MATH 425a ASSIGNMENT 1
FALL 2016 Prof. Alexander
Due Friday September 2.

Rudin Chapter 1 #4 plus the problems (I) - (VIII) below:

(I) Take the set of all real numbers (not complex!) as the universe. Explain why each of the following is true or false. Note you can prove falsity of a "$\forall x \dots$" statement by giving an example where it fails. If it's true you don't need a formal proof, just say informally how you know it's true.
(a) $\forall x\ \exists y : y < x^2$
(b) $\exists x\ \forall y : x + y^2 > 0$
(c) $\exists x\ \forall y : y + x^2 > 0$
(d) $\forall a\ \forall b\ \forall c\ \exists x : ax^2 + bx + c = 0$

(II) Write out each statement in (I)(b)(c)(d) as a complete English sentence with no symbols. For example: (a) can be "for every number, there is a number smaller than its square."

(III) Let $p(x, y)$ be a statement about the variables x and y. The statement "$\forall x\ \exists y : p(x, y)$ is true" is logically equivalent to which of the following? Explain your answer briefly.
(a) "$\neg(\exists x\ \forall y : p(x, y)$ is false)." Note this means the negation of the statement in parentheses.
(b) "$\exists y\ \forall x : p(x, y)$ is true."

(IV) Take the real number system as the universe, and consider the sets

$$A = \{x \in \mathbb{R} : x > 0\}, \quad B = \{2, 4, 8, 16, 32\}, \quad C = \{2, 4, 6, 8, 10, 12, 14\},$$

$$D = \{x : -3 < x < 9\}, \quad E = \{x : x \leq 1\}.$$

Calculate the following:
(a) $B \cap C$
(b) $B \cup C$
(c) $A \cap (D \cup E)$
(d) $(A \cap C) \cup (B \cap D)$

(V) Show that $\sqrt{45}$ is irrational.

(VI) (Harder problem. Do Ch. 1 #2 first!) In lecture and in Chapter 1 #2, you have shown that $\sqrt{2}$ and $\sqrt{45}$ are irrational, by using properties of the prime factorization. But some integers have rational square roots, for example $\sqrt{144} = 12$ is rational, and the same is true if you use the square of any integer, in place of 144. In general, think about what aspect of the prime factorization makes the proof of irrationality work for $\sqrt{2}$ and $\sqrt{45}$, but not for $\sqrt{144}$. For example, if $n = 2^a 3^b 7^c$, for what a, b, c does the proof of irrationality work for $\sqrt{n}$? Note $144 = 2^4 \cdot 3^2$. Use your analysis to show that in fact, if a positive integer n is not the square of an integer, then $\sqrt{n}$ is irrational.

(VII) Suppose $q \neq 0$ is rational and x is irrational. Show that $q - x$ and x/q are irrational.

(VIII) Let $A \subset \mathbb{R}$ be nonempty and bounded below, with $\alpha = \inf A$. Show that $\sup(-A) = -\alpha$, where $-A = \{-x : x \in A\}$.

HINTS:

(V) Use the prime factorization $45 = 3^2 \cdot 5$.

(VIII) The definition of sup tells you what you need to show. You need (i) $-\alpha$ is an upper bound for $-A$, and (ii) there is no smaller upper bound for $-A$, that is, if $\gamma < -\alpha$ then γ is not an upper bound for $-A$.