Chapter 13
The Black–Scholes–Merton Model

SOLUTIONS TO QUESTIONS AND PROBLEMS

**Problem 13.1.**
The Black–Scholes option pricing model assumes that the probability distribution of the stock price in 1 year (or at any other future time) is lognormal. It assumes that the continuously compounded rate of return on the stock during the year is normally distributed.

**Problem 13.2.**
The standard deviation of the percentage price change in time Δt is σ√Δt where σ is the volatility. In this problem σ = 0.3 and, assuming 252 trading days in one year, Δt = 1/252 = 0.004 so that σ√Δt = 0.3√0.004 = 0.019 or 1.9%.

**Problem 13.3.**
The price of an option or other derivative when expressed in terms of the price of the underlying stock is independent of risk preferences. Options therefore have the same value in a risk-neutral world as they do in the real world. We may therefore assume that the world is risk neutral for the purposes of valuing options. This simplifies the analysis. In a risk-neutral world all securities have an expected return equal to risk-free interest rate. Also, in a risk-neutral world, the appropriate discount rate to use for expected future cash flows is the risk-free interest rate.

**Problem 13.4.**
In this case $S_0 = 50$, $K = 50$, $r = 0.1$, $σ = 0.3$, $T = 0.25$, and
\[
d_1 = \frac{\ln(50/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.2417
\]
\[
d_2 = d_1 - 0.3\sqrt{0.25} = 0.0917
\]
The European put price is
\[
50N(-0.0917)e^{-0.1\times0.25} - 50N(-0.2417) = 50 \times 0.4634e^{-0.1\times0.25} - 50 \times 0.4045 = 2.37
\]
or $2.37$. 

As before, in all problems, please ensure that all numbers are expressed to at least 3 significant figures.

**Problem 13.5.**
The value of $50N(-0.0917)e^{-0.1\times0.25}$ is $2.37$, and the value of $50N(-0.2417)$ is $2.37$, so the option is worth $2.37$.

**Problem 13.6.**
In this case $S_0 = 50$, $K = 50$, $r = 0.1$, $σ = 0.3$, $T = 0.25$, and
\[
d_1 = \frac{\ln(50/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.2417
\]
\[
d_2 = d_1 - 0.3\sqrt{0.25} = 0.0917
\]
The European put price is
\[
50N(-0.0917)e^{-0.1\times0.25} - 50N(-0.2417) = 50 \times 0.4634e^{-0.1\times0.25} - 50 \times 0.4045 = 2.37
\]
or $2.37$. 

As before, in all problems, please ensure that all numbers are expressed to at least 3 significant figures.
Problem 13.5.

In this case we must subtract the present value of the dividend from the stock price before using Black–Scholes. Hence the appropriate value of \( S_0 \) is

\[
S_0 = 50 - 1.50e^{-0.1667 \times 0.1} = 48.52
\]

As before \( K = 50, \ r = 0.1, \ \sigma = 0.3, \) and \( T = 0.25. \) In this case

\[
d_1 = \frac{\ln(48.52/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.0414
\]

\[
d_2 = d_1 - 0.3\sqrt{0.25} = -0.1086
\]

The European put price is

\[
50N(0.1086)e^{-0.1 \times 0.25} - 48.52N(-0.0414)
\]

\[
= 50 \times 0.5432e^{-0.1 \times 0.25} - 48.52 \times 0.4835 = 3.03
\]

or $3.03.

Problem 13.6.

The implied volatility is the volatility that makes the Black–Scholes price of an option equal to its market price. It is calculated using an iterative procedure.

Problem 13.7.

In this case \( \mu = 0.15 \) and \( \sigma = 0.25. \) From equation (13.7) the probability distribution for the rate of return over a 2-year period with continuous compounding is:

\[
\phi \left( 0.15 - \frac{0.25^2}{2}, \frac{0.25}{\sqrt{2}} \right)
\]

i.e.,

\[
\phi(0.11875, 0.1768)
\]

The expected value of the return is 11.875% per annum and the standard deviation is 17.68% per annum.

Problem 13.8.

(a) The required probability is the probability of the stock price being above $40 in six months’ time. Suppose that the stock price in six months is \( S_T \)

\[
\ln S_T \sim \phi(\ln 38 + (0.16 - \frac{0.35^2}{2})0.5, 0.35\sqrt{0.5})
\]
Chapter 13. The Black–Scholes–Merton Model

i.e.,

$$\ln S_T \sim \phi(3.687, 0.247)$$

Since \( \ln 40 = 3.689 \), the required probability is

$$1 - N\left(\frac{3.689 - 3.687}{0.247}\right) = 1 - N(0.008)$$

From normal distribution tables \( N(0.008) = 0.5032 \) so that the required probability is 0.4968. In general the required probability is \( N(d_2) \). (See Problem 13.22).

(b) In this case the required probability is the probability of the stock price being less than $40 in six months’ time. It is

$$1 - 0.4968 = 0.5032$$

Problem 13.9.

From equation (13.3):

$$\ln S_T \sim \phi[\ln S_0 + (\mu - \frac{\sigma^2}{2})T, \sigma \sqrt{T}]$$

95% confidence intervals for \( \ln S_T \) are therefore

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T - 1.96\sigma \sqrt{T}$$

and

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T + 1.96\sigma \sqrt{T}$$

95% confidence intervals for \( S_T \) are therefore

$$e^{\ln S_0 + (\mu - \frac{\sigma^2}{2})T - 1.96\sigma \sqrt{T}}$$

and

$$e^{\ln S_0 + (\mu - \frac{\sigma^2}{2})T + 1.96\sigma \sqrt{T}}$$

i.e.

$$S_0 e^{(\mu - \frac{\sigma^2}{2})T - 1.96\sigma \sqrt{T}}$$

and

$$S_0 e^{(\mu - \frac{\sigma^2}{2})T + 1.96\sigma \sqrt{T}}$$

Problem 13.10.

The statement is misleading in that a certain sum of money, say $1000, when invested for 10 years in the fund would have realized a return (with annual compounding) of less than 20% per annum.

The average of the returns realized in each year is always greater than the return per annum (with annual compounding) realized over 10 years. The first is an arithmetic average of the returns in each year; the second is a geometric average of these returns.
Problem 13.12.

This problem is related to Problem 12.10.

(a) If \( G(S,t) = h(t,T)S^n \) then \( \partial G/\partial t = h_tS^n, \partial G/\partial S = hS^{n-1}, \) and \( \partial^2 G/\partial S^2 = h(n-1)S^{n-2} \) where \( h_t = \partial h/\partial t. \) Substituting into the Black–Scholes differential equation we obtain
\[
h_t + rhS + \frac{1}{2}\sigma^2 h(n-1)S^{n-2} = rh
\]

(b) The derivative is worth \( S^n \) when \( t = T. \) The boundary condition for this differential equation is therefore \( h(T,T) = 1 \)

(c) The equation
\[
h(t,T) = e^{\left[0.5\sigma^2 n(n-1) + r(n-1)\right](T-t)}
\]
satisfies the boundary condition since it collapses to \( h = 1 \) when \( t = T. \) It can also be shown that it satisfies the differential equation in (a). Alternatively we can solve the differential equation in (a) directly. The differential equation can be written
\[
\frac{h_t}{h} = -r(n-1) - \frac{1}{2}\sigma^2 n(n-1)
\]
The solution to this is
\[
\ln h = \left[-r(n-1) - \frac{1}{2}\sigma^2 n(n-1)\right]t + k
\]
where \( k \) is a constant. Since \( \ln h = 0 \) when \( t = T \) it follows that
\[
k = \left[r(n-1) + \frac{1}{2}\sigma^2 n(n-1)\right]T
\]
so that
\[
\ln h = \left[r(n-1) + \frac{1}{2}\sigma^2 n(n-1)\right](T-t)
\]
or
\[
h(t,T) = e^{0.5\sigma^2 n(n-1) + r(n-1)[T-t]}
\]

Problem 13.13.

In this case \( S_0 = 52, K = 50, r = 0.12, \sigma = 0.30 \) and \( T = 0.25. \)

\[
d_1 = \frac{\ln(52/50) + (0.12 + 0.3^2/2)0.25}{0.30\sqrt{0.25}} = 0.5365
\]
\[
d_2 = d_1 - 0.30\sqrt{0.25} = 0.3865
\]
The price of the European call is

\[ 52N(0.5365) - 50e^{-0.12 \times 0.25}N(0.3865) \]
\[ = 52 \times 0.7042 - 50e^{-0.03} \times 0.6504 \]
\[ = 5.06 \]

or $5.06.

**Problem 13.14.**

In this case \( S_0 = 69, K = 70, r = 0.05, \sigma = 0.35 \) and \( T = 0.5 \).

\[
d_1 = \frac{\ln(69/70) + (0.05 + 0.35^2/2) \times 0.5}{0.35\sqrt{0.5}} = 0.1666
\]
\[
d_2 = d_1 - 0.35\sqrt{0.5} = -0.0809
\]

The price of the European put is

\[ 70e^{-0.05 \times 0.5}N(0.0809) - 69N(-0.1666) \]
\[ = 70e^{-0.025} \times 0.5323 - 69 \times 0.4338 \]
\[ = 6.40 \]

or $6.40.

**Problem 13.15.**

Using the notation of Section 13.12, \( D_1 = D_2 = 1, K(1 - e^{-r(T-t_2)}) = 65(1 - e^{-0.1 \times 0.1667}) = 1.07 \), and \( K(1 - e^{-r(T-t_1)}) = 65(1 - e^{-0.1 \times 0.25}) = 1.60 \). Since

\[ D_1 < K(1 - e^{-r(T-t_2)}) \]

and

\[ D_2 < K(1 - e^{-r(T-t_1)}) \]

It is never optimal to exercise the call option early. DerivaGem shows that the value of the option is 10.94.

**Problem 13.16.**

In the case \( c = 2.5, S_0 = 15, K = 13, T = 0.25, r = 0.05 \). The implied volatility must be calculated using an iterative procedure.

A volatility of 0.2 (or 20% per annum) gives \( c = 2.20 \). A volatility of 0.3 gives \( c = 2.32 \). A volatility of 0.4 gives \( c = 2.507 \). A volatility of 0.39 gives \( c = 2.487 \). By interpolation the implied volatility is about 0.397 or 39.7% per annum.
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<th>Week</th>
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<th>Daily return</th>
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</table>

\[
s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (u_i - \bar{u})^2} = 0.027787 \text{ or } 2.7787\%.
\]

\[
\tau = 1/52
\]

\[
\Rightarrow \hat{\sigma} = \frac{s}{\sqrt{\tau}} = 0.200376
\]

And the standard error of this estimate is \( \frac{\hat{\sigma}}{\sqrt{2n}} = 0.03658 \).