Chapter 9
Properties of Stock Options

SOLUTIONS TO QUESTIONS AND PROBLEMS

Problem 9.1.

The six factors affecting stock option prices are the stock price, strike price, risk-free interest rate, volatility, time to maturity, and dividends.

Problem 9.2.

The lower bound is

$$28 - 25e^{-0.08 \times 0.3333} = 3.66$$

Problem 9.3.

The lower bound is

$$15e^{-0.06 \times 0.0833} - 12 = 2.93$$

Problem 9.4.

Delaying exercise delays the payment of the strike price. This means that the option holder is able to earn interest on the strike price for a longer period of time. Delaying exercise also provides insurance against the stock price falling below the strike price by the expiration date. Assume that the option holder has an amount of cash $$K$$ and that interest rates are zero. Exercising early means that the option holder’s position will be worth $$S_T$$ at expiration. Delaying exercise means that it will be worth max $$(K, S_T)$$ at expiration.

Problem 9.5.

An American put when held in conjunction with the underlying stock provides insurance. It guarantees that the stock can be sold for the strike price, $$K$$. If the put is exercised early, the insurance ceases. However, the option holder receives the strike price immediately and is able to earn interest on it between the time of the early exercise and the expiration date.

Problem 9.6.

An American call option can be exercised at any time. If it is exercised its holder gets the intrinsic value. It follows that an American call option must be worth at least its intrinsic value. A European call option can be worth less than its intrinsic value. Consider, for example, the situation where a stock is expected to provide a very high dividend during the life of an option.
The price of the stock will decline as a result of the dividend. Because the European option can be exercised only after the dividend has been paid, its value may be less than the intrinsic value today.

**Problem 9.7.**

In this case $c = 1$, $T = 0.25$, $S_0 = 19$, $K = 20$, and $r = 0.04$. From put–call parity

$$p = c + Ke^{-rT} - S_0$$

or

$$p = 1 + 20e^{-0.04 \times 0.25} - 19 = 1.80$$

so that the European put price is $1.80.

**Problem 9.8.**

When early exercise is not possible, we can argue that two portfolios that are worth the same at time $T$ must be worth the same at earlier times. When early exercise is possible, the argument falls down. Suppose that $P + S > C + Ke^{-rT}$. This situation does not lead to an arbitrage opportunity. If we buy the call, short the put, and short the stock, we cannot be sure of the result because we do not know when the put will be exercised.

**Problem 9.9.**

The lower bound is

$$80 - 75e^{-0.1 \times 0.5} = 8.66$$

**Problem 9.10.**

The lower bound is

$$65e^{-0.05 \times 2/12} - 58 = 6.46$$

**Problem 9.11.**

The present value of the strike price is $60e^{-0.12 \times 4/12} = 57.65$. The present value of the dividend is $0.80e^{-0.12 \times 1/12} = 0.79$. Because

$$5 < 64 - 57.65 - 0.79$$

the condition in equation (9.5) is violated. An arbitrageur should buy the option and short the stock. This generates $64 - 5 = 59$. The arbitrageur invests $0.79 of this at 12% for one month to pay the dividend of $0.80 in one month. The remaining $58.21 is invested for four months at 12%. Regardless of what happens a profit will materialize.

If the stock price declines below $60 in four months, the arbitrageur loses the $5 spent on the option but gains on the short position. The arbitrageur shorts when the stock price is $64, has to pay dividends with a present value of $0.79, and closes out the short position when the stock price is $60 or less. Because $57.65 is the present value of $60, the short position
Problem 9.15.

If the put price is $3.00, it is too high relative to the call price. An arbitrageur should buy the call, short the put and short the stock. This generates $-2 + 3 + 29 = $30 in cash which is invested at 10%. Regardless of what happens a profit with a present value of $3.00 - 2.51 = $0.49 is locked in.

If the stock price is above $30 in six months, the call option is exercised, and the put option expires worthless. The call option enables the stock to be bought for $30, or $30 e^{-0.10 \times 6/12} = $28.54 in present value terms. The dividends on the short position cost $0.5 e^{-0.1 \times 5/12} + 0.5 e^{-0.1 \times 5/12} = $0.97 in present value terms so that there is a profit with a present value of $30 - 28.54 - 0.97 = $0.49.

If the stock price is below $30 in six months, the put option is exercised and the call option expires worthless. The short put option leads to the stock being bought for $30, or $30 e^{-0.10 \times 6/12} = $28.54 in present value terms. The dividends on the short position cost $0.5 e^{-0.1 \times 5/12} + 0.5 e^{-0.1 \times 5/12} = $0.97 in present value terms so that there is a profit with a present value of $30 - 28.54 - 0.97 = $0.49.

Problem 9.16.

From equation (9.4)

\[ S_0 - K \leq C - P \leq S_0 - Ke^{-rT} \]

In this case

\[ 31 - 30 \leq 4 - P \leq 31 - 30 e^{-0.08 \times 0.25} \]

or

\[ 1.00 \leq 4.00 - P \leq 1.59 \]

or

\[ 2.41 \leq P \leq 3.00 \]

Upper and lower bounds for the price of an American put are therefore $2.41 and $3.00.

Problem 9.17.

If the American put price is greater than $3.00 an arbitrageur can sell the American put, short the stock, and buy the American call. This realizes at least $3 + 31 - 4 = $30 which can be invested at the risk-free interest rate. At some stage during the 3-month period either the American put or the American call will be exercised. The arbitrageur then pays $30, receives the stock and closes out the short position. The cash flows to the arbitrageur are +$30 at time zero and -$30 at some future time. These cash flows have a positive present value.

Problem 9.18.

As in the text we use \( c \) and \( p \) to denote the European call and put option price, and \( C \) and \( P \) to denote the American call and put option prices. Because \( P \geq p \), it follows from put–call parity that

\[ P \geq c + Ke^{-rT} - S_0 \]
Chapter 11
Binomial Trees

SOLUTIONS TO QUESTIONS AND PROBLEMS

Problem 11.1.

Consider a portfolio consisting of:

- 1 : call option
+ \Delta : shares

If the stock price rises to $42, the portfolio is worth $42\Delta - 3$. If the stock price falls to $38$, it is worth $38\Delta$. These are the same when

$$42\Delta - 3 = 38\Delta$$

or $\Delta = 0.75$. The value of the portfolio in one month is $28.5$ for both stock prices. Its value today must be the present value of $28.5$, or $28.5e^{-0.08 \times 0.08333} = 28.31$. This means that

$$-f + 40\Delta = 28.31$$

where $f$ is the call price. Because $\Delta = 0.75$, the call price is $40 \times 0.75 - 28.31 = $1.69. As an alternative approach, we can calculate the probability, $p$, of an up movement in a risk-neutral world. This must satisfy:

$$42p + 38(1 - p) = 40e^{0.08 \times 0.08333}$$

so that

$$4p = 40e^{0.08 \times 0.08333} - 38$$

or $p = 0.5669$. The value of the option is then its expected payoff discounted at the risk-free rate:

$$[3 \times 0.5669 + 0 \times 0.4331]e^{-0.08 \times 0.08333} = 1.69$$

or $1.69$. This agrees with the previous calculation.

Problem 11.2.

In the no-arbitrage approach, we set up a riskless portfolio consisting of a position in the option and a position in the stock. By setting the return on the portfolio equal to the risk-free interest rate, we are able to value the option. When we use risk-neutral valuation, we first
choose probabilities for the branches of the tree so that the expected return on the stock equals the risk-free interest rate. We then value the option by calculating its expected payoff and discounting this expected payoff at the risk-free interest rate.

Problem 11.3.

The delta of a stock option measures the sensitivity of the option price to the price of the stock when small changes are considered. Specifically, it is the ratio of the change in the price of the stock option to the change in the price of the underlying stock.

Problem 11.4.

Consider a portfolio consisting of:

\[-1 : \text{put option}\
\[\Delta : \text{shares}\

If the stock price rises to $55, this is worth 55\Delta. If the stock price falls to $45, the portfolio is worth 45\Delta - 5. These are the same when

\[45\Delta - 5 = 55\Delta\]

or \(\Delta = -0.50\). The value of the portfolio in one month is \(-27.5\) for both stock prices. Its value today must be the present value of \(-27.5\), or \(-27.5 \times e^{-0.1 \times 0.5} = -26.16\). This means that

\[-f + 50\Delta = -26.16\]

where \(f\) is the put price. Because \(\Delta = -0.50\), the put price is $1.16. As an alternative approach we can calculate the probability, \(p\), of an up movement in a risk-neutral world. This must satisfy:

\[55p + 45(1 - p) = 50e^{0.1 \times 0.5}\]

so that

\[10p = 50e^{0.1 \times 0.5} - 45\]

or \(p = 0.7564\). The value of the option is then its expected payoff discounted at the risk-free rate:

\[0 \times 0.7564 + 5 \times 0.2436 \times e^{-0.1 \times 0.5} = 1.16\]

or $1.16. This agrees with the previous calculation.

Problem 11.5.

In this case \(u = 1.10, d = 0.90, \Delta t = 0.5,\) and \(r = 0.08,\) so that

\[p = \frac{e^{0.08 \times 0.5} - 0.90}{1.10 - 0.90} = 0.7041\]
The tree for stock price movements is shown in Figure S11.1. We can work back from the end of the tree to the beginning, as indicated in the diagram, to give the value of the option as $9.61. The option value can also be calculated directly from equation (11.10):

\[
[0.7041^2 \times 21 + 2 \times 0.7041 \times 0.2959 \times 0 + 0.2959^2 \times 0]e^{-2 \times 0.08 \times 0.5} = 9.61
\]

or $9.61.

**Problem 11.6.**

Figure S11.2 shows how we can value the put option using the same tree as in Problem 11.5. The value of the option is $1.92. The option value can also be calculated directly from equation (11.10):

\[
e^{-2 \times 0.08 \times 0.5} [0.7041^2 \times 0 + 2 \times 0.7041 \times 0.2959 \times 1 + 0.2959^2 \times 19] = 1.92
\]

or $1.92. The stock price plus the put price is $100 + 1.92 = $101.92. The present value of the strike price plus the call price is $100e^{-0.08 \times 1} + 9.61 = $101.92. These are the same, verifying that put–call parity holds.

**Problem 11.7.**

\[
u = e^{\sigma \sqrt{N}} \text{ and } d = e^{-\sigma \sqrt{N}}.
\]

**Problem 11.8.**

The riskless portfolio consists of a short position in the option and a long position in \( \Delta \) shares. Because \( \Delta \) changes during the life of the option, this riskless portfolio must also change.

**Problem 11.9.**

At the end of two months the value of the option will be either $4 (if the stock price is $53) or $0 (if the stock price is $48). Consider a portfolio consisting of:

\[
\begin{align*}
+\Delta & : \text{ shares} \\
-1 & : \text{ option}
\end{align*}
\]

The value of this portfolio is $1.92, i.e., the value of the put. This is greater than the present value of the riskless portfolio, which only has a value of $9.61, so that

\[
\text{The value of the put is } 1.92 > 0.5 \times 1.92 = 0.96.
\]

**Problem 11.10.**

At the end of two months the value of the option will be either $4 (if the stock price is $53) or $0 (if the stock price is $48). Consider a portfolio consisting of:

\[
\begin{align*}
+\Delta & : \text{ shares} \\
-1 & : \text{ option}
\end{align*}
\]
\[ \Delta = -0.5 \]

the value of the portfolio is certain to be 22.5. For this value of \( \Delta \) the portfolio is therefore riskless. The current value of the portfolio is

\[ -40\Delta + f \]

where \( f \) is the value of the option. Since the portfolio must earn the risk-free rate of interest

\[(40 \times 0.5 + f) \times 1.02 = 22.5\]

Hence

\[ f = 2.06 \]

i.e., the value of the option is $2.06.

This can also be calculated using risk-neutral valuation. Suppose that \( p \) is the probability of an upward stock price movement in a risk-neutral world. We must have

\[ 45p + 35(1 - p) = 40 \times 1.02 \]

i.e.,

\[ 10p = 5.8 \]

or:

\[ p = 0.58 \]

The expected value of the option in a risk-neutral world is:

\[ 0 \times 0.58 + 5 \times 0.42 = 2.10 \]

This has a present value of

\[ \frac{2.10}{1.02} = 2.06 \]

This is consistent with the no-arbitrage answer.

**Problem 11.12.**

A tree describing the behavior of the stock price is shown in Figure S11.3. The risk-neutral probability of an up move, \( p \), is given by

\[ p = \frac{e^{0.05 \times 3/12} - 0.95}{1.06 - 0.95} = 0.5689 \]

There is a payoff from the option of \( 56.18 - 51 = 5.18 \) for the highest final node (which corresponds to two up moves) zero in all other cases. The value of the option is therefore

\[ 5.18 \times 0.5689^2 \times e^{-0.05 \times 6/12} = 1.635 \]
This can also be calculated by working back through the tree as indicated in Figure S11.1. The value of the call option is the lower number at each node in the figure.

Problem 11.13.

The tree for valuing the put option is shown in Figure S11.4. We get a payoff of 51 - 50.35 = 0.65 if the middle final node is reached and a payoff of 51 - 45.125 = 5.875 if the lowest final node is reached. The value of the option is therefore

\[
(0.65 \times 2 \times 0.5689 \times 0.4311 + 5.875 \times 0.4311^2) e^{-0.05 \times 6/12} = 1.376
\]

This can also be calculated by working back through the tree as indicated in Figure S11.2.

The value of the put plus the stock price is from Problem 11.12

\[
1.376 + 50 = 51.376
\]

The value of the call plus the present value of the strike price is

\[
1.635 + 51 e^{-0.05 \times 6/12} = 51.376
\]

This verifies that put–call parity holds.
To test whether it worth exercising the option early we compare the value calculated for the option at each node with the payoff from immediate exercise. At node C the payoff from immediate exercise is \(51 - 47.5 = 3.5\). Because this is greater than 2.8664, the option should be exercised at this node. The option should not be exercised at either node A or node B.

**Problem 11.14.**

At the end of two months the value of the derivative will be either 529 (if the stock price is 23) or 729 (if the stock price is 27). Consider a portfolio consisting of:

\[ +\Delta : \text{shares} \]
\[ -1 : \text{derivative} \]

The value of the portfolio is either \(27\Delta - 729\) or \(23\Delta - 529\) in two months. If

\[ 27\Delta - 729 = 23\Delta - 529 \]

i.e.,

\[ \Delta = 50 \]

the value of the portfolio is certain to be 621. For this value of \(\Delta\) the portfolio is therefore riskless. The current value of the portfolio is:

\[ 50 \times 25 - f \]