This exam is closed to books and notes, but you may use a calculator. You have 80 minutes. Your exam contains 3 questions and 6 pages. Please make sure you print your name and sign the honor code below.

I acknowledge that I have neither given nor received aid on this examination nor have I concealed any violation of the Honor Code.

(SIGNED) Solutions
1. Consider a portfolio of 10-year deferred whole life annuities of $1 per year payable continuously. All policies have level benefit premiums payable continuously during the deferral period. The composition of the portfolio on January 1, 2008 is following:

<table>
<thead>
<tr>
<th>Age at Issue</th>
<th>Issue Date</th>
<th>Face Amount</th>
<th>Number of Policies</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>January 1, 2003</td>
<td>1</td>
<td>60</td>
</tr>
<tr>
<td>50</td>
<td>January 1, 1998</td>
<td>4</td>
<td>40</td>
</tr>
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The force of mortality is $\mu(x) = 0.03$ for $x \geq 0$ and the force of interest rate is $\delta = 0.05$.

(a) (10 pts.) Calculate $\bar{P}_{(10|\bar{a}_{40})}$.

Solution.

\[
\begin{align*}
\bar{a}_{40} &= \int_0^\infty e^{-0.05t} \cdot e^{-0.03t} dt = \frac{1}{0.08} e^{-0.8} \\
\bar{a}_{40:10} &= \int_0^{10} e^{-0.05t} \cdot e^{-0.03t} dt = \frac{1}{0.08} (1 - e^{-0.8}) \\
\therefore \bar{P}_{(10|\bar{a}_{40})} &= \frac{10 \bar{a}_{40}}{\bar{a}_{40:10}} = 0.81597
\end{align*}
\]

(b) (12 pts.) Calculate the benefit reserve on January 1, 2007 for one such annuity issued on (50).

Solution.

\[
\begin{align*}
4 \cdot g \bar{V}_{(10|\bar{a}_{50})} &= 4 \left( E_{50} \cdot a_{60} - \bar{P}_{(10|\bar{a}_{50})} \cdot \bar{a}_{50:10} \right) \\
&= 4 \left( e^{-0.08} \cdot \frac{1}{0.08} - 0.81597 \cdot \frac{1}{0.08} (1 - e^{-0.08}) \right) \\
&= 43.01909
\end{align*}
\]

or

\[
\begin{align*}
4 \cdot g \bar{V}_{(10|\bar{a}_{50})} &= 4 \cdot \frac{1}{9} E_{50} \cdot \bar{P}_{(10|\bar{a}_{50})} \cdot \bar{a}_{50:10} \\
&= 4e^{0.08-9} \cdot 0.81597 \cdot \frac{1}{0.08} (1 - e^{-0.08-9}) = 43.01909
\end{align*}
\]
1. Consider a portfolio of 10-year deferred whole life annuities of $1 per year payable continuously. All policies have level benefit premiums payable continuously during the deferral period. The composition of the portfolio on January 1, 2008 is following:

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The force of mortality is $\mu(x) = 0.03$ for $x \geq 0$ and the force of interest rate is $\delta = 0.05$.

(c) (10 pts.) Calculate the aggregate benefit reserve on January 1, 2008.

Solution.

$$5\bar{V}_{(10)}(\bar{a}_{40}) = \frac{1}{5}E_{40} \cdot P_{(10)}(\bar{a}_{40}) \cdot \bar{a}_{40.\infty}$$

$$= e^{0.08 \cdot 5} \cdot 0.81597 \cdot \frac{1}{0.08} (1 - e^{-0.08 \cdot 5})$$

$$= 5.10640$$

$$10\bar{V}_{(10)}(\bar{a}_{50}) = \bar{a}_{60} = \frac{1}{0.08}$$

$$\therefore 60 \cdot 5\bar{V}_{(10)}(\bar{a}_{40}) + 40 \cdot 4 \cdot 10\bar{V}_{(10)}(\bar{a}_{50}) = 2300.984$$

(d) (10 pts.) Given one annuitant, age 50 at issue, dies on January 1, 2009, calculate the insurer’s loss at issue on this annuity.

Solution.

$$0L = 4 \left( v^{10} \cdot \bar{a}_{\infty} - \bar{P}_{(10)}(\bar{a}_{50}) \cdot \bar{a}_{\infty} \right)$$

$$= 4 \left( e^{-0.5} \left( \frac{1 - e^{-0.05}}{0.05} \right) - 0.81597 \left( \frac{1 - e^{-0.05 \cdot 10}}{0.05} \right) \right)$$

$$= -23.31814$$
2. A whole life insurance policy payable at the end of the year of death is issued to a select life, age 41. The death benefit is $100,000 and the level annual premium is $900. The table below is an excerpt from the 2-year select-and-ultimate table used. The interest rate is $i = 0.08$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$l_x$</th>
<th>$l_{x+1}$</th>
<th>$l_{x+2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>93,738</td>
<td>93,477</td>
<td>93,193</td>
</tr>
<tr>
<td>41</td>
<td>93,383</td>
<td>93,100</td>
<td>92,794</td>
</tr>
<tr>
<td>42</td>
<td>93,007</td>
<td>92,701</td>
<td>92,371</td>
</tr>
<tr>
<td>43</td>
<td>92,608</td>
<td>92,279</td>
<td>91,923</td>
</tr>
<tr>
<td>44</td>
<td>92,186</td>
<td>91,831</td>
<td>91,448</td>
</tr>
</tbody>
</table>

(a) (12 pts.) Calculate the terminal benefit reserve for year 3.

Solution.

\[
1V = \frac{900 \cdot 1.08 - 100000 \cdot q^{41}}{p^{41}} = 670.98040
\]

\[
2V = \frac{(1V + 900) \cdot 1.08 - 100000 \cdot q^{41+1}}{p^{41+1}} = 1372.49108
\]

\[
3V = \frac{(2V + 900) \cdot 1.08 - 100000 \cdot q^{43}}{p^{43}} = 2007.59352
\]

where the probabilities can be obtained by

\[p^{41} = \frac{93100}{93383} = 0.99696\]

\[p^{41+1} = \frac{92794}{93100} = 0.99671\]

\[p^{43} = \frac{92371}{92794} = 0.99544\]

(b) (10 pts.) Calculate the initial benefit reserve for year 5.

Solution.

\[
4V = \frac{(3V + 900) \cdot 1.08 - 100000 \cdot q^{44}}{p^{44}} = 2668.14079
\]

where \[p^{44} = \frac{91923}{92371} = 0.99514\].

\[\therefore 4V + 900 = 3568.14079\]
3. For a special fully discrete whole life insurance of $1,000 issued on the life of (75), increasing premiums $\pi_k$ are payable at time $k$ for $k = 0, 1, 2, \cdots$. The interest rate is $i = 0.05$. The annual premiums are given by $\pi_k = \pi_0(1 + r)^k$, where the rate is $r = 0.05$. Mortality follows de Moivres law with $\omega = 105$. Premiums are calculated in accordance with the equivalence principle.

(a) (12 pts.) Determine $\pi_0$.

Solution.

\[ 0V = 1000A_{75} - \pi_0 \sum_{k=0}^{29} (1 + r)^k \cdot v^k \cdot kP_{75} \]
\[ = 1000A_{75} - \pi_0(1 + e_{75}) = 0 \]

where

\[ A_{75} = \sum_{k=0}^{29} v^{k+1} \cdot k|q_{75} = \frac{v(1 - v^{30})}{1 - v} \cdot \frac{1}{30} = \frac{512.41502}{1000} \]

\[ 1 + e_{75} = \sum_{k=0}^{29} kP_{75} = \sum_{k=0}^{29} \left(1 - \frac{k}{30}\right) = \frac{30 + 29 + \cdots + 1}{30} = 15.5 \]

\[ \therefore \pi_0 = \frac{1000A_{75}}{1 + e_{75}} = 33.05903 \]

(b) (12 pts.) Calculate the benefit reserve for this insurance at the end of year 5.

Solution.

\[ A_{80} = \sum_{k=0}^{24} v^{k+1} \cdot k|q_{80} = \frac{v(1 - v^{25})}{1 - v} \cdot \frac{1}{25} = \frac{563.75778}{1000} \]

\[ 1 + e_{80} = \sum_{k=0}^{24} kP_{80} = \sum_{k=0}^{24} \left(1 - \frac{k}{25}\right) = \frac{25 + 24 + \cdots + 1}{25} = 13 \]

\[ \therefore 5V = 1000A_{80} - \pi_0(1.05)^5(1 + e_{80}) = 15.25351 \]
3. For a special fully discrete whole life insurance of $1,000 issued on the life of (75), increasing premiums $\pi_k$ are payable at time $k$ for $k = 0, 1, 2, \ldots$. The interest rate is $i = 0.05$. The annual premiums are given by $\pi_k = \pi_0 (1 + r)^k$, where the rate is $r = 0.05$. Mortality follows de Moivre's law with $\omega = 105$. Premiums are calculated in accordance with the equivalence principle.

(c) (12 pts.) Calculate the variance of the loss at the end of year 27.

**Solution.**

\[ 27L = \begin{cases} 
1000v - \pi_0(1.05)^{27} = 828.95, & K(75) = 27 \\
1000v^2 - \pi_0(1.05)^{27}(1 + 1) = 660.18, & K(75) = 28 \\
1000v^3 - \pi_0(1.05)^{27}(1 + 1 + 1) = 493.56, & K(75) = 29 
\end{cases} \]

\[
E[27L|K(75) \geq 27] = \frac{1}{3}(828.95 + 660.18 + 493.56) \\
= 660.90
\]

\[
E[27L^2|K(75) \geq 27] = \frac{1}{3}(828.95^2 + 660.18^2 + 493.56^2) \\
= 455537.62
\]

\[
\therefore \text{Var}[27L|K(75) \geq 27] = E[27L^2|K(75) \geq 27] - E[27L|K(75) \geq 27]^2 \\
= 18748.25
\]