n and $p = \frac{\lambda_1}{\lambda}$. That is, given that the total number of occurrences is n, the probability that a given occurrence is of type 1 is $\frac{\lambda_1}{\lambda}$ independently of the rest of the occurrences.

Proof.
We have

$$\begin{aligned}
\Pr(N_1(t)=k|N(t)=n) =& \frac{\Pr([N_1(t)=k]\cap[N(t)=n])}{\Pr(N(t)=n)}\\
=& \frac{\Pr([N_1(t)=k]\cap[N_2(t)=n-k])}{\Pr(N(t)=n)}\\
=& \frac{\Pr(N_1(t)=k)\Pr(N_2(t)=n-k)}{\Pr(N(t)=n)}\\
=& \frac{e^{-\lambda_1 t}\frac{(\lambda_1 t)^k}{k!}e^{-\lambda_2 t}\frac{(\lambda_2 t)^{n-k}}{(n-k)!}}{e^{-(\lambda_1+\lambda_2)t}\frac{[(\lambda_1+\lambda_2)t]^n}{n!}}\\
=& \left(\begin{array}{c} n \\ k \end{array}\right) p^k(1-p)^{n-k} \blacksquare
\end{aligned}$$

Example 78.2
A building can be accessed from two different entrances: The west entrance and the east entrance. The flows of people arriving to the building from these two entrances are independent Poisson processes with rates $\lambda_W = 0.5$ per minute and $\lambda_E = 1.5$ per minute, respectively.
What is the probability that a given person actually entered from the East entrance?

Solution.
The probability that a given person actually entered from the East entrance is $\frac{\lambda_E}{\lambda_E+\lambda_W} = \frac{1.5}{0.5+1.5} = 0.75$ $\blacksquare$

Next, we consider the question of **splitting** or **thinning** a Poisson process. Suppose that $\{N(t) : t \geq 0\}$ is a Poisson process with rate λ. Suppose additionally that each event, independently of the others, is of one of two types: Type 1 with probability p, Type 2 with probability $q = 1 - p$. This is sometimes referred to as **splitting** a Poisson process. For example, the arrivals are customers at a service station and each customer is classified as either male (type I) or female (type II).
Let $\{N_i(t) : t \geq 0\}$ be the number of type i events in the interval $[0, t]$ where $i = 1, 2$. We have

Theorem 78.3
$\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ are independent Poisson processes with respective rates λp and $\lambda(1 - p)$.

Proof.
First, since $N(0) = N_1(0) + N_2(0) = 0$, we have $N_1(0) = N_2(0) = 0$. Next, note that for any given non-negative integers n and m and for $j \neq n+m$ we have

$$\Pr(N_1(t+s) - N_1(s) = n, N_2(t+s) - N_2(s) = m | N(t+s) - N(s) = j) = 0$$

where $t, s \geq 0$. Thus, we have

$$\begin{aligned}
&\Pr(N_1(t+s) - N_1(s)) = n, N_2(t+s) - N_2(s) = m) = \\
&\quad \sum_{j=0}^{\infty} \Pr(N_1(t+s) - N_1(s) = n, N_2(t+s) - N_2(s) = m | N(t+s) - N(s) = j) \\
&\times \Pr(N(t+s) - N(s) = j) \\
&= \Pr(N_1(t+s) - N_1(s) = n, N_2(t+s) - N_2(s) = m | N(t+s) - N(s) = n+m) \\
&\times \Pr(N(t+s) - N(s) = n+m) \\
&= \left(\begin{array}{c} n+m \\ n \end{array} \right) p^n (1-p)^m e^{-\lambda t} \frac{(\lambda t)^{n+m}}{(n+m)!} \\
&= e^{-\lambda t} \frac{(\lambda t)^n}{n!} e^{-\lambda(1-p)} \frac{[\lambda(1-p)]^m}{m!}.
\end{aligned}$$

Hence,

$$P(N_1(t+s) - N_1(s) = n) = \sum_{m=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} e^{-\lambda(1-p)} \frac{[\lambda(1-p)]^m}{m!} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

Likewise,

$$P(N_2(t+s) - N_2(s) = m) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} e^{-\lambda(1-p)} \frac{[\lambda(1-p)]^m}{m!} = e^{-\lambda(1-p)} \frac{[\lambda(1-p)]^m}{m!}.$$

It follows that $N_1(t+s) - N_1(s)$ is a Poisson random variable with parameter λt and $N_2(t+s) - N(s)$ is a Poisson random variable with parameter $\lambda(1-p)$. Moreover, the two variables are independent as shown above.
Finally, let $0 < s < t < u$. Since all events are independent of each other, $N_1(t) - N_1(s)$ and $N_1(u) - N_1(t)$ are independent random variables. Likewise, $N_2(t) - N_2(s)$ and $N_2(u) - N_2(t)$ are independent random variables. It follows that $\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ are Poisson processes ■

Remark 78.1
This "thinning" can easily be generalized to many event type occurring independently with probabilities $p_1, p_2, \cdots$.

Example 78.3
Consider an insurance company that has two types of policy: Policy A and Policy B. Total claims from the company arrive according to a Poisson process at the rate of 9 per day. A randomly selected claim has a 1/3 chance that it is of policy A.
(a) Calculate the probability that claims from policy A will be fewer than 2 on a given day.
(b) Calculate the probability that claims policy B will be fewer than 2 on a given day.
(c) Calculate the probability that total claims from the company will be fewer than 2 on a given day.

Solution.
Let $N_A(t)$ be the number of policy A claims $N_B(t)$ be the number of policy B claims. N_A and N_B are two independent Poisson processes with rates 3 and 6 respectively.
(a) We have

$$\Pr(N_A(1) < 2) = \Pr(N_A(1) = 0) + \Pr(N_A(1) = 1) = e^{-3} + 3e^{-3} = 0.19915.$$

(b) We have

$$\Pr(N_B(1) < 2) = \Pr(N_B(1) = 0) + \Pr(N_B(1) = 1) = e^{-6} + 6e^{-6} = 0.01735.$$

(c) We have We have

$$\Pr(N(1) < 2) = \Pr(N(1) = 0) + \Pr(N(1) = 1) = e^{-9} + 9e^{-9} = 0.00123 \blacksquare$$

Example 78.4 ‡
Workers' compensation claims are reported according to a Poisson process with mean 100 per month. The number of claims reported and the claim amounts are independently distributed. 2% of the claims exceed 30,000.
Calculate the number of complete months of data that must be gathered to have at least a 90% chance of observing at least 3 claims each exceeding 30,000.

Solution.
Let $N_1(t)$ be the number of claims exceeding 30000 received until time t months. Then $\{N_1(t) : t \geq 0\}$ is a Poisson process with rate $\lambda p = 0.02(100) = 2$. We have

$$\begin{aligned}
\Pr(N_1(2) \geq 3) =& 1 - \Pr(N(2) = 0) - \Pr(N(2) = 1) - \Pr(N(2) = 2)\\
=& 1 - e^{-4} - e^{-4}\frac{4}{1!} - e^{-4}\frac{4^2}{2!} = 0.761 < 90\%\\
\Pr(N_1(3) \geq 3) =& 1 - \Pr(N(3) = 0) - \Pr(N(3) = 1) - \Pr(N(3) = 2)\\
=& 1 - e^{-6} - e^{-6}\frac{6}{1!} - e^{-6}\frac{6^2}{2!} = 0.93 > 90\%
\end{aligned}$$

Thus, the number of months is 2 ■

Consider two independent Poisson processes $\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ with respective rates λ_1 and λ_2. Let $S_n^{(I)}$ be the time of the n^{th} event for the first Poisson process. Let $S_m^{(II)}$ be the time of the m^{th} arrival for the second Poisson process. What is the probability that the n^{th} event of process I occurs before the m^{th} event of process II?
At the time of the $(n+m-1)^{\text{th}}$ event for both Poisson processes, we have observed n or more of the first Poisson process. Hence,

$$\Pr(S_n^{(I)} < S_m^{(II)}) = \sum_{k=n}^{n+m-1} \left(\begin{array}{c} n \\ k \end{array} \right) p^k (1-p)^{n+m-1-k}$$

where

$$p = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Example 78.5
An insurance company receives two type of claims: car and home. The number of car insurance claims received follows a Poisson process distribution with rate 20 claims per day. The number of home insurance claims received follows a Poisson process distribution with rate 5 claims per day. Both processes are independent.
Calculate the probability that at least two car insurance claims arrive before three home insurance claims arrive.

Solution.
The probability that a claim is a car insurance is $p = \frac{20}{20+5} = 0.8$. Let $S_2^{(C)}$ be the time of receiving the second car claims. Let $S_3^{(H)}$ be the time of receiving the third home claims. Then

$$\Pr(S_2^{(C)} < S_5^{(H)}) = \sum_{k=2}^{4} \left(\begin{array}{c} 4 \\ k \end{array} \right) (0.8)^k (0.2)^{4-k} \approx 0.9728 \blacksquare$$

Practice Problems

Problem 78.1
A building can be accessed from two different entrances: The west entrance and the east entrance. The flows of people arriving to the building from these two entrances are independent Poisson processes with rates $\lambda_W = 0.5$ per minute and $\lambda_E = 1.5$ per minute, respectively.
What is the probability that no one will enter the building during a fixed three-minute time interval?

Problem 78.2
An insurance company receives two type of claims: car and home. The number of car insurance claims received follows a Poisson process distribution with rate 20 claims per day. The number of home insurance claims received follows a Poisson process distribution with rate 5 claims per day. Both processes are independent. Suppose that in a given day five claims are received. Calculate the probability that exactly three claims are car insurance claims.

Problem 78.3
Customers arrive to a store according with a Poisson process with rate $\lambda =$ 20 arrivals per hour. Suppose that the probability that a customer buys something is p = 0.30.
(a) Find the expected number of sales made during an eight-hour business day.
(b) Find the probability that 10 or more sales are made in a period of one hour.
(c) The store opens at 8 a.m. find the expected time of the tenth sale of the day.

Problem 78.4 ‡
Lucky Tom finds coins on his way to work at a Poisson rate of 0.5 coins per minute. The denominations are randomly distributed:
(i) 60% of the coins are worth 1;
(ii) 20% of the coins are worth 5;
(iii) 20% of the coins are worth 10.
Calculate the variance of the value of the coins Tom finds during his one-hour walk to work.

Problem 78.5 ‡
A Poisson claims process has two types of claims, Type I and Type II.
(i) The expected number of claims is 3000.
(ii) The probability that a claim is Type I is 1/3.
(iii) Type I claim amounts are exactly 10 each.
(iv) The variance of aggregate claims is 2,100,000.
Calculate the variance of aggregate claims with Type I claims excluded.

Problem 78.6
Customers arrive at a service facility according to a Poisson process with rate 10 per hour. The service facility classifies arriving customers according to three types, with the probability of the types being $p_1 = 0.5,\ p_2 = 0.3,\ p_3 = 0.2$.
Find the probability that there are 2 type one arrivals before 2 arrivals of types 2 and 3 combined.

Problem 78.7 ‡
Subway trains arrive at a station at a Poisson rate of 20 per hour. 25% of the trains are express and 75% are local. The type of each train is independent of the types of preceding trains. An express gets you to the stop for work in 16 minutes and a local gets you there in 28 minutes. You always take the first train to arrive. Your co-worker always takes the first express. You both are waiting at the same station.
Calculate the probability that the train you take will arrive at the stop for work before the train your co-worker takes.

Problem 78.8 ‡
Lucky Tom finds coins on his way to work at a Poisson rate of 0.5 coins/minute. The denominations are randomly distributed:
(i) 60% of the coins are worth 1 each
(ii) 20% of the coins are worth 5 each
(iii) 20% of the coins are worth 10 each.
Calculate the probability that in the first ten minutes of his walk he finds at least 2 coins worth 10 each, and in the first twenty minutes finds at least 3 coins worth 10 each.

Problem 78.9 ‡
Kings of Fredonia drink glasses of wine at a Poisson rate of 2 glasses per day.

Assassins attempt to poison the king's wine glasses. There is a 0.01 probability that any given glass is poisoned. Drinking poisoned wine is always fatal instantly and is the only cause of death.
The occurrences of poison in the glasses and the number of glasses drunk are independent events.
Calculate the probability that the current king survives at least 30 days.

Problem 78.10 ‡
Job offers for a college graduate arrive according to a Poisson process with mean 2 per month. A job offer is acceptable if the wages are at least 28,000. Wages offered are mutually independent and follow a lognormal distribution with $\mu = 10.12$ and $\sigma = 0.12$. For the lognormal distribution,

$$F(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$$

where $\Phi(z)$ is the cdf of the standard normal distribution.
Calculate the probability that it will take a college graduate more than 3 months to receive an acceptable job offer.

Problem 78.11 ‡
Subway trains arrive at your station at a Poisson rate of 20 per hour. 25% of the trains are express and 75% are local. The types and number of trains arriving are independent. An express gets you to work in 16 minutes and a local gets you there in 28 minutes. You always take the first train to arrive. Your co-worker always takes the first express. You are both waiting at the same station.
Calculate the conditional probability that you arrive at work before your co-worker, given that a local arrives first.

Problem 78.12 ‡
Lucky Tom deposits the coins he finds on the way to work according to a Poisson process with a mean of 22 deposits per month.
5% of the time, Tom deposits coins worth a total of 10.
15% of the time, Tom deposits coins worth a total of 5.
80% of the time, Tom deposits coins worth a total of 1.
The amounts deposited are independent, and are independent of the number of deposits. Calculate the variance in the total of the monthly deposits.

Problem 78.13 ‡
A casino has a game that makes payouts at a Poisson rate of 5 per hour and the payout amounts are $1, 2, 3, \cdots$ without limit. The probability that any given payout is equal to i is $\frac{1}{2^i}$. Payouts are independent.
Calculate the probability that there are no payouts of 1, 2, or 3 in a given 20 minute period.

79 Non-Homogeneous Poisson Process

Thus far we have considered a Poisson process with a constant intensity λ. This can be generalized to a so called non-homogeneous Poisson process by letting the intensity to vary in time.
A counting process $\{N(t) : t \geq 0\}$ is called a **non-homogeneous Poisson process with intensity function** $\lambda(t),\ t \geq 0$, if
(i) $N(0) = 0$
(ii) The process has independent increments.
(iii) $N(s+t) - N(s)$, the number of events occurring in a time interval from time s to time $t+s$ has a Poisson distribution with mean $\int_s^{s+t} \lambda(u)du$.
The function $m(t)$, called the **mean value function** of the process, is defined to be

$$m(t) = \int_0^t \lambda(s)ds$$

so that

$$E[N(s+t) - N(s)] = m(s+t) - m(s).$$

Some Observations:
• In the non-homogeneous case, the rate parameter $\lambda(t)$ now depends on t.
• When $\lambda(t) = \lambda$, constant, then it reduces to the homogeneous case.
• A non-homogeneous Poisson process will not have stationary increments, in general.

Example 79.1
For a non-homogenous Poisson process the intensity function is given by $\lambda(t) = \frac{1}{1+t},\ t \geq 0$. Find the probability that the number of observed occurrences in the time period $[0, 1]$ is more than 1.

Solution.
We first find the mean function

$$m(1) = \int_0^1 \frac{dt}{1+t} = \ln(1+t)|_0^1 = \ln 2.$$

Thus,

$$\Pr(N(1) \geq 1) = 1 - \Pr(N(1) = 0) = 1 - e^{-\ln 2} = \frac{1}{2} \blacksquare$$

Example 79.2
Suppose that a nonhomogeneous Poisson process has intensity function

$$\lambda(t) = \begin{cases} 10 & \text{if } t \text{ is in } (0, 1/2], (1, 3/2], \cdots \\ 2 & \text{if } t \text{ is in } (1/2, 1], (3/2, 2], \cdots \end{cases}$$

Find the probability that the number of observed occurrences in the time period $(1.5, 4]$ is more than three.

Solution.
$N(4) - N(1.5)$ has a Poisson distribution with mean

$$m(4)-m(1.5) = \int_0^4 \lambda(t)dt = \int_{1.5}^2 2dt+\int_2^{2.5} 10dt+\int_{2.5}^3 2dt+\int_3^{3.5} 10dt+\int_{3.5}^4 2dt = 13.$$

Thus,

$$\begin{aligned}\Pr(N(4) - N(1.5) > 3) =& 1 - \Pr(N(4) - N(1.5) = 0) - \Pr(N(4) - N(1.5) = 1) \\ & - \Pr(N(4) - N(1.5) = 2) - \Pr(N(4) - N(1.5) = 3) \\ =& 1 - e^{-13}\left[1 + 13 + \frac{13^2}{2} + \frac{13^3}{6}\right] = 0.9989 \blacksquare\end{aligned}$$

Let S_n be the time of the n^{th} occurrence. Then

$$\Pr(S_n > t) = \Pr(N(t) \le n - 1) = \sum_{k=0}^{n-1} e^{-m(t)} \frac{(m(t))^k}{k!}.$$

Hence,

$$F_{S_n}(t) = 1 - \Pr(S_n > t) = 1 - \sum_{k=0}^{n-1} e^{-m(t)} \frac{(m(t))^k}{k!}$$

and

$$\begin{aligned}f_{S_n}(t) =& -\sum_{k=0}^{n-1} \frac{d}{dt}\left(e^{-m(t)}\frac{(m(t))^k}{k!}\right) \\ =& -\sum_{k=1}^{n-1} k\lambda(t)e^{-m(t)}\frac{(m(t))^{k-1}}{k!} + \sum_{k=0}^{n-1} \lambda(t)e^{-m(t)}\frac{(m(t))^k}{k!} \\ =& e^{-m(t)}\frac{(m(t))^{n-1}}{(n-1)!}.\end{aligned}$$

Example 79.3
Suppose that a nonhomogeneous Poisson process has intensity function

$$\lambda(t) = \begin{cases} 10 & \text{if } t \text{ is in } (0, 1/2], (1, 3/2], \cdots \\ 2 & \text{if } t \text{ is in } (1/2, 1], (3/2, 2], \cdots \end{cases}$$

If $S_{10} = 0.45$ is given, calculate the probability that $S_{11} > 0.75$.

Solution.
We have

$$\begin{aligned} \Pr(S_{11} > 0.75|S_{10} = 0.45) =&\Pr(N(0.75) = 10|S_{10} = 0.45) \\ =&\Pr(N(0.75) - N(0.45) = 0|S_{10} = 0.45) \\ =&\Pr(N(0.75) - N(0.45) = 0|N(0.45) - N(0) = 10) \\ =&\Pr(N(0.75) - N(0.45) = 0) = e^{-[m(0.75)-m(0.45)]} = e^{-1} \end{aligned}$$

since

$$m(0.75) - m(0.45) = \int_{0.45}^{0.75} \lambda(t)dt = \int_{0.45}^{0.5} 10dt + \int_{0.5}^{0.75} 2dt = 1 \blacksquare$$

Example 79.4 ‡
For a claims process, you are given:
(i) The number of claims $\{N(t) : t \geq 0\}$ is a nonhomogeneous Poisson process with intensity function:

$$\lambda(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 2, & 1 \leq t < 2 \\ 3, & 2 \leq t \end{cases}$$

(ii) Claims amounts Y_i are independently and identically distributed random variables that are also independent of $N(t)$.
(iii) Each Y_i is uniformly distributed on [200,800].
(iv) The random variable P is the number of claims with claim amount less than 500 by time $t = 3$.
(v) The random variable Q is the number of claims with claim amount greater than 500 by time $t = 3$.
(vi) R is the conditional expected value of P, given $Q = 4$.
Calculate R.

Solution.
Since P and Q are independent random variables, we have $R = E[P|Q = 4] = E[P]$. $N(3)$ is a Poisson random variable with mean

$$m(3) = \int_0^3 \lambda(t)dt = \int_0^1 dt + \int_1^2 2dt + \int_2^3 3dt = 6.$$

Since

$$P(Y_i < 500) = \int_{200}^{500} \frac{1}{600}dt = \frac{1}{2}$$

we obtain that P is a Poisson random variable with mean $\lambda p = 3$. Hence, $R = 3$ ■

Practice Problems

Problem 79.1
Suppose that claims arrive at an insurance company according to a Poisson process with intensity function $\lambda(t) = 1 + t$ where t is measured in hours. Calculate the probability there will be exactly 3 claims arriving within the first two hours.

Problem 79.2
For a non-homogenous Poisson process the intensity function is given by $\lambda(t) = \frac{1}{1+t},\ t \geq 0$. Find the expected time until the first event.

Problem 79.3 ‡
Assume that the customers in a department store arrive at a Poisson rate that increases linearly from 6 per hour at 1 PM, to 9 per hour at 2 PM. Calculate the probability that exactly 2 customers arrive between 1 PM and 2 PM.

Problem 79.4
An insurance company finds that for a certain group of insured drives, the number of accidents over each 24-hour period rises from midnight to noon, and then declines until the following midnight.
Suppose that the number of accidents can be modeled by a non-homogeneous Poisson process where the intensity at time t is given by

$$\lambda(t) = \frac{1}{6} - \frac{(12-t)^2}{1152}$$

where t is the number of hours since midnight.
(a) Calculate the expected number of daily accidents.
(b) Calculate the probability that there will be exactly one accident between 6:00 AM and 6:00 PM.

Problem 79.5
Claims from an insurance company arrive, within a one-month period according to a non-homogeneous Poisson process.
The intensity function, $\lambda(t)$, gives the number of claims per day and varies with t, the number of days in the month. Assume 30 days in one month. You are given:

$$\lambda(t) = \begin{cases} 0.1 & 0 \leq t < 8 \\ 0.05t & 8 \leq t < 15 \\ 0.02 & 15 \leq t \leq 30 \end{cases}$$

Calculate the probability that there will be fewer than 2 claims in one month.

Problem 79.6 ‡
Beginning with the first full moon in October deer are hit by cars at a Poisson rate of 20 per day. The time between when a deer is hit and when it is discovered by highway maintenance has an exponential distribution with a mean of 7 days. The number hit and the times until they are discovered are independent.
Calculate the expected number of deer that will be discovered in the first 10 days following the first full moon in October.

80 Compound Poisson Process

A counting process $\{X(t) : t \geq 0\}$ is a **compound Poisson process** if

$$X(t) = \sum_{i=1}^{N(t)} Y_i(t)$$

where $\{N(t) : t \geq 0\}$ is a Poisson process with intensity λ and $\{Y_i : i = 1, 2, \cdots\}$ are independent identically distributed random variables independent of $\{N(t) : t \geq 0\}$.

Theorem 80.1
If $\{X(t) : t \geq 0\}$ is a compound Poisson process then

$$E[X(t)] = E[N]E[Y_1] = \lambda t E[Y_1] \text{ and}$$
$$\text{Var}[X(t)] = E(N)\text{Var}(Y_1) + (E(Y_1))^2\text{Var}(N) = \lambda t E[Y_1^2].$$

Proof.
Using the double expectation theorem, we have that

$$\begin{aligned}
E[X(t)|N(t) = n] =& E[\sum_{i=1}^{n} Y_i|N(t) = n] = nE[Y_1] \\
E[X(t)] =& E[E[X(t)|N(t) = n]] = E[nE[Y_1]] = E[N(t)E[Y_1]] \\
=& E[N(t)]E[Y_1] = \lambda t E[Y_1] \\
\text{Var}[X(t)|N(t) = n] =& \text{Var}[\sum_{i=1}^{n} Y_i|N(t) = n] = n\text{Var}(Y_1) \\
\text{Var}[X(t)] =& E[\text{Var}[X(t)|N(t) = n]] + \text{Var}[E[X(t)|N(t) = n]] \\
=& E[N(t)\text{Var}(Y_1)] + \text{Var}[N(t)E[Y_1]] \\
=& \lambda t\text{Var}(Y_1) + \lambda t(E[Y_1])^2 \\
=& \lambda t\text{Var}(Y_1) + \lambda t[E[Y_1^2] - \text{Var}(Y_1)] \\
=& \lambda t E[Y_1^2] \ \blacksquare
\end{aligned}$$

We next discuss an important application of the concept of compound Poisson process to insurance. Let $N(t)$ denote the number of claims that an insurance company receives in the time interval $[0, t]$. Assume that $\{N(t) : t \geq 0\}$ is a Poisson process with rate λ. Let $\{Y_i : i = 1, 2, \cdots\}$ denote a sequence of claims that the company gets. Assume that $\{Y_i : i = 1, 2, \cdots\}$ is a set of independent identically distributed random variable that is independent of $\{N(t) : t \geq 0\}$. Let $X(t) = \sum_{i=1}^{N(t)} Y_i$ be the total amount of claims received until time t. $X(t)$ is called the **aggregate claims**.

Example 80.1 ‡
The claims department of an insurance company receives envelopes with claims for insurance coverage at a Poisson rate of $\lambda = 50$ envelopes per week. For any period of time, the number of envelopes and the numbers of claims in the envelopes are independent. The numbers of claims in the envelopes have the following distribution:

# of Claims	Probability
1	0.20
2	0.25
3	0.40
4	0.15

Find the mean and variance of the aggregate claims that occur by time 13.

Solution.
Let $N(t)$ be the number of envelopes received until time t. Then $N(t)$ is a Poisson process with rate $\lambda = 50$. Let $\{Y_i : i = 1, 2, \cdots\}$ be the number of claims received in each envelope. $\{Y_i : i = 1, 2, \cdots\}$ is an idenpent identically distributed set of random variables independent of $\{N(t) : t \geq 0\}$. The total number of claims received by time 13 is

$$X(13) = \sum_{i=1}^{N(13)} Y_i.$$

We have

$$\begin{aligned}
E[Y_1] =& 1(0.20) + 2(0.25) + 3(0.40) + 4(0.15) = 2.5\\
E[Y_1^2] =& 1^2(0.20) + 2^2(0.25) + 3^2(0.40) + 4^2(0.15) = 7.2\\
E[X(13)] =& (50)(13)(2.5) = 1625
\end{aligned}$$

$\text{Var}[X(13)] = (50)(13)(7.2) = 4680$ ■

Example 80.2
In the example above, estimate $\Pr(X(13) > 1800)$ assuming that $X(13)$ can be approximated by a normal random variable with mean 1625 and variance 4680^2.

Solution.
The usual Normal-variable approximation method gives

$$\begin{aligned}\Pr(X(13) > 1800) =&\Pr\left(Z > \frac{1800 - 1625}{\sqrt{4680}}\right)\\ =&\Pr(Z > 2.57) = 1 - 0.9948 = 0.0052 \blacksquare\end{aligned}$$

Example 80.3 ‡
Bob is an overworked underwriter. Applications arrive at his desk at a Poisson rate of 60 per day. Each application has a 1/3 chance of being a "bad" risk and a 2/3 chance of being a "good" risk.
Since Bob is overworked, each time he gets an application he flips a fair coin. If it comes up heads, he accepts the application without looking at it. If the coin comes up tails, he accepts the application if and only if it is a "good" risk. The expected profit on a "good" risk is 300 with variance 10,000. The expected profit on a "bad" risk is −100 with variance 90,000.
Calculate the variance of the profit on the applications he accepts today.

Solution.
Let N be the number of applications received today. Then N is a Poisson process with rate $\lambda = 60$ per day. Let N_G denote the number of "good" risk applications accepted. Then N_G is a Poisson process with rate $\lambda_G = \frac{2}{3}(60) = 40$. Let X_G be the profit per "good" received and S_G be the aggregate profit of "good" received. We have

$$\begin{aligned}\text{Var}(S_G) =&E[N_G]\text{Var}(X_G) + \text{Var}(N_G)(E(X_G))^2\\ =&(40)(10,000) + 40(300)^2 = 4,000,000.\end{aligned}$$

Likewise, let N_{AB} denote the number of "bad" risk applications accepted. Then N_{AB} is a Poisson process with rate $\lambda_{AB} = \frac{1}{2}\frac{1}{3}(60) = 10$. Let X_{AB} be the profit per "bad" accepted and S_{AB} be the aggregate profit of "bad" accepted. We have

$$\begin{aligned}\text{Var}(S_{AB}) =&E[N_{AB}]\text{Var}(X_{AB}) + \text{Var}(N_{AB})(E(X_{AB}))^2\\ =&10(90,000) + 10(-100)^2 = 1,000,000.\end{aligned}$$

Let S be the profit the applications accepted today. Then $S = S_G + S_{AB}$ with S_G and S_{AB} being independent. Thus,

$$\text{Var}(S) = \text{Var}(S_G) + \text{Var}(S_{AB}) = 4,000,000 + 1,000,000 = 5,000,000 \blacksquare$$

Example 80.4 ‡
Insurance losses are a compound Poisson process where:
(i) The approvals of insurance applications arise in accordance with a Poisson process at a rate of 1000 per day.
(ii) Each approved application has a 20% chance of being from a smoker and an 80% chance of being from a non-smoker.
(iii) The insurances are priced so that the expected loss on each approval is -100.
(iv) The variance of the loss amount is 5000 for a smoker and is 8000 for a non-smoker.
Calculate the variance for the total losses on one day's approvals.

Solution.
Let N be the number of approved applications in a day. Then $\{N(t) : t \geq 0\}$ is a Poisson process with rate 1000 per day. Let $N_s(t)$ be the number of approved smoker applications and N_{ns} that for non-smokers. Then $\{N_s(t) : t \geq 0\}$ is a Poisson process with rate $\lambda_s = (0.2)(1000) = 200$ per day and $\{N_{ns}(t) : t \geq 0\}$ is a Poisson process with rate $\lambda_{ns} = (0.8)(1000) = 800$ per day. Both processes are independent. Let S be the total loss random variable for one day approval for smokers and NS that for non-smokers. Then

$$S = Y_1^1 + Y_2^1 + \cdots + Y_{N_s}^1$$

where Y_i^1 is the loss for smoker i. Likewise, we have

$$NS = Y_1^2 + Y_2^2 + \cdots + Y_{N_{ns}}^2.$$

Hence,

$$\text{Var}(S) = E[N_s]\text{Var}(Y_i^1) + E[Y_i^1]^2\text{Var}(N_s) = 200[5000 + (-100)^2] = 3,000,000$$

and

$$\text{Var}(NS) = E[N_{ns}]\text{Var}(Y_i^2) + E[Y_i^2]^2\text{Var}(N_{ns}) = 800[8000 + (-100)^2] = 14,400,000$$

Since S and NS are independent, we have

$$\text{Var}(\text{Total loos per day}) = \text{Var}(S) + \text{Var}(NS) = 17,400,000 \blacksquare$$

Example 80.5 ‡
Customers arrive at a bank according to a Poisson process at the rate of 100 per hour. 20% of them make only a deposit, 30% make only a withdrawal and the remaining 50% are there only to complain. Deposit amounts are distributed with mean 8000 and standard deviation 1000. Withdrawal amounts have mean 5000 and standard deviation 2000.
The number of customers and their activities are mutually independent.
Using the normal approximation, calculate the probability that for an 8-hour day the total withdrawals of the bank will exceed the total deposits.

Solution.
Let $N(t)$ be the number of customers arriving at the bank after t hours. Then $N(t)$ is a Poisson process with rate 100 per hour. Let $N_d(t)$ be the number of customers making deposits. Then $N_d(t)$ is a Poisson process with rate $\lambda_d = 100(0.2) = 20$ per hour. Let $N_w(t)$ be the number of customers making withdrawals. Then $N_w(t)$ is a Poisson process with rate $\lambda_w = 100(0.3) = 30$ per hour.
For $t = 8$, $N_d(8)$ is a Poisson distribution with mean $\lambda_d t = 20(8) = 160$ while $N_w(8)$ is a Poisson distribution with mean $30(8) = 240$. Let S_D be the aggregate deposits. That is,

$$S_D = \sum_{i=1}^{N_d(8)} D_i$$

where D_i is the i^{th} deposit. We have

$$\begin{aligned}
E(S_D) =& E(D)E[N_d(8)] = 8000(160) = 1,280,000\\
\text{Var}(S_D) =& E[N_d(8)]\text{Var}(D) + [E(D)]^2\text{Var}[N_d(8)]\\
=&(160)(1000)^2 + (8000)^2(160) = 1.04 \times 10^{10}.
\end{aligned}$$

Likewise, let S_W be the aggregate withdrawals where $S_W = \sum_{i=1}^{N_w(8)} W_i$ with $E(W) = 5000$ and $\text{Var}(W) = 2000^2$. Then

$$\begin{aligned}
E(S_W) =& E(W)E[N_w(8)] = 5000(240) = 1,200,000\\
\text{Var}(S_W) =& E[N_w(8)]\text{Var}(W) + [E(W)]^2\text{Var}[N_w(8)]\\
=&(240)(2000)^2 + (5000)^2(240) = 0.696 \times 10^{10}.
\end{aligned}$$

Hence,

$$\begin{aligned}
E(S_W - S_D) =& E(S_W) - E(S_D) = 1,200,000 - 1,280,000 = -80,000 \\
\text{Var}(S_W - S_D) =& \text{Var}(S_W) + \text{Var}(S_D) = 1.736 \times 10^{10} \\
\sigma =& \sqrt{1.736 \times 10^{10}} = 131,757.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\Pr(S_W - S_D > 0) =& \Pr\left(\frac{S_W - S_D + 80,000}{131,757} > \frac{80,000}{131,57}\right) \\
=& \Pr(Z > 0.607) = 1 - \Phi(0.607) = 0.27 \blacksquare
\end{aligned}$$

Practice Problems

Problem 80.1
The number of dental claims received by an insurance company follows a Poisson process with rate $\lambda = 50$ claims/day. The claim amounts are independent and uniformly distributed over $[0, 300]$.
Find the mean and the standard deviation of the total claim amounts received in a 30 days period.

Problem 80.2
A company provides insurance to a concert hall for losses due to power failure. You are given:
(i) The number of power failures in a year has a Poisson distribution with mean 1.
(ii) The distribution of ground up losses due to a single power failure is:

x	Probability
10	0.30
20	0.30
50	0.40

Find the mean of total amount of claims paid by the insurer in one year.

Problem 80.3
Suppose that families migrate to an area at a Poisson rate 2 per week. If the number of people in each family is independent and takes on the values 1, 2, 3, 4 with respective probabilities 1/6, 1/3, 1/3, 1/6, then what is the expected value and variance of the number of individuals migrating to this area during a fixed five-week period?

Problem 80.4
An insurance company pays out claims on its life insurance policies in accordance with a Poisson Process having rate lambda = 5 per week. If the amount of money paid on each policy is exponentially distributed with mean \$2000, what is the mean and variance of the amount of money paid by the insurance company in a four week span?

Problem 80.5
Suppose that health claims are filed with a health insurer at the Poisson rate

$\lambda = 20$ per day, and that the independent severities Y of each claim are Exponential random variables with mean 500. Let $X(10)$ be the aggegate claim during the first 10 days.
(a) Find the mean and the variance of $X(10)$.
(b) Suppose that $X(10)$ can be approximated by a Normal random variable with mean and variance of those of $X(10)$. Estimate $\Pr(X(10) > 120,000)$.

Problem 80.6 ‡
The RIP Life Insurance Company specializes in selling a fully discrete whole life insurance of 10,000 to 65 year olds by telephone. For each policy:
(i) The annual contract premium is 500.
(ii) Mortality follows the Illustrative Life Table.
(iii) $i = 0.06$
The number of telephone inquiries RIP received follows a Poisson process with mean 50 per day. 20% of the inquiries result in the sale of a policy.
The number of inquiries and the future lifetimes of all the insureds who purchase policies on a particular day are independent.
Using the normal approximation, calculate the probability that S, the total prospective loss at issue for all the policies sold on a particular day, will be less than zero.

81 Conditional Poisson Processes

Let Λ be a continuous positive random variable. Let $\{N(t) : t \geq 0\}$ be a counting process such that $\{[N(t)|\Lambda = \lambda], t \geq 0\}$ is a Poisson process with rate λ. We call $\{N(t) : t \geq 0\}$ a **conditional Poisson process.**
If $f_\Lambda(\lambda)$ is the pdf of Λ, we have

$$\begin{aligned}
\Pr(N(s+t) - N(s) = n) =& \int_0^\infty [\Pr(N(s+t) - N(s) = n|\Lambda = \lambda)] f_\Lambda(\lambda) d\lambda \\
=& \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} f_\Lambda(\lambda) d\lambda.
\end{aligned}$$

It follows that a conditional Poisson process has stationary increments. However, a conditional Poisson process does not have independent increments and thus is not generally a Poisson process.
Now, suppose that Λ is a Gamma random variable with parameters θ and m with m a positive integer. The pdf of Λ is given by

$$f_\Lambda(\lambda) = \frac{\theta e^{-\theta\lambda} (\theta\lambda)^{m-1}}{(m-1)!}.$$

In this case, We have

$$\begin{aligned}
\Pr(N(t) = n) =& \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} \frac{\theta e^{-\theta\lambda} (\theta\lambda)^{m-1}}{(m-1)!} d\lambda \\
=& \frac{t^n \theta^m}{n!(m-1)!} \int_0^\infty e^{-(t+\theta)\lambda} \lambda^{n+m-1} d\lambda \\
=& \frac{t^n \theta^m (n+m-1)!}{n!(m-1)!(t+\theta)^{n+m}(n+m-1)!} \int_0^\infty (t+\theta) e^{-(t+\theta)\lambda} \frac{[(t+\theta)\lambda]^{n+m-1}}{(n+m-1)!} d\lambda.
\end{aligned}$$

Note that the integrand is the density function of a Gamma random variable with parameters $n+m$ and $t+\theta$ so that the integral value is 1. Hence,

$$\Pr(N(t) = n) = \frac{t^n \theta^m (n+m-1)!}{n!(m-1)!(t+\theta)^{n+m}(n+m-1)!} = \left(\begin{array}{c} n+m-1 \\ n \end{array} \right) p^m (1-p)^n$$

where $p = \frac{\theta}{t+\theta}$. Therefore, the number of events in an interval of length t has the same distribution of the number of failures that occur before reaching a total of m successes, where the probability of a success is p. In other words, $N(t)$ is a negative binomial random variable with parameters (m, p).

Example 81.1
Compute the mean and the variance of a conditional Poisson process $\{N(t) : t \geq 0\}$.

Solution.
By the law of double expectation, we have

$$E(N(t)) = E[E[N(t)|\Lambda]] = E[t\Lambda] = tE[\Lambda].$$

Also, by the law of total variance, we have

$$\begin{aligned}\text{Var}(N(t)) =& E[\text{Var}(N(t)|\Lambda)] + \text{Var}[E[N(t)|\Lambda]]\\ =& E[t\Lambda] + \text{Var}[t\Lambda]\\ =& tE[\Lambda] + t^2\text{Var}(\Lambda) \blacksquare\end{aligned}$$

Example 81.2 ‡
On his walk to work, Lucky Tom finds coins on the ground at a Poisson rate. The Poisson rate, expressed in coins per minute, is constant during any one day, but varies from day to day according to a gamma distribution with mean 2 and variance 4.
Calculate the probability that Lucky Tom finds exactly one coin during the sixth minute of today's walk.

Solution.
We have

$$f_\Lambda(\lambda) = \frac{1}{2}e^{-\frac{\lambda}{2}}.$$

Using integration by parts, we find

$$\begin{aligned}\Pr(N(6) - N(5) = 1) =& \int_0^\infty e^{-\lambda}\lambda\left(\frac{1}{2}e^{-\frac{\lambda}{2}}\right)d\lambda\\ =& \int_0^\infty \frac{\lambda}{2}e^{-\frac{3}{2}\lambda}d\lambda\\ =& \frac{1}{2}\left[-\frac{2}{3}\lambda e^{-\frac{3}{2}\lambda} - \frac{4}{9}e^{-\frac{3}{2}\lambda}\right]_0^\infty\\ =& \frac{2}{9} = 0.222 \blacksquare\end{aligned}$$

Answer Key

Section 18

18.1 No

18.2 III

18.3 $A = 1, \ B = -1$

18.4 0.033

18.5 0.9618

18.6 0.04757

18.7 $s(0) = 1, \ s'(x) < 0, \ s(\infty) = 0$

18.8 0.149

18.9 $1 - e^{-0.34x}, \ x \geq 0$

18.10 $\frac{x^2}{100}, \ x \geq 0$

18.11 I

18.12 (a 0.3 (b) 0.3

18.13 $1 - \frac{x}{108}, \ x \geq 0$

18.14 (a)

18.15 $(x+1)e^{-x}$

18.16 $0.34e^{-0.34x}$

18.17 $\lambda e^{-\lambda x}$

18.18

$$f(x) = \begin{cases} \frac{7}{16}, & 0 < x < 1 \\ \frac{3x}{8}, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$$

18.19 Both functions represent the density of death at age x. The probability density function is unconditional (i.e., given only existence at age 0) whereas $\mu(x)$ is conditional on survival to age x

18.20 $\frac{1}{2}(1-x)^{-1}$

18.21 $f(x) = \mu(x)S_X(x) = \mu(x)e^{-\Lambda(x)}$

18.22 $\int_0^\infty \mu(x)dx = \lim_{R\to\infty}\int_0^R \mu(x)dx = -\lim_{R\to\infty}\ln s(x) = -(-\infty) = \infty$

18.23 0.34

18.24 $s(x) = e^{-\int_0^x \mu(s)ds} = e^{-\mu x}, F(x) = 1 - s(x) = 1 - e^{-\mu x}$, and $f(x) = F'(x) = \mu e^{-\mu x}$

18.25 $\frac{1}{480}$

18.26 $\ln(x+1),\ x \geq 0$

18.27 $\frac{2x}{4-x^2},\ 0 \leq x < 2$

18.28

$$
\begin{aligned}
s(x) =& e^{-\Lambda_X(x)} = e^{-\mu x} \\
F(x) =& 1 - e^{-\mu x} \\
f(x) =& -S'_X(x) = -\mu e^{-\mu x}
\end{aligned}
$$

18.29 1.202553

18.30 0.2

18.31 (I) and (II)

18.32 2

18.33 $24(2+x)^{-4}$

18.34 2

18.35 4

18.36 kp

18.37 $\frac{3}{4}k$

18.38 $45\sqrt{2}$

18.39 $\mathring{e}_0 = 60$ and $\text{Var}(X) = 450$

18.40 (a) $\frac{720}{7}$ (b) 0.062

18.41 median = 0.51984 and mode = 0

Section 19

19.1 675

19.2 50

19.3 (a) $\mu(x) = -\frac{s'(x)}{s(x)} = \frac{1}{90-x}$. (b) $F(x) = 1 - s(x) = \frac{x}{90}$. (c) $f(x) = F'(x) = \frac{1}{90}$. (d) $\Pr(20 < X < 50) = s(20) - s(50) = \frac{50}{90} - \frac{20}{90} = \frac{1}{3}$

19.4 (a) $F(x) = 1 - s(x) = 1 - \left(1 - \frac{x}{\omega}\right)^{\alpha}$
(b) $f(x) = F'(x) = \frac{\alpha}{\omega}\left(1 - \frac{x}{\omega}\right)^{\alpha-1}$
(c) $\mu(x) = \frac{f(x)}{s(x)} = \frac{\alpha}{\omega}\left(1 - \frac{x}{\omega}\right)^{-1}$

19.5 $\frac{t}{\omega-x},\ 0 \le t \le \omega - x.$

19.6 $1 - \frac{t}{\omega-x},\ 0 \le t \le \omega - x.$

19.7 40

19.8 4

19.9 $\ln\left(\frac{\omega}{\omega-x}\right)$

19.10 0.449

19.11 0,1481

19.12 0.5

19.13 $\mu = 0.3054$ and median $= 2.27$

19.14 $1 - e^{-\mu t}$

19.15 46.67

19.16 $\mathring{e}_0 = 60$ and $\mathrm{Var}(X) = 3600$

19.17 (III)

19.18 0.01837

19.19 $s(x) = e^{-\int_0^x Bc^t dt} = e^{\frac{B}{\ln c}(1-c^x)}$ and $F(x) = 1 - e^{\frac{B}{\ln c}(1-c^x)}$

19.20 $f(x) = -s'(x) = Bc^x e^{\frac{B}{\ln c}(1-c^x)}$

19.21 $\Lambda(x) = \int_0^x Bc^t dt = \left.\frac{Bc^t}{\ln c}\right|_0^x = \frac{B}{\ln c}(c^x - 1)$

19.22 6.88

19.23 $-3.008(1.05)^x$

19.24 $1 - e^{\frac{0.0004}{\ln 1.07}(1-1.07^x)}$

19.25 $f(x) = \mu(x)s(x) = (A + Bc^x)e^{-Ax-m(c^x-1)}$ where $m = \frac{B}{\ln c}$

19.26 $1 - e^{-Ax-m(c^x-1)}$

19.27 $f(x) = (0.31{+}0.45(2^x))e^{-0.31x-\frac{0.43}{\ln 2}(2^x-1)}$ and $F(x) = 1{-}e^{-0.31x-\frac{0.43}{\ln 2}(2^x-1)}$

19.28 0.0005131

19.29 $\mu(x) = 0.31 + 0.43(2^x)$

19.30 0.111395

19.31 $f(x) = kx^n e^{-k\frac{x^{n+1}}{n+1}}$

19.32 $k = 2$ and $n = 1$

19.33 30

19.34 e^{-225}

19.35 e^{-16}

19.36 0.009831

19.37 $n = 5.1285$ and $k = 1.5198 \times 10^{-11}$.

Section 20

20.1 $s_{T(x)}(t) = 1 - \frac{t}{\omega - x},\ 0 \le t \le \omega - x.$

20.2 $s_{T(x)}(t) = 1 - \frac{t}{75-x},\ 0 \le t \le 75 - x$ and $f_T(t) = \frac{1}{75-x}$

20.3 ${}_{m+n}p_x = \frac{s(x+m+n)}{s(x)} = \frac{s(x+m+n)}{s(x+m)} \cdot \frac{s(x+m)}{s(x)} = {}_np_{x+m} \cdot {}_mp_x$

20.4 0.9215

20.5 Induction on n and Problem 20.3

20.6 (a) ${}_{17}p_{35} - {}_{38}p_{35}$ (b) 0.323

20.7 $\frac{4}{t+4}$

20.8 0.9559

20.9 We have

$$\int_x^{x+t} \mu(y)dy = \int_x^{x+t} -[\ln s(y)]'dy = \ln\left(\frac{s(x)}{s(x+t)}\right)$$

so that

$${}_tp_x = \frac{s(x+t)}{s(x)} = e^{-\int_x^{x+t}\mu(y)dy}$$

20.10 0.59049

20.11 (a) 0.8795 (b) 0.9359

20.12 We have

$$\begin{aligned}\frac{\partial}{\partial t}{}_tp_x =& \frac{\partial}{\partial t}\left(\frac{s(x+t)}{s(x)}\right) = \frac{s'(x+t)}{s(x)} \\ =& \frac{s'(x+t)}{s(x+t)}\frac{s(x+t)}{s(x)} = -{}_tp_x\mu(x+t)\end{aligned}$$

20.13 $\mu(x) = -0.04 + 0.00189(1.056)^x$

20.14 We have

$$
\begin{aligned}
{}_{t|u}q_x =&\Pr(t < T(x) \leq t+u)\\
=&F_{T(x)}(t+u) - F_{T(x)}(t)\\
=&{}_{t+u}q_x - {}_tq_x\\
=&(1 - {}_{t+u}p_x) - (1 - {}_tp_x)\\
=&{}_tp_x - {}_{t+u}p_x
\end{aligned}
$$

20.15 We have

$$
\begin{aligned}
{}_{t|u}q_x =&{}_tp_x - {}_{t+u}p_x\\
=&\frac{s(x+t)}{s(x)} - \frac{s(x+t+u)}{s(x)}\\
=&\frac{s(x+t) - s(x+t+u)}{s(x)}\\
=&\left[\frac{s(x+t)}{s(x)}\right]\left[\frac{s(x+t) - s(x+t+u)}{s(x+t)}\right]\\
=&{}_tp_{x\,u}q_{x+t}
\end{aligned}
$$

20.16 0.5714

20.17 0.9841

20.18 0.025

20.19 This follows from ${}_tq_x = 1 - {}_tp_x$ and Problem 20.12

20.20 $1 - \left(\frac{\alpha}{\alpha+t}\right)^{\beta}$

20.21 0.9913

20.22 ${}_{2|}q_1$ is the probability that a life currently age 1 will die between ages 3 and 4

20.23 0.23915

20.24 $1 - \left(\frac{120}{120+t}\right)^{1.1}$

20.25 0.1694

20.26 0.6857

20.27 $s_{T(x)}(t) = \frac{90-x-t}{90-x}$ and $f_{T(x)}(t) = \frac{1}{90-x},\ 0 \le t \le 90 - x$

20.28 $f_{T(x)}(t) = \frac{1}{90-x},\ 0 \le t \le 90 - x$

20.29 0.633

20.30 $f_{T(36)}(t) = \frac{0.0625}{(64-t)^{\frac{1}{2}}}$

20.31 $f_{T(2)}(t) = \frac{2+t}{48}$

20.32 0.01433

20.33 $\frac{d}{dt}(1 - {}_tp_x) = \frac{d}{dt}({}_tq_x) = {}_tp_x\mu(x+t)$

20.34 $\int_0^\infty {}_tp_x\mu(x+t)dx = \int_0^\infty f_{T(x)}(x)dx = 1$

20.35 $\mu_{T(x)}(t) = \frac{f_{T(x)}(t)}{{}_tp_x} = \frac{F'_{T(x)}(t)}{1-{}_tq_x} = \frac{F'_{T(x)}(t)}{1-F_{T(x)}(t)}$

20.36 $\mu(x+t) = \frac{1}{100-x-t},\ 0 \le t \le 100 - x$

20.37 $\mu(x+t) = \mu(x) = \mu$

20.38 0.015

20.39 5.25

20.40 0.3783

20.41 49.8

20.42 10510.341

20.43 300

20.44 50

20.45 $p_{K_x}(k) = {}_{k-1}p_x \cdot q_{x+k-1} = {}_{k-1|}q_x$

20.46 $\Pr(K_x \geq k) = \Pr(T(x) > k-1) = s_{T(x)}(k-1) = {}_{k-1}p_x$

20.47 $p_{K(x)}(k) = {}_kp_x - {}_{k+1}p_x = \left(\frac{100-x-k}{100-x}\right)^{0.5} - \left(\frac{100-x-k-1}{100-x}\right)^{0.5}$

20.48 $e_x = \frac{99-x}{2}$ and $\overset{\circ}{e}_x = \frac{100-x}{2}$

20.49 1

20.50 2

20.51 We have

$$\begin{aligned}
e_x =& \sum_{k=1}^{\infty} {}_kp_x = p_x + \sum_{k=2}^{\infty} {}_kp_x \\
=& p_x + \sum_{k=2}^{\infty} p_x {}_{k-1}p_{x+1} \\
=& p_x + \sum_{k=1}^{\infty} p_x {}_kp_{x+1} \\
=& p_x(1 + e_{x+1})
\end{aligned}$$

20.52 $T(x) = K_x - 1 + S_x = K(x) + S_x$

20.53 1.07

20.54 9.5

Section 21

21.1 $\frac{1}{99.5-x}$

21.2 We have

$$
\begin{aligned}
{}_nm_x =& \frac{\int_x^{x+n} f(y)dy}{\int_x^{x+n} s(y)dy} \\
=& - \frac{\int_x^{x+n} s'(y)dy}{\int_x^{x+n} s(y)dy} \\
=& \frac{s(x) - s(x+n)}{\int_x^{x+n} s(t)dt}
\end{aligned}
$$

21.3 ${}_nm_x = \frac{2n}{200-2nx-1}$

21.4 0.6039

21.5 $\frac{1}{75}$

21.6 $m_{40} = 0.0096864$ and ${}_{10}m_{75} = 0.044548$.

Section 22

22.1 (a) $\ell_x = 10 - x$ (b) $p_2 = \frac{7}{8},\ q_3 = \frac{1}{7},\ {}_3p_7 = 0,\ {}_2q_7 = \frac{2}{3}$

22.2 $F(x) = \frac{\ell_0 - \ell_x}{\ell_0}$

22.3 ${}_{t|u}q_x = \frac{\ell_{x+t} - \ell_{x+t+u}}{\ell_x}$

22.4 We have

Age	ℓ_x	d_x	p_x	q_x
0	100,000	501	0.99499	0.00501
1	99,499	504	0.99493	0.00506
2	98,995	506	0.99489	0.00511
3	98,489	509	0.99483	0.00517
4	97,980	512	0.99477	0.00523
5	97,468	514	NA	NA

22.5 ℓ_{t+x}

22.6 (a) 9734 (b) 50 (c) 200 (d) 0.0211 (e) 0.0055

Section 23

23.1 $\mu(x) = \frac{1}{\omega - x},\ 0 \le x < \omega$

23.2 0.05

23.3 Let $u = x + t$. Then

$$\begin{aligned}\mu(x+t) =& \mu(u) = -\frac{\frac{d\ell_u}{du}}{\ell_u}\\ =& -\frac{\frac{d\ell_u}{dt}\cdot\frac{dt}{du}}{\ell_u}\\ =& -\frac{\frac{d\ell_{x+t}}{dt}}{\ell_{x+t}}\end{aligned}$$

23.4 $\ell_x = 100 - x$

23.5 $\mu(x) = \frac{1}{3}(90 - x)^{-1}$

23.6 $\ell_x - \ell_{x+n} = \int_x^{x+n}\left[-\frac{d}{dy}\ell_y\right]dy = \int_x^{x+n}\ell_y\mu(y)dy$

23.7 $\frac{d}{dx}\ell_x\mu(x) = \frac{d}{dx}\left[-\frac{d}{dx}\ell_x\right] = -\frac{d^2}{dx^2}\ell_x$

23.8 $f(x) = 0.95(100 - x)^{-1}$

23.9 5000

23.10 Using integration by parts we find

$$\begin{aligned}\int_0^\infty xf(x)dx =& -\frac{1}{\ell_0}\int_0^\infty x\ell_x' dx\\ =& -\frac{[1}{\ell_0}\left[x\ell_x|_0^\infty - \int_0^\infty \ell_x dx\right]\\ =& \frac{1}{\ell_0}\int_0^\infty \ell_x dx\end{aligned}$$

where we used the fact that $\ell_\infty = \ell_0 s(\infty) = 0$

23.11 Using integration by parts we find

$$\begin{aligned}\int_0^\infty x^2 f(x)dx =& -\frac{1}{\ell_0}\int_0^\infty x^2\ell'_x dx\\ =& -\frac{[1}{\ell_0}\left[x^2\ell_x\Big|_0^\infty - 2\int_0^\infty x\ell_x dx\right]\\ =&\frac{2}{\ell_0}\int_0^\infty x\ell_x dx\end{aligned}$$

where we used the fact that $\ell_\infty = \ell_0 s(\infty) = 0$

23.12 $f_{T(x)}(t) = \frac{d}{dt}{}_tq_x = \frac{d}{dt}\left[1 - \frac{\ell_{x+t}}{\ell_x}\right] = -\frac{1}{\ell_x}\frac{d}{dt}\ell_{x+t}$

23.13 $T_x = \int_x^{10} 100(10-y)^{0.85}dy = -100\left[\frac{(10-y)^{1.85}}{1.85}\right]_x^{10} = \frac{100(10-x)^{1.85}}{1.85}$

23.14 5.405

23.15 48.881

23.16 $5000(1+x)^{-2}$

23.17 0.1

23.18 0.123

23.19 ${}_np_x$ is the probability of surviving to age $x+n+m$. If we remove ${}_{n+m}p_x$, which is the probability of surviving to $x+n+m$ years, then we have the probability of surviving to age $x+n$ but dying by the age of $x+n+m$ which is ${}_{n|m}q_x$

23.20 ${}_{6|10}q_{64}$

23.21 $\frac{m}{\omega - x}$

23.22 $e^{-n\mu} - e^{-(n+m)\mu}$

23.23 0.3064

23.24 20

23.25 800

23.26 400

23.27 $\frac{1+x}{3}$

23.28 $\frac{(1+x)^2}{3}$

23.29 $\frac{2}{9}(1+x)^2$

23.30 $\frac{137}{3}$

23.31 352.083

23.32 The expected number of years (60) is expected to live in the next 25 years is 17.763

23.33 We have

$$\begin{aligned}
\mathring{e}_{x:\overline{m+n}|} &= \int_0^{m+n} {}_tp_x dt = \int_0^m {}_tp_x dt + \int_m^{m+n} {}_tp_x dt \\
&= \int_0^m {}_tp_x dt + \int_0^n {}_{m+y}p_x dy = \int_0^m {}_tp_x dt + \int_0^n {}_mp_x \cdot {}_yp_{x+m} dy \\
&= \mathring{e}_{x:\overline{m}|} + {}_mp_x \cdot \mathring{e}_{x+m:\overline{n}|}
\end{aligned}$$

23.34 We have

$$\begin{aligned}
\mathring{e}_x &= \int_0^\infty {}_tp_x dt = \int_0^n {}_tp_x dt + \int_n^\infty {}_tp_x dt \\
&= \int_0^n {}_tp_x dt + \int_0^\infty {}_{y+n}p_x dt = \int_0^n {}_tp_x dt + \int_0^\infty {}_np_x \cdot {}_yp_{x+n} dt \\
&= \mathring{e}_{x:\overline{n}|} + {}_np_x \cdot \mathring{e}_{x+n}
\end{aligned}$$

23.35 6.968

23.36 15.6

23.37 $E[K(x)^2] = \sum_{k=1}^{\infty}(2k-1)_k p_x = \frac{1}{\ell_x}\sum_{k=1}^{\infty}(2k-1)\ell_{x+k}$

23.38 7.684

23.39 2.394

23.40 $p_{K(20)}(k) = {}_kp_{20} - {}_{k+1}p_{20} = e^{-0.05k} - e^{-0.05(k+1)} = e^{-0.05k}(1-e^{-0.05})$

23.41 0.905

23.42 We have $T_x = \int_x^{\infty} \ell_y dy = \sum_{k=x}^{\infty}\int_k^{k+1} \ell_y dy = \sum_{k=x}^{\infty}\int_0^1 \ell_{k+t}dt = \sum_{k=x}^{\infty} L_k$

23.43 ${}_nL_x = T_x - T_{x+n} = 200e^{-0.05x}(1-e^{-0.05n})$

23.44 ${}_nL_x = T_x - T_{x+n} = 1000(x+1)^{-3} - 1000(x+n+1)^{-3}$

23.45 577.190

23.46 Recall that

$$L_x = -(x-\omega) - \frac{1}{2}$$

so that

$$\sum_{k=1}^{n} L_k = L_1 - L_{n+1} = n$$

23.47 We have

$$L_x = \int_x^{x+1} \ell_y dy = -\int_0^x \ell_y dy + \int_0^{x+1} \ell_y dy$$

and therefore

$$\frac{d}{dx}L_x = \ell_{x+1} - \ell_x = -d_x$$

23.48 We have

$$\frac{d}{dt}L_t = -d_t = -m_t L_t.$$

Separating the variables and integrating both sides from x to $x+1$ we obtain

$$\ln\left(\frac{L_{x+1}}{L_x}\right) = -\int_x^{x+1} m_y dy.$$

Solving for L_{x+1} we find

$$L_{x+1} = L_x e^{-\int_x^{x+1} m_y dy}$$

23.49 (a) We have

$$\begin{aligned} d_x =& \ell_x - \ell_{x+1} = 1 \\ L_x =& \int_0^1 \ell_{x+t} dt = \int_0^1 (\omega - x - t) dt = \omega - x - \frac{1}{2} \\ m_x =& \frac{d_x}{L_x} = \frac{1}{\omega - x - 0.5} \end{aligned}$$

(b) For DeMoivre's Law RV we have $\mu(x) = \frac{1}{\omega - x}$.
(c)We have

$$\frac{m_x}{1 + 0.5 m_x} = \frac{1}{\omega - x - 0.5} \cdot \frac{\omega - x - 0.5}{\omega - x} = \frac{1}{\omega - x} = \mu(x)$$

23.50 (a) 2502.357 (b) 0.0159

23.51 (a) ${}_{10}L_{20} = 750,\ {}_{10}d_{20} = 10$ (b) $\frac{1}{75}$

23.52 100

Section 24

24.1

$$\ell_t = \begin{cases} 100,000-501t & 0\le t\le 1\\ 99,499-504(t-1) & 1\le t\le 2\\ 98,995-506(t-2) & 2\le t\le 3\\ 98,489-509(t-3) & 3\le t\le 4\\ 97,980-512(t-4) & 4\le t\le 5\\ 97,468-514(t-5) & 5\le t\le 6\end{cases}$$

24.2

$$ {}_tp_0 = \begin{cases} \frac{100,000-501t}{100,000} & 0\le t\le 1\\ \frac{99,499-504(t-1)}{100,000} & 1\le t\le 2\\ \frac{98,995-506(t-2)}{100,000} & 2\le t\le 3\\ \frac{98,489-509(t-3)}{100,000} & 3\le t\le 4\\ \frac{97,980-512(t-4)}{100,000} & 4\le t\le 5\\ \frac{97,468-514(t-5)}{100,000} & 5\le t\le 6\end{cases}$$

24.3 $e_0 = 4.92431$ and $\mathring{e}_0 = 5.42431$

24.4 We have

$$\begin{aligned} {}_tq_{x+s} =& tq_{x+s} = \frac{d_{x+s}}{\ell_{x+s}} = t\frac{\ell_{x+s}-\ell_{x+s+1}}{\ell_{x+1}}\\ =& t\frac{d_x}{\ell_x - sd_x} = t\frac{\frac{d_x}{\ell_x}}{1-s\frac{d_x}{\ell_x}} = \frac{tq_x}{1-sq_x}\end{aligned}$$

24.5 0.95

24.6 (I) and (II)

24.7 1/9

24.8 ${}_{r|h}q_x = \frac{\ell_{x+r}-\ell_{x+r+h}}{\ell_x} = \frac{(\ell_x - rd_x)-(\ell_x-(r+h)d_x)}{\ell_x} = h\frac{d_x}{\ell_x} = hq_x$

24.9 0.813

24.10 0.2942

24.11 0.5447

24.12 We have

$$
\begin{aligned}
{}_{t-s}q_{x+s} =& 1 - {}_{t-s}p_{x+s} \\
=& 1 - e^{-\int_{x+s}^{x+t}\mu(y)dy} \\
=& 1 - e^{-\int_s^t \mu(x+r)dr} \\
=& 1 - e^{-(t-s)\mu_x}
\end{aligned}
$$

24.13 ${}_{\frac{1}{12}}q_{90} = 0.02369$ and ${}_{\frac{1}{12}}q_{90+\frac{11}{12}} = 0.02369$

24.14 ${}_{0.5}q_x = 0.0513$ and ${}_{0.5}q_{x+0.5} = 0.0513$

24.15 $L_{95} = 690.437$ and $m_{95} = \frac{\ell_{95}-\ell_{96}}{L_{95}} = \frac{200}{690.437} = 0.28967$

24.16 We have

$$
\begin{aligned}
{}_{s-t}q_{x+t} =& \frac{s(x+t) - s(x+s)}{s(x+t)} = 1 - \frac{s(x+s)}{s(x+t)} \\
=& 1 - \frac{{}_sp_x}{{}_tp_x} = 1 - \frac{\frac{p_x}{s+(1-s)p_x}}{\frac{p_x}{t+(1-t)p_x}} \\
=& 1 - \frac{t+(1-t)p_x}{s+(1-s)p_x} \\
=& \frac{(s-t)(1-p_x)}{s+(1-s)p_x} \\
=& \frac{(s-t)q_x}{1-(1-s)q_x}
\end{aligned}
$$

24.17 ${}_{0.75}p_{80} = 0.95857$ and ${}_{2.25}p_{80} = 0.87372$

24.18 13440

24.19 0.00057

24.20 (i) 1.475801 (ii) 1.475741

Section 25

25.1 84

25.2 8.2

25.3 8056

25.4 0.4589

25.5 0.0103

Section 26

26.1 0.03125

26.2 $\bar{A}_{20} = 0.4988,\ {}^2\bar{A}_{20} = 0.2998,\ \mathrm{Var}(\bar{Z}_{20}) = 0.0501$

26.3 3.75

26.4 0.04

26.5 116.09

26.6 14.10573

26.7 $\bar{A}^{\,1}_{25:\overline{10}|} = 0.0885,\ {}^2\bar{A}^{\,1}_{25:\overline{10}|} = 0.0685,\ \mathrm{Var}(\bar{Z}^{\,1}_{25:\overline{10}|}) = 0.0607$

26.8 $\bar{A}^{\,1}_{30:\overline{20}|} = 0.3167,\ {}^2\bar{A}^{\,1}_{30:\overline{20}|} = 0.1987,\ \mathrm{Var}(\bar{Z}^{\,1}_{30:\overline{20}|}) = 0.0984$

26.9 0.2378

26.10 0.4305

26.11 $\frac{1}{1+e^{-(\mu+\delta)}}$

26.12 0.05

26.13 $A_{30:\overset{\,1}{\overline{20}|}} = 0.2628,\ {}^2A_{30:\overset{\,1}{\overline{20}|}} = 0.0967,\ \mathrm{Var}(\bar{Z}_{30:\overset{\,1}{\overline{20}|}}) = 0.0276$

26.14 $A_{30:\overset{\,1}{\overline{20}|}} = 0.2231,\ {}^2A_{30:\overset{\,1}{\overline{20}|}} = 0.0821,\ \mathrm{Var}(\bar{Z}_{30:\overset{\,1}{\overline{20}|}}) = 0.0323$

26.15 0.02497

26.16 0.7409

26.17 mean = 1051.43 and the standard deviation is 197.94

26.18 (a) 590.41 (b) 376.89

26.19 0.2793

26.20 0.4775

26.21 0.73418

26.22 ${}^2\bar{A}_{x:\overline{n}|} = {}^2\bar{A}^1_{x:\overline{n}|} + {}^2A_{x:\overset{1}{\overline{n}|}} = \int_0^n \nu^{2t}{}_tp_x\mu(x+t)dt + \nu^{2n}{}_np_x$

26.23 ${}^2{}_{m|}\bar{A}_x = \nu^{2m}{}_mp_x{}^2\bar{A}_{x+m}$

26.24 0.1647

26.25 0.0253

26.26 0.0873

26.27 0.0154

Section 27

27.1 $A_{30} = 0.3168,\ {}^2A_{30} = 0.1805,\ \text{Var}(Z_{30}) = 0.0801$

27.2 $A^{\ 1}_{30:\overline{10|}} = 0.2461,\ {}^2A^{\ 1}_{30:\overline{10|}} = 0.1657,\ \text{Var}(Z^{\ 1}_{30:\overline{10|}}) = 0.1051$

27.3 ${}_{10|}A_{30} = 0.1544,\ {}^2{}_{10|}A_{30} = 0.02981,\ \text{Var}({}_{10|}Z_{30}) = 0.00597$

27.4 $A_{30:\overline{10|}} = 0.4692,\ {}^2A_{30:\overline{10|}} = 0.2478,\ \text{Var}(Z_{30:\overline{10|}}) = 0.0277$

27.5 1730.10

27.6 0.19026

27.7 0.9396

Section 28

28.1 0.671

28.2 We have

$$\begin{aligned}
A_x =&\sum_{k=0}^{\infty} \nu^{k+1}{}_kp_xq_{x+k} = A^1_{x:\overline{n}|} + \sum_{k=n}^{\infty} \nu^{k+1}{}_kp_xq_{x+k}\\
=&A^1_{x:\overline{n}|} + \sum_{k=0}^{\infty} \nu^{k+1+n}{}_{k+n}p_xq_{x+k+n}\\
=&A^1_{x:\overline{n}|} + \nu^n{}_np_x\sum_{k=0}^{\infty} \nu^{k+1}{}_kp_{x+n}q_{x+k+n}\\
=&A^1_{x:\overline{n}|} + \nu^n{}_np_xA_{x+n}
\end{aligned}$$

28.3 0.0081

28.4 0.00242

28.5 Using (i) and (ii), we can rewrite the given relation as

$$u(k-1) = u(k)\nu p_{k-1} + \nu q_{k-1}.$$

Now, we have

$$\begin{aligned}
u(70) =&1\\
u(69) =&\nu p_{69} + \nu q_{69} = A_{69:\overline{1}|}\\
u(68) =&[\nu p_{69} + \nu q_{69}]\nu p_{68} + \nu q_{68} = \nu^2 p_{68}p_{69} + \nu^2 p_{68}q_{69} + \nu q_{68} = A_{68:\overline{2}|}\\
u(67) =&[\nu^2 p_{68}p_{69} + \nu^2 p_{68}q_{69} + \nu q_{68}]\nu p_{67} + \nu q_{67}\\
=&\nu^3 p_{67}p_{68}p_{69} + \nu^3 p_{67}p_{68}q_{69} + \nu^2 p_{67}q_{68} + \nu q_{67} = A_{67:\overline{3}|}\\
\vdots =&\vdots\\
u(40) =&A_{40:\overline{30}|}
\end{aligned}$$

28.6 We have

$$\begin{aligned}
{}^2A_x - \nu^n {}_nE_x {}^2A_{x+n} + \nu^n {}_nE_x =& \sum_{k=0}^{\infty} \nu^{2(k+1)} {}_kp_x q_{x+k} - \sum_{k=0}^{\infty} \nu^{2(k+1+n)} {}_np_x {}_kp_{x+n} q_{x+n+k} + \nu^{2n} {}_np_x \\
=& \sum_{k=0}^{\infty} \nu^{2(k+1)} {}_kp_x q_{x+k} - \sum_{k=0}^{\infty} \nu^{2(k+1+n)} {}_{n+k}p_x q_{x+n+k} + \nu^{2n} {}_np_x \\
=& \sum_{k=0}^{\infty} \nu^{2(k+1)} {}_kp_x q_{x+k} - \sum_{k=n}^{\infty} \nu^{2(k+1)} {}_kp_x q_{x+k} + \nu^{2n} {}_np_x \\
=& \sum_{k=0}^{n-1} \nu^{2(k+1)} {}_kp_x q_{x+k} + \nu^{2n} {}_np_x = {}^2A_{x:\overline{n|}}
\end{aligned}$$

28.7 0.02544

28.8 2.981%

Section 29

29.1 11772.61

29.2 10416.22

29.3 $E(Z) = 20.3201,\ E(Z^2) = 2683.7471,\ \text{Var}(Z) = 2270.8406$

29.4 87.35

29.5 12.14

29.6 4

29.7 0.3403

29.8 1

29.9 $(I\bar{A})_{30} = -\sum_{k=0}^{29}(k+1)e^{-0.02(k+1)}$

29.10 $(\bar{I}\bar{A})_x = \int_0^\infty e^{-\delta t}e^{-\mu t}(\mu)dt = \frac{\mu}{(\mu+\delta)^2} = (\mu+\delta)^{-1}\bar{A}_x$

29.11 1.9541

29.12 We have

$$\begin{aligned}
(\bar{I}\bar{A})_x =& \int_0^\infty t\nu^t{}_tp_x\mu(x+t)dt \\
=& \int_0^\infty \left(\int_0^t ds\right)\nu^t{}_tp_x\mu(x+t)dt \\
=& \int_0^\infty \int_s^\infty \nu^t{}_tp_x\mu(x+t)dtds \\
=& \int_0^\infty {}_{s|}\bar{A}_x ds
\end{aligned}$$

29.13 We have

$$\begin{aligned}(I\bar{A})^{1}_{x:\overline{n}|}+(D\bar{A})^{1}_{x:\overline{n}|}=&\int_0^n\lfloor t+1\rfloor\nu^t{}_tp_x\mu(x+t)dt+\int_0^n(n-\lfloor t\rfloor)\nu^t{}_tp_x\mu(x+t)dt\\=&(n+1)\int_0^n\nu^t{}_tp_x\mu(x+t)dt\\=&(n+1)\bar{A}^{1}_{x:\overline{n}|}\end{aligned}$$

where we used the fact that $\lfloor t+1\rfloor-\lfloor t\rfloor=1$ for $k-1\le t\le k$.

29.14 We have

$$\begin{aligned}(IA)^{1}_{x:\overline{n}|}=&\sum_{k=0}^{n-1}(k+1)\nu^{k+1}{}_{k|}q_x=\sum_{k=0}^{n-1}(k+1)\nu^{k+1}{}_kp_xq_{x+k}\\=&\sum_{k=0}^{n-1}k\nu^{k+1}{}_kp_xq_{x+k}+\sum_{k=0}^{n-1}\nu^{k+1}{}_kp_xq_{x+k}\\=&\sum_{k=0}^{n-1}\nu^{k+1}{}_kp_xq_{x+k}+\nu\sum_{k=1}^{n-1}k\nu^kp_{xk-1}p_{x+1}q_{x+k}\\=&\sum_{k=0}^{n-1}\nu^{k+1}{}_kp_xq_{x+k}+\nu p_x\sum_{k=0}^{n-2}(k+1)\nu^{k+1}{}_kp_{x+1}q_{x+k+1}\\=&\sum_{k=0}^{\infty}\nu^{k+1}{}_kp_xq_{x+k}+\nu p_x\sum_{k=0}^{n-2}(k+1)\nu^{k+1}{}_{k|}q_{x+1}\\=&A^{1}_{x:\overline{n}|}+\nu p_x(IA)^{\;1}_{x+1:\overline{n-1}|}.\end{aligned}$$

29.15 5.0623

29.16 This follows from the two recursion relations

$$A^{1}_{x:\overline{n}|}=\nu q_x+\nu p_xA^{\;1}_{x+1:\overline{n-1}|}$$

and

$$(IA)^{1}_{x:\overline{n}|}=A^{1}_{x:\overline{n}|}+\nu p_x(IA)^{\;1}_{x+1:\overline{n-1}|}.$$

29.17 We have

$$\begin{aligned}(IA)^1_{x:\overline{n}|} + (DA)^1_{x:\overline{n}|} =& \sum_{k=0}^{n-1}(k+1)\nu^{k+1}{}_{k|}q_x + \sum_{k=0}^{n-1}(n-k)\nu^{k+1}{}_{k|}q_x \\ =& \sum_{k=0}^{n-1}[k+1+n-k]\nu^{k+1}{}_{k|}q_x \\ =&(n+1)\sum_{k=0}^{n-1}\nu^{k+1}{}_{k|}q_x \\ =&(n+1)\sum_{k=0}^{n-1}\nu^{k+1}{}_{k}p_x q_{x+k} \\ =&(n+1)A^1_{x:\overline{n}|}.\end{aligned}$$

29.18 12.2665.

Section 30

30.1 115.10

30.2 543.33

30.3 2758.99

30.4 We have

$$\begin{aligned}(I\bar{A})_x =&E[\lfloor T+1\rfloor \nu^T] = E[(K+1)\nu^{K+1}\nu^{S-1}]\\ =&E[(K+1)\nu^{K+1}]E[\nu^{S-1}]\\ =&\frac{i}{\delta}(IA)_x.\end{aligned}$$

30.5 We have

$$\begin{aligned}E[(S-1)\nu^{S-1}] =& \int_0^1 (s-1)(1+i)^{1-s}ds\\ =&-\int_0^1 s(1+i)^s ds\\ =&-\left.\frac{1}{\delta}e^{s\delta}\right|_0^1 + \frac{1}{\delta}\int_0^1 e^{s\delta}ds\\ =&-\left(\frac{1+i}{\delta}-\frac{i}{\delta^2}\right).\end{aligned}$$

30.6 We have

$$\begin{aligned}(\bar{I}\bar{A})_x =&E(T\nu^T) = E[(K+1+S-1)\nu^T]\\ =&E[(K+1)\nu^T] + E[(S-1)\nu^T]\\ =&E[\lfloor T+1\rfloor\nu^T] + E[(S-1)\nu^T]\\ =&(I\bar{A})_x + E[(S-1)\nu^{K+1}\nu^{S-1}]\\ =&\frac{i}{\delta}(IA)_x + E[\nu^{K+1}]E[(S-1)\nu^{S-1}]\\ =&\frac{i}{\delta}(IA)_x + A_xE[(S-1)\nu^{S-1}]\\ =&\frac{i}{\delta}(IA)_x - \left(\frac{1+i}{\delta}-\frac{i}{\delta^2}\right)A_x.\end{aligned}$$

Section 31

31.1 $A_{69}^{(2)} = 0.5020$ is the actuarial present value of a whole life insurance of \$1 issued to (69) with death benefit paid at the end of the semiannual in the year of death.

31.2 0.0695

31.3 0.9137

31.4 0.5217

31.5 0.8494

Section 32

32.1 280.65

32.2 248.67

32.3 0.6614

32.4 $F_{T(20)}(t) = 1 + \frac{\ln t}{4}$.

32.5 0.8187

32.6 1,430,000

Section 33

33.1 12

33.2 $E(\bar{Y}_x^2) = \frac{1}{\delta^2}[1 - 2\bar{A}_x + {}^2\bar{A}_x]$

33.3 2.8

33.4 7.217

33.5 $\Pr(Y_x > \bar{a}_x) = \left(\frac{\mu}{\mu+\delta}\right)^{\frac{\mu}{\delta}}$

33.6 13.027

33.7 $\frac{1}{12\mu^2}$

33.8 13.96966

33.9 0.7901

33.10 0.8

33.11 65098.637

33.12 19.0042586

Section 34

34.1 $\bar{a}_{x:\overline{n}|} = \frac{1-e^{-(\mu+\delta)n}}{\mu+\delta}$

34.2 2.16166

34.3 We have

$$\begin{aligned}
\bar{a}_x =& \int_0^\infty {}_tE_x dt \\
=& \int_0^n {}_tE_x dt + \int_n^\infty {}_tE_x dt \\
=& \bar{a}_{x:\overline{n}|} + \int_n^\infty \nu^t {}_np_x {}_{t-n}p_{x+n} dt \\
=& \bar{a}_{x:\overline{n}|} + \nu^n p_x \int_0^\infty \nu^t {}_tp_{x+n} dt \\
=& \bar{a}_{x:\overline{n}|} + \nu^n {}_np_x \bar{a}_{x+n}
\end{aligned}$$

34.4 7.8202

34.5 We have

$$\begin{aligned}
\bar{a}_{x:\overline{m+n}|} =& \int_0^{m+n} \nu^t {}_tp_x dt \\
=& \int_0^m \nu^t {}_tp_x dt + \int_m^{m+n} \nu^t {}_tp_x dt \\
=& \bar{a}_{x:\overline{m}|} + \int_m^{m+n} \nu^t {}_{t-m}p_{x+m} {}_mp_x dt \\
=& \bar{a}_{x:\overline{m}|} + \nu^m {}_mp_x \int_m^{m+n} \nu^{t-m} {}_{t-m}p_{x+m} dt \\
=& \bar{a}_{x:\overline{m}|} + \nu^m {}_mp_x \int_0^n \nu^t {}_tp_{x+m} dt \\
=& \bar{a}_{x:\overline{m}|} + {}_mE_x \bar{a}_{x+m:\overline{n}|}
\end{aligned}$$

Section 35

35.1 We have

$$\begin{aligned}{}_{n|}\bar{a}_x = & \int_n^\infty e^{-\delta t}e^{-\mu t}dt = \int_n^\infty e^{-t(\delta+\mu}dt \\ = & -\left.\frac{e^{-(\mu+\delta)t}}{\mu+\delta}\right|_n^\infty = \frac{e^{-n(\mu+\delta)}}{\mu+\delta}\end{aligned}$$

35.2 0.3319

35.3 For $T(x) \le n$, we have

$${}_{n|}\bar{Y}_x = Z_{x:\overline{n|}}^{\ 1} = {}_{n|}\bar{Z}_x = 0.$$

For $T(x) > n$ we have $Z_{x:\overline{n|}}^{\ 1} = \nu^n$ and ${}_{n|}\bar{Z}_x = \nu^T$. Thus,

$$\frac{Z_{x:\overline{n|}}^{\ 1} - {}_{n|}\bar{Z}_x}{\delta} = \frac{\nu^n - \nu^T}{\delta} = \nu^n\frac{1-\nu^{T-n}}{\delta} = {}_{n|}\bar{Y}_x.$$

35.4 We have

$$\begin{aligned}E[({}_{n|}\bar{Y}_x)^2] = & \int_n^\infty \nu^{2n}(\bar{a}_{\overline{t-n|}})^2{}_tp_x\mu(x+t)dt \\ = & \nu^{2n}\left[-\left.(\bar{a}_{\overline{t-n|}})^2{}_tp_x\right|_n^\infty + 2\int_n^\infty \nu^{t-n}{}_tp_xdt\right] \\ = & 2\nu^{2n}{}_np_x\int_0^\infty \nu^t\bar{a}_{\overline{t|}}{}_tp_{x+n}dt\end{aligned}$$

35.5 ${}_{20|}\bar{a}_x = 1.2235$ and $\mathrm{Var}({}_{20|}\bar{Y}_x) = 0.9753$

Section 36

36.1 ${}_{20|}\bar{a}_{50} = 0.3319$ and $\bar{a}_{\overline{50:\overline{20}|}} = 8.9785$

36.2 We have

$$\begin{aligned}\bar{a}_{\overline{x:\overline{n}|}} =& E(\bar{Y}_{\overline{x:\overline{n}|}})\\ =& \int_0^n \bar{a}_{\overline{n}|}\,{}_tp_x\mu(x+t)dt + \int_n^\infty \bar{a}_{\overline{t}|}\,{}_tp_x\mu(x+t)dt\\ =& \bar{a}_{\overline{n}|}\left[-{}_tp_x\right]_0^n + \int_n^\infty \bar{a}_{\overline{t}|}\,{}_tp_x\mu(x+t)dt\\ =& \bar{a}_{\overline{n}|}\,{}_nq_x + \int_n^\infty \bar{a}_{\overline{t}|}\,{}_tp_x\mu(x+t)dt.\end{aligned}$$

36.3 We have

$$\begin{aligned}\bar{a}_{\overline{x:\overline{n}|}} =& \bar{a}_{\overline{n}|}\,{}_nq_x + \int_n^\infty \bar{a}_{\overline{t}|}\,{}_tp_x\mu(x+t)dt\\ =& \bar{a}_{\overline{n}|}\,{}_nq_x - \bar{a}_{\overline{t}|}\,{}_tp_x\big|_n^\infty + \int_n^\infty \nu^t\,{}_tp_xdt\\ =& \bar{a}_{\overline{n}|} + + \int_n^\infty \nu^t\,{}_tp_xdt.\end{aligned}$$

36.4 This follows from

$$\text{Var}(\bar{Y}_{\overline{x:\overline{n}|}}) = \text{Var}(\bar{a}_{\overline{n}|} + {}_{n|}\bar{Y}_x) = \text{Var}({}_{n|}\bar{Y}_x)$$

36.5 From Section 35, we have that

$${}_{n|}\bar{a}_x = {}_nE_x\bar{a}_{x+n}.$$

Therefore,

$$\bar{a}_{\overline{x:\overline{n}|}} = \bar{a}_{\overline{n}|} + {}_nE_x\bar{a}_{x+n}.$$

36.6 From Problem 34.3, we have that

$$\bar{a}_x = \bar{a}_{x:\overline{n}|} + {}_nE_x\bar{a}_{x+n}.$$

Therefore,

$$\bar{a}_{\overline{x:\overline{n}|}} = \bar{a}_{\overline{n}|} + (\bar{a}_x - \bar{a}_{x:\overline{n}|}).$$

Section 37

37.1 0.5235

37.2 13.78

37.3 We have

$$\begin{aligned}\ddot{a}_x =& \sum_{k=0}^{\infty} \nu^k {}_kp_x = 1 + \sum_{k=1}^{\infty} \nu^k {}_kp_x = 1 + \nu p_x \sum_{k=1}^{\infty} \nu^{k-1} {}_{k-1}p_{x+1} \\ =& 1 + \nu p_x \sum_{k=0}^{\infty} \nu^k {}_kp_{x+1} = 1 + \nu p_x \ddot{a}_{x+1}.\end{aligned}$$

37.4 0.364

37.5 7%

37.6 150,000

37.7 52,297.43

37.8 1296.375

37.9 We have

$$\begin{aligned}\ddot{a}_{x:\overline{n}|} =& \sum_{k=0}^{n-1} \nu^k {}_kp_x \\ =& 1 + \sum_{k=1}^{n-1} \nu^k {}_kp_x \\ =& 1 + \nu p_x \sum_{k=1}^{n-1} \nu^{k-1} {}_{k-1}p_{x+1} \\ =& 1 + \nu p_x \sum_{k=0}^{n-2} \nu^k {}_kp_{x+1} \\ =& 1 + \nu p_x \ddot{a}_{x+1:\overline{n-1}|}.\end{aligned}$$

37.10 264.2196

37.11 114.1785

37.12 0.2991

37.13 280.41

37.14 49.483

37.15 10.3723

37.16 Recall the following

$$A_x = A^1_{x:\overline{n}|} + {}_nE_xA_{x+n}$$

and

$$A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + {}_nE_x.$$

Thus,

$$\begin{aligned}
\ddot{a}_x =& \frac{1 - A_x}{d} = \frac{1 - A^1_{x:\overline{n}|} - {}_nE_xA_{x+n}}{d} \\
=& \frac{1 - A_{x:\overline{n}|} + {}_nE_x - {}_nE_xA_{x+n}}{d} \\
=& \frac{1 - A_{x:\overline{n}|}}{d} + {}_nE_x\left(\frac{1 - A_{x+n}}{d}\right) \\
=& \ddot{a}_{x:\overline{n}|} + {}_nE_x\ddot{a}_{x+n}.
\end{aligned}$$

37.17 Recall that

$$Z_{x:\overset{\ 1}{\overline{n}|}} = \begin{cases} 0 & T \leq n \\ \nu^n & T > n \end{cases}$$

and

$${}_{n|}Z_x = \begin{cases} 0 & K \leq n - 1 \\ \nu^{K+1} & K \geq n \end{cases}$$

Thus, if $K \leq n - 1$ then $T \leq n$ and therefore $Z_{x:\overset{\ 1}{\overline{n}|}} = {}_{n|}Z_x = {}_{n|}\ddot{Z}_x = 0$. If $K \geq n$ then $T > n$ so that $Z_{x:\overset{\ 1}{\overline{n}|}} = \nu^n$ and ${}_{n|}Z_x = \nu^{K+1}$. Thus, $\frac{Z_{x:\overset{\ 1}{\overline{n}|}} - {}_{n|}Z_x}{d} = \frac{\nu^n - \nu^{K+1}}{d} = \nu\ddot{a}_{\overline{K+1-n}|} = {}_{n|}\ddot{Z}_x$.

37.18 This follows from the previous problem by taking expectation of both sides.

37.19 ${}_{n|}\ddot{a}_x = {}_nE_x\ddot{a}_{x+n} = \frac{\nu^n(p_x)^n}{1-e^{-(\delta+\mu)}} = \frac{e^{-n(\delta+\mu)}}{1-e^{-(\delta+\mu)}}$.

37.20 0.4151

37.21 0.45

37.22 16.6087

37.23 15.2736

37.24 58.36

37.25 3.30467

37.26 We have

$$e_x = \sum_{k=1}^{\infty} {}_kp_x = \sum_{k=1}^{\infty} p_{x\,k-1}p_{x+1} = p_x + p_x \sum_{k=2}^{\infty} {}_{k-1}p_{x+1} = p(x)(1+e_{x+1}).$$

(b) 0.0789.

37.27 $a_x = \frac{e^{-(\mu+\delta)}}{1-e^{-(\mu+\delta)}}$

37.28 We have

$$\begin{aligned} a_x =& \sum_{k=1}^{\infty} \nu^k {}_kP_x = \sum_{k=1}^{\infty} \nu^k p_{x\,k-1}p_{x+1} = \nu p_x \sum_{k=1}^{\infty} \nu^{k-1}{}_{k-1}p_{x+1} \\ =& \nu p_x(1+\sum_{k=2}^{\infty} \nu^{k-1}{}_{k-1}p_{x+1}) = \nu p_x(1+\sum_{k=1}^{\infty} \nu^k {}_kp_{x+1}) = \nu p_x(1+a_{x+1}) \end{aligned}$$

37.29 7.6

37.30 0.1782

37.31 $a_{x:\overline{n}|} = e^{-(\mu+\delta)}\left(\frac{1-e^{-n(\mu+\delta)}}{1-e^{-(\mu+\delta)}}\right)$

37.32 11.22

Section 38

38.1 12.885

38.2 13.135

38.3 $A_{80} = 0.8162$ and $\bar{a}_{80} = 2.5018$

38.4 15.5

38.5 8.59

38.6 We have

$$\begin{aligned}
\ddot{a}_{x:\overline{n}|}^{(m)} =& \frac{1}{m}\sum_{k=0}^{mn-1}\nu^{\frac{k}{m}}\,{}_{\frac{k}{m}}p_x\\
=& \frac{1}{m} + \frac{1}{m}\sum_{k=1}^{mn}\nu^{\frac{k}{m}}\,{}_{\frac{k}{m}}p_x - \frac{1}{m}\nu^n\,{}_np_x\\
=& a_{x:\overline{n}|}^{(m)} + \frac{1}{m}(1 - {}_nE_x).
\end{aligned}$$

38.7 We have

$$\begin{aligned}
\ddot{a}_{x:\overline{n}|}^{(m)} =& \ddot{a}_x^{(m)} - {}_{n|}\ddot{a}_x^{(m)}\\
\approx& \frac{i}{i^{(m)}}\frac{d}{d^{(m)}}\ddot{a}_x - \frac{i-i^{(m)}}{i^{(m)}d^{(m)}} - \frac{i}{i^{(m)}}\frac{d}{d^{(m)}}\,{}_{n|}\ddot{a}_x + \frac{i-i^{(m)}}{i^{(m)}d^{(m)}}\,{}_nE_x\\
=& \frac{i}{i^{(m)}}\frac{d}{d^{(m)}}(\ddot{a}_x - {}_{n|}\ddot{a}_x) - \frac{i-i^{(m)}}{i^{(m)}d^{(m)}}(1 - {}_nE_x)\\
=& \frac{i}{i^{(m)}}\frac{d}{d^{(m)}}\ddot{a}_{x\overline{n}|} - \frac{i-i^{(m)}}{i^{(m)}d^{(m)}}(1 - {}_nE_x).
\end{aligned}$$

38.8 (a) We have

$$\begin{aligned} a_x^{(m)} =&\ddot{a}_x^{(m)} - \frac{1}{m}\\ \approx&\frac{i}{i^{(m)}}\frac{d}{d^{(m)}}\ddot{a}_x - \frac{i-i^{(m)}}{i^{(m)}d^{(m)}} - \frac{1}{m}\\ =&\frac{i}{i^{(m)}}\frac{d}{d^{(m)}}(a_x+1) - \frac{i-i^{(m)}}{i^{(m)}d^{(m)}} - \frac{1}{m}\\ =&\frac{i}{i^{(m)}}\frac{d}{d^{(m)}}a_x + \frac{i}{i^{(m)}}\frac{d}{d^{(m)}} - \frac{i-i^{(m)}}{i^{(m)}d^{(m)}} - \frac{(1-\nu^{\frac{1}{m}})i^{(m)}}{i^{(m)}d^{(m)}}\\ =&\frac{i}{i^{(m)}}\frac{d}{d^{(m)}}a_x + \frac{d^{(m)}-d}{i^{(m)}d^{(m)}}. \end{aligned}$$

(b) We have

$$\begin{aligned} a_{x:\overline{n}|}^{(m)} =&\ddot{a}_{x:\overline{n}|}^{(m)} - \frac{1}{m}(1-{}_nE_x)\\ \approx&\frac{i}{i^{(m)}}\frac{d}{d^{(m)}}\ddot{a}_{x\overline{n}|} - \frac{i-i^{(m)}}{i^{(m)}d^{(m)}}(1-{}_nE_x) - \frac{1}{m}(1-{}_nE_x)\\ =&\frac{i}{i^{(m)}}\frac{d}{d^{(m)}}(1-{}_nE_x+a_{x:\overline{n}|}) - \frac{i-i^{(m)}}{i^{(m)}d^{(m)}}(1-{}_nE_x) - \frac{1}{m}(1-{}_nE_x)\\ =&\frac{i}{i^{(m)}}\frac{d}{d^{(m)}}a_{x:\overline{n}|}\frac{d^{(m)}-d}{i^{(m)}d^{(m)}}(1-{}_nE_x). \end{aligned}$$

38.9 (a) We have

$$\begin{aligned} \ddot{a}_{x:\overline{n}|}^{(m)} =&\ddot{a}_x^{(m)} - {}_{n|}\ddot{a}_x^{(m)}\\ \approx&\ddot{a}_x - \frac{m-1}{2m} - {}_{n|}\ddot{a}_x + \frac{m-1}{2m}{}_nE_x\\ =&\ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1-{}_nE_x). \end{aligned}$$

(b) We have

$$\begin{aligned} a_x^{(m)} =&\ddot{a}_x^{(m)} - \frac{1}{m}\\ \approx&\ddot{a}_x - \frac{m-1}{2m} - \frac{1}{m}\\ =&a_x + 1 - \frac{m-1}{2m} - \frac{1}{m}\\ =&a_x + \frac{m-1}{2m}. \end{aligned}$$

(c) We have

$$
\begin{aligned}
{}_{n|}a_x^{(m)} =&{}_nE_xa_{x+n}^{(m)}\\
\approx&{}_nE_x\left(a_{x+n}+\frac{m-1}{2m}\right)\\
=&{}_nE_xa_{x+n}+\frac{m-1}{2m}{}_nE_x\\
=&{}_{n|}a_x+\frac{m-1}{2m}{}_nE_x.
\end{aligned}
$$

(d) We have

$$
\begin{aligned}
a_{x:\overline{n}|}^{(m)} =&a_x^{(m)}-{}_{n|}a_x\\
\approx&a_x+\frac{m-1}{2m}-{}_{n|}a_x-\frac{m-1}{2m}{}_nE_x\\
=&a_{x:\overline{n}|}+\frac{m-1}{2m}(1-{}_nE_x).
\end{aligned}
$$

38.10 (a) We have

$$
\begin{aligned}
\ddot{a}_{x:\overline{n}|}^{(m)} =&\ddot{a}_x^{(m)}-{}_nE_x\ddot{a}_{x+n}^{(m)}\\
\approx&\ddot{a}_x-\frac{m-1}{2m}-\frac{m^2-1}{12m}(\mu(x)+\delta)\\
&-{}_nE_x(\ddot{a}_{x+n}-\frac{m-1}{2m}-\frac{m^2-1}{12m}(\mu(x+n)+\delta)\\
=&\ddot{a}_{x:\overline{n}|}-\left(\frac{m-1}{m}\right)(1-{}_nE_x)-\frac{m^2-1}{12m}(\delta+\mu(x)-{}_nE_x(\delta+\mu(x+n))).
\end{aligned}
$$

(b) The result follows by letting $m\to\infty$ in the 3-term Woolhouse formula.
(c) The result follows by letting $m\to\infty$ in (a)

Section 39

39.1 218.79

39.2 8.56

39.3 5.1029

39.4 5.7341

39.5 5.3465

39.6 204.08

39.7 4.4561

39.8 $(I\bar{a})_x = \int_0^\infty \lceil t \rceil \nu^t {}_ip_x dt$

39.9 $(I\bar{a})_x = \int_0^n \lceil t \rceil \nu^t {}_ip_x dt$

39.10 $(D\bar{a})_{x:\overline{n|}} = \int_0^n \lceil n - t \rceil \nu^t {}_ip_x dt$

Section 40

40.1 0.2

40.2 $\bar{P}(\bar{A}_{75}) = 0.02901$ and $\mathrm{Var}(\bar{L}_x) = 0.15940$

40.3 0.2

40.4 0.7125

40.5 0.1

40.6 0.05137

40.7 $\bar{P}(\bar{A}^1_{x:\overline{n}|}) = 0.02$ and $\mathrm{Var}(\bar{L}^1_{75:\overline{20}|}) = 0.1553$

40.8 $\bar{P}(\bar{A}^1_{x:\overline{n}|}) = \frac{\frac{1}{\delta(\omega-x)}(1-e^{-n\delta})}{\frac{1}{\delta}\left(1-\frac{1}{\delta(\omega-x)}(1-e^{-n\delta})-e^{-n\delta}\left(1-\frac{n}{\omega-x}\right)\right)}$ and

$$\mathrm{Var}(\bar{L}^1_{x:\overline{n}|}) = \left(1+\frac{\frac{1}{\delta(\omega-x)}(1-e^{-n\delta})}{\left(1-\frac{1}{\delta(\omega-x)}(1-e^{-n\delta})-e^{-n\delta}\left(1-\frac{n}{\omega-x}\right)\right)}\right)^2 \times\left[\frac{1}{2\delta(\omega-x)}(1-e^{-n\delta})-\frac{1}{\delta^2(\omega-x)^2}(1-e^{-n\delta})^2\right]$$

40.9 $\bar{P}(\bar{A}^1_{75:\overline{20}|}) = 0.02402$ and $\mathrm{Var}(\bar{L}^1_{75:\overline{20}|}) = 0.13694$

40.10 0.25285

40.11 0.47355

40.12 0.04291

40.13 0.09998

40.14 We have

$$\bar{P}(\bar{A}_{x:\overline{n|}}) = \frac{\left[\frac{1}{\delta(\omega-x)}(1-e^{-n\delta}) + e^{-n\delta}\left(1-\frac{n}{\omega-x}\right)\right]\delta}{1-\frac{1}{\delta(\omega-x)}(1-e^{-n\delta}) - e^{-n\delta}\left(1-\frac{n}{\omega-x}\right)}$$

$$\mathrm{Var}(\bar{L}_{x:\overline{n|}}) = \frac{\frac{1}{2\delta(\omega-x)}(1-e^{-2n\delta}) + e^{-2n\delta}\left(1-\frac{n}{\omega-x}\right) - \left(\frac{1}{\delta(\omega-x)}(1-e^{-n\delta}) + e^{-n\delta}\left(1-\frac{n}{\omega-x}\right)\right)^2}{\left(1-\frac{1}{\delta(\omega-x)}(1-e^{-n\delta}) - e^{-n\delta}\left(1-\frac{n}{\omega-x}\right)\right)^2}$$

40.15 0.04498

40.16 0.10775

40.17 0.06626

40.18 0.4661

40.19 0.0229

40.20 0.42341

40.21 We have

$$\begin{aligned}
\bar{P}(\bar{A}^1_{x:\overline{n|}}) + \bar{P}(\bar{A}_{x:\overset{1}{\overline{n|}}})\bar{A}_{x+n} =& \frac{\bar{A}^1_{x:\overline{n|}}}{\bar{a}_{x:\overline{n|}}} + \frac{\bar{A}_{x:\overset{1}{\overline{n|}}}}{\bar{a}_{x:\overline{n|}}}\bar{A}_{x+n} \\
=& \frac{\bar{A}^1_{x:\overline{n|}} + {}_{n|}\bar{A}_x}{\bar{a}_{x:\overline{n|}}}\bar{A}_{x+n} \\
=& \frac{\bar{A}_x}{\bar{a}_{x:\overline{n|}}} \\
=& {}_n\bar{P}(\bar{A}_x)
\end{aligned}$$

40.22 We have

$$\begin{aligned}\bar{P}(\bar{A}_{x:\overline{n}|}) =&\frac{\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}\\ =&\frac{\bar{A}^{1}_{x:\overline{n}|}+\bar{A}_{x:\overset{1}{\overline{n}|}}}{\bar{a}_{x:\overline{n}|}}\\ =&\frac{\bar{A}^{1}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}+\frac{\bar{A}_{x:\overset{1}{\overline{n}|}}}{\bar{a}_{x:\overline{n}|}}\\ =&\bar{P}(\bar{A}^{1}_{x:\overline{n}|})+\bar{P}(\bar{A}_{x:\overset{1}{\overline{n}|}}) \blacksquare\end{aligned}$$

40.23 0.01657

40.24 0.03363

40.25 0.0498

40.26 1.778

40.27 0.7696

40.28 -5.43

40.29 -14.09

40.30 0.005

Section 41

41.1 12381.06

41.2 124.33

41.3 $P(A_x)=\frac{A_x}{\ddot{a}_x}=\frac{\frac{q_x}{q_x+i}}{\frac{1+i}{q_x+i}}=\nu q_x$

41.4 16076.12

41.5 33.15

41.6 $\frac{4}{105}$

41.7 From the definition of $P(A_x)$ and the relation $A_x+da_x=1$ we can write

$$\begin{aligned}
P(A_x)=&\frac{A_x}{a_x}=\frac{1-da_x}{a_x}\\
P(A_x)a_x=&1-da_x\\
a_x(P(A_x)+d)=&1\\
a_x=&\frac{1}{P(A_x)+d}
\end{aligned}$$

41.8 33.22

41.9 We have

$$\begin{aligned}
L^1_{x:\overline{n}|}=&Z^1_{x:\overline{n}|}-P\ddot{Y}_{x:\overline{n}|}\\
=&Z^1_{x:\overline{n}|}-P\left(\frac{1-Z_{x:\overline{n}|}}{d}\right)\\
=&Z^1_{x:\overline{n}|}-P\left(\frac{1-Z^1_{x:\overline{n}|}-Z_{x:\overset{1}{\overline{n}|}}}{d}\right)\\
=&\left(1+\frac{P}{d}\right)Z^1_{x:\overline{n}|}+\frac{P}{d}Z_{x:\overset{1}{\overline{n}|}}-\frac{P}{d}
\end{aligned}$$

41.10 0.317

41.11 2410.53

41.12 0.0368

41.13 281.88

41.14 -10877.55

41.15 261.14

41.16 0.2005

41.17 0.087

41.18 This follows easily by dividing

$$A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|}) + A_{x:\overset{\,1}{\overline{n}|}}$$

by $\ddot{a}_{x:\overline{n}|}$

41.19 We have

$$\begin{aligned}
P(A^{1}_{x:\overline{n}|}) + P(A_{x:\overset{\,1}{\overline{n}|}})A_{x+n} =& \frac{A^{1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} + \frac{A_{x:\overset{\,1}{\overline{n}|}}}{\ddot{a}_{x:\overline{n}|}}A_{x+n}\\
=& \frac{A^{1}_{x:\overline{n}|} + A_{x:\overset{\,1}{\overline{n}|}}A_{x+n}}{\ddot{a}_{x:\overline{n}|}}\\
=& \frac{A_x}{\ddot{a}_{x:\overline{n}|}} = {}_nP(A_x)
\end{aligned}$$

41.20 0.00435

41.21 0.03196

41.22 0.03524

41.23 0.51711

41.24 We have

$$\begin{aligned}{}_{n|}L_x =& {}_{n|}Z_x - P\left(\frac{1-Z_x}{d}\right) = {}_{n|}Z_x - \frac{P}{d} + \frac{P}{d}\left(Z^1_{x:\overline{n}|} + {}_{n|}Z_x\right)\\ =&\left(1+\frac{P}{d}\right)n|Z_x + \frac{P}{d}Z^1_{x:\overline{n}|} - \frac{P}{d}\end{aligned}$$

41.25 Note first that

$$Z^1_{x:\overline{n}|}{}_{n|}Z_x = \nu^{K+1}\mathbf{I}(K\geq n)\nu^{K+1}\mathbf{I}(K\leq n-1) = 0.$$

Thus,

$$\begin{aligned}E\left[\left({}_{n|}L_x + \frac{P}{d}\right)^2\right] =& E\left[\left(\frac{P}{d}\right)^2 (Z^1_{x:\overline{n}|})^2 + \left(1+\frac{P}{d}\right)^2 ({}_{n|}Z_x)^2\right]\\ =&\left(\frac{P}{d}\right)^2 ({}^2A^1_{x:\overline{n}|} + \left(1+\frac{P}{d}\right)^2 {}^2{}_{n|}A_x\end{aligned}$$

41.26 The loss random variable is

$$\nu^{K+1}\mathbf{I}(K\geq n) - P\ddot{a}_{\overline{\min{(K+1,t)}}|} = {}_{n|}Z_x - P\ddot{Y}_{x:\overline{t}|}.$$

The actuarial present value is

$${}_{n|}A_x - P\ddot{a}_{x:\overline{t}|}$$

41.27 The benefit premium which satisfies the equivalence principle is

$${}_tP({}_{n|}A_x) = \frac{{}_{n|}A_x}{\ddot{a}_{x:\overline{t}|}}$$

41.28 0.01567

41.29 13092.43

41.30 0.024969

Section 42

42.1 0.0193

42.2 0.0256

42.3 0.0347

42.4 This is the benefit premium for a 20-payment, semi-continuous whole life insurance issued to (40) with face value of 1000

42.5 0.04575

42.6 0.0193

42.7 0.0289

42.8 0.829

42.9 0.0069

42.10 11.183

42.11 -12972.51

42.12 0.0414

42.13 0.0620

42.14 0.0860

42.15 We have

$$\begin{aligned}
\frac{P(\bar{A}_{x:\overline{n}|} - {}_nP(\bar{A}_x)}{P(A_{x:\overset{1}{\overline{n}|}})} =&\frac{\bar{A}_{x:\overline{n}|} - \bar{A}_x}{A_{x:\overset{1}{\overline{n}|}}}\\
=&\frac{\bar{A}^1_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}} - \bar{A}_x}{A_{x:\overset{1}{\overline{n}|}}}\\
=&\frac{A_{x:\overset{1}{\overline{n}|}} - {}_n\bar{A}_x}{A_{x:\overset{1}{\overline{n}|}}}\\
=&\frac{A_{x:\overset{1}{\overline{n}|}} - A_{x:\overset{1}{\overline{n}|}}\bar{A}_{x+n}}{A_{x:\overset{1}{\overline{n}|}}} = 1 - \bar{A}_{x+n}.
\end{aligned}$$

42.16 This follows from the formula $\bar{A}_{x:\overline{n}|} = \bar{A}^1_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}$

42.17 0.0096

42.18 0.0092

42.19 We have

$$\begin{aligned}
P({}_{n|}\bar{A}_x) =&\frac{A_{x:\overset{1}{\overline{n}|}}\bar{A}_{x+n}}{\ddot{a}_x}\\
=&\frac{A_{x:\overset{1}{\overline{n}|}}\bar{A}_{x+n}}{\ddot{a}_{x:\overline{n}|} + {}_nE_x\ddot{a}_{x+n}}.
\end{aligned}$$

42.20 77079

Section 43

43.1 231.64

43.2 122.14

43.3 331.83

43.4 493.58

43.5 94.83

43.6 224.45

43.7 117.52

43.8 325.19

43.9 484.32

Section 44

44.1 7.747π

44.2 102

44.3 0.078π

44.4 0.88π

44.5 15.02

44.6 5.1

44.7 19.07

44.8 73.66

44.9 397.41

44.10 1.276

44.11 478.98

44.12 3362.51

44.13 900.20

44.14 17.346

44.15 3.007986

44.16 15513.82

Section 45

45.1 0.0363

45.2 0.0259

45.3 0.049

45.4 0.07707

45.5 0.02174

45.6 (a) $E(L_x) = bA_x - \pi\ddot{a}_x$ (b) $\text{Var}(L_x) = \left(b + \frac{\pi}{d}\right)^2 [{}^2A_x - (A_x)^2]$

45.7 33023.89

45.8 27

45.9 0.208765

45.10 36.77

Section 46

46.1 We have

$$\begin{aligned}
{}_t\bar{V}(\bar{A}_x) =&\bar{A}_{x+t} - \bar{P}(\bar{A}_x)\bar{a}_{x+t}\\
=&(1-\delta\bar{a}_{x+t}) - \left(\frac{1-\delta\bar{a}_x}{\bar{a}_x}\right)\bar{a}_{x+t}\\
=&1-\delta\bar{a}_{x+t} - \frac{\bar{a}_{x+t}}{\bar{a}_x} + \delta\bar{a}_{x+t}\\
=&1-\frac{\bar{a}_{x+t}}{\bar{a}_x}.
\end{aligned}$$

46.2 8.333

46.3 0.04

46.4 0.0654

46.5 1.6667

46.6 0.1667

46.7 0.47213

46.8 0.20

46.9 0.14375

46.10 0.3

46.11 0.1184

46.12 0.1667

46.13 0.1183

46.14 0.1183

46.15 we have

$$\begin{aligned}
{}_t\bar{V}(\bar{A}_x) =& \bar{A}_{x+t} - \bar{P}(\bar{A}_x)\bar{a}_{x+t}\\
=& \bar{a}_{x+t}\left(\frac{\bar{A}_{x+t}}{\bar{a}_{x+t}} - \bar{P}(\bar{A}_x)\right)\\
=& \bar{a}_{x+t}\left(\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)\right)
\end{aligned}$$

46.16 0.1183

46.17 0.0654

46.18 The prospective formula is

$${}_{10}\bar{V}(\bar{A}_{50}) = \bar{A}_{60} - \bar{P}(\bar{A}_{50})\bar{a}_{60}.$$

The retrospective formula is

$${}_{10}\bar{V}(\bar{A}_{50}) = \frac{\bar{P}(\bar{A}_{50})\bar{a}_{50:\overline{10}|} - \bar{A}^{\,1}_{50:\overline{10}|}}{{}_{10}E_{50}}$$

46.19 We have

$$\begin{aligned}
{}_t\bar{V}(\bar{A}_x) =& \frac{\bar{P}(\bar{A}_x)\bar{a}_{x:\overline{n}|} - \bar{A}^{1}_{x:\overline{n}|}}{{}_nE_x}\\
=& \frac{\bar{P}(\bar{A}_x) - \frac{\bar{A}^{1}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}}{\frac{{}_nE_x}{\bar{a}_{x:\overline{n}|}}}\\
=& \frac{\bar{P}(\bar{A}_x) - \bar{P}(\bar{A}^{1}_{x:\overline{n}|})}{\bar{P}(A_{x:\overline{n}|}^{\ \ 1})}
\end{aligned}$$

46.20 True

46.21 0.0851

46.22 We have

$$
\begin{aligned}
{}_t\bar{V}(\bar{A}^1_{x:\overline{n}|}) =& \bar{A}^{\ 1}_{x+t:\overline{n-t}|} - \bar{P}(\bar{A}^1_{x:\overline{n}|})\bar{a}_{x+t:\overline{n-t}|}\\
=&\frac{\mu}{\mu+\delta}(1-e^{(n-t)(\mu+\delta)}) - \mu\left[\frac{1-\bar{A}_{x:\overline{n}|}}{\delta}\right]\\
=&\frac{\mu}{\mu+\delta}(1-e^{(n-t)(\mu+\delta)}) - \mu\left[\frac{1-\frac{\mu}{\mu+\delta}(1-e^{(n-t)(\mu+\delta)}) - e^{(n-t)(\mu+\delta)}}{\delta}\right]\\
=&\left(\frac{\mu}{\mu+\delta} - \frac{\mu}{\delta} + \frac{\mu^2}{\delta(\mu+\delta)}\right)(1-e^{(n-t)(\mu+\delta)})\\
=&0\times(1-e^{(n-t)(\mu+\delta)}) = 0
\end{aligned}
$$

46.23 Follows from the previous problem.

46.24 0.0294

46.25 ${}_t\bar{V}(\bar{A}^1_{x:\overline{n}|}) = \bar{a}_{x+t,\overline{n-t}|}[\bar{P}(\bar{A}^{\ 1}_{x+t:\overline{n-t}|}) - \bar{P}(\bar{A}^1_{x:\overline{n}|})]$

46.26 ${}_t\bar{V}(\bar{A}^1_{x:\overline{n}|}) = \bar{A}^{\ 1}_{x+t:\overline{n-t}|}\left[1 - \frac{\bar{P}(\bar{A}^1_{x:\overline{n}|})}{\bar{P}(\bar{A}^{\ 1}_{x+t:\overline{n-t}|})}\right]$

46.27 0.4207

46.28 0.3317

46.29 Recall that

$$\bar{a}_{x:\overline{n}|} = \frac{1-\bar{A}_{x:\overline{n}|}}{\delta}.$$

Thus,

$$
\begin{aligned}
{}_t\bar{V}(\bar{A}_{x:\overline{n}|}) =& \bar{A}_{x+t:\overline{n-t}|} - \bar{P}(\bar{A}_{x:\overline{n}|})\bar{a}_{x+t:\overline{n-t}|}\\
=&\bar{A}_{x+t:\overline{n-t}|} - \frac{\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}\bar{a}_{x+t:\overline{n-t}|}\\
=&\bar{A}_{x+t:\overline{n-t}|} - \frac{\bar{A}_{x:\overline{n}|}}{\frac{1-\bar{A}_{x:\overline{n}|}}{\delta}}\cdot\frac{1-\bar{A}_{x+t:\overline{n-t}|}}{\delta}\\
=&\frac{\bar{A}_{x+t:\overline{n-t}|} - \bar{A}_{x:\overline{n}|}}{1-\bar{A}_{x:\overline{n}|}}
\end{aligned}
$$

46.30 0.3431

46.31

$$ {}_t\bar{V}(A_{x:\overline{n}|}^{\ \ 1}) = \begin{cases} A_{x+t:\overline{n-t}|}^{\ \ \ \ \ \ 1} - \bar{P}(A_{x:\overline{n}|}^{\ \ 1})\bar{a}_{x+t:\overline{n-t}|}, & t < n \\ 1, & t = n. \end{cases} $$

46.32 0.7939

46.33 1

46.34 0.3088

46.35 0.2307

46.36 This is the 10th year benefit reserve for a fully continuous 20-year pure endowment of unit benefit issued to (75).

46.37 We have

$$ \begin{aligned} {}_t\bar{V}(\bar{A}_{x:\overline{n}|}^{\ \ 1}) =& \bar{A}_{x+t:\overline{n-t}|}^{\ \ \ \ \ \ 1} - \bar{P}(A_{x:\overline{n}|}^{\ \ 1})\bar{a}_{x+t:\overline{n-t}|} \\ =& \bar{A}_{x+t:\overline{n-t}|} - \bar{A}^{\ 1}_{x+t:\overline{n-t}|} - [\bar{P}(\bar{A}_{x:\overline{n}|}) - \bar{P}(\bar{A}^{1}_{x:\overline{n}|})]\bar{a}_{x+t:\overline{n-t}|} \\ =& [\bar{A}_{x+t:\overline{n-t}|} - \bar{P}(\bar{A}_{x:\overline{n}|})\bar{a}_{x+t:\overline{n-t}|}] - [\bar{A}^{\ 1}_{x+t:\overline{n-t}|} - \bar{P}(\bar{A}^{1}_{x:\overline{n}|})\bar{a}_{x+t:\overline{n-t}|}] \\ =& {}_t\bar{V}(\bar{A}_{x:\overline{n}|}) - {}_t\bar{V}(\bar{A}^{1}_{x:\overline{n}|}) \ \blacksquare \end{aligned} $$

46.38 This follows from the previous problem and Problem 46.22.

46.39 24

46.40 4.6362

46.41 5.9055

46.42 14.2857

46.43 14.2857

Section 47

47.1 (a) 0.0533 (b) 0.1251

47.2 We have

$$\begin{aligned}
{}_kV(A_x) =& A_{x+k} - P(A_x)\ddot{a}_{x+k} \\
=& 1 - d\ddot{a}_{x+k} - \frac{(1 - d\ddot{a}_x)}{\ddot{a}_x}\ddot{a}_{x+k} \\
=& 1 - d\ddot{a}_{x+k} - \frac{\ddot{a}_{x+k}}{\ddot{a}_x} + d\ddot{a}_{x+k} \\
=& 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x}
\end{aligned}$$

47.3 0.053

47.4 We have

$$\begin{aligned}
{}_kV(A_x) =& A_{x+k} - P(A_x)\ddot{a}_{x+k} \\
=& P(A_{x+k})\ddot{a}_{x+k} - P(A_x)\ddot{a}_{x+k} \\
=& (P(A_{x+k}) - P(A_x))\ddot{a}_{x+k}
\end{aligned}$$

47.5 0.0534

47.6 We have

$$\begin{aligned}
{}_kV(A_x) =& A_{x+k} - P(A_x)\ddot{a}_{x+k} \\
=& A_{x+k}\left(1 - P(A_x)\frac{\ddot{a}_{x+k}}{A_{x+k}}\right) \\
=& A_{x+k}\left(1 - \frac{P(A_x)}{P(A_{x+k})}\right)
\end{aligned}$$

47.7 0.0534

47.8 We have

$$
\begin{aligned}
{}_kV(A_x) =& 1-\frac{\ddot{a}_{x+k}}{\ddot{a}_x}\\
=& 1-\frac{\frac{1-A_{x+k}}{d}}{\frac{1-A_x}{d}}\\
=& 1-\frac{1-A_{x+k}}{1-A_x}\\
=& \frac{A_{x+k}-A_x}{1-A_x}
\end{aligned}
$$

47.9 0.053

47.10 We have

$$
\begin{aligned}
{}_k\bar{V}(A_x) =& A_{x+k}-P(A_x)\ddot{a}_{x+k}\\
=& A_{x+k}-P(A_x)\ddot{a}_{x+k}+\frac{P(A_x)\ddot{a}_x-A_x}{{}_kE_x}\\
=& P(A_x)\left(\frac{\ddot{a}_x-{}_kE_x\ddot{a}_{x+k}}{{}_kE_x}\right)-\left(\frac{A_x-{}_kE_xA_{x+k}}{{}_kE_x}\right)\\
=& P(A_x)\left(\frac{\ddot{a}_{x:\overline{k}|}}{{}_kE_x}\right)-\left(\frac{A^1_{x:\overline{k}|}}{{}_kE_x}\right)\\
=& P(A_x)\ddot{s}_{x:\overline{k}|}-\frac{A^1_{x:\overline{k}|}}{{}_kE_x}
\end{aligned}
$$

47.11 0.053

47.12 We have

$$
P(A_{x+k})=\frac{A_{x+k}}{d^{-1}(1-A_{x+k})}\Longrightarrow\frac{A_{x+k}}{P(A_{x+k})}=\frac{1}{P(A_{x+k})+d}.
$$

Thus,

$$
\begin{aligned}
{}_kV(A_x) =& A_{x+k}\left(1-\frac{P(A_x)}{P(A_{x+k})}\right)\\
=& \frac{[P(A_{x+k})-P(A_x)]A_{x+k}}{P(A_{x+k})}\\
=& \frac{P(A_{x+k})-P(A_x)}{P(A_{x+k})+d}
\end{aligned}
$$

47.13 305.651

47.14 114.2984

47.15 0.0851

47.16 171.985

47.17 4420.403

47.18 0.0042

47.19 -0.0826

47.20 0.1587

47.21 0.2757

47.22 0.0138

47.23 629.89

47.24 528.48

47.25 (a) For a fully discrete $n-$year pure endowment, the insurer's prospective loss at time k (or at age $x+k$) is:

$$ {}_kL(A_{x:\overline{n}|}^{\ \ 1}) = \nu^{n-k}\mathbf{I}(K \geq n) - P(A_{x:\overline{n}|}^{\ \ 1})\ddot{a}_{\overline{\min\{(K-k+1,n-k)\}}|},\ k < n $$

and ${}_nL(A_{x:\overline{n}|}^{\ \ 1}) = 1$.
(b) The prospective benefit reserve is

$$ {}_kV(A_{x:\overline{n}|}^{\ \ 1}) = \begin{cases} A_{x+k:\overline{n-k}|}^{\ \ \ \ \ \ 1} - P(A_{x:\overline{n}|}^{\ \ 1})\ddot{a}_{x+k:\overline{n-k}|} & k < n \\ 1 & k = n. \end{cases} $$

47.26 0.23426

47.27 8119.54

47.28 7.2170

47.29 (a) The prospective formula is

$$ {}_3V({}_{15|}\ddot{a}_{65}) = {}_{12}E_{68}\ddot{a}_{80} - P({}_{15|}\ddot{a}_{65})\ddot{a}_{68:\overline{12|}}. $$

(b) The retrospective formula is

$$ {}_3V({}_{15|}\ddot{a}_{65}) = \frac{P({}_{15|}\ddot{a}_{65})\ddot{a}_{65:\overline{3|}}}{{}_3E_{65}} $$

47.30 ${}_kV({}_{n|}\ddot{a}_x) = \frac{P({}_{n|}\ddot{a}_x)\ddot{a}_{x:\overline{n|}}}{{}_kE_x} - \frac{{}_n\ddot{a}_{x:\overline{k-n|}}}{{}_kE_x}$

47.31 3.3086

Section 48

48.1 0.0828

48.2 (a) The k^{th} terminal prospective loss random variable for an $n-$year term insurance contract

$$ {}_kL(\bar{A}^{1}_{x:\overline{n}|}) = \bar{Z}^{\ 1}_{x+k:\overline{n-k}|} - P(\bar{A}^{1}_{x:\overline{n}|})\ddot{Y}_{x+k:\overline{n-k}|}. $$

(b) The k^{th} terminal prospective reserve is given by

$$ {}_kV(\bar{A}^{1}_{x:\overline{n}|}) = \bar{A}^{\ 1}_{x+k:\overline{n-k}|} - P(\bar{A}^{1}_{x:\overline{n}|})\ddot{a}_{x+k:\overline{n-k}|} $$

48.3 (a) The prospective loss random variable is

$$ {}^h_kL(\bar{A}^{1}_{x:\overline{n}|}) = \begin{cases} \bar{Z}^{\ 1}_{x+k:\overline{n-k}|} - {}_hP(\bar{A}^{1}_{x:\overline{n}|})\ddot{Y}_{x+k:\overline{h-k}|} & k < h < n \\ \bar{Z}^{\ 1}_{x+k:\overline{n-k}|} & h < k < n. \end{cases} $$

(b) The k^{th} terminal prospective reserve for this contract

$$ {}^h_kV(\bar{A}^{1}_{x:\overline{n}|}) = \begin{cases} \bar{A}^{\ 1}_{x+k:\overline{n-k}|} - {}_hP(\bar{A}^{1}_{x:\overline{n}|})\ddot{a}_{x+k:\overline{h-k}|} & k < h < n \\ \bar{A}^{\ 1}_{x+k:\overline{n-k}|} & h < k < n \end{cases} $$

48.4 (a) The prospective loss random variable is

$$ {}_kL(\bar{A}_{x:\overline{n}|}) = \bar{Z}_{x+k:\overline{n-k}|} - P(\bar{A}_{x:\overline{n}|})\ddot{Y}_{x+k:\overline{n-k}|}. $$

(b) The k^{th} terminal prospective reserve for this contract

$$ {}_kV(\bar{A}_{x:\overline{n}|}) = \bar{A}_{x+k:\overline{n-k}|} - P(\bar{A}_{x:\overline{n}|})\ddot{a}_{x+k:\overline{n-k}|}. $$

48.5 (a) The prospective loss random variable is

$$ {}^h_kL(\bar{A}_{x:\overline{n}|}) = \begin{cases} \bar{Z}_{x+k:\overline{n-k}|} - {}_hP(\bar{A}_{x:\overline{n}|})\ddot{Y}_{x+k:\overline{h-k}|} & k < h < n \\ \bar{Z}_{x+k:\overline{n-k}|} & h \le k < n \\ 1 & k = n. \end{cases} $$

(b) The k^{th} terminal prospective reserve for this contract is

$$ {}^h_kV(\bar{A}_{x:\overline{n}|}) = \begin{cases} \bar{A}_{x+k:\overline{n-k}|} - {}_hP(\bar{A}_{x:\overline{n}|})\ddot{a}_{x+k:\overline{h-k}|} & k < h < n \\ \bar{A}_{x+k:\overline{n-k}|} & h \le k < n \\ 1 & k = n \end{cases} $$

48.6 Recall that under UDD, we have

$$\begin{aligned}
\bar{A}_{x+k:\overline{n-k}|} =& \frac{i}{\delta}A_{x+k:\overline{n-k}|} + {}_{n-k}E_{x+k}\\
{}_hP(\bar{A}_{x:\overline{n}|}) =& \frac{\bar{A}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{h}|}} = \frac{\bar{A}^1_{x:\overline{n}|} + {}_nE_x}{\ddot{a}_{x:\overline{h}|}}\\
=& \frac{\frac{i}{\delta}A^1_{x:\overline{n}|} + {}_nE_x}{\ddot{a}_{x:\overline{h}|}}\\
=& \frac{i}{\delta}{}_hP(A^1_{x:\overline{n}|}) + {}_hP(A_{x:\overset{1}{\overline{n}|}}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\bar{A}_{x+k:\overline{n-k}|} - {}_hP(\bar{A}_{x:\overline{n}|})\ddot{a}_{x+k:\overline{h-k}|} =& \frac{i}{\delta}A_{x+k:\overline{n-k}|} + {}_{n-k}E_{x+k}\\
&-(\frac{i}{\delta}{}_hP(A^1_{x:\overline{n}|}) + {}_hP(A_{x:\overset{1}{\overline{n}|}}))\ddot{a}_{x+k:\overline{h-k}|}\\
=& \frac{i}{\delta}{}^h_kV(A^1_{x:\overline{n}|}) + {}^h_kV(A_{x:\overset{1}{\overline{n}|}}).
\end{aligned}$$

48.7 Recall the following expressions:

$$\begin{aligned}
\bar{A}_x =& \bar{A}^1_{x:\overline{k}|} + {}_kE_x\bar{A}_{x+k}\\
\ddot{a}_x =& \ddot{a}_{x:\overline{k}|} + {}_kE_x\ddot{a}_{x+k}.
\end{aligned}$$

Thus,

$$\begin{aligned}
{}_kV(\bar{A}_x) =& \bar{A}_{x+k} - P(\bar{A}_x)\ddot{a}_{x+k}\\
=& \bar{A}_{x+k} - P(\bar{A}_x)\ddot{a}_{x+k} + \frac{P(\bar{A}_x)\ddot{a}_x - \bar{A}_x}{{}_kE_x}\\
=& \bar{A}_{x+k} - P(\bar{A}_x)\ddot{a}_{x+k} + \frac{P(\bar{A}_x)[\ddot{a}_{x:\overline{k}|} + {}_kE_x\ddot{a}_{x+k}]}{{}_kE_x} - \frac{[\bar{A}^1_{x:\overline{k}|} + {}_kE_x\bar{A}_{x+k}]}{{}_kE_x}\\
=& P(\bar{A}_x)\frac{\ddot{a}_{x:\overline{k}|}}{{}_kE_x} - \frac{\bar{A}^1_{x:\overline{k}|}}{{}_kE_x}\\
=& P(\bar{A}_x)\ddot{s}_{x:\overline{k}|} - \frac{\bar{A}^1_{x:\overline{k}|}}{{}_kE_x}.
\end{aligned}$$

Section 49

49.1 0.342035

49.2 0.0840

49.3 We will prove (b) and leave (a) to the reader. We have

$$\begin{aligned}
{}_kV^{(m)}(A_x) - {}_kV(A_x) =& P(A_x)\ddot{a}_{x+k} - P^{(m)}(A_x)[\alpha(m)\ddot{a}_{x+k} - \beta(m)]\\
=& P^{(m)}(A_x)\frac{\ddot{a}_x^{(m)}}{\ddot{a}_x}\ddot{a}_{x+k} - P^{(m)}(A_x)[\alpha(m)\ddot{a}_{x+k} - \beta(m)]\\
=& P^{(m)}\left[\frac{\ddot{a}_x^{(m)}}{\ddot{a}_x}\ddot{a}_{x+k} - \alpha(m)\ddot{a}_{x+k} + \beta(m)\right]\\
=& P^{(m)}\left[\frac{\alpha(m)\ddot{a}_x - \beta(m)}{\ddot{a}_x}\ddot{a}_{x+k} - \alpha(m)\ddot{a}_{x+k} + \beta(m)\right]\\
=& \beta(m)P^{(m)}\left[1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x}\right]\\
=& \beta(m)P^{(m)}(A_x){}_kV(A_x).
\end{aligned}$$

49.4 We have

$$\begin{aligned}
\frac{{}_kV^{(m)}(A_x) - {}_kV(A_x)}{{}_kV^{(m)}(\bar{A}_x) - {}_kV(\bar{A}_x)} =& \frac{P^{(m)}(A_x)}{P^{(m)}(\bar{A}_x)} = \frac{\frac{A_x}{\ddot{a}_x^{(m)}}}{\frac{\bar{A}_x}{\ddot{a}_x^{(m)}}}\\
=& \frac{A_x}{\bar{A}_x} = \frac{A_x}{\frac{i}{\delta}A_x} = \frac{\delta}{i}
\end{aligned}$$

49.5 We have

$$\begin{aligned}
{}_kV(A_x) =& A_{x+k} - P^{(m)}(A_x)\ddot{a}_{x+k}^{(m)} \\
=& A_{x+k} - P^{(m)}(A_x)\ddot{a}_{x+k}^{(m)} + \frac{P^{(m)}(A_x)\ddot{a}_x^{(m)} - A_x}{{}_kE_x} \\
=& A_{x+k} - P^{(m)}(A_x)\ddot{a}_{x+k}^{(m)} + \frac{P^{(m)}(A_x)[\ddot{a}_{x:\overline{k}|}^{(m)} + {}_kE_x\ddot{a}_{x+k}^{(m)}]}{{}_kE_x} - \frac{[\bar{A}_{x:\overline{k}|}^{1} + {}_kE_xA_{x+k}]}{{}_kE_x} \\
=& P^{(m)}(A_x)\frac{\ddot{a}_{x:\overline{k}|}^{(m)}}{{}_kE_x} - \frac{\bar{A}_{x:\overline{k}|}^{1}}{{}_kE_x} \\
=& P^{(m)}(A_x)\ddot{s}_{x:\overline{k}|}^{(m)} - \frac{\bar{A}_{x:\overline{k}|}^{1}}{{}_kE_x}
\end{aligned}$$

Section 50

50.1 The insurer's prospective loss random variable is

$$ {}_hL = \begin{cases} 0 & K(x) < h \\ b_{K(x)+1+h}\nu^{K(x)+1-h} - \sum_{j=h}^{K(x)} \pi_j \nu^{j-h} & K(x) \geq h \end{cases} $$

50.2 564.46

50.3 1027.42

50.4 30.395

50.5 (a) 30.926 (b) 129.66 (c) 382.44

50.6 255.064

50.7 31.39

50.8 499.102

Section 51

51.1 The prospective loss of this contract at time t is

$$ {}_t\bar{L} = \text{PVFB} - \text{PVFP} = \begin{cases} 0 & T(x) \le t \\ b_{T(x)}\nu^{T(x)-t} - \int_t^{T(x)} \pi_u \nu^{u-t} du & T(x) > t. \end{cases} $$

51.2 We have

$$ \begin{aligned} {}_t\bar{V} =& E[{}_t\bar{L}|T(x) > t] = E\left[b_{(T(x)-t)+t}\nu^{T(x)-t} - \int_0^{T(x)-t} \pi_{t+r}\nu^r dr|T(x) > t\right] \\ =& E\left[b_{T(x+t)+t}\nu^{T(x+t)} - \int_0^{T(x+t)} \pi_{t+r}\nu^r dr\right] \\ =& \int_0^\infty \left[b_{u+t}\nu^u - \int_0^u \pi_{t+r}\nu^r dr\right] f_{T(x+t)}(u)du \\ =& \int_0^\infty \left[b_{u+t}\nu^u - \int_0^u \pi_{t+r}\nu^r dr\right] {}_up_{x+t}\mu(x+t+u)du \\ =& \int_0^\infty b_{u+t}\nu^u{}_up_{x+t}\mu(x+t+u)du - \int_0^\infty \int_0^u \pi_{t+r}\nu^r dr{}_up_{x+t}\mu(x+t+u)du \\ =& \int_0^\infty b_{u+t}\nu^u{}_up_{x+t}\mu(x+t+u)du + \int_0^\infty \int_0^u \pi_{t+r}\nu^r dr\frac{d}{du}[{}_up_{x+t}]du \\ =& \int_0^\infty b_{u+t}\nu^u{}_up_{x+t}\mu(x+t+u)du - \int_0^\infty \pi_{t+r}\nu^r{}_up_{x+t}dr \\ =& \text{APV of future benefits} - \text{APV of future benefit premiums} \end{aligned} $$

51.3 95.96

51.4 1055.79

51.5 we have

$$ 0 = {}_0\bar{V} = \int_0^\infty b_u\nu^u{}_up_x\mu(x+u)du - \int_0^\infty \pi_u\nu^u{}_up_x du. $$

Thus,

$$ \int_0^t (\pi_u\nu^u{}_up_x - b_u\nu^u{}_up_x\mu(x+u))du = \int_t^\infty (b_u\nu^u{}_up_x\mu(x+u) - \pi_u\nu^u{}_up_x)du. $$

Letting $u = t + y$ on the right integral, we obtain

$$\begin{aligned}
\int_0^t (\pi_u \nu^u {}_u p_x - b_u \nu^u {}_u p_x \mu(x+u)) du =& \int_0^\infty (b_{t+y} \nu^{t+y} {}_{t+y} p_x \mu(x+t+y) - \pi_{t+y} \nu^{t+y} {}_{t+y} p_x) dy \\
=& \nu^t {}_t p_x \left[\int_0^\infty (b_{t+y} \nu^y {}_y p_{x+t} \mu(x+t+y) - \pi_{t+y} \nu^y {}_y p_{x+y}) dy \right] \\
=& {}_t E_x \left[\int_0^\infty (b_{t+y} \nu^y {}_y p_{x+t} \mu(x+t+y) - \pi_{t+y} \nu^y {}_y p_{x+y}) dy \right].
\end{aligned}$$

Now, the result follows by dividing both sides by ${}_t E_x$

Section 52

52.1 (a) 564.470 (b) 2000

52.2 324.70

52.3 77.66

52.4 −4.33

52.5 0.015

52.6 36657.31

52.7 0.017975

52.8 0.028

52.9 355.87

52.10 For any n, we have

$$({}_nV+\pi)(1+i)=q_{25+n}{}_{n+1}V+p_{25+n}{}_{n+1}V={}_{n+1}V.$$

Thus,

$$\sum_{n=0}^{34}({}_nV+\pi)(1+i)^{35-n}=\sum_{n=0}^{34}{}_{n+1}V(1+i)^{34-n}$$

which implies

$${}_0V(1+i)^{35}+\pi\ddot{s}_{\overline{35|}}={}_{35}V.$$

But ${}_0V=0$ and ${}_{35}V=\ddot{a}_{\overline{60|}}$ (actuarial present value of future benefits; there are no future premiums). Thus,

$$\pi=\frac{\ddot{a}_{\overline{60|}}}{\ddot{s}_{\overline{35|}}}.$$

Likewise,

$$\sum_{n=0}^{19}({}_nV+\pi)(1+i)^{20-n}=\sum_{n=0}^{19}{}_{n+1}V(1+i)^{19-n}$$

which implies

$$ {}_0V(1+i)^{20} + \pi \ddot{s}_{\overline{20|}} = {}_{20}V. $$

Hence,

$$ {}_{20}V = \left(\frac{\ddot{a}_{\overline{60|}}}{\ddot{s}_{\overline{35|}}}\right)\ddot{s}_{\overline{20|}}. $$

52.11 5.28

52.12 9411.01

52.13 296.08

52.14 1027.42

52.15 286.04

52.16 (a) 0.091 (b) 101.05

Section 53

53.1 0

Section 54

54.1 0.25904

54.2 302.31

54.3 1799.037

54.4 697.27

54.5 495.80

54.6 (a) 0.0505 (b) 110.85

Section 55

55.1 The expected value is 0.37704 and the variance is 0.03987

55.2 0.458431

55.3 5.4

55.4 0.1296

Section 56

56.1 We have

$$\begin{aligned}{}_tq_{xy} =&1 - {}_tp_{xy} = 1 - {}_tp_x\ {}_tp_y\\ =&1 - (1 - {}_tq_x)(1 - {}_tq_y)\\ =&1 - (1 - {}_tq_x - {}_tq_y + {}_tq_x\ {}_tq_y)\\ =&{}_tq_x + {}_tq_y - {}_tq_x\ {}_tq_y\end{aligned}$$

56.2 We have

$$\begin{aligned}\Pr[(T(x) > n) \cup (T(y) > n)] =&\Pr[T(x) > n] + \Pr[T(y) > n] - \Pr[(T(x) > n) \cap (T(y) > n)]\\ =&{}_np_x + {}_np_y - np_{xn}p_y = {}_np_x + {}_np_y - np_{xy}\end{aligned}$$

56.3 0.2

56.4 $\frac{1}{3}$

56.5 0.067375

56.6 We have

$$\begin{aligned}{}_tq_{xy} =&1 - {}_tp_{xy} = 1 - {}_tp_x\ {}_tp_y\\ =&1 - (1 - {}_tq_x)(1 - {}_tq_y)\\ =&1 - (1 - {}_tq_x - {}_tq_y + {}_tq_x\ {}_tq_y)\\ =&{}_tq_x + {}_tq_y - {}_tq_x\ {}_tq_y\end{aligned}$$

56.7 0.10969

56.8 ${}_{n|m}q_{xy} = {}_{n+m}q_{xy} - {}_tq_{xy} = {}_tp_{xy} - {}_{n+m}p_{xy}$

56.9 0.03436

56.10 We have

$$\begin{aligned}
{}_{\frac{1}{3}}q_{xy} =& 1 - {}_{\frac{1}{3}}p_{xy} = 1 - (1 - {}_{\frac{1}{3}}q_x)(1 - {}_{\frac{1}{3}}q_y) \\
=& 1 - \left(1 - \frac{1}{3}q_x\right)\left(1 - \frac{1}{3}q_y\right) \\
=& \frac{1}{3}q_x + \frac{1}{3}q_y - \frac{1}{9}q_xq_y \\
{}_{\frac{1}{2}}q_{xy} =& \frac{1}{2}q_x + \frac{1}{2}q_y - \frac{1}{4}q_xq_y.
\end{aligned}$$

Thus,

$$18{}_{\frac{1}{3}}q_{xy} - 12{}_{\frac{1}{2}}q_{xy} = 6q_x + 6q_y - 2q_xq_y - 6q_x - 6q_y + 3q_xq_y = q_xq_y$$

56.11 0.08

56.12 $\frac{2}{3}$

56.13 4×10^{-8}

56.14 0.06

56.15 0.10

56.16 10.42

56.17 12.5

56.18 0.21337

56.19 0.36913

56.20 2.916667

56.21 160.11

Section 57

57.1 54.16667

57.2 0.9167

57.3 5.41667

57.4 0.05739

57.5 0.961742

57.6 0.24224

57.7 34

57.8 40.8333

57.9 0.05982

57.10 We have

$$\begin{aligned}\mu_{\overline{xy}}(t) =& -\frac{\frac{d}{dt}{}_tp_{\overline{xy}}}{{}_tp_{\overline{xy}}} = \frac{\frac{d}{dt}{}_tq_{\overline{xy}}}{1-{}_tq_{\overline{xy}}}\\ =&\frac{\frac{d}{dt}{}_tq_{x\,t}q_y}{1-{}_tq_{\overline{xy}}} = \frac{{}_tp_{x\,t}q_y\mu(x+t)+{}_tp_{y\,t}q_x\mu(y+t)}{1-{}_tq_{x\,t}q_y}\end{aligned}$$

57.11 0.0023

57.12 13.17

57.13 30.33

57.14 5

57.15 28.5585

57.16 1/14

57.17 (a) 0.155 (b) 30

57.18 1.25

Section 58

58.1 We have

$$\begin{aligned}
\mathrm{Cov}(T(xy),T(\overline{xy})) =& E[T(xy)\cdot T(\overline{xy})]-E[T(xy)]E[T(\overline{xy})]\\
=& E[T(x)T(y)]-E[T(xy)]E[T(x)+T(y)-T(xy)]\\
=& E[T(x)T(y)]-E[T(x)E[T(y)]-E[T(xy)](E[T(x)]+E[T(y)]-E[T(xy)\\
&+E[T(x)E[T(y)]\\
=& \mathrm{Cov}(T(x),T(y))-\mathring{e}_{xy}(\mathring{e}_x+\mathring{e}_y-\mathring{e}_{xy})+\mathring{e}_x\mathring{e}_y\\
=& \mathrm{Cov}(T(x),T(y))+(\mathring{e}_x-\mathring{e}_{xy})(\mathring{e}_y-\mathring{e}_{xy})
\end{aligned}$$

58.2 3.7

58.3 4.3

58.4 400

58.5 We have

$$\begin{aligned}
\mathrm{Cov}(T(xy),T(\overline{xy})) =& (\mathring{e}_x-\mathring{e}_{xy})(\mathring{e}_y-\mathring{e}_{xy})\\
=& \mathring{e}_x\mathring{e}_y-\mathring{e}_{xy}(\mathring{e}_x+\mathring{e}_y-\mathring{e}_{xy})\\
=& \mathring{e}_x\mathring{e}_y-\mathring{e}_{xy}\mathring{e}_{\overline{xy}}
\end{aligned}$$

Section 59

59.1 0.6

59.2 0.030873

59.3 We have

$$\begin{aligned}
{}_nq_{xy}^{1} + {}_n\,q_{xy}^{\;\;1} =& \int_0^n {}_tp_{xy}\mu(x+t)dt + \int_0^{10} {}_tp_{xy}\mu(y+t)dt \\
=& \int_0^n {}_tp_{xy}[\mu(x+t)+\mu(y+t)]dt = \int_0^n {}_tp_{xy}\mu_{xy}(t)dt \\
=& \int_0^n \int_0^n f_{T(xy)}(t)dt = {}_nq_{xy}
\end{aligned}$$

59.4 0.0099

59.5 0.0001467

59.6 0.141

59.7 $\frac{1-e^{n\mu}}{\mu(95-x)}$

59.8 0.0134

Section 60

60.1 4.2739

60.2 0.9231

60.3 0.06

60.4 0.1345

60.5 11.27

60.6 0.0549

60.7 0.0817

60.8 0.18

60.9 27927.51

60.10 (a) 115,714.29 (b) $14.4\bar{P}$.

60.11 0.38

60.12 600

60.13 0.191

Section 61

61.1 12.8767

61.2 12.7182

61.3 5.95238

61.4 4.7

61.5 We know that

$$\bar{a}_y = \frac{1 - \bar{A}_y}{\delta}$$

and

$$\bar{a}_{xy} = \frac{1 - \bar{A}_{xy}}{\delta}.$$

Hence,

$$\bar{a}_{x|y} = \frac{1 - \bar{A}_y}{\delta} - \frac{1 - \bar{A}_{xy}}{\delta} = \frac{\bar{A}_{xy} - \bar{A}_y}{\delta}$$

Section 62

62.1 0.069944

62.2 1.441188

62.3 0.082667

62.4 1691.92

62.5 0.736

Section 63

63.1 $\frac{t_y}{36}+\frac{1}{12}$

63.2 2.75

63.3 We have

$$f_{T(x)}(t_x)f_{T(y)}(t_y)=\left(\frac{t_x}{36}+\frac{1}{12}\right)\left(\frac{t_y}{36}+\frac{1}{12}\right)\neq\frac{t_x+t_y}{216}=f_{T(x)T(y)}(t_x,t_y)$$

63.4 4.4375

63.5 1.6136

63.6 1

63.7 $\frac{(6-n)^2(6+n)}{216}$

63.8 0.8102

Section 64

64.1 $e^{-0.06t}$

64.2 (a) We have

$$f_{T,J}(t,j) = {}_tp_{50}^{(\tau)}\mu^{(j)}(x+t) = \frac{j}{50^3}(50-t)^2, \quad j = 1,2.$$

(b) $f_T(t) = \sum_{j=1}^{2} f_{T(x),J(x)}(t,j) = \frac{3}{50^3}(50-t)^2$.
(c) $f_J(j) = \int_0^{50} f_{s,J(x)}(t,j)ds = \frac{j}{50^3}\int_0^{50}(50-t)^2dt = \frac{j}{3},\ j = 1,2$.
(d) $f_{J|T}(j|t) = \frac{\mu^{(j)}(50+t)}{\mu^{(\tau)}(50+t)} = \frac{1}{3}j,\ j = 1,2$ **64.3** 0.12

64.4 11.11

64.5 (a) 0.00446 (b) 1/3

64.6 0.259

64.7 0.0689

Section 65

65.1 0.60

65.2 0.4082483

65.3 0.12531

65.4 0.0198

65.5 0.216

65.6 0.644

65.7 0.512195

Section 66

66.1 A decrease of 10 in the value of $d_{26}^{(1)}$.

66.2 119

66.3 0.05

66.4 0.0555

66.5 0.2634

66.6 0.0426

66.7 7.6

66.8 803

66.9 0.38

Section 67

67.1 0.154103

67.2 0.04525

67.3 0.0205

67.4 0.02214

67.5 25.537

67.6 0.053

67.7 14.1255563

67.8 0.09405

67.9 0.0766

67.10 0.1802

Section 68

68.1 3000

68.2 1.90

68.3 1

68.4 53,045.10

68.5 40.41

68.6 457.54

68.7 7841.28

Section 69

69.1 18,837.04

69.2 2.5

69.3 120 is payable for the next 10 years and 100 is payable after 10 years

69.4 0

69.5 14.7

69.6 11,194.0199

69.7 922.014

Section 70

70.1 19.88

70.2 (a) 10.8915 (b) 17.6572 (c) 6.7657 (d) 104.297 (e) 104.5549 (f) 0.00027

70.3 8.8932

70.4 10.0094

70.5 888.225

70.6 0.472

70.7 -445.75

70.8 92.82

70.9 1371.72

70.10 2302.52

70.11 1177.23

Section 71

71.1 (a) 23.88 (b) 5.655

71.2 30.88

71.3 (a) 883.9871 (b) 903.9871

71.4 887.145

71.5 4.2379

71.6 $G = \frac{1000_{10|20}A_{30}+20+10a_{30:\overline{9}|}}{0.85\ddot{a}_{30:\overline{5}|}-0.15}$

Section 72

72.1 1750.03

72.2 414.82

72.3 16.8421

72.4 Multiplying the equation

$$ {}_{k+1}AS\ell^{(\tau)}_{x+k+1} = ({}_kAS + G - c_kG - e_k)(1+i)\ell^{(\tau)}_{x+k} - b_{k+1}d^{(d)}_{x+k} - {}_{k+1}CVd^{(w)}_{x+k}. $$

by ν^{k+1} we obtain

$$ {}_{k+1}AS\nu^{k+1}\ell^{(\tau)}_{x+k+1} - {}_kAS\nu^k\ell^{(\tau)}_{x+k} = G(1-c_k)\nu^k\ell^{(\tau)}_{x+k} - e_k\nu^k\ell^{(\tau)}_{x+k} - (b_{k+1}d^{(d)}_{x+k} + {}_{k+1}CVd^{(w)}_{x+k})\nu^{k+1}. $$

Using the fact that ${}_0AS = 0$ and summing this telescoping series gives

$$ \begin{aligned} {}_nAS\nu^n\ell^{(\tau)}_{x+n} =& \sum_{k=0}^{n-1}[{}_{k+1}AS\nu^{k+1}\ell^{(\tau)}_{x+k+1} - {}_kAS\nu^k\ell^{(\tau)}_{x+k}] \\ =& G\sum_{k=0}^{n-1}(1-c_k)\nu^k\ell^{(\tau)}_{x+k} - \sum_{k=0}^{n-1} e_k\nu^k\ell^{(\tau)}_{x+k} \\ &- \sum_{k=0}^{n-1}(b_{k+1}d^{(d)}_{x+k} + {}_{k+1}CVd^{(w)}_{x+k})\nu^{k+1} \end{aligned} $$

72.5 ${}_{10}AS_1 - {}_{10}AS_2 = (G_1 - G_2)\sum_{k=0}^{9}(1-c_k)\nu^{k-9}\left(\frac{\ell^{(\tau)}_{x+k}}{\ell^{(\tau)}_{x+10}}\right)$

72.6 1627.63

72.7 1.67

Section 73

73.1 We have

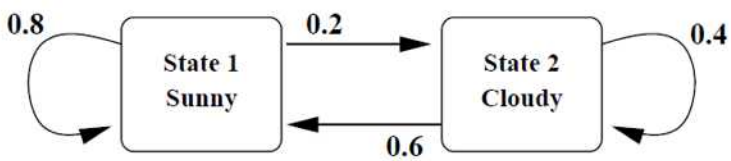
$$\sum_{j\in E} Q_n(i,j) = \sum_{j\in E} \Pr(X_{n+1} = j|X_n = i) = \Pr(X_{n+1} \in E|X_n = i) = 1$$

73.2 The entries in the second row do not sum up to 1. Therefore, the given matrix can not be a transition matrix.

73.3

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$$

73.4 The transition diagram is



The transition matrix is

$$Q = \begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix}$$

73.5

$$Q_n = \begin{pmatrix} p_{x+n} & q_{x+n} \\ 0 & 1 \end{pmatrix}$$

73.6 The transition probabilities are $Q_n(0,0) = p_{x+n}^{(\tau)},\ Q_n(0,j) = q_{x+n}^{(j)}$ for $j = 1, 2, \cdots, m,\ Q_n(j,j) = 1$ for $j = 1, 2, \cdots, m,\ Q_n(i,j) = 0$ for all other values of i and j

73.7

$$Q_{61} = \begin{bmatrix} 0.20 & 0.10 & 0.70 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Section 74

74.1 At time $t = 1$ we have

$$Q = \begin{pmatrix} 0.92 & 0.05 & 0.03 \\ 0.00 & 0.76 & 0.24 \\ 0.00 & 0.00 & 1.00 \end{pmatrix}$$

At time $t = 2$ we have

$${}_2Q = Q^2 = \begin{pmatrix} 0.8464 & 0.084 & 0.0696 \\ 0.00 & 0.5776 & 0.4224 \\ 0.00 & 0.00 & 1.00 \end{pmatrix}$$

At time $t = 2$ we have

$${}_3Q = Q^3 = \begin{pmatrix} 0.778688 & 0.106160 & 0.115152 \\ 0.00 & 0.438976 & 0.561024 \\ 0.00 & 0.00 & 1.00 \end{pmatrix}$$

74.2 (a) 0.70 (b) 0.3125

74.3 0.056

74.4 0.892

74.5 0.0016

74.6 0.489

74.7 4.40

Section 75

75.1 16.82

75.2 170.586

75.3 1960

75.4 185.11

75.5 0.34

75.6 10,694.64

75.7 160

Section 76

76.1 Since $N(s)$ and $N(t)-N(s)$ are independent, we have $\text{Cov}(N(s), N(t)-N(s)) = 0$. Thus,

$$\begin{aligned}\text{Cov}(N(s), N(t)) =& \text{Cov}(N(s), N(s) + N(t) - N(s)) = \text{Cov}(N(s), N(s)) + \text{Cov}(N(s), N(t) - N\\ =& \text{Cov}(N(s), N(s)) = \text{Var}(N(s)) = \lambda s\end{aligned}$$

76.2 8.338×10^{-4}

76.3 $e^{-6}\frac{6^5}{5!}$

76.4 $E[2N(3) - 4N(5)] = -28$ and $\text{Var}[2N(3) - 4N(5)] = 88$

76.5 We have

$$\begin{aligned}\Pr(N(t) = k|N(s+t) = n) =& \frac{\Pr(N(t) = k, N(s+t) = n)}{\Pr(N(s+t) = n)}\\ =& \frac{\Pr(N(t) = k, N(s+t) - N(t) = n-k)}{\Pr(N(s+t) = n)}\\ =& \frac{e^{-\lambda t}\frac{(\lambda t)^k}{k!}e^{-\lambda s}\frac{(\lambda s)^{n-k}}{(n-k)!}}{e^{-\lambda(s+t)}\frac{[\lambda(s+t)]^n}{n!}}\\ =& \left(\begin{array}{c} n \\ k \end{array}\right)\left(\frac{t}{t+s}\right)^k\left(\frac{s}{t+s}\right)^{n-k}\end{aligned}$$

76.6 0.2963

76.7 0.593994

76.8 (a) 0.503 (b) 18805

Section 77

77.1 If $S_n \leq n$ then the n^{th} event happens before time t. This means that there are n or more events in the interval $[0, t]$ which implies that $N(t) \geq n$

77.2 0.0000393

77.3 The expected value is 1 and the variance is 1/3

77.4 $f_{T_5}(t) = 3e^{-3t}, t \geq 0$

77.5 1/9

77.6 0.6321

77.7 Both expected arrival time are the same.

Section 78

78.1 0.0025

78.2 0.2048

78.3 (a) 48 (b)0.04262 (c) 100 minutes

78.4 768

78.5 2,000,000

78.6 0.5

78.7 0.276

78.8 0.1965

78.9 0.55

78.10 0.3859

78.11 0.3679

78.12 210.10

78.13 0.23

Section 79

79.1 0.1954

79.2 ∞

79.3 0.016

79.4 (a) 3 (b) 1.875

79.5 0.03642

79.6 93.55

Section 80

80.1 The mean is 225,000 and the variance is 45,000,000

80.2 29

80.3 The mean is 25 and the variance is 215/3

80.4 The mean is 40,000 and the variance is 160,000,000

80.5 (a) 100,000,000 (b) 0.0228

80.6 0.6712

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