THE IMPACT OF DIFFERENT DISTRIBUTIONAL HYPOTHESES ON RETURNS IN ASSET ALLOCATION

Marida Bertocchi¹
Rosella Giacometti¹
Sergio Ortobelli²
Svetlozar Rachev³

¹University of Bergamo, Department MSIA, Bergamo, Italy
²Contact autor: University of Bergamo, Department MSIA, Via dei Caniana, 2, 24127 Bergamo, Italy. Tel: +39-035-2052564 E-mail:sol@unibg.it
³University of Karlsruhe, Institut für Statistik und Mathematische Wirtschaftstheorie, Germany and University of California Santa Barbara, California
This paper discusses and analyzes some portfolio allocation problems with autoregressive processes. Firstly, we examine the distributional behavior of some international indexes. Then, we propose an AR(1)-GARCH(1,1) model to describe the evolution of the portfolio returns over time. In particular, we assume that investors wish to maximize some expected utility functionals of final wealth and we evaluate the impact of different distributional hypotheses on investor’s choices. Finally, we describe the properties of some optimal choices.

Key words: Stable distributions, generalized Student’s $t_3$, AR(1)-GARCH(1,1) models.

2000 AMS Subject Classification: 91B28, 91B84, 62P05, 62M10, 60G52, 60E07.
1. Introduction

Many of the concepts developed in theoretical and empirical finance over the past decades rest upon the assumption that asset returns follow a normal distribution. In particular, this distributional assumption was applied to the one period strategic allocation problem in order to justify the mean variance analysis. Such an approach, called myopic, was partially motivated by Samuelson [31] and Merton [21] who proved that when we assume a perfect market, the investors with constant relative risk aversion maintain constant their portfolio compositions independently of the temporal horizon. On the other hand, since the space of feasible consumption bundles is generally a linear space (as Ross [29], Cox and Leland [8], Rubinstein [30] and many others have emphasized), the original dynamic problem can be replaced with an equivalent one-period problem which has appropriate terminal state prices, if consumption takes place at the end. More generally, if preferences are time separable and if we treat consumption at each date separately, the analysis is unchanged. However, the myopic approach presents several drawbacks. First of all, it does not consider the return predictability empirically observed. Secondly, this approach cannot be used for long term approaches when transaction costs are allowed.

In order to evaluate how the investors’ choices change when the return predictability is modeled, Wilkie [34], [35], Hodrick [13], Boender et al. [5] have proposed several autoregressive models to describe the evolution of the asset returns over time. However, most of these models assume that returns or their stochastic innovations are Gaussian distributed. This distributional assumption is against the empirical evidence. The excess kurtosis, found in Mandelbrot’s [19], [20] and Fama’s [12] investigations, led them to reject the normal assumption and to propose the stable Paretian distribution as a statistical model for asset returns. In addition, many other empirical analyses (see Rachev and Mittnik [28] and the references therein) have shown that residuals of autoregressive models applied to daily financial series are not Gaussian distributed.

In this paper we relax the Gaussian distributional hypothesis on the stochastic innovations of returns and we evaluate the impact of different distributional hypotheses on multiperiod portfolio choices. Our analysis is the starting point for further developments of the most recent and used multiperiod stochastic portfolio choice models (see Dupacová [10], [11] Ziemba [37] Zenios and Ziemba [36]). In particular, we assume that every portfolio of risky returns evolves as an AR(1)-GARCH(1,1) model. Therefore, we consider the autoregressive component of the returns and the heteroschedasticity of their volatility. In addition, we distinguish three different distributional hypotheses for the standardized residuals and we assume that they could be either Gaussian or $t_3$ or alpha-stable distributed. Consequently, we estimate the parameters associated at every distributional hypothesis of residuals using the maximum likelihood method. We generate several scenarios for each portfolio and distributional hypothesis in order to compute the expected utility of the final wealth. Specifically, we analyze three investment allocation problems. The investment allocation problems consist of the maximization of the mean minus a measure of portfolio risk. The
first and the second allocation problem are typical problems of the safety first analysis where we assume as the risk measures respectively the value at risk (VaR) and the conditional value at risk (CVaR). The third problem considers the expected value of a power absolute deviation as the risk measure. Using the Simulated Annealing algorithm we choose the optimal composition among a set of admissible portfolios.

From a financial point of view, we typically propose and solve a fixed-mix portfolio choice problem following Boender [4] and Tokat, Rachev and Schwartz [33] analysis but differently we compare the allocation obtained with the Gaussian, the stable non-Gaussian and the Student distributional assumption for the residuals of AR(1)-GARCH(1,1) approximation of risky portfolios. To select the portfolio we consider the daily observations of the stock indexes S&P500, DAX30 and CAC40.

2. Modeling Uncertainty in Returns

As it is well known in literature and suggested by many authors, see [32], [33] financial modelling often requires information about the past movements of financial variables as prices or returns. Let us suppose that our portfolio horizon is T and the number of risky assets involved in the portfolio is n. Let $P_{i,t}$ be the price of the asset $i$ at time $t$, we define the return $R_{i,t}$ of the asset $i$ at time $t$ as

$$R_{i,t} = \log \frac{P_{i,t+1}}{P_{i,t}}, \quad i = 1, \ldots, n; \quad t = 1, \ldots, T. \quad (1)$$

The contribution of each asset $i$ in the portfolio is represented by the quantity $x_i$. The portfolio return, $R_{p,t}$ at time $t$, is defined as

$$R_{p,t} = x' \mathbf{R}_t \quad (2)$$

where $\mathbf{R}_t = [R_{1,t}, \ldots, R_{n,t}]'$ and $x' = [x_1, \ldots, x_n]$. There are many alternative methods to describe the uncertainty in asset returns, we decided to take into consideration those where the future return distribution is conditioned by information sets based on past returns and the time series exhibit some kind of volatility cluster.

2.1. AR-GARCH modeling

Here we refer to the generalized autoregressive conditional heteroskedastic model (GARCH) of [6] combined with an autoregressive model (AR). The choice of an AR(1)-GARCH(1,1) for our portfolio arises from the analysis of the time series of the asset returns in our portfolio. A detailed description will be given in the next Section. Using an AR(1)-GARCH(1,1) to shape the behavior of $R_{p,t}$, we derive the following model

$$R_{p,t} = \alpha_0 + \alpha_1 R_{p,t-1} + \epsilon_t \quad (3)$$
\[ \sigma_i^2 = \gamma_0 + \gamma_1 \epsilon_{t-1}^2 + \gamma_2 \sigma_{t-1}^2 \quad (4) \]

In the classical model the innovations \( \epsilon_t \) are normally distributed with zero mean and variance \( \sigma_i^2 \), i.e. \( \epsilon_t | I_{t-1} \sim N(0, \sigma_i^2) \), where \( I_{t-1} \) is the information set at time \( t-1 \). However, many studies have shown that the residuals from GARCH models are strongly inconsistent with normality and exhibit heavy tails and skewness. Thus, alternatively to the Gaussian hypothesis, we assume \( \alpha \)-stable and \( t_3 \) distributions for residuals. Observe that all the parameters in equations (3), (4) have to be estimated using the observed data available on the market. The standard procedure to carry out the estimation is the well-known maximum-likelihood methodology and it consists in maximizing the logarithm of the conditional likelihood function built on the basis of innovations distributions whose properties are recalled here in the following.

2.1.1. The \( \alpha \)-stable distribution

The \( \alpha \)-stable distributions describe a general class of distribution functions. A random variable \( X \) is \( \alpha \)-stable distributed if it has a domain of attraction, that is there exists a sequence of i.i.d. random variables \( \{Y_i\}_{i \in \mathbb{N}} \), a sequence of positive real values \( \{d_i\}_{i \in \mathbb{N}} \) and a sequence of real values \( \{a_i\}_{i \in \mathbb{N}} \) such that, as \( n \to +\infty \)

\[ \frac{1}{d_n} \sum_{i=1}^{n} Y_i + a_n \xrightarrow{d} X \quad (5) \]

where \( \sim \) points out the convergence in the distribution. Thus, the \( \alpha \)-stable random variables describe a general class of distributions including the leptokurtic and asymmetric ones. The \( \alpha \)-stable distribution is identified by four parameters: the index of stability \( \alpha \in (0,2] \) which is the parameter of the kurtosis, the skewness parameter \( \beta \in [-1,1] \); \( \delta \in \mathbb{R} \) and \( \gamma \in \mathbb{R}^+ \) which are, respectively, the location and the dispersion parameter. If \( X \) is a random variable whose distribution is \( \alpha \)-stable, we use the following notation to underline the parameter dependence

\[ X \overset{d}{=} S_\alpha(\gamma, \beta, \delta). \quad (6) \]

When \( \alpha = 2 \) the stable distribution is distributed as a Normal. As Figure 1 shows \( \alpha \)-stable distributions with \( \alpha < 2 \) are leptokurtotic and present fat tails. Whereas Figure 2 considers stable distributions for different levels of \( \beta \). A positive skewness \( (\beta > 0) \) identifies distributions with right fat tails, while a negative skewness \( (\beta < 0) \) typically characterizes distributions with left fat tails. Therefore, the stable density functions synthesize the distributional forms empirically observed in the real data. The estimation of stable distribution parameters takes place by maximizing the likelihood function. Unfortunately the density of stable distributions cannot be expressed in closed form. As a
matter of fact, the stable distributions are described with their characteristic function

\[
\Phi_R(t) = \begin{cases} 
\exp \left[ -\gamma^\alpha |t|^\alpha \left( 1 - i \beta \text{sgn}(t) \tan \left( \frac{\pi \alpha}{2} \right) \right) + i \delta t \right] & \text{if } \alpha \neq 1 \\
\exp \left[ -\gamma |t| \left( 1 + i \beta \frac{2}{\alpha} \text{sgn}(t) \log |t| \right) + i \delta t \right] & \text{if } \alpha = 1
\end{cases}
\]  

(7)

Thus, in order to evaluate the density function \( f(x|\alpha, \gamma, \beta, \delta) \), it is necessary to invert the characteristic function (7). The MLE procedure used to approximate stable parameters is described by Rachev and Mittnik (2000). Observe that the stable non-Gaussian distributions exhibit infinite variance. However, empirical evidence suggests that financial series distributions show tails that are heavier than those with finite variance distributions, i.e.

\[
P(\{|e| > x\}) \sim x^{-\alpha} L(x) \quad \text{if } x \to \infty,
\]  

(8)

where \( 0 < \alpha < 2 \) and \( L(x) \) is a slowly varying function at infinity, i.e.,

\[
\lim_{x \to \infty} \frac{L(cx)}{L(x)} \to 1 \quad \text{for all } c > 0,
\]  

(9)

see Rachev and Mittnik [28] and the references therein. Therefore, the stable laws are justified by the stationary behavior of returns and of the Central Limit Theorem for normalized sums of i.i.d. random variables which determines the domain of attraction of each stable law (see Zolatorev [38]). In addition, the so-called pre-limit theorem provides an approximation for distribution functions in case the number of observation \( T \) is “large” but not too “large” (see Klebanov, Rachev, Szekely [16] and Klebanov, Rachev, Safarian [17]). In particular the “pre-limiting” approach helps to overcome the drawback of the Lévy-type central limit theorems. We can assume that residuals are bounded, say daily residuals cannot be outside the interval \([-0.5, 0.5]\). Thus, we can assume that the residuals are truncated \( \alpha \)-stable distributed with support \([-0.5, 0.5]\). Even if the residuals will be attracted by the CLT to the Gaussian law, pre-limit theorems show that for any reasonable \( T \) the truncated stable laws will be attracted to the stable laws. Therefore, we are justified to use \( \alpha \)-stable distributions even if we assume that our historical series have finite variance.

2.1.2. \( t_3 \) distribution

Alternatively to the stable distributed standardized residuals, we consider the so-called \( t_3 \) distribution (see [26]). The \( t_3 \) distribution contains three parameters and it is characterized by the following density

\[
f(z; d, \theta, \nu) = K \cdot \begin{cases} 
\left( 1 + \left( \frac{z \theta}{\nu} \right)^{d} \right)^{-(\nu+\frac{1}{2})} & \text{if } z < 0 \\
\left( 1 + \left( \frac{\theta}{z} \right)^{d} \right)^{-(\nu+\frac{1}{2})} & \text{if } z \geq 0
\end{cases}
\]  

(10)
where $K^{-1} = (\theta + \theta^{-1})d^{-1} \nu^{\frac{1}{2}} B(d^{-1}, \nu)$ and $B(\cdot)$ denotes the beta function $B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}$ and $\Gamma(w) = \int_0^\infty x^{w-1} \exp^{-x} dx$. It has been shown that the distribution is unimodal, the parameter $\nu > 0$ represents the degrees of freedom that together with $d > 0$ measures the heaviness of the distribution tails and $\theta > 0$ represents the skewness of the distribution.

The $t_3$ distribution is a generalization of the generalized exponential as well as the Student’s $t$, Laplace, Cauchy and the Gaussian as $\nu$ tends to infinity. For $d = 2$, $\theta = 1$ we obtain the centered Student’s-$t$ distribution with $\nu$ degrees of freedom and for opportune parameters we could obtain the other distributions (see [26]).

As shown in Figure 3, the $t_3$ distributions present fatter tails than the Gaussian. In particular, $vd$ provides the maximal existing moment of the unobservable innovations process. Thus, if the estimated product $vd$ is less than 2, we implicitly assume that the $t_3$ distributed residuals have infinite variance (see Mittnik and Paolella (2000)).

Figure 4 presents $t_3$ distributions for different levels of $\theta$. A positive skewness ($\theta > 1$) identifies distributions with right fat tails, while a negative skewness ($\theta < 1$) typically characterizes distributions with left fat tails.

3. Empirical evidence

In this section we employ different criteria for comparing candidate distributional assumptions. In our comparison we use daily data on three index-daily returns: DAX 30, CAC 40, S&P 500 quoted on the stock market from January 1994 to January 1998.

Recall that in literature there exist many tests to verify the normality of random variables. Some of the most used tests are based on the Fisher skewness and the Pearson kurtosis coefficients. The coefficient of skewness is given by

$$\phi = \frac{1}{T} \sum_{t=1}^{T} \left( \frac{R_t - E(R_t)}{\sigma_t} \right)^3,$$  \hspace{1cm} (11)

where $\sigma_t$ is the standard deviation relative to the return $R_t$. When the skewness coefficient is positive (negative) the right (left) tail is generally fatter than the left tail (right). Thus values of $\phi$ different from zero point out the presence of skewness in the distribution examined. The Pearson kurtosis coefficient is given by

$$\kappa = \frac{1}{T} \sum_{t=1}^{T} \left( \frac{R_t - E(R_t)}{\sigma_t} \right)^4.$$  \hspace{1cm} (12)

The kurtosis coefficient for a Gaussian distribution is equal to three. Distributions with a kurtosis greater (smaller) than 3 are defined as leptokurtotic (platikurtotic) distributions and they generally are more (less) concentrated around the mean and have fat tails (thin tails). Similarly, in order to test the normality, we can propose other statistics and observe how far they are from
those obtained from the best approximating Gaussian distribution. In particular, we consider four different risk measures of the risky indexes used in our portfolio analysis: \(VaR_{99\%},VaR_{95\%},CVaR_{99\%},CVaR_{95\%}\) (see Table I).

From the analysis of these distributional parameters we observe that all the indexes present kurtosis and skewness far from the normal values. Moreover, even if S&P500 dominates the other indexes in a mean-variance framework, this index presents the greatest kurtosis and conditional value at risk at 95% confidence level.

A first empirical analysis on the historical return series indicates a considerable deviation from the normality. Table I reports the maximum likelihood estimation of \(\alpha\)-stable and \(t_3\) parameters of return series. Considering that the index of stability is always below two and that \(\alpha\) is not very high, then the return series present heavy tails. Moreover, the stable index of skewness \(\beta\) is always different from zero and the series presenting skewness do not present the strong asymmetry \(\theta \approx 1\). In Table I we can observe that the return series present kurtosis and skewness that are different from the Gaussian ones. In addition, we employ two tests which are based on the whole empirical distribution. Other goodness-of-fit measures can also be used, see [22] and [28]. In particular, the distribution of index returns has been analyzed using the Kolmogorov-Smirnov (K-S) test

\[
\sup_{x \in \mathbb{R}} |F_E(x) - F(x)|, \tag{13}
\]

and the Anderson-Darling (A-D) statistic

\[
AD = \max_{x \in \mathbb{R}} \frac{|F_E(x) - F(x)|}{\sqrt{F(x)(1 - F(x))}},
\]

where \(F_E\) is the empirical distribution and \(F\) the chosen model distribution.

The A-D statistic weighs discrepancies appropriately across the whole support of the distribution and it is particularly important if one is interested in determining tails of return distributions. Therefore, first we present additional empirical evidence comparing Gaussian and non-Gaussian distributional assumptions on the unconditional return distributions. In addition, the K-S test and the A-D statistic (see Table III) show that the \(t_3\) distribution presents optimal performance in approximating the empirical distribution. The Stable distribution presents a better approximation to the tails among the three distributions proposed (see A-D results). Therefore, the proposed tests agree upon in rejecting the hypothesis of normality.

Next we analyze conditional homoskedastic and heteroskedastic distributions. Thus, we could compare the empirical cumulative distribution \(F_E(x)\) of several standardized portfolio residuals with either a simulated Gaussian or a non-Gaussian distribution \(F(x)\).

3.1. Autoregressive dependency
The study and the analysis conducted on the empirical behavior of data (see, among others, Akigiray (1989) Campbell (1987), Fama and French (1988) French, Schwert and Stambaugh (1987)) have reported evidence that conditional first and second moments of stock returns are time varying and potentially persistent especially when returns are measured over long horizons. If financial modeling involves information on past market movements, it is not the unconditional return distribution which is of interest but the conditional distribution on information contained in past return data, or a more general information set. The class of auto-regressive moving average (ARMA) models is a natural candidate for conditioning the past of some return series. A peculiar feature of ARMA models is that the conditional volatility is independent of past realizations. The assumption of conditional homoskedasticity is often violated in financial data where we could observe volatility clusters implying that a large absolute return is often followed by larger absolute returns, which more or less slowly decay. Such a behavior can be captured by conditional heteroskedastic models, such as Engle’s (Engle (1982)) auto-regressive conditional heteroskedastic (ARCH) models and the Bollerslev’s generalization (GARCH models, see Bollerslev (1986)).

In order to test the ARMA and GARCH dependencies we followed the standard Box-Jenkins identification techniques (see Brockwell and Davis (1991)). More specifically, we inspected the sample autocorrelation functions (SACF) and the sample partial autocorrelation functions (PACF) of the return series $r_t$ to test the ARMA dependencies and of the return series $r^2_t$ to describe the GARCH dependencies. The exponentially decaying SACF and the single large spike in PACF corresponding to our three historical series suggest the appropriateness of an AR(1) structure while there is not strong evidence of the moving average MA(1) component (see Table IV). In addition, Table IV shows that the standard Box-Jenkins methodology would suggest the need for a mixed model. That is, we find that GARCH(1,1) is adequate in capturing the correlation structure of these squared series.

For these reasons we assume an AR(1)-GARCH(1,1) to model the portfolio returns. However, the residuals of an AR(1)-GARCH(1,1) model generally cannot be considered Gaussian distributed. As a matter of fact, we observe that $t_3$ and $\alpha-$stable distributions present the best performance to approximate the empirical distribution of residuals, as confirmed by Kolmogorov-Smirnov and Anderson-Darling distance in Table V. From this empirical analysis it results that $\alpha-$stable or $t_3$ distributed innovations would be a reasonable assumption. Davis and Resnik (1986, 1996), Davis, Mikosch and Basrak (2000), Mikosch and Starica (1998) studied the asymptotic behavior of sample auto-correlation function of linear and non-linear stationary sequences with heavy-tailed distributions (\textit{regularly varying finite-dimensional distributions}). Panorska, Mittnik and Rachev (1996), Mittnik, Rachev and Paolella (1996) and ( [26] ) proposed the stable and the $t_3$ GARCH model, they derived necessary and sufficient conditions for the existence and uniqueness of the stationary stable or the $t_3$ GARCH processes. In particular, an AR(1)-GARCH(1,1) with finite mean and infinite variance takes the following form
\[ R_{p,t} = \alpha_0 + \alpha_1 R_{p,t-1} + \epsilon_t \]

and

\[ \sigma_t = \gamma_0 + \gamma_1 |\epsilon_{t-1}| + \gamma_2 \sigma_{t-1}. \]

4. **The asset allocation problem**

In order to evaluate the impact of different distributional approximations for future returns in the investor’s portfolio choice, we consider a dynamic asset allocation approach very similar to that proposed by Boender [4] and Tokat, Rachev and Schwartz [33]. We generate different initial asset allocations then simulated in the future by using the economic scenarios generated under the Gaussian, the stable and the \( t \) assumptions for the innovations of the time series models. We are interested in maximizing the expected utility of the final wealth, \( W_T \), at time horizon \( T \).

4.1. **The choice of utility functions**

We assume that investors wish to maximize three different functionals of final wealth which differ for their risk measures and for the risk aversion coefficient \( c \).

1. **The mean-VaR utility functional**

\[ U_{(1)}(W_T) = E(W_T) - cVaR_\alpha (W_T) \] (14)

where \( VaR_\alpha (W_T) = -\inf \{ z/P (W_T \leq z) > \alpha \} \) where \( \alpha \) represents the maximum probability of loss that the investor would accept.

2. **The mean-CVaR utility functional**

\[ U_{(2)}(W_T) = E(W_T) - cCVaR_\alpha (W_T) \] (15)

where \( CVaR_\alpha (W_T) = E(-W_T / -W_T \geq VaR_\alpha (W_T)) \). In particular we recall that this risk measure is coherent in the sense of Artzner et al. [2].

3. **The mean-dispersion utility functional**

\[ U_{(3)}(W_T) = E(W_T) - cE(|W_T - E(W_T)|^q) \] (16)

with power \( q > 0 \) is equivalent to maximizing the utility functional

\[ aE(W_T) - bE(|W_T - E(W_T)|^q), \] (17)

assuming \( c = \frac{b}{a} \) for every \( a, b > 0 \). Thus, \( E(|W_T - E(W_T)|^q)^{1/q} \) represents a particular risk measure of portfolio loss which satisfies the main
characteristics of the classical dispersion measures. By solving optimally this allocation problem, the investor implicitly maximizes the expected mean of the increment wealth $aW$ as well as minimizes the individual risk $bE((W_T - E(W_T))^q)$. In particular, when $q = 2$, the maximization of the utility functional motivates the mean variance approach in terms of preference relations.

4.2. Solving an optimization model

We shall consider that the decision makers with utility functions (14), (15), (16) invest their wealth among the three indexes S&P500, DAX30 and CAC40 plus a riskless asset (that we assume to be equal to 6% a.r. ). The investor calibrates her/his portfolio at every decision stage using the fixed-mix allocation rule. This strategy is often used by financial practitioners, even if it is an optimal strategy only under some regularity conditions. The fixed-mix strategy maintains a fixed exposure on the different assets by requiring the purchase of stocks as they decrease in value and the sale of stocks as they increase in value.

Therefore, among the initial fixed-mix portfolios the investor will choose the portfolio $x^* = [x_1, \ldots, x_n]$ which maximizes the utility functional associated $U_{(k)}(W_T^x)$, i.e.

$$max_x U_{(k)}(W_T^x) \quad s.t. \quad x_i \geq 0, \quad \sum_i x_i = 1,$$

(18)

where $U_{(k)}(W_T^x)$, is given, in turn, by equations (14), (15), (16)

$$W_{s,T}^x = \sum_{t=1}^T R_{x,t}^s$$

(19)

is the final wealth for portfolio $x$ under the scenario $s \in \{1, \ldots, S\}$ and

$$R_{x,t}^s = x'R_{s,t}$$

(20)

is the return of portfolio $x$ under the scenario $s$ at time $t$. $R_{s,t} = [R_{1,s,t}, \ldots, R_{n,s,t}]$ where $R_{i,s,t}$ is the rate of return of $i$-th asset under the scenario $s$ at time $t$;

$$E(W_T^x) = \frac{1}{S} \sum_{s=1}^S W_{s,T}^x$$

(21)

is the expected value of the final wealth for portfolio $x$; and

1. $VaR_{\alpha}(W_T^x)$ is the opposite of the $[\alpha S] - th$ observation among the $S$ ordinate scenarios of final wealth $W_T^x$.

2. $CVaR_{\alpha}(W_T^x) = \frac{1}{[\alpha S]} \sum_{s=1}^{[\alpha S]} W_{s,T}^x$.

3. $E(\|W_T^x - E(W_T^x)\|^q) = \frac{1}{S} \sum_{s=1}^S |W_{s,T}^x - E(W_T^x)|^q$. 

11
These are the three possible risk measures associated to portfolio $x$ for the three functionals. We assume that the investor has a temporal horizon of 5 periods, i.e. $T = 5$.

5. Computational results and scenario generation

In order to solve portfolio problem (18), we consider two admissible procedures:

1. using a grid of admissible portfolios,
2. using the simulated annealing procedure in order to obtain the global maximum.

Grid methodology

We consider $21^3$ initial asset allocations corresponding to 21 different admissible weights $x_i$ for the $i$-th asset $i = 1, 2, 3, 4$. For each portfolio $R_{t,j} = x^T R_t$ we estimate the AR(1)-GARCH(1,1) parameters. Then, we simulate the allocations in the future by using the economic scenarios generated by sampling randomly from Gaussian, $\alpha$-stable and $t_3$ innovation distributions. For each sample, we simulate 5000 future economic scenarios at daily intervals. Thus, every set of scenarios is generated by assuming that residuals of the variables are iid normal, iid stable, iid $t_3$ distributed. The horizon of interest is 5 days. Finally, we solve the optimization problem (18). This analysis gives us an approximation of the optimal expected utility considering the three different distribution families. It is interesting to observe that the volatility of scenarios changes deeply as the distributional assumption for the residuals of portfolios changes. Figure 5 underlines these differences for an equally distributed portfolio. In fact, it appears clearly that $\alpha$-stable scenarios are more volatile than $t_3$ scenarios, and $t_3$ scenarios are more volatile than the Gaussian ones.

Simulated annealing methodology

The previous methodology considers only a grid of all admissible portfolios. Observe that the assumption of an AR(1)-GARCH(1,1) for each portfolio and each distributional assumption implies that every expected utility function could present more than a local maximum. Thus, the determination of the optimal portfolio is not an easy issue from the computational point of view and what we consider to be the optimum it may not be the global optimum. In order to overcome this problem, we do not fix the initial asset allocations and we propose to choose the optimal expected utility among simulated scenarios using the simulated annealing algorithm, first introduced by Kirkpatrick [15]. In this case, the computational complexity increases sensibly and for this reason we suggest this approach only when the number of assets is small. In particular, we try to simplify the computation using the simulated annealing program starting from the point found by the grid analysis. Other methods may be used to find the global optimum, as recently mentioned by Biggs (see Biggs (2004)) [3] or LGO software package (see Pinter (1996)) may be useful.
5.1. Distributional analysis on some optimal portfolios

As a consequence of the previous analysis we obtain a different optimal portfolio for each utility function (14), (15), (16) as well as different risk aversion coefficients “c”. In Tables VI, VII and VIII we report some of the optimal portfolios obtained under the different distributional hypotheses of the return residuals. These tables show that for each utility function the portfolio choice consisting of α-stable scenarios is the most diversified and it always considers the index CAC 40 while the other set of scenarios (t₃ and Gaussian) does not. Besides there are strong differences among the allocation choices based on different approaches. The main reason of this different diversification is probably due to the fact that CAC 40 presents the less fat tails among the three risky indexes. Thus, in order to minimize the risk due to the tail distributions, the "stable investor" assumes a positive position on this index, even if CAC 40 is less appealing in terms of the expected return. Note that the investor with t₃ distributed innovations chooses the less diversified optimal portfolios and he/she invests all the wealth only in the riskless and in the S&P 500 index.

In order to evaluate the skewness and the kurtosis of optimal portfolio choices, we consider the MLE t₃ and the α-stable parameters (see Tables IX and X) of optimal portfolios. Tables IX and X show that all optimal portfolios obtained with α-stable or Gaussian residuals generally present a large index of stability α (about α = 1.91) and large degrees of freedom v. These optimal portfolios are less risky than those obtained assuming t₃ innovations.

To help in distinguishing among competing models, we use two goodness-of-fit measures, i.e the Kolmogorov-Smirnoff statistic (13) and the Anderson-Darling statistic [1]. Therefore we can compare the empirical cumulative distribution $F_E(x)$ of several portfolio residuals with either a simulated Gaussian or a non-Gaussian distribution $F(x)$. The Kolmogorov-Smirnoff test does not show big differences among the three distributional hypotheses (see Table XI). Indeed, the Anderson Darling statistic agrees upon in rejecting the hypothesis of normality (see Table XII), while t₃ and α-stable distributed residuals generally cannot be rejected.

6. Conclusions

This paper proposes and compares alternative distributional assumptions for portfolio selection models. First, we analyze the historical series considered in the empirical portfolio selection proposed. The observed skewness and kurtosis together with other empirical tests point out that the returns are generally far to be Gaussian distributed. Moreover, we observe that there is a strong evidence that the indexes returns follow an AR(1)-GARCH(1,1) model. In particular, we consider different distributional assumptions on return innovations because we observe that also residuals of the AR(1)-GARCH(1,1) model generally cannot be considered as Gaussian distributed.
Secondly, we propose a comparison made among the $\alpha-$stable, Gaussian and the $t_3$ distributed portfolio residuals assuming three different allocation problems. We find that the tail behavior of different distributional approaches could imply substantial differences in the asset allocation. This analysis indicates that the $\alpha-$stable allocation is more risk preserving than the other ones because it presents the greatest diversification in the portfolio composition and it considers the component of risk due to the fat tails. On the other hand, all the statistical tests have pointed out that the $\alpha-$stable and the $t_3$ present the best approximation to the data. Taken into account that the $\alpha$ stable and $t_3$ approaches are more adherent to the reality of the market, we have obtained models that improve the performance measurements with the distributional assumption: the $\alpha$ stable approach which is more risk preserving and the $t_3$ approach which is riskier but takes into more account the final expected wealth.

Acknowledgements. We are grateful to Stoyan Stoyanov for the computational analysis and helpful comments. Bertocchi, Giacometti, Ortobelli acknowledge the support given by research projects Murst 40%, 60% 2002, 2003 and CNR-MIUR-Legge 95/95. S. Rachev’s research was supported by grants from the Division of Mathematical, Life and Physical Sciences, College of Letters and Science, University of California, Santa Barbara and the Deutschen Forschsgemeinschaft.

References


Figure 1
Gaussian and symmetric standardized stable laws with $\beta=0$ and $\alpha=0.8; 1.1; 1.4; 1.7; 2$
Figure 2
Asymmetric standardized stable laws with $\alpha=1.5$ and $\beta=-1; -0.5; 0; 0.5; 1$

Figure 3
Gaussian and symmetric standardized Student’s $t_\nu$ with $\theta=1, \nu=3, d=1; 1.5; 2; 2.5$
Figure 4
Gaussian and asymmetric standardized Student’s $t$ with $d=1.5$, $v=3$, $\theta=0.3; 0.6; 1.67; 3.33$.

Figure 4:
Figure 5
Scenario generation with Gaussian, Student’s $t_3$ and stable innovations.
Table I

This table summarizes parameter estimates of three international indexes. In particular, we consider: the mean, the standard deviation, the skewness, the kurtosis, the VaR99%, the VaR95%, the CVaR99%, the CVaR95%. All parameters are computed on series of daily returns between January 1994 and January 1998 for a total of 948 observations.

<table>
<thead>
<tr>
<th>Index</th>
<th>Mean</th>
<th>Sigma</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>VaR99%</th>
<th>VaR95%</th>
<th>CVaR99%</th>
<th>CVaR95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>DAX 30</td>
<td>0.0746</td>
<td>1.1339</td>
<td>-0.5353</td>
<td>7.6844</td>
<td>-3.1902</td>
<td>-1.9542</td>
<td>-4.0757</td>
<td>-2.745</td>
</tr>
<tr>
<td>CAC 40</td>
<td>0.0481</td>
<td>1.1048</td>
<td>0.0887</td>
<td>5.308</td>
<td>-2.7602</td>
<td>-1.7976</td>
<td>-3.3539</td>
<td>-2.531</td>
</tr>
<tr>
<td>S&amp;P 500</td>
<td>0.0945</td>
<td>0.8326</td>
<td>-0.7287</td>
<td>12.0318</td>
<td>-2.2179</td>
<td>-1.1776</td>
<td>-3.2157</td>
<td>-3.004</td>
</tr>
</tbody>
</table>

Table II

This table summarizes parameter estimates of three international indexes. In particular, we consider: the maximum likelihood estimates (MLE) of the stable distribution parameters and of Student’s $t_3$ parameters. All parameters are computed on series of daily returns between January 1994 and January 1998 for a total of 948 observations.

<table>
<thead>
<tr>
<th>Index</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
<th>$\phi$</th>
<th>$\nu$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DAX 30</td>
<td>1.815</td>
<td>-0.668</td>
<td>0.692</td>
<td>0.05</td>
<td>1.641</td>
<td>4</td>
<td>1.002</td>
</tr>
<tr>
<td>CAC 40</td>
<td>1.838</td>
<td>-0.185</td>
<td>0.707</td>
<td>0.038</td>
<td>1.354</td>
<td>19</td>
<td>0.997</td>
</tr>
<tr>
<td>S&amp;P 500</td>
<td>1.705</td>
<td>-0.088</td>
<td>0.465</td>
<td>0.1</td>
<td>1.806</td>
<td>3</td>
<td>1.009</td>
</tr>
</tbody>
</table>

Figure 6:

Figure 7:
Table III
Kolmogorov-Smirnoff test and Anderson-Darling test for return series.

<table>
<thead>
<tr>
<th></th>
<th>Kolmogorov-Smirnoff</th>
<th>Anderson-Darling</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Gaussian Stable $t_3$</td>
<td>Gaussian Stable $t_3$</td>
</tr>
<tr>
<td>DAX 30</td>
<td>0.0608 0.0464 0.0376</td>
<td>931.99 0.1121 0.1227</td>
</tr>
<tr>
<td>CAC 40</td>
<td>0.0451 0.0674 0.026</td>
<td>60.63 0.1353 0.2392</td>
</tr>
<tr>
<td>S&amp;P 500</td>
<td>0.0708 0.0556 0.0256</td>
<td>800000 0.1158 0.1529</td>
</tr>
</tbody>
</table>

Figure 8:

Table IV
AR(1)-GARCH(1,1) parameters of DAX30, CAC40 and S&P500 return series.

<table>
<thead>
<tr>
<th></th>
<th>AR(1)</th>
<th>GARCH(1)</th>
<th>ARCH(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha_0$ $\alpha_1$ $\gamma_0$</td>
<td>$\gamma_1$ $\gamma_2$</td>
<td></td>
</tr>
<tr>
<td>DAX 30</td>
<td>0.075736 -0.17654 0.027456</td>
<td>0.86582 0.11766</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2.2719) (-4.9226) (2.3775)</td>
<td>(39.5341) (6.8878)</td>
<td></td>
</tr>
<tr>
<td>CAC 40</td>
<td>0.050762 -0.6265 0.008936</td>
<td>0.95577 0.03777</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1.4146) (-1.7819) (1.3806)</td>
<td>(74.9731) (3.8432)</td>
<td></td>
</tr>
<tr>
<td>S&amp;P 500</td>
<td>0.09912 0.064278 0.006227</td>
<td>0.92287 0.074339</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3.916) (1.5632) (2.2765)</td>
<td>(72.1109) (6.8565)</td>
<td></td>
</tr>
</tbody>
</table>

Figure 9:

Table V
Kolmogorov-Smirnoff test and Anderson-Darling test for residuals of return series.

<table>
<thead>
<tr>
<th></th>
<th>Kolmogorov-Smirnoff</th>
<th>Anderson-Darling</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Gaussian Stable $t_3$</td>
<td>Gaussian Stable $t_3$</td>
</tr>
<tr>
<td>DAX 30</td>
<td>0.0442 0.0526 0.0584</td>
<td>0.5215 0.1053 0.1343</td>
</tr>
<tr>
<td>CAC 40</td>
<td>0.0424 0.059 0.035</td>
<td>0.735 0.1215 0.0797</td>
</tr>
<tr>
<td>S&amp;P 500</td>
<td>0.0598 0.0553 0.0274</td>
<td>65.302 0.1245 0.1517</td>
</tr>
</tbody>
</table>

Figure 10:
<table>
<thead>
<tr>
<th>Parameter $c^*$</th>
<th>Gaussian generation</th>
<th>Stable generation</th>
<th>$t_3$ generation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DAX 30</td>
<td>CAC 40</td>
<td>S&amp;P 500</td>
</tr>
<tr>
<td>0.04</td>
<td>60%</td>
<td>0%</td>
<td>40%</td>
</tr>
<tr>
<td>0.06</td>
<td>58.75%</td>
<td>0%</td>
<td>41.25%</td>
</tr>
<tr>
<td>0.08</td>
<td>58.44%</td>
<td>0%</td>
<td>41.56%</td>
</tr>
<tr>
<td>0.2</td>
<td>0%</td>
<td>0%</td>
<td>100%</td>
</tr>
<tr>
<td>0.04</td>
<td>16.33%</td>
<td>34.54%</td>
<td>49.13%</td>
</tr>
<tr>
<td>0.06</td>
<td>5.93%</td>
<td>40.07%</td>
<td>53.92%</td>
</tr>
<tr>
<td>0.08</td>
<td>8.70%</td>
<td>41.87%</td>
<td>47.14%</td>
</tr>
<tr>
<td>0.2</td>
<td>0.00%</td>
<td>0.00%</td>
<td>4.99%</td>
</tr>
<tr>
<td>0.04</td>
<td>0.00%</td>
<td>0.00%</td>
<td>100.00%</td>
</tr>
<tr>
<td>0.06</td>
<td>0.00%</td>
<td>0.00%</td>
<td>100.00%</td>
</tr>
<tr>
<td>0.08</td>
<td>0.00%</td>
<td>0.00%</td>
<td>99.99%</td>
</tr>
<tr>
<td>0.2</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
</tbody>
</table>

Table VI
Optimal allocation for the optimization problem

\[
\max E(W_t) - c \cdot \text{VaR}_{0.95}(W_t)
\]
when Gaussian, Student’s $t_3$ and stable non-Gaussian distributional assumptions are considered.

Figure 11:
<table>
<thead>
<tr>
<th>Parameter(c)</th>
<th>DAX 30</th>
<th>CAC 40</th>
<th>S&amp;P 500</th>
<th>Riskfree</th>
<th>Expected utility</th>
<th>Mean</th>
<th>Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Gaussian generation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.04</td>
<td>59.85%</td>
<td>0.00%</td>
<td>40.15%</td>
<td>0.00%</td>
<td>0.2631</td>
<td>0.4211</td>
<td>3.9503</td>
</tr>
<tr>
<td>0.06</td>
<td>60.02%</td>
<td>0.00%</td>
<td>39.98%</td>
<td>0.00%</td>
<td>0.1840</td>
<td>0.4211</td>
<td>3.9517</td>
</tr>
<tr>
<td>0.08</td>
<td>58.81%</td>
<td>0.00%</td>
<td>41.17%</td>
<td>0.02%</td>
<td>0.1051</td>
<td>0.4207</td>
<td>3.9449</td>
</tr>
<tr>
<td><strong>Stable generation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.04</td>
<td>15.69%</td>
<td>34.51%</td>
<td>49.80%</td>
<td>0.00%</td>
<td>0.1770</td>
<td>0.3637</td>
<td>4.667</td>
</tr>
<tr>
<td>0.06</td>
<td>5.89%</td>
<td>40.07%</td>
<td>53.95%</td>
<td>0.09%</td>
<td>0.0918</td>
<td>0.3515</td>
<td>4.328</td>
</tr>
<tr>
<td>0.08</td>
<td>0.89%</td>
<td>2.76%</td>
<td>4.46%</td>
<td>91.89%</td>
<td>0.0803</td>
<td>0.1049</td>
<td>0.307</td>
</tr>
<tr>
<td><strong>t(_3) generation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.04</td>
<td>0.00%</td>
<td>0.00%</td>
<td>100.00%</td>
<td>0.00%</td>
<td>0.2335</td>
<td>0.4252</td>
<td>4.793</td>
</tr>
<tr>
<td>0.06</td>
<td>0.00%</td>
<td>0.00%</td>
<td>99.60%</td>
<td>0.40%</td>
<td>0.1394</td>
<td>0.4266</td>
<td>4.786</td>
</tr>
<tr>
<td>0.08</td>
<td>0.00%</td>
<td>0.00%</td>
<td>5.00%</td>
<td>95.00%</td>
<td>0.0864</td>
<td>0.0993</td>
<td>0.161</td>
</tr>
<tr>
<td>0.2</td>
<td>0.00%</td>
<td>0.00%</td>
<td>100.00%</td>
<td>0.00%</td>
<td>0.0822</td>
<td>0.0822</td>
<td></td>
</tr>
</tbody>
</table>

Table VII
Optimal allocation for the optimization problem

\[
\text{max } \mathbb{E}(W_T) - c \cdot \text{CVaR}_{0.05}(W_T)
\]

when Gaussian, Student’s \(t\_3\) and stable non-Gaussian distributional assumptions are considered.

Figure 12:
Table VIII
Optimal allocation for the optimization problem
\[
\max E(W_T) - cE(|WT - E(W_T)|^{1.5})
\]
when Gaussian, Student’s \( t_3 \) and stable non-Gaussian distributional assumptions are considered.

Table IX
Stable maximum likelihood estimates of the optimal portfolio risky component. The optimal portfolios are obtained by maximizing the utility functionals (17), (18), (19) and considering different coefficients "c" under different distributional hypothesis.
<table>
<thead>
<tr>
<th>Coeff.</th>
<th>VaR95%</th>
<th>Coeff.</th>
<th>CVaR95%</th>
<th>Coeff.</th>
<th>MAD q=1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>d</td>
<td>v</td>
<td>θ</td>
<td>Normal</td>
<td>d</td>
</tr>
<tr>
<td>0.04</td>
<td>1.316</td>
<td>0.99</td>
<td>0.04</td>
<td>1.316</td>
<td>0.99</td>
</tr>
<tr>
<td>0.06</td>
<td>1.315</td>
<td>0.99</td>
<td>0.06</td>
<td>1.316</td>
<td>0.99</td>
</tr>
<tr>
<td>0.08</td>
<td>1.315</td>
<td>0.991</td>
<td>0.08</td>
<td>1.315</td>
<td>0.99</td>
</tr>
<tr>
<td>Stable</td>
<td>d</td>
<td>v</td>
<td>θ</td>
<td>Normal</td>
<td>d</td>
</tr>
<tr>
<td>0.04</td>
<td>1.308</td>
<td>0.994</td>
<td>0.04</td>
<td>1.309</td>
<td>0.99</td>
</tr>
<tr>
<td>0.06</td>
<td>1.315</td>
<td>0.995</td>
<td>0.06</td>
<td>1.305</td>
<td>0.995</td>
</tr>
<tr>
<td>0.08</td>
<td>1.315</td>
<td>0.995</td>
<td>0.08</td>
<td>1.329</td>
<td>0.996</td>
</tr>
<tr>
<td>$t_3$</td>
<td>d</td>
<td>v</td>
<td>θ</td>
<td>Normal</td>
<td>d</td>
</tr>
<tr>
<td>0.04</td>
<td>1.536</td>
<td>5.286</td>
<td>1.007</td>
<td>0.04</td>
<td>1.536</td>
</tr>
<tr>
<td>0.06</td>
<td>1.538</td>
<td>5.264</td>
<td>1.007</td>
<td>0.05</td>
<td>1.536</td>
</tr>
<tr>
<td>0.08</td>
<td>1.536</td>
<td>5.286</td>
<td>1.007</td>
<td>0.08</td>
<td>1.536</td>
</tr>
</tbody>
</table>

Table X

$t_3$ maximum likelihood estimates of the optimal portfolio risky component. The optimal portfolios are obtained by maximizing the utility functionals (17), (18), (19) and considering different risk aversion coefficients $c$ under different distributional hypothesis.

Figure 15:

<table>
<thead>
<tr>
<th>Coeff.</th>
<th>VaR95%</th>
<th>Coeff.</th>
<th>CVaR95%</th>
<th>Coeff.</th>
<th>MAD q=1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>d</td>
<td>v</td>
<td>θ</td>
<td>Normal</td>
<td>d</td>
</tr>
<tr>
<td>0.04</td>
<td>0.043</td>
<td>0.054</td>
<td>0.051</td>
<td>0.04</td>
<td>0.043</td>
</tr>
<tr>
<td>0.06</td>
<td>0.04</td>
<td>0.051</td>
<td>0.049</td>
<td>0.06</td>
<td>0.04</td>
</tr>
<tr>
<td>0.08</td>
<td>0.039</td>
<td>0.05</td>
<td>0.047</td>
<td>0.08</td>
<td>0.04</td>
</tr>
<tr>
<td>Stable</td>
<td>d</td>
<td>v</td>
<td>θ</td>
<td>Normal</td>
<td>d</td>
</tr>
<tr>
<td>0.04</td>
<td>0.049</td>
<td>0.057</td>
<td>0.032</td>
<td>0.04</td>
<td>0.05</td>
</tr>
<tr>
<td>0.06</td>
<td>0.048</td>
<td>0.056</td>
<td>0.034</td>
<td>0.06</td>
<td>0.048</td>
</tr>
<tr>
<td>0.08</td>
<td>0.05</td>
<td>0.057</td>
<td>0.034</td>
<td>0.08</td>
<td>0.046</td>
</tr>
<tr>
<td>$t_3$</td>
<td>d</td>
<td>v</td>
<td>θ</td>
<td>Normal</td>
<td>d</td>
</tr>
<tr>
<td>0.04</td>
<td>0.06</td>
<td>0.055</td>
<td>0.027</td>
<td>0.04</td>
<td>0.06</td>
</tr>
<tr>
<td>0.06</td>
<td>0.06</td>
<td>0.055</td>
<td>0.027</td>
<td>0.06</td>
<td>0.06</td>
</tr>
<tr>
<td>0.08</td>
<td>0.08</td>
<td>0.055</td>
<td>0.027</td>
<td>0.08</td>
<td>0.08</td>
</tr>
</tbody>
</table>

Table XI

Kolmogorov-Smirnoff test for residuals of optimal portfolio return series. The optimal portfolios are obtained by maximizing the utility functionals (17), (18), (19) and considering different risk aversion coefficients $c$ under different distributional hypothesis.

Figure 16:
<table>
<thead>
<tr>
<th>Coeff. &quot;c&quot;</th>
<th>VaR95%</th>
<th>CVaR95%</th>
<th>MAD q=1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal generation</td>
<td>0.04 0.034 0.011 0.012</td>
<td>0.04 0.034 0.011 0.012</td>
<td>0.024 0.034 0.011 0.012</td>
</tr>
<tr>
<td>Stable generation</td>
<td>0.06 0.033 0.011 0.011</td>
<td>0.06 0.034 0.011 0.012</td>
<td>0.05 0.034 0.011 0.012</td>
</tr>
<tr>
<td>Normal generation</td>
<td>0.08 0.032 0.011 0.011</td>
<td>0.08 0.033 0.011 0.011</td>
<td>0.1 0.033 0.011 0.011</td>
</tr>
<tr>
<td>Stable generation</td>
<td>0.03 1.44 0.12 0.1</td>
<td>1.55 0.12 0.17 0.24</td>
<td>1.91 0.14 0.15</td>
</tr>
<tr>
<td>Stable generation</td>
<td>0.06 1.09 0.12 0.13</td>
<td>1.11 0.12 0.13 0.05</td>
<td>1.11 0.12 0.13</td>
</tr>
<tr>
<td>Stable generation</td>
<td>0.08 0.88 0.12 0.13</td>
<td>0.88 0.12 0.13 0.08</td>
<td>4.02 0.12 0.17 0.1</td>
</tr>
<tr>
<td>Stable generation</td>
<td>0.04 605.3 0.12 0.15</td>
<td>0.04 605.3 0.12 0.15</td>
<td>0.024 605.3 0.12 0.15</td>
</tr>
<tr>
<td>Stable generation</td>
<td>0.06 605.3 0.12 0.15</td>
<td>0.06 605.3 0.12 0.15</td>
<td>0.05 605.3 0.12 0.15</td>
</tr>
<tr>
<td>Stable generation</td>
<td>0.08 605.29 0.12 0.15</td>
<td>0.08 605.29 0.12 0.15</td>
<td>0.1 605.4 0.12 0.15</td>
</tr>
</tbody>
</table>

Table XII

Anderson-Darling test for residuals of optimal portfolio return series. The optimal portfolios are obtained by maximizing the utility functionals (17), (18), (19) and considering different risk aversion coefficients "c" under different distributional hypothesis.

Figure 17: