MATH 425a ASSIGNMENT 10
FALL 2016 Prof. Alexander
Due Wednesday November 30.

Rudin Chapter 5 #15, Chapter 6 #1, 2, 3a, plus the problems (I)-(VI) below. Problems 1, 2, and (VI) should be relatively "quick" ones, perhaps the best type to practice for exams.

(I) Typically, if a continuous function $f : [a, b] \to \mathbb{R}$ has a local maximum at some $c \in [a, b]$, then f is increasing in $(c - \delta, c]$ and decreasing in $[c, c + \delta)$ for some $\delta > 0$. Find an example, though, in which this is false.

(II) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere, with $f'(x) < 3$ for all $x < 0$ and $f'(x) > 3$ for all $x > 0$. Show that $f'(0) = 3$.

(III) Let $f(x) = 2 + 3x$ and

$$\alpha(x) = \begin{cases} x^2, & 0 \le x < 1 \\ 2, & 1 \le x < 2 \\ 2x, & x \ge 2. \end{cases}$$

Calculate $\int_0^3 f \, d\alpha$. (Just a calculation, you don't have to prove the steps.)

(IV) In the definition of derivative, the difference quotient $\frac{f(x+h)-f(x)}{h}$ for calculating $f'(x)$ is the slope of the secant line corresponding to two points on the graph, one of which is always at x. Let us consider what happens if we take two points both close to x, but neither equal to x. We will do this for $x = 0$.

Suppose $f : \mathbb{R} \to \mathbb{R}$, $f'(0)$ exists, and $s_n < 0 < t_n$ with $s_n \to 0, t_n \to 0$. Show that

$$\lim_n \frac{f(t_n) - f(s_n)}{t_n - s_n} = f'(0).$$

(V) For $x \in [1, 4]$ let $f(x) = 2x$ and

$$\alpha(x) = \begin{cases} 0 & \text{if } x < 2, \\ \frac{1}{2} & \text{if } 2 \le x \le 3, \\ \frac{3}{2} & \text{if } x > 3. \end{cases}$$

(a) Let P be a partition of $[1, 4]$ such that some $x_i = 2$ and some $x_j = 3$. Find $U(P, f, \alpha)$. The only unspecified quantities in your answer should be some or all of the points $x_0, \ldots, x_n$.

(b) Find $\overline{\int_1^4} f \, d\alpha$.

(c) Similarly to (a) and (b), find $L(P, f, \alpha)$ and $\underline{\int_1^4} f \, d\alpha$.

(d) Is $f \in \mathcal{R}(\alpha)$? How can you tell from (a)–(c) alone?

(VI) Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable and bounded, say $|f(t)| \le M$ for all t. Let

$$F(x) = \int_a^x f(t)\, dt, \qquad x \in [a, b].$$

Show that F is uniformly continuous.

HINTS:

(15) You can omit the vector-valued part. The bound on $|f'(x)|$ is valid for *all* $h > 0$. How can you take advantage of this?

What follows isn't really a hint, just an informal explanation of what problem 15 shows. If M_1 is large, this means $|f'(x)|$ is large for some x. This means one of two things must happen. One possibility is that $|f'(x)|$ remains large for points in the vicinity of x, in which case $f(x)$ must climb to a very large value, meaning M_0 is large. The other possibility is that $|f'(x)|$ does not remain large for points in the vicinity of x, in which case $f'(x)$ must change rapidly, forcing M_2 to be large. Either way, if M_1 is large then the product $M_0 M_2$ must be large. The bound in the problem quantifies this exactly.

(1) For a partition $a = t_0 < \cdots < t_n = b$, focus on the interval $[t_{i-1}, t_i]$ containing x_0.

(2) If f is continuous and $f(x) > 0$ for some x, what must be true for values close to x?

(3) Part (c) was done in lecture; (a) and (b) are similar.

(8) Compare $f(n)$, $f(n+1)$ and $\int_n^{n+1} f(x)\, dx$.

(I) Take a function which has a known local maximum, for example $g(x) = -x^2$ at $x = 0$. Add something to it so that it's no longer monotone on either side of $x = 0$, but it still has a local maximum at $x = 0$.

(II) The problem would be easier if 3 were replaced everywhere by 0. Add something to f to make a new function g, so that the problem converts to this easier question, for g.

(IV) Relate $\frac{f(t_n)-f(s_n)}{t_n-s_n}$ to $\frac{f(t_n)-f(0)}{t_n-0}$ and $\frac{f(0)-f(s_n)}{0-s_n}$. Notice that $\frac{t_n-0}{t_n-s_n}$ and $\frac{0-s_n}{t_n-s_n}$ are positive numbers that sum to 1.