QUIZ THREE

Clearly show all reasoning and calculations. Questions 1-3 are multiple choice.

1. Which quantity does not appear in the Black-Scholes-Merton (BSM) pde? (2 pts)
   a) volatility  b) risk-free rate  c) expected return  d) current stock price

2. The BSM model assumes that the terminal stock price follows which distribution? (2pts)
   a) normal  b) lognormal  c) weibull  d) exponential  e) none of these

3. A long position in a call option has the following delta and gamma exposure: (2pts)
   a) long delta, long gamma
   b) long delta, short gamma
   c) short delta, long gamma
   d) short delta, short gamma

4. Given the Black-Scholes-Merton assumptions, derive the BSM partial differential equation (pde).
   Follow the outline, filling in all details for full credit.

   a) Step 1: Write the process for the return of the stock. Hint: geometric Brownian motion (2 pts)
      \[
      \frac{dS_t}{S_t} = \mu dt + \sigma dW_t
      \]

   b) Step 2: Construct the riskless portfolio, and write its value at time t (2 pts)
      \[
      \Pi_t = -f_t + \frac{df}{ds} S_t
      \]
      (short one option, long \( \frac{df}{ds} \) shares)
c) Step 3: Write the change in portfolio value over time step $dt$. Hint: You’ll need to use Ito’s Lemma here (3 pts)

$$d\Pi_t = -\left(\frac{\partial f}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} (dS_t^2)\right) dt$$

Use Ito’s Lemma: $dS_t = \frac{\partial f}{\partial S} dt + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} dS_t^2$; $dS_t^2 = \sigma^2 S_t^2 dt$

Then $d\Pi_t = -\left(\frac{\partial f}{\partial S} dt + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S_t^2 dt\right) + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S_t^2 dt = -\frac{\partial f}{\partial S} dt - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S_t^2 dt$

(1)

d) Step 4: Think about the return of the portfolio over this time step to make a key observation (1 pt)

Part c tells us that $\Pi_t$ is deterministic, and therefore riskless.
Portfolio should earn risk-free rate: $d\Pi_t = r \Pi_t dt$ (2)

e) Step 5: Simplify the expression, and write the pde (2 pts)

Set (1) = (2): $d\Pi_t = -\frac{\partial f}{\partial S} dt - \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2} dt = r \Pi_t dt$

$$-\frac{\partial f}{\partial S} dt - \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2} dt = r \Pi_t dt$$

Cancel $dt$ and rearrange: \[
\begin{align*}
\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2} + \rho S_t \frac{\partial f}{\partial S} &= rf_t
\end{align*}
\]

BSM PDE

5a. Cliff’s portfolio is long 1000 deltas and short 5000 gammas. He may take long or short positions in the underlying stock, and another traded option. The delta and gamma of the traded option is 0.7 and 2.0, respectively. What trades are necessary to make the portfolio delta and gamma neutral? Clearly show all reasoning. (5 pts)

Begin by making portfolio gamma neutral.
Portfolio long 1000 deltas: $\Delta = 1000$
short 5000 gammas; $\Gamma = -5000$

For gamma neutrality, take position $\Gamma_t$ in traded option: $\Gamma + \Gamma_t = 0$

Then $\Gamma_t = -\frac{\Gamma}{100} \frac{100}{200} = \frac{-5000}{100} = 5000$

25 contracts; so go long 25 option contracts.

Tell me a joke (keep in clean)

Delta of portfolio now $\Delta + \Gamma_t = 1000 + 25 \cdot 100$. So do $\Delta = 2750$

Need negative delta, so short 2750 shares of underlying.