FILLING THE GAP BETWEEN AMERICAN AND RUSSIAN OPTIONS: ADJUSTABLE REGRET

SAVAS DAYANIK AND MICHAEL LUDKOVSKI

1. Introduction

Let X be a (geometric) Brownian motion whose initial state is x, and $r \geq 0$ be a constant discount rate. We study the optimal multiple-stopping problem

$$\sup_{\tau_1,\tau_2,\ldots,\tau_n} \mathbb{E}_x \left[e^{-r\tau_n} \left(\max_{1\leq i\leq n} X_{\tau_i} \right) \right], \tag{1}$$

where the supremum is taken over $n \geq 1$ stopping times $\tau_1, \ldots, \tau_n$ of the process X.

The value of (1) can be thought as the value of a perpetual financial option, which gives its holder n rights to mark the price X of a stock and pays her the (discounted) maximum of those n recorded marks at the final exercise time. This closely resembles the Russian option

$$\sup_{\tau} \mathbb{E}_x \left[e^{-r\tau} \left(\max_{0\leq t\leq \tau} X_t \right) \right], \tag{2}$$

where the supremum is taken over stopping times τ of the process X. The Russian option problem was introduced and solved by Shepp and Shiryaev [6, 7] for a geometric Brownian motion X with drift μ and volatility σ; see also Duffie and Harrison [4] for the related arbitrage pricing problem.

L. Shepp and A. Shiryaev argued that, compared to a standard perpetual American option in a Black-Scholes model, the Russian option can reduce its holder's regret for not having stopped earlier. Indeed, unlike an American option which pays the holder the stock price at the exercise time, the Russian option pays the historical maximum of the stock price at the exercise time since the contract has been entered.

However, they also showed that this option's value equals infinity if the drift μ and discount rate r are the same. In other words, if the stock pays no dividend, then the price to be paid for the Russian option's "little or no regret" feature is unaffordable.

This raises the following interesting questions: is there an option that provides the holder a *range* of comfort/regret levels at affordable prices? Is there a family of options with various levels of regret that can be obtained by a potential holder for cheaper prices?

A multiple-stopping option as in (1) can provide an affirmative answer to both questions. Its value is always finite, and it is always cheaper than the Russian option. It reduces to a standard American option for $n = 1$, and its value increases to that of the Russian option as the number of exercise rights n increases to infinity. Hence, the family of multiple-stopping options spans the range between American and Russian options in terms of price and reduced regret.

Some of the above facts are immediate, and we establish others after solving the problem in (1). We give explicit formula for its value function and describe an explicit optimal multiple-stopping strategy whenever one exists.

We also solve the problem in (1) when $r = 0$ and X is a standard Brownian motion on the unit interval $\mathcal{I} = [0, 1]$ with absorbing boundary points. This is an example of best-choice problems with several rights to choose; see, e.g., Samuels [5].

Decision maker is likely to spare some of her rights for future, hoping that X will reach some day the highest possible value at the right boundary point. However, if she skips every opportunity to mark a new record by X and if X is absorbed eventually at the left boundary, then the remaining rights are not useful any more. We give explicitly both the value function and an optimal multiple-stopping rule in this case, and also in the case that the terminal payoff for a strategy $(\tau_1, \ldots, \tau_n) \in \mathcal{S}^{(n)}$ in (1) is not "$\max_{1\le i\le n} X_{\tau_i}$" but "$\max_{1\le i\le n}(K - X_{\tau_i})^+$" for some constant $K \in [0, 1]$, mimicking a Russian Put.

2. Problem formulation and main results

Let X be a linear regular diffusion with state space $\mathcal{I} \subseteq \mathbb{R}$. Let $\mathcal{S} \equiv \mathcal{S}^{(1)}$ and

$$\mathcal{S}^{(n)} \triangleq \{(\tau_1, \ldots, \tau_n) : \text{each } \tau_i \text{ is a stopping time of } X \text{ and } \tau_1 \le \ldots \le \tau_n\},\ n \ge 1$$

be the collection of all multiple-stopping rules for every fixed number of exercise rights n. Importantly, we allow strategies with multiple simultaneous exercises, $\tau_k = \tau_{k+1}$. Define the discounted optimal multiple-stopping problems

$$V^{(n)}(x, m) \triangleq \sup_{(\tau_1,\ldots,\tau_n)\in\mathcal{S}^{(n)}} \mathbb{E}_x\left[e^{-r\tau_n}\left(m \vee \max_{1\le i\le n} X_{\tau_i}\right)\right], \qquad x, m \in \mathcal{I},\ n \ge 1 \tag{3}$$

where $x \in \mathcal{I}$ is the initial value of the process X, and $m \in \mathcal{I}$ is an initial lower cap on the terminal payoff. Note that the supremum in (1) equals $V^{(n)}(x, 0)$.

A multiple-stopping rule $(\tau, \tau_1, \ldots, \tau_{n-1}) \in \mathcal{S}^{(n)}$ is optimal for $V^{(n)}$ if and only if (i) the rule $(\tau_1, \ldots, \tau_{n-1})$ is optimal for $V^{(n-1)}$, and (ii) the optimal stopping rule τ maximizes the expected discounted future payoff $V^{(n-1)}(X_\tau, m \vee X_\tau)$ obtained by following the optimal rule $(\tau_1, \ldots, \tau_{n-1})$. This application of the dynamic programming principle can be made rigorous as in Carmona and Dayanik [1], Carmona and Touzi [2], and gives the relation

$$V^{(n)}(x, m) = \sup_{\tau\in\mathcal{S}} \mathbb{E}_x\left[e^{-r\tau}\, V^{(n-1)}(X_\tau, m \vee X_\tau)\right], \qquad x, m \in \mathcal{I},\ n \ge 1. \tag{4}$$

We set $V^{(0)}(x, m) = m$ for every $x, m \in \mathcal{I}$. Thus, the optimal multiple-stopping problem can be addressed by sequentially solving a family of optimal stopping problems.

2.1. **Brownian motion.** We begin by stating the following two simple results about a standard Brownian motion B on the interval $\mathcal{I} = [0,1]$, absorbed at the endpoints. For the first problem, define

$$V^{(n)}(x,m) = \sup_{\tau_1,\tau_2,\dots,\tau_n} \mathbb{E}_x[m \vee \max_i B_{\tau_i}]. \tag{5}$$

Proposition 1. *We have*

$$V^{(n)}(x,m) = \begin{cases} m + n(1-m^{1/n})x & x < m^{(n-1)/n} \\ nx - (n-1)x^{n/(n-1)} & x \geqslant m^{(n-1)/n}. \end{cases} \tag{6}$$

The optimal strategy with n initial exercise opportunities is to stop at the first hitting times of the thresholds $x_k^{(n)}(m)$, $k = 1,2,\dots,n$ where

$$x_k^{(n)}(m) = m^{(n-k)/n} \tag{7}$$

As a second example, define

$$V^{(n)}(x,m) = \sup_{\tau_1,\tau_2,\dots,\tau_n} \mathbb{E}_x[m \vee \max_i (K - B_{\tau_i})^+]. \tag{8}$$

Then

Proposition 2. *Let $0 < m < K \leq 1$. Then,*

$$V^{(n)}(x,m) = \begin{cases} (K-x) + (n-1)(1-x)[1 - 2^{-2/(n-1)}(1-x)^{1/(n-1)}] & x < x_n^* \\ m + n(1-x)(1 - 2^{-(n-1)/n}(\alpha_m)^{1/n}) & x \geqslant x_n^* \end{cases} \tag{9}$$

where $\alpha_m = (1 + m - K)$ and the threshold $x_n^* = x_n^*(m)$ satisfies $x_n^* = 1 - 2^{2/n}(\alpha_m)^{(n-1)/n}$.

Suppose that $r > 0$. Let

$$X_t = x + \mu t + \sigma W_t, \qquad t \geq 0,\ x \in \mathcal{I} = \mathbb{R}, \tag{10}$$

be a Brownian motion with drift μ and volatility σ. Denote by $k_{1,2}$ the roots of the quadratic equation $(\sigma^2/2)k^2 + \mu k - r = 0$; namely,

$$-k_1 = -\frac{\mu}{\sigma^2} - \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} < 0 < k_2 = -\frac{\mu}{\sigma^2} + \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}}; \quad \text{and} \quad k = k_2 + k_1.$$

In this case we have

$$\psi(x) = e^{k_2 x}, \qquad \varphi(x) = e^{k_1 x}, \qquad F(x) = \frac{\psi(x)}{\varphi(x)} = e^{kx}, \qquad x \in \mathbb{R}.$$

Accordingly, $g_m^{(1)}(y) = \max(my^{k_1/k}, \frac{1}{k}\log(y) \cdot y^{k_1/k})$. It is easy to check that each of the two pieces of $g_m^{(1)}(y)$ is concave and we have a valley around $y = e^{km}$. This valley is therefore bridged by an affine function $\ell_m^{(1)}(y)$ from some $y_{m,1}^{(1)}$ to $y_{m,2}^{(1)}$ such that $\ell_m^{(1)}(y)$ is tangent to $my^{k_1/k}$ at $y_{m,1}^{(1)}$ and to $\frac{1}{k}\log(y) \cdot y^{k_1/k}$ at $y_{m,2}^{(1)}$. Let z solve the equation

$$(1 - \frac{k}{k_1})z - \left((mk_1)^{-k/k_2} - m^{k/k_2} k k_1^{k_1/k_2}\right) z^{-k_1/k_2} = \frac{k}{k_1}$$

Then $y^{(1)}_{m,2} = e^{\frac{k}{k_1}(z-1)}$ and $y^{(1)}_{m,1} = y^{(1)}_{m,2}\left(\frac{z}{mk_1}\right)^{-k/k_2}$.

Therefore, the concave majorant of $g^{(1)}_m(y)$ is

$$W^{(1)}(y,m) = \begin{cases} my^{k_1/k} & y < y^{(1)}_{m,1} \\ \ell^{(1)}_m(y) & y^{(1)}_{m,1} < y < y^{(1)}_{m,2} \\ \frac{1}{k}\log(y)\cdot y^{k_1/k} & y > y^{(1)}_{m,2} \end{cases} \tag{11}$$

where

$$\ell^{(1)}_m(y) = \frac{\frac{1}{k}\log(y^{(1)}_2)(y^{(1)}_2)^{k_1/k} - m(y^{(1)}_2)^{k_1/k}(\frac{z}{mk_1})^{-k_1/k_2}}{y^{(1)}_2 - y^{(1)}_1}(y - y^{(1)}_1) + m(y^{(1)}_1)^{k_1/k}$$

The next step would be

$$g^{(2)}_m(y) = W^{(1)}(y, \frac{1}{k}\log(y) \vee m) \tag{12}$$

but this is not tractable in the sense of there being no explicit formulas for $g^{(2)}$ or any higher order terms (we verified this with Maple and Mathematica symbolic solvers). However, it is easy to see that at each step $W^{(n)}(y,m)$ has three pieces, namely an initial one of the form $my^{k_1/k}$ for $y \le y^{(n)}_{m,1}$, then an affine segment for $y^{(n)}_{m,1} < y < y^{(n)}_{m,2}$ and then a concave piece on $y \ge y^{(n)}_{m,2}$. In fact, this allows us to obtain some information about the limit as $n \to \infty$.

Proposition 3.

$$V^{(\infty)}(x,m) = \begin{cases} m & x \le \frac{1}{k}\log(\hat{y}) \\ l_m(x) = f(m)e^{k_2 x} + (m\hat{y}^{k_1/k} - f(m)\hat{y})e^{-k_1 x} & \frac{1}{k}\log(\hat{y}) < x < m \\ L(x) & x > m \end{cases} \tag{13}$$

where

$$L(x) = l_x(x) = f(x)e^{k_2 x} + (x\hat{y}^{k_1/k} - f(x)\hat{y})e^{-k_1 x}. \tag{14}$$

and $f(m)$ solves

$$f'(m) = \frac{(\frac{f(m)}{m}\frac{k}{k_1})^{-k_1/k_2}}{(\frac{f(m)}{m}\frac{k}{k_1})^{-k/k_2} - e^{km}} \tag{15}$$

while $\hat{y}(m) = (\frac{f(m)}{m}\frac{k}{k_1})^{-k/k_2}$.

2.2. **Geometric Brownian motion.** Suppose that $r > 0$. Let

$$X_t = x \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t\right\}, \qquad t \geq 0,\ x \in \mathcal{I} = [0, \infty) \tag{16}$$

be a geometric Brownian motion with drift μ and volatility σ. Let us denote by $-k_1 < 0 < 1 \leq k_2$ the roots of the quadratic equation $f(k) \triangleq (\sigma^2/2)k^2 + [\mu - (\sigma^2/2)]k - r = 0$; namely,

$$-k_1, k_2 = -\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right) \mp \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}; \quad \text{and} \quad k \triangleq k_1 + k_2.$$

Proposition 4. *Suppose that $\mu = r$. Define the increasing sequence*

$$a_1 = 1, \qquad a_{n+1} = a_n + \frac{k_1^{k_1}}{k^k} a_n^{-k_1},\ n = 1, 2, \ldots, \tag{17}$$

Then, for every $n \geq 1$ and $m \in \mathbb{R}_+$, we have the following:

(i) *The value function $V^{(n)}(x, m)$ in (3) is given by*

$$V^{(n)}(x, m) = \left\{\begin{array}{ll} m, & 0 \leq x < x_m^{(n)} \triangleq \dfrac{k_1}{k} \cdot \dfrac{m}{a_n} \\ a_n x + \dfrac{k_1^{k_1}}{k^k} a_n^{-k_1} m^k x^{-k_1}, & x \geq x_m^{(n)} \end{array}\right\}, \quad x \in \mathbb{R}_+. \tag{18}$$

(ii) *The sequence $(x_m^{(n)})_{n \geq 1}$ is decreasing, and $x_m^{(n)} < m$.*

(iii) *For the problem in (4), immediate stopping is optimal if $m = 0$, and no optimal stopping time exists if $m > 0$, but for every $\varepsilon > 0$, the first exit time*

$$\tau_m^{(n)}(\varepsilon) \triangleq \inf\{t \geq 0 : X_t \notin (x_m^{(n)}, x_m^{(n)}(\varepsilon))\}$$

of the process X from the open interval $(x_m^{(n)}, x_m^{(n)}(\varepsilon))$ is always ε-optimal, where

$$x_m^{(n)}(\varepsilon) = \max\left\{m, \frac{k_1}{a_n}\left(\frac{m^k}{\varepsilon k^k}\right)^{1/k_1}\right\}.$$

Corollary 1. *Suppose that $\mu = r$. The value of the option with $n \geq 1$ exercise rights equals $V^{(n)}(x, 0) = a_n x$ at every initial stock price $x \in \mathbb{R}_+$. There is no optimal multiple-stopping strategy. However, for every $\varepsilon > 0$, the strategy $(\tau_1(\varepsilon), \ldots, \tau_n(\varepsilon)) \in \mathcal{S}^{(n)}$ is ε-optimal, if*

$$\tau_1(\varepsilon) \equiv 0, \quad \textit{and} \quad \tau_{k+1}(\varepsilon) = \tau_m^{(n-k)}(\varepsilon) \circ \theta_{\tau_k(\varepsilon)}\big|_{m = M_k(\varepsilon)}, \quad k = 1, \ldots, n-1$$

is the first exit time after $\tau_k(\varepsilon)$ of the process X from the interval $(x_m^{(n-k)}, x_m^{(n-k)}(\varepsilon))$ for $m = M_k(\varepsilon)$; here, $M_k(\varepsilon) \triangleq \max_{1 \leq i \leq k} X_{\tau_i(\varepsilon)}$ is the running maximum of $X_{\tau_1(\varepsilon)}, \ldots, X_{\tau_n(\varepsilon)}$, and θ is the shift-operator: $X_t \circ \theta_s = X_{s+t}$ for every $t, s \in \mathbb{R}_+$.

Remark 1. Since the sequence $(a_n)_{n \geq 1}$ of (17) is increasing, its limit as $n \to \infty$ exists and is greater than one. Taking limit as $n \to \infty$ of the recursion's both sides in (17) implies that $\lim_{n \to \infty} a_n = +\infty$. Therefore, for every $m \in \mathbb{R}_+$, we have $\lim_{n \to \infty} x_m^{(n)} = 0$ and $\lim_{n \to \infty} V^{(n)}(x, m) = +\infty$, which is in agreement with the infinite value of the Russian option in the case that $\mu = r$; see Shepp and Shiryaev [7].

Proposition 5. *Suppose that $\mu < r$. Define the increasing sequence*

$$a_1 = 1, \quad a_{n+1} = \frac{k_1}{k}\left[\left(\frac{k_1}{1+k_1}\right)^{1+k_1}\left(\frac{k_2}{k_2-1}\right)^{k_2-1}\right]^{-k_2/k} a_n^{k_2} + \frac{k_2}{k}\left[\left(\frac{k_1}{1+k_1}\right)^{1+k_1}\left(\frac{k_2}{k_2-1}\right)^{k_2-1}\right]^{k_1/k} a_n^{-k_1}, \quad n = 1, 2, \ldots. \tag{19}$$

Then, for every $n \geq 1$ and $m \in \mathbb{R}_+$, we have the following:

(i) *The value function $V^{(n)}(x, m)$ in (3) is given by*

$$V^{(n)}(x,m) = \begin{Bmatrix} m, & 0 \leq x < x^{(n)}_{m,1} \\ \ell^{(n)}_m \triangleq \frac{m}{k}\left[k_1\left(\frac{x}{x^{(n)}_{m,1}}\right)^{k_2} + k_2\left(\frac{x}{x^{(n)}_{m,1}}\right)^{-k_1}\right], & x^{(n)}_{m,1} \leq x < x^{(n)}_{m,2} \\ a_n x, & x \geq x^{(n)}_{m,2} \end{Bmatrix}, \quad x \in \mathbb{R}_+, \tag{20}$$

where

$$x^{(n)}_{m,1} \triangleq \left(\frac{k_1}{1+k_1}\right)^{(1+k_1)/k}\left(\frac{k_2}{k_2-1}\right)^{(k_2-1)/k}\frac{m}{a_n}, \quad x^{(n)}_{m,2} \triangleq \left(\frac{k_1}{1+k_1}\right)^{k_1/k}\left(\frac{k_2}{k_2-1}\right)^{k_2/k}\frac{m}{a_n} = \left[\frac{(1+k_1)k_2}{k_1(k_2-1)}\right]^{1/k} x^{(n)}_{m,1}. \tag{21}$$

(ii) *The sequences $(x^{(n)}_{m,1})_{n\geq 1}$ and $(x^{(n)}_{m,2})_{n\geq 1}$ are decreasing, and $x^{(n)}_{m,1} < m < x^{(n)}_{m,2}$.*

(iii) *The first exit time*

$$\tau^{(n)}_m \triangleq \inf\{t \geq 0 : \; X_t \notin (x^{(n)}_{m,1}, x^{(n)}_{m,2})\}$$

of the process X from the interval $(x^{(n)}_{m,1}, x^{(n)}_{m,2})$ is an optimal stopping time for the problem in (4).

Remark 2. If $\mu = r$, then $k_2 = 1$ and $k = 1 + k_1$. As $\mu \nearrow r$, we have $k_2 \searrow 1$, and the sequence $(a_n)_{n\geq 1}$ in (19) reduces to that in (17). Therefore, the (a_n) in (17) and (19) are the same sequence, whose form is determined implicitly by the relation between r and μ.

Corollary 2. *Suppose that $\mu < r$. The value of the option with $n \geq 1$ exercise rights equals $V^{(n)}(x, 0) = a_n x$ at every initial stock price $x \in \mathbb{R}_+$. If*

$$\tau_1 \equiv 0, \quad \text{and} \quad \tau_{k+1} = \tau^{(n-k)}_m \circ \theta_{\tau_k}\big|_{m=M_k}, \quad k = 1, \ldots, n-1$$

is the first exit time after τ_k of the process X from the interval $(x^{(n-k)}_{m,1}, x^{(n-k)}_{m,2})$ for $m = M_k \triangleq \max_{1\leq i\leq k} X_{\tau_i}$, then the strategy $(\tau_1, \ldots, \tau_n) \in \mathcal{S}^{(n)}$ is optimal for the problem in (3).

Figure 1 illustrates the implementation of the optimal policy with $n \geq 5$ exercise rights along a sample path of the process X. One can make the following observations:

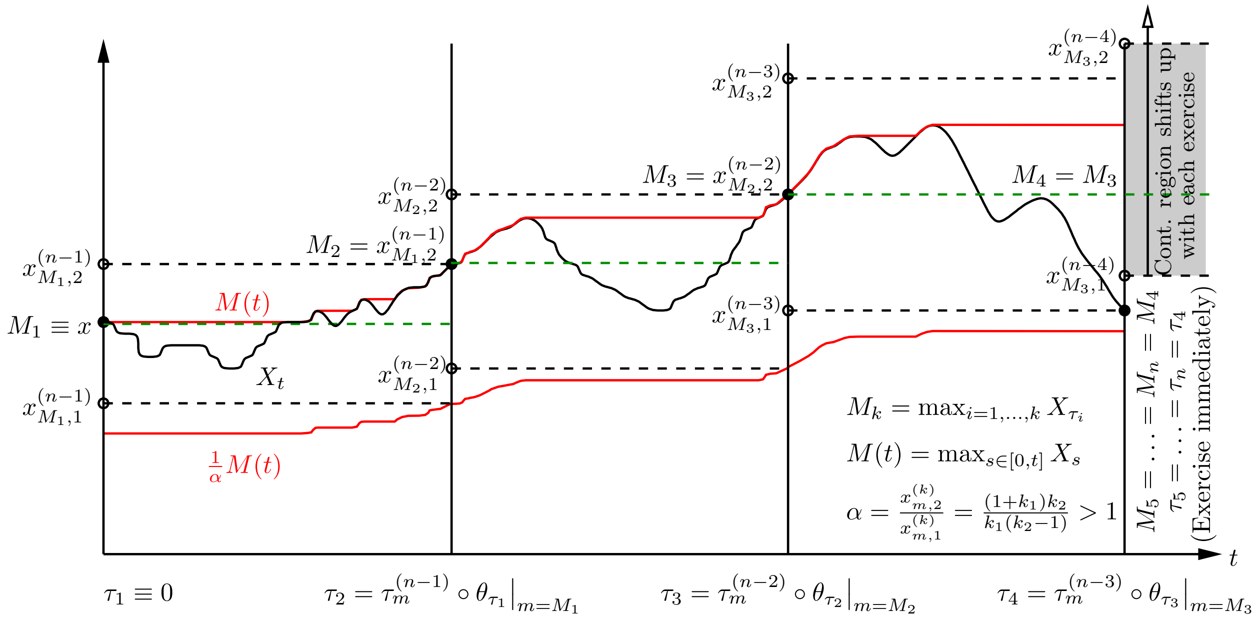

FIGURE 1. Geometric Brownian motion with $\mu < r$. Execution of the optimal policy when $n \geq 5$ along a sample path

(i) Between successive exercises, the continuation region is a bounded interval about the running maximum of $\{X_{\tau_1}, \ldots, X_{\tau_n}\}$.

(ii) Waiting time between two exercises is positive if and only if the process X leaves the current continuation region from its upper boundary. Note that in Figure 1 $\tau_1 < \tau_2 < \tau_3 = \tau_4 = \ldots = \tau_n$. As soon as the process leaves one of the continuation regions from its lower boundary, all of the remaining rights are exercised instantaneously.

Hence, left-boundaries of continuation regions provide protection again deteriorating time-value of the option, while right-boundaries enhance the terminal payoff by marking new records of the process.

(iii) Recall that an optimal exercise rule for the Russian option in (2) is the first time τ_R that the process X reaches to the $(1/\alpha)$th-fraction of its running maximum $M(t) \triangleq \max_{s\in[0,t]} X_t$; see, e.g., Shepp and Shiryaev [7]:

$$\tau_R \triangleq \inf\left\{t \geq 0 : X_t \leq \frac{M(t)}{\alpha}\right\}, \qquad \text{where} \qquad \alpha \triangleq \frac{(1+k_1)k_2}{k_1(k_2-1)} > 1.$$

The number α is also the ratio of upper and lower boundaries $x^{(n)}_{m,2}$ and $x^{(n)}_{m,1}$ in (21) of continuation region of the optimal multiple-stopping problem for every $n \geq 1$ and $m \in \mathbb{R}_+$.

In Figure 1, the lower boundaries $x^{(n-j)}_{M_j,1}$, $j = 1, \ldots, n$ of continuation regions always lay above the exercise boundary $t \mapsto M(t)/\alpha$ of the Russian option. In other words, the continuation region of the multiple optimal-stopping is contained in that of the

Russian option. If $(\tau_1, \ldots, \tau_n)$ is the multiple optimal-stopping strategy described in Corollary 2, then it is easy to show that $\tau_1 \le \cdots \le \tau_n \le \tau_R$.

Proposition 6. *Suppose that $\mu < r$. Then, for every $m \in \mathbb{R}_+$, we have*

$$a \triangleq \lim_{n\to\infty} a_n = \left(\frac{k_1}{1+k_1}\right)^{k_1/k} \left(\frac{k_2}{k_2-1}\right)^{k_2/k},$$

$$x_{m,1} \triangleq \lim_{n\to\infty} x_{m,1}^{(n)} = \left[\frac{k_1(k_2-1)}{(1+k_1)k_2}\right]^{1/k} m < m = \lim_{n\to\infty} x_{m,2}^{(n)},$$

and the limit $V(x,m) \triangleq \lim_{n\to\infty} V^{(n)}(x,m)$ exists and equals

$$V(x,m) = \left\{ \begin{array}{ll} m, & 0 \le x < x_{m,1} \\ \dfrac{m}{k}\left[k_1\left(\dfrac{x}{x_{m,1}}\right)^{k_2} + k_2\left(\dfrac{x}{x_{m,1}}\right)^{-k_1}\right], & x_{m,1} \le x < m \\ ax, & x \ge m \end{array} \right\}, \qquad x \in \mathbb{R}_+. \tag{22}$$

Remark 3. For every $s \in \mathbb{R}_+$, the function $V(x,s)$, $x \in [0,s]$ in (22) coincides with the value function of the Russian option calculated by Shepp and Shiryaev [7, Equation 2.4] (our $-k_1, k_2, x_{m,1}/m$ are their $\gamma_1, \gamma_2, \alpha$, respectively).

Moreover, the identity $V(x,0) = V(x,x)$, $x \in \mathbb{R}_+$ shows that, if the initial stock price is x, then the limiting value function $\lim_{n\to\infty} V^{(n)}(x,0) = V(x,0)$ of multiple-stopping option agrees with the Russian option's value function $V(x,x)$. Hence, as the number of exercise rights increases the regret of the holder for buying finite number of exercise rights instead of a full lookback option reduces to zero.

Finally, since the upper exercise boundary $x_{m,2}^{(n)}$ of the optimal multiple-stopping problem converges as $n \to \infty$ to the "running maximum" m, the Russian option may be thought loosely as a multiple-stopping option with unlimited number of exercise rights, which are used to mark every time the underlying process breaks a record.

3. Method of Solution

The relation (4) lets us calculate the functions $V^{(n)}, n = 1, 2, \ldots$ recursively. After $V^{(n-1)}$ is calculated for some $n = 1, 2, \ldots$, let

$$g_m^{(n)}(x) \triangleq V^{(n-1)}(x, x \vee m), \qquad x, m \in \mathcal{I}.$$

Then (4) becomes a discounted optimal stopping problem with terminal payoff function $g_m^{(n)}$. In order to solve it, let us introduce the functions

$$\psi(x) = \left\{ \begin{array}{ll} \mathbb{E}_x[\exp\{-r\tau_c\}], & x < c \\ 1/\mathbb{E}_c[\exp\{-r\tau_x\}], & x \ge c \end{array} \right\}, \qquad \varphi(x) = \left\{ \begin{array}{ll} 1/\mathbb{E}_c[\exp\{-r\tau_x\}], & x < c \\ \mathbb{E}_x[\exp\{-r\tau_c\}], & x \ge c \end{array} \right\}, \quad \text{and}$$

$$F(x) = \frac{\psi(x)}{\varphi(x)}, \quad x \in \mathcal{I},$$

where c is an arbitrary but fixed point in the interior of the state space $\mathcal{I}$, and the random variable τ_y is the first passage time of X to the level $y \in \mathcal{I}$. The functions $\psi(\cdot)$ and $\varphi(\cdot)$ are increasing and decreasing, respectively, and are the only (up to multiplication by positive constants) monotonic solutions of the differential equation

$$\mathcal{A}u(x) - ru(x) = 0.$$

Here $\mathcal{A}$ is the infinitesimal generator of the process X and coincides on smooth functions with the differential operators

$$\mathcal{A}u(x) = \frac{\sigma^2}{2}u''(x) + \mu u'(x) \quad \text{and} \quad \mathcal{A}u(x) = \frac{\sigma^2}{2}x^2u''(x) + \mu x u'(x)$$

for arithmetic and geometric Brownian motion, respectively. Both of these processes are linear regular diffusions, and the boundaries, denoted by $a < b$, of their state-spaces are natural. Therefore, the results of Dayanik and Karatzas [3, Subsection 5.2] apply, and we summarize here their direct implications for the problem in (4):

Proposition 7. *(i) The function $x \mapsto V^{(n)}(x,m)$ is either identically infinite or finite everywhere. It is finite if and only if the limits*

$$\limsup_{x\downarrow a} \frac{g_m^{(n)} \vee 0}{\varphi} \circ F^{-1}(x) \qquad \textit{and} \qquad \limsup_{x\uparrow b} \frac{g_m^{(n)} \vee 0}{\psi} \circ F^{-1}(x) \quad \textit{are finite.}$$

(ii) If the function $x \mapsto V^{(n)}(x,m)$ is finite, then $V^{(n)}(x,m) = \varphi(x)\cdot W_m^{(n)}(F(x))$ for every $x, m \in \mathcal{I}$, where $W_m^{(n)}$ is the smallest nonnegative concave majorant of the function

$$G_m^{(n)}(y) \triangleq \left(\frac{g_m^{(n)}}{\varphi}\right) \circ F^{-1}(y), \qquad y \in F(\mathcal{I}).$$

(iii) If an optimal stopping time exists for (4), then the first exit time of the process X from the continuation region $C_m = \{x \in \mathcal{I} : V^{(n)}(x,m) > g_m^{(n)}(x)\}$ is also optimal.

(iv) An optimal stopping time exists if the limits in part (i) are zero. If one of the limits is positive, then an optimal stopping time exists if and only if the associated boundary point is a limit point of the stopping region $\mathcal{I} \setminus C_m$.

Since $V^{(n)}(F^{-1}(y),m)/\varphi(F^{-1}(y)) = W_m^{(n)}(y)$ by Proposition 7(ii), we have

$$G_m^{(n+1)}(y) = \begin{cases} W_m^{(n)}(y), & 0 \le y < F(m) \\ W_{F^{-1}(y)}^{(n)}(y), & y \ge F(m) \end{cases} \tag{23}$$

Consequently, we can replace (4) in the original $V(x)$-space with (23) in the transformed $W(y)$-space. This idea forms the basis of our explicit calculations for all the special cases below.

3.1. **Case Study: Proposition 1.** As a case study we analyze the problem (5). This is related to the classical problem of finding the Russian option value

$$V^{(n)}(x,0) = \sup_{\tau_1,\tau_2,\dots,\tau_n} \mathbb{E}_x[\max_i B_{\tau_i}], \tag{24}$$

where (B_t) is a Brownian motion on the interval $\mathcal{I} = [0,1]$, killed at the endpoints. First observe that the value function is trivially bounded $0 \leqslant V^{(n)}(x,0) \leqslant 1$. Moreover, with probability 1, (B_t) will eventually be absorbed, so the horizon is finite a.s.

In this setting we do not to make any transformations (see original paper of Dynkin), and the value function of any optimal stopping problem is simply the concave majorant of the reward function. As in (4) we introduce the standard optimal stopping problems:

$$V^{(n)}(x,m) = \sup_{\tau} \mathbb{E}_x[V^{(n-1)}(B_\tau, B_\tau \vee m)]. \tag{25}$$

Fixing m and letting $g_m^{(n)}(x) = V^{(n-1)}(x, x \vee m)$ it follows that $V^{(n)}(x,m)$ is the smallest concave majorant on $[0,1]$ of $g_m^{(n)}(x)$. This fact together with direct geometric reasoning allows us to give a complete description of $V^{(n)}(x,m)$ as in Proposition 1.

To give the reader some intuition, let us carry out explicitly the first couple of steps. First, if no stopping times are left then trivially $V^{(0)}(x,m) = m$. Hence $g_m^{(1)}(x) = x \vee m$ and it is easy to see that the concave majorant of $g_m^{(1)}(x)$ is $m + (1-m)x = V^{(1)}(x,m)$. Also since $V^1(x,m) > g_m^{(1)}(x)$ for all x it follows that the stopping region is empty. This shows that given one exercise, we always wait until hitting (and being killed upon) 1. Next,

$$g_m^{(2)}(x) = \begin{cases} m + (1-m)x & x < m \\ x + (1-x)x & x \geqslant m \end{cases} \tag{26}$$

Hence, $g^{(2)}(x)$ is linear for small x, and then concave for large x. Note that $\partial_x g_m^{(2)}(x)|_{x=m-} = 1-m$ while $\partial_x g_m^{(2)}(x)|_{x=m+} = 2(1-m) > (1-m)$. Therefore, we have a 'corner' at $x = m$. As Figure 2 illustrates, the concave majorant of $g_m^{(2)}(x)$ is given by

$$V^{(2)}(x,m) = \begin{cases} m + 2(1-\sqrt{m})x & x < \sqrt{m} \\ 2x - x^2 & x \geqslant \sqrt{m} \end{cases}$$

Moreover, we see that the continuation region is $[0, x_2^*)$ where the threshold $x_2^*(m) = \sqrt{m}$ solves the slope-matching equation $(g_m^{(2)}(x) - m)/x = (g_m^{(2)})'(x)$. Continuing in the same manner,

$$g_m^{(3)}(x) = V^{(2)}(x, x \vee m) = \begin{cases} m + 2(1-\sqrt{m})x & x < m, \\ 3x - 2x^{3/2} & x \geqslant m. \end{cases}$$

and its concave majorant is

$$V^{(3)}(x,m) = \begin{cases} m + 3(1-\sqrt[3]{m})x & x < m^{2/3} \\ 3x - 2x^{3/2} & x \geqslant m^{2/3} \end{cases}$$

The new continuation region is $[0, m^{2/3})$. Proceeding in this way one may now guess the result of Proposition 1 and in fact (6) follows easily from the preceding arguments by induction. We have already confirmed it for $n = 1, 2, 3$. Assuming (6) is true for $n' = n$, we obtain (note $m < m^{(n-1)/n}$ so we are always in the first case of (6) when computing g_{n+1})

$$g_m^{(n+1)}(x) = V^{(n)}(x, x \vee m) = \begin{cases} m + n(1 - m^{1/n})x & x < m \\ (n+1)x - nx^{(n+1)/n} & x \geqslant m \end{cases}$$

Again, the concave majorant of $g_m^{(n+1)}(x)$ consists of a straight line from $(0, m)$ to $(x^*_{n+1}(m), (n+1)x - n(x^*_{n+1})^{(n+1)/n}$ where the derivative of $g_m^{(n+1)}(x^*)$ must match the slope of the line. Therefore,

$$(n+1) - (n+1)x^{1/n} = \frac{(n+1)x - nx^{(n+1)/n} - m}{x}$$

or $x^*_{n+1}(m) = m^{1/(n+1)}$ as claimed and $V^{(n+1)}(x, m)$ is a line on $[0, x^*)$ and $g_m^{(n+1)}(x)$ on $[x^*, 1]$. Plugging this in we exactly recover (6) for the case $n' = n+1$, QED. The continuation region is connected because $g_m^{(n+1)}(x)$ is concave on $[x^*, 1] \subset [m, 1]$.

In particular, the following is the optimal policy to follow for the original problem $V^{(n)}(x, 0)$. Starting at x, stop immediately. This resets $m = x$ so that from now on we only need to look at the thresholds of $V^{(k)}(x, x)$. Stop when first hitting $x^{(n-2)/(n-1)}$. Next stop when first hitting $x^{(n-3)/(n-1)}$, and so on, stopping each time a level $x^{(n-k)/(n-1)}$, $k = 2, 3, \ldots, n-2$ is reached. The last stopping time will be at $x^{0/(n-1)} = 1$. For example, starting at $x = 1/4$ and having 5 stopping opportunities, we will stop at first passage time of $1/4, (1/4)^{3/4} = 0.3536, (1/4)^{2/4} = 0.5, (1/4)^{1/4} = 0.707$ and 1. Figure 1b illustrates $V^{(4)}(x, 1/4)$ which comes up as part of obtaining $V^{(5)}(x, 0)$ above.

As implied by (6), $V^{(n)}(x, 0) = nx - (n-1)x^{n/(n-1)} = x + (n-1)x(1 - x^{1/(n-1)})$. Observe that as $n \to \infty$, $\lim_{n\to\infty} V^{(n)}(x, 0) = x - x \cdot \log x$. The next proposition shows that this is in fact the expected value of the maximum of B_t:

Lemma 1. *$\mathbb{E}_x[\max_t B_t] = x(1 - \log x)$.*

Proof. Observing that $\max_t B_t > a \;\Leftrightarrow \tau_a < \tau_0$ where $\tau_a = \inf_s B_s = a$ is the hitting time of a level a (because B_t exits the interval $[0, 1]$ with probability 1 in finite time, $\tau_a, \tau_0 < \infty$ almost surely) we compute

$$\begin{aligned} \mathbb{E}_x[\max_t B_t] &= \int_0^\infty P_x(\max_t B_t > a)da \\ &= x + \int_x^{(1)} P_x(\tau_a < \tau_0)da \\ &= x + \int_x^{(1)} \frac{x}{a} da = x - x \log x. \end{aligned}$$

Figure 2 shows graphically the convergence of $V^{(n)}(x, 0)$ to $x(1 - \log x)$. ∎

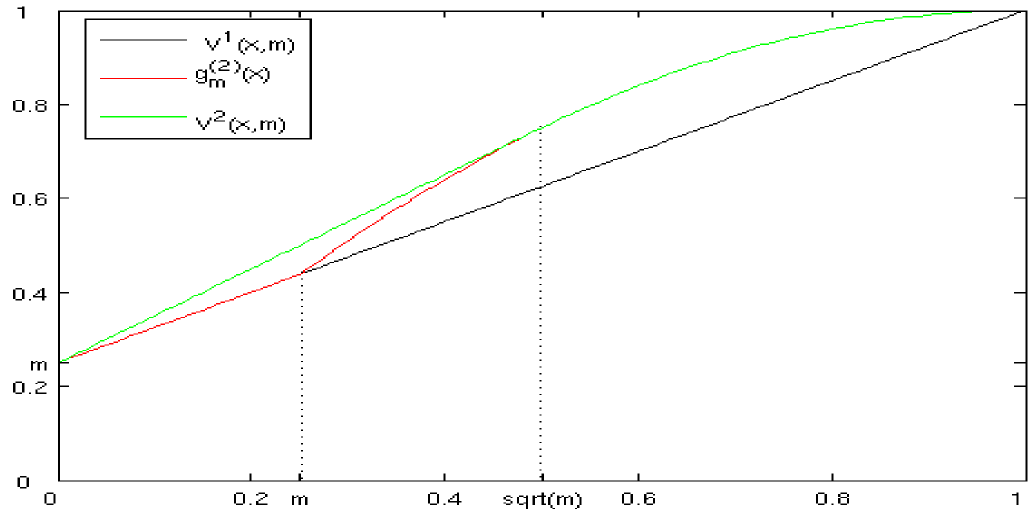

FIGURE 2. Plot of $V^{(1)}(x,1/4)$, $g^{(2)}_{1/4}(x)$ and $V^{(2)}(x,1/4)$ for the standard killed Brownian motion on $[0,1]$.

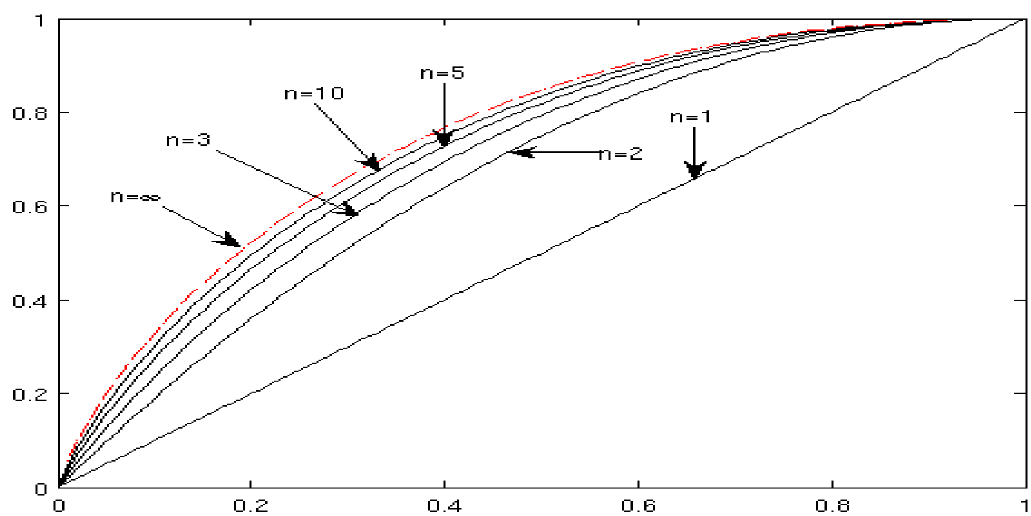

FIGURE 3. Convergence of $V^{(n)}$ to $V^{(\infty)}$ for the standard Brownian motion on $[0,1]$.

3.1.1. *A Second Example.* A second example we study is $\sup_{\tau_1,\ldots,\tau_n} \mathbb{E}_x[\max_i(K-B_{\tau_i})_+]$ where (B_t) is again a standard Brownian motion on $[0,1]$ killed at the endpoints. Carrying out steps analogous to the first example above we find (note that now $0 \le m \le K$)

$$V^{(1)}(x,m) = K - (K-m)x,$$

then $V^{(2)}(x,m)$ is the smallest concave majorant of $V^{(1)}(x,(K-x)_+ \vee m)$, i.e.

$$V^{(2)}(x,m) = \begin{cases} K - x^2 x < 1 - \sqrt{1+m-K} \\ m + 2(1-x)(1-\sqrt{1+m-K}) x \geqslant 1 - \sqrt{1+m-K} \end{cases}$$

Again, Proposition 2 now easily follows by induction. Observe that $V^{(n)}(x,m)$ is linear on $[x_n^*, 1]$ and concave on $[0, x_n^*]$. The limit is

$$\mathbb{E}_x[\sup_\tau (K - B_\tau)_+] = \begin{cases} (x-1)\log(1-K) & x \geq K \\ (K-x) + (x-1)\log(1-x) & x < K \end{cases}$$

3.2. **Value of Russian Option as Fixed Point.** As proven above, in the limit $n \to \infty$, the multiple-stop value function $V^{(n)}(x,0)$ converges to the value of the Russian option $V(x)$. This provides an alternative method of obtaining $V(x)$ as a fixed point of the iteration performed in (4). Indeed, it follows that $V^{(\infty)}(x,m)$ is the concave majorant of $g_m^{(\infty)}(x)$ for any $x, m \in \mathcal{I}$. In turn, this leads to an ordinary differential equation (ODE) that must be satisfied by $V^{(\infty)}(x,m)$ from the slope-matching conditions.

To illustrate, consider the first case problem of maximizing (24). From Figure 2 it is easy to see that $V^{(\infty)}(x,m)$ must consist of a linear segment between $[0, \hat{x}(m)]$ and the curve $V(x) \equiv V^{(\infty)}(x,0)$ on $[\hat{x}(m), 1]$. Moreover, since with infinite amount of exercise rights we exercise whenever a new record is achieved, we must have $\hat{x}(m) = m$. The slope-matching condition at $\hat{x}$ now implies that

$$(V(m) - m)/m = V'(m). \tag{27}$$

Moreover, we have the boundary condition $V(1) = 1$. Solving the simple first order ODE of (27) we immediately get $V(x) = x(1 - \log(x))$ as already shown before.

Applying the same method to the second case study, we obtain $\hat{x}(m) = K - m$ and $V^{(\infty)}(x,m)$ is $V(x)$ on $[0, \hat{x}(m)]$ and linear on $[\hat{x}(m), 1]$. $V(x)$ itself is linear on $[K, 1]$, since on the latter interval the reward is identically zero. The slope-matching at $\hat{x}$ reduces to the first-order ODE

$$\frac{V(K-m) - m}{(K-m) - 1} = V'(K-m) \tag{28}$$

with the boundary condition $V(0) = K$. Solving we directly obtain

$$V(x) = \begin{cases} (K-x) + (x-1)\log(1-x) & x < K \\ (x-1)\log(1-K) & x \geq K \end{cases} \tag{29}$$

that can be verified to equal $V(x) = \mathbb{E}_x[(K - \min_t B_t)^+]$.

In general, let $H(y) = \frac{1}{\phi(F^{-1}(y))}$. Then in the canonical situation where the stopping region is determined by an upper and lower boundary, we have

$$W^{(\infty)}(y,m) = \begin{cases} mH(y) & y < \hat{y} \\ l_m(y) = f(m)y + mH(\hat{y}) - f(m)\hat{y} & \hat{y} < y < F(m) \\ L_m(y) & y > F(m) \end{cases} \tag{30}$$

where $L_m(y) = f(F^{-1}(y))y + F^{-1}(y)H(\hat{y}) - f(F^{-1}(y))\hat{y}$. Namely, for x very small, $V(x,m) = m$ which implies $W(y,m) = mH(y)$. In the continuation region, W is affine, while for large x we again stop immediately and the reward is $W(x,x)$. The lower boundary $\hat{y}$ is computed below, while the upper boundary is naturally taken to be $F(m)$ since with unlimited number of exercises we stop immediately once a new record is set, i.e. as soon as we enter $[F(m),\infty)$.

Since $W^{(\infty)}$ is a fixed-point of the iterations, the slope at the upper threshold $F(m)$ must equal the slope at the slope of the lower threshold $\hat{y}$, as well as the slope of the affine segment. It follows that the only unknowns above is the function $f(\cdot)$ which gives the slope of the linear portion of $W^{(\infty)}(y,m)$ as a function of m and the threshold $\hat{y}(m)$ which indicates where this segment begins. These slope-matching equations give

$$\frac{d}{dy}L_m(y)\Big|_{y=F(m)} = f(m) \qquad \text{and} \tag{31}$$

$$f(m) = m\frac{d}{dy}H(y)\Big|_{y=\hat{y}} \tag{32}$$

The second equation can be solved to find the lower threshold $\hat{y}(m)$, while the other equation can be simplified to obtain

$$f'(m)(\hat{y} - F(m)) = H(\hat{y}). \tag{33}$$

This implies the result of Proposition 3 after plugging-in the given $\varphi(x), F(x)$ in that case.

4. Proofs of Propositions 4, 5, and 6

4.1. **Proof of Proposition 4.** We shall start by proving (i). By an induction on n, we will establish simultaneously the identities (18) and

$$W_m^{(n)}(y) = \begin{cases} my^{k_1/k}, & 0 \le y \le y_m^{(n)} \triangleq \left(\frac{k_1}{k}\cdot\frac{m}{a_n}\right)^k < m^k, \\ L_m^{(n)}(y) \triangleq a_n y + \frac{k_1^{k_1}}{k^k}a_n^{-k_1}m^k, & y > y_m^{(n)}. \end{cases} \tag{34}$$

For $n=1$, we have $g_m^{(1)}(x) = x \vee m$, and

$$\limsup_{x\downarrow 0}\frac{g_m^{(1)}(x)}{\varphi(x)} = \lim_{x\downarrow 0}\frac{m}{x^{-k_1}} = 0, \quad \limsup_{x\uparrow\infty}\frac{g_m^{(1)}(x)}{\psi(x)} = \lim_{x\uparrow\infty}\frac{x}{x} = 1.$$

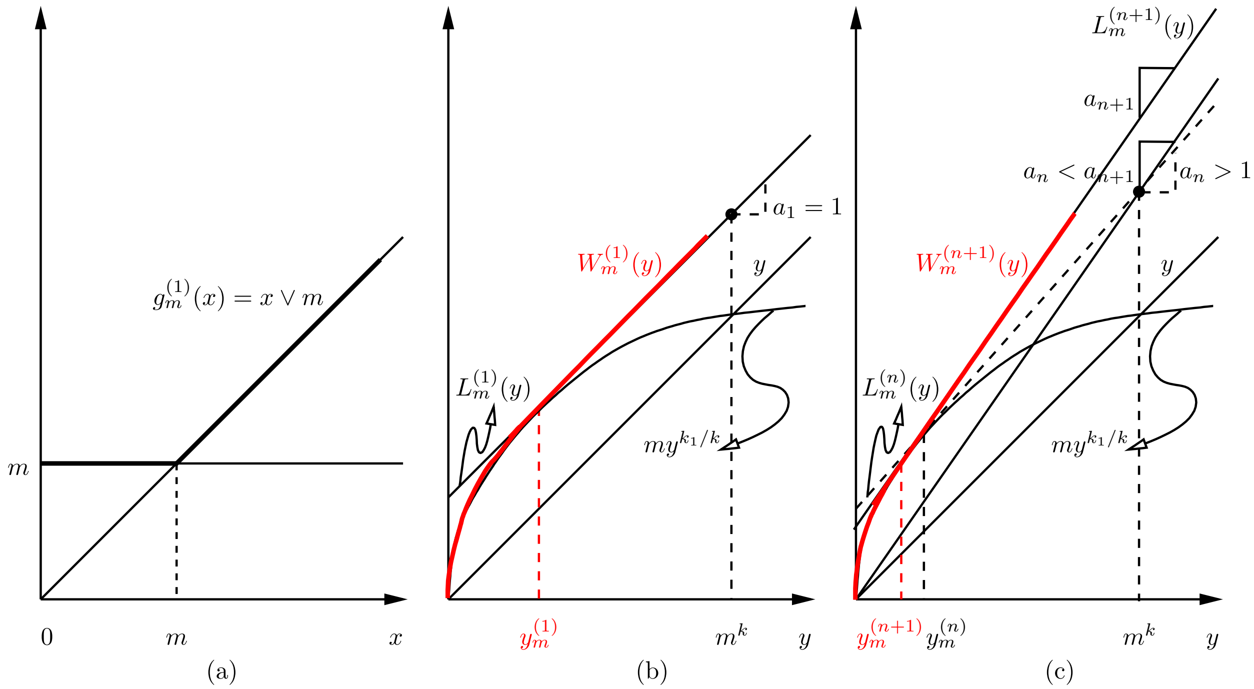

FIGURE 4. Illustrations for the proof of Proposition 4. In (b) and (c), $n \geq 1$, and $L_m^{(n)}(y)$ is the straight line, which is parallel to $y \mapsto a_n y$ and tangent to $y \mapsto my^{k_n/k}$ at $y = y_m^{(n)}$. This line intersects with $y \mapsto a_{n+1}y$ at $y = m^k$. In (b), $W_m^{(1)}(y)$ is the smallest nonnegative concave majorant of $G_m^{(1)}(y)$, which coincides on $[0, m^k]$ with $my^{k_1/k}$ and on $[m^k, \infty)$ with y. In (c), $W_m^{(n+1)}(y)$ is the same majorant of $G_m^{(n+1)}(y)$, which coincides on $[0, y_m^{(n)}]$ with $my^{k_1/k}$, on $[y_m^{(n)}, m^k]$ with $L_m^{(n)}(y)$, and on $[m^k, \infty)$ with $a_{n+1}y$.

By Proposition 7(i), the function $V^{(1)}(\cdot, m)$ is finite. The smallest nonnegative concave majorant $W_m^{(1)}(y)$ of the function

$$G_m^{(1)}(y) = \frac{g_m^{(1)}}{\varphi} \circ F^{-1}(y) = \begin{cases} my^{k_1/k}, & 0 \leq y \leq m^k, \\ y, & y > m^k \end{cases}$$

coincides on $[0, y_m^{(1)}]$ with $ym^{k_1/k}$, and on $[y_m^{(1)}, \infty)$ with the affine function $L_m^{(1)}(y)$ which has slope one and is tangent to $y \mapsto my^{k_1/k}$ at $y = y_m^{(1)}$; see Figure 4(a,b). The equations

$$1 = \frac{d}{dy}\left(my^{k_1/k}\right)\Big|_{y=y_m^{(1)}}, \quad L_m^{(1)}(y_m^{(1)}) = my^{k_1/k}\Big|_{y=y_m^{(1)}}$$

imply that

$$W_m^{(1)}(y) = \begin{cases} my^{k_1/k}, & 0 \leq y < y_m^{(1)} \equiv \left(\dfrac{k_1}{k} \cdot m\right)^k < m^k, \\[2ex] L_m^{(1)}(y) = y + \dfrac{k_1^{k_1}}{k^k} m^k, & y \geq y_m^{(1)}. \end{cases}$$

Finally, if we define $x^{(1)} = F^{-1}(y_m^{(1)}) = (k_1/k)m < m$ and $\ell_m^{(1)}(x) = \varphi(x)L_m^{(1)}(F(x))$, then Proposition 7 (ii) and (iv) imply that

$$V^{(1)}(x,m) = \varphi(x)W_m^{(1)}(F(x)) = \begin{cases} m, & 0 \le x < x_m^{(1)} = \dfrac{k_1}{k}m < m, \\ \ell_m^{(1)}(x) = x + \dfrac{k_1^{k_1}}{k^k}m^k x^{-k_1}, & x \ge x_m^{(1)}. \end{cases}$$

Hence, both (18) and (34) hold for $n = 1$ since $a_1 = 1$ by definition. Note also that, since the second limit in Proposition 7(i) is positive, and the right boundary $+\infty$ is not a limit point of the "stopping region" $\{x \in \mathbb{R}_+ : V^{(1)}(x,m) = g_m^{(1)}(x)\}$ unless $m = 0$, there is no optimal stopping time by Proposition 7(iv) if and only if $m > 0$.

Suppose now that (18) and (34) hold for some $n \ge 1$. Let us show that they are also correct for $n+1$. Recall that

$$V^{(n+1)}(x,m) = \sup_{\tau \in \mathcal{S}} \mathbb{E}_x\left[e^{-r\tau} g_m^{(n+1)}(X_\tau)\right], \qquad x, m \in \mathbb{R}_+,$$

where $g_m^{(n+1)}(x) \triangleq V^{(n)}(x, m \vee x)$ equals

$$g_m^{(n+1)}(x) = \left\{ \begin{array}{ll} V^{(n)}(x,m), & 0 \le x < m \\ \left[a_n + \dfrac{k_1^{k_1}}{k^k}a_n^{-k_1}\right]x, & x \ge m \end{array} \right\} = \left\{ \begin{array}{ll} V^{(n)}(x,m), & 0 \le x < m \\ a_{n+1}x, & x \ge m \end{array} \right\}$$

by induction hypothesis. Indeed, for every $x \in \mathbb{R}_+$, we can calculate $V^{(n)}(x,x) = V^{(n)}(m,m)|_{m=x}$ from (18), and the second equality follows from the definition of a_{n+1} in (17). Note that

$$\limsup_{x \downarrow 0} \frac{g_m^{(n+1)}(x)}{\varphi(x)} = \lim_{x\downarrow 0} \frac{m}{x^{-k_1}} = 0, \qquad \limsup_{x\uparrow\infty} \frac{g_m^{(n+1)}(x)}{\varphi(x)} = \lim_{x\uparrow\infty}\frac{a_{n+1}x}{x} = a_{n+1} > 0.$$

Since a_{n+1} is finite, the function $V^{(n+1)}(x,m)$ is finite by Proposition 7(i). Let $W_m^{(n+1)}(y)$ be the smallest nonnegative concave majorant of $G_m^{(n+1)}(y) \triangleq [g_m^{(n+1)}/\varphi] \circ F^{-1}(y)$, $y \in \mathbb{R}_+$. Since $V^{(n)}(F^{-1}(y),m)/\varphi(F^{-1}(y)) = W_m^{(n)}(y)$ by Proposition 7(ii), we have

$$G_m^{(n+1)}(y) = \left\{ \begin{array}{ll} W_m^{(n)}(y), & 0 \le y < m^k \\ a_{n+1}\dfrac{y^{1/k}}{y^{-k_1/k}}, & y \ge m^k \end{array} \right\} = \left\{ \begin{array}{ll} my^{k_1/k}, & 0 \le y < y_m^{(n)} \\ L_m^{(n)}(y), & y_m^{(n)} \le y < m^k \\ a_{n+1}y, & y \ge m^k \end{array} \right\},$$

where the second equality follows from (34) by induction hypothesis, and $L_m^{(n)}(y) = a_n y + (k_1^{k_1}/k^k)a_n^{-k_1}m^k$.

Let us now find $W_m^{(n+1)}$. It is easy to check that the straight line $y \mapsto L_m^{(n)}(y)$ is tangent to the strictly concave and increasing curve $y \mapsto my^{k_1/k}$ at $y = y_m^{(n)} < m^k$. Moreover, the same line intersects with $y \mapsto a_{n+1}y$ at $y = m^k$; see Figure 4(c). Let $y = y_m^{(n+1)}$ be the point where the derivative of $y \mapsto my^{k_1/k}$ equals a_{n+1}. Since this curve is strictly concave and has infinite right-derivative at $y = 0$, the number $y_m^{(n+1)}$ exists and is unique and positive. Moreover, $y_m^{(n+1)} < y_m^{(n)}$, since the derivative of the same concave curve at $y = y_m^{(n)}$ equals $a_n < a_{n+1}$.

Now it is clear that $W_m^{(n+1)}(y)$ coincides on $[0, y_m^{(n+1)}]$ with $my^{k_1/k}$ and on $[y_m^{(n+1)}, \infty)$ with the straight line $L_m^{(n+1)}(y)$, which has slope a_{n+1} and is tangent to the curve $y \mapsto my^{k_1/k}$ at $y = y_m^{(n+1)}$. The equations

$$a_{n+1} = \frac{d}{dy}\left(my^{k_1/k}\right)\Big|_{y=y_m^{(n+1)}}, \qquad L_m^{(n+1)}(y_m^{(n+1)}) = my^{k_1/k}\Big|_{y=y_m^{(n+1)}}$$

imply that

$$W_m^{(n+1)}(y) = \begin{cases} my^{k_1/k}, & 0 \le y < y_m^{(n+1)} = \left(\dfrac{k_1}{k}\dfrac{m}{a_{n+1}}\right)^k, \\ L_m^{(n+1)}(y) = a_{n+1}y + \dfrac{k_1^{k_1}}{k^k}a_{n+1}^{-k_1}m^k, & y \ge y_m^{(n+1)}. \end{cases}$$

Finally, if we define $x_m^{(n+1)} \triangleq F^{-1}(y_m^{(n+1)})$, then $V^{(n+1)}(x,m) = \varphi(x)W_m^{(n+1)}(F(x))$ gives

$$V^{(n+1)}(x,m) = \begin{cases} m, & 0 \le x < x_m^{(n+1)} = \dfrac{k_1}{k}\cdot\dfrac{m}{a_{n+1}}, \\ a_{n+1}x + \dfrac{k_1^{k_1}}{k^k}m^k x^{-k_1}, & x \ge x_m^{(n+1)} \end{cases}$$

by Proposition 7(ii). Note that $V_m^{(n+1)}$ and $W_m^{(n+1)}$ have the same form as in (18) and (34), and the proof of Proposition 4(i) is complete.

Moreover, the second limit in Proposition 7(i) is positive, and the right boundary is again not a limit point of the "stopping region" $\{x \in \mathbb{R}_+ : V^{(n+1)}(x,m) = g_m^{(n+1)}(x)\}$ unless $m = 0$. By Proposition 7(iv), there is no optimal stopping time if and only if $m > 0$.

Since $(a_n)_{n\ge1}$ is increasing, Proposition 4(ii) is obvious. For the proof of (iii), suppose $m > 0$ and fix $\varepsilon > 0$. Notice that

$$x_m^{(n)}(\varepsilon) = \min\{x \ge m;\ V^{(n)}(x,m) - g_m^{(n)}(x) \le \varepsilon\} = \min\left\{x \ge m;\ \frac{k_1^{k_1}}{k^k}a^{-k_1}m^k x^{-k_1} \le \varepsilon\right\}.$$

The first exit time $\tau_m^{(n)}(\varepsilon)$ of the process X from the open interval $(x_m^{(n)}, x_m^{(n)}(\varepsilon))$ is finite a.s., and the function $x \mapsto V^{(n)}(x,m)$ is r-harmonic on the continuation region $(x_m^{(n)}, \infty) \supset (x_m^{(n)}, x_m^{(n)}(\varepsilon))$. Therefore,

$$\mathbb{E}_x\left[e^{-r\tau_m^{(n)}(\varepsilon)}g_m^{(n)}\left(X_{\tau_m^{(n)}(\varepsilon)}\right)\right] \ge \mathbb{E}_x\left[e^{-r\tau_m^{(n)}(\varepsilon)}\left(V^{(n)}\left(X_{\tau_m^{(n)}(\varepsilon)}, m\right)\right)\right] - \varepsilon = V^{(n)}(x,m) - \varepsilon,$$

and Proposition 4(iii) is proved. □

4.2. **Proof of Proposition 5.** We will prove (i) and (ii) first. By an induction on n, we will establish simultaneously the equations (22) and

$$W_m^{(n)}(y) = \begin{cases} my^{k_1/k}, & 0 \le y < y_{m,1}^{(n)} < m^k, \\ L_m^{(n)}(y) = m\dfrac{k_1}{k}\left[y_{m,1}^{(n)}\right]^{-k_2/k} y + m\dfrac{k_2}{k}\left[y_{m,1}^{(n)}\right]^{k_1/k}, & y_{m,1}^{(n)} \le y < y_{m,2}^{(n)}, \\ a_n y^{(1+k_1)/k}, & y \ge y_{m,2}^{(n)} > m^k, \end{cases} \tag{35}$$

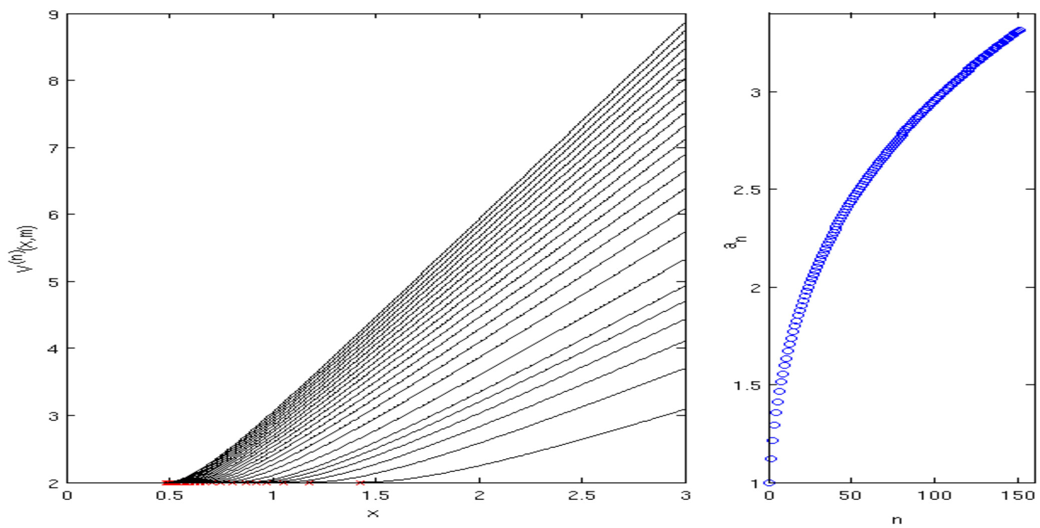

FIGURE 5. Behavior of $V^{(n)}(x,m)$ as a function of n. We take $\sigma = 0.2, \mu = r = 0.04$, $m = 2$. The left panel shows $V^{(n)}(x,m)$ for a range of x around m with $n = 1, 3, 5, \ldots, 11, 15, 20, \ldots, 60$. The thresholds $x_1^{(n)}(m)$ are marked with a red cross and are seen to be decreasing as expected. The right panel shows the sequence (a_n) from (17). As proven the sequence grows without limit.

where

$$y_{m,1}^{(n)} = \left(\frac{k_1}{1+k_1}\right)^{1+k_1} \left(\frac{k_2}{k_2-1}\right)^{k_2-1} \left(\frac{m}{a_n}\right)^k, \quad y_{m,2}^{(n)} = \left(\frac{k_1}{1+k_1}\right)^{k_1} \left(\frac{k_2}{k_2-1}\right)^{k_2} \left(\frac{m}{a_n}\right)^k$$

are the tangent points of the straight line $y \mapsto L_m^{(n)}(y)$ to $y \mapsto G_m^{(n)}(y)$ defined in Proposition 7(ii).

For $n = 1$, we have $g_m^{(1)}(x) = m \vee x$. Since

$$\limsup_{x \downarrow 0} \frac{g_m^{(1)}(x)}{\varphi(x)} = \limsup_{x \uparrow \infty} \frac{g_m^{(1)}(x)}{\psi(x)} = 0,$$

the value function $V^{(1)}(\cdot, m)$ is finite and admits an optimal stopping time by Proposition 7 (i) and (iv). Let us calculate the smallest nonnegative concave majorant $W_m^{(1)}(y)$ of the function

$$G_m^{(1)}(y) = \frac{g^{(1)}}{\varphi} \circ F^{-1}(y) = \begin{cases} m y^{k_1/k}, & 0 \le y < m^k, \\ y^{(1+k_1)/k}, & y \ge m^k. \end{cases}$$

The function $y \mapsto G_m^{(1)}(y)$ is the maximum of the strictly concave increasing functions $y \mapsto m y^{k_1/k}$ and $y \mapsto y^{(1+k_1)/k}$, which intersect at $y = m^k$; see Figure 6(b). The valley centered at $y = m^k$ can be bridged by a straight line $L_m^{(1)}$ which is tangent to $y \mapsto m y^{k_1/k}$ and $y \mapsto y^{(1+k_1)/k}$ at $y = y_{m,1}^{(1)}$ and $y = y_{m,2}^{(1)}$, respectively, and majorizes $G_m^{(1)}$ everywhere. The

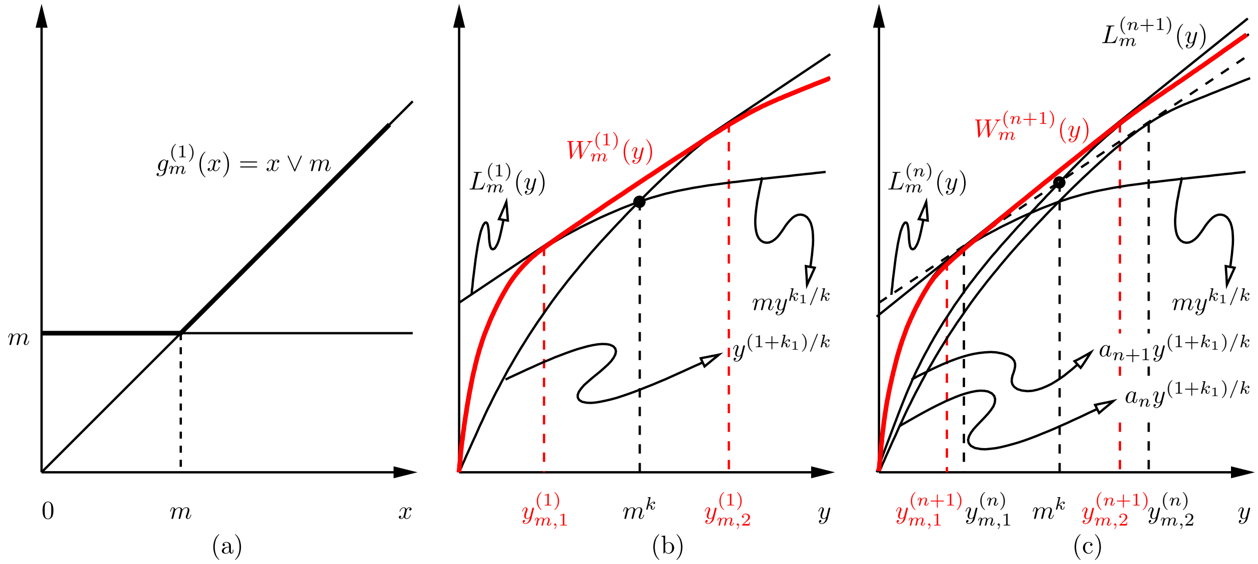

FIGURE 6. Illustrations for the proof of Proposition 5. In (b) and (c), $L_m^{(n)}(y)$ is the straight line which is tangent to strictly increasing and concave curves $y \mapsto my^{k_1/k}$ and $y \mapsto a_n y^{(1+k_1)/k}$ at $y = y_{m,1}^{(n)}$ and $y = y_{m,2}^{(n)}$, respectively. In (b), $W_n^{(1)}(y)$ is the smallest nonnegative concave majorant of the function $G_m^{(1)}(y)$, which is the maximum of the curves $y \mapsto my^{k_1/k}$ and $y \mapsto y^{(1+k_1)/k}$. In (c), $W_m^{(n+1)}$ is the same majorant of $G_m^{(n+1)}(y)$, which coincides on $[0, y_{m,1}^{(n)}]$ with $my^{k_1/k}$, on $[y_{m,1}^{(n)}, m^k]$ with $L_m^{(n)}(y)$, and on $[m^k, \infty)$ with $a_{n+1}y^{(1+k_1)/k}$. The curves $y \mapsto a_n y^{(1+k_1)/k}$ and $y \mapsto a_{n+1}y^{(1+k_1)/k}$ intersect at $y = m^k$.

points $y_{m,1}^{(1)}$ and $y_{m,2}^{(1)}$ are unique solutions $u < m^k < v$ of the system of equations

$$\frac{d}{dy}\left(my^{k_1/k}\right)\Big|_{y=u} = \frac{v^{(1+k_1)/k} - mu^{k_1/k}}{v-u} = \frac{d}{dy}\left(y^{(1+k_1)/k}\right)\Big|_{y=v}.$$

The straight-forward calculations give

$$y_{m,1}^{(1)} = \left(\frac{k_1}{1+k_1}\right)^{1+k_1}\left(\frac{k_2}{k_2-1}\right)^{k_2-1} m^k < m^k < y_{m,2}^{(1)} = \left(\frac{k_1}{1+k_1}\right)^{k_1}\left(\frac{k_2}{k_2-1}\right)^{k_2} m^k.$$

Moreover, the equation of the straight line $L_m^{(1)}(y)$ becomes

$$G_m^{(1)}\left(y_{m,1}^{(1)}\right) + \left(y - y_{m,1}^{(1)}\right)\cdot \frac{d}{dy}G_m^{(1)}(y)\big|_{y=y_{m,1}^{(1)}} = m\frac{k_1}{k}\left[y_{m,1}^{(1)}\right]^{-k_2/k} y + m\frac{k_2}{k}\left[y_{m,1}^{(1)}\right]^{k_1/k}.$$

Therefore, we have

$$W_m^{(1)}(y) = \begin{cases} my^{k_1/k}, & 0 \le y < y_{m,1}^{(1)} < m^k, \\ L_m^{(1)}(y), & y_{m,1}^{(1)} \le y < y_{m,2}^{(1)}, \\ y^{(1+k_1)/k}, & y \ge y_{m,2}^{(1)} > m^k. \end{cases}$$

If we define $x^{(1)}_{m,j} = F^{(-1)}(y^{(1)}_{m,j}) = (y^{(1)}_{m,j})^{1/k}$ for $j = 1, 2$, then Proposition 7(ii) implies that $V^{(1)}(x, m) = \varphi(x) W^{(1)}_m(F(x))$; i.e., with $\ell^{(1)}_m(x) = x^{-k_1} L^{(1)}_m(x^k)$ we have

$$V^{(1)}(x,m) = \begin{cases} m, & 0 \le x < x^{(1)}_{m,1} < m^k, \\ \ell^{(1)}_m(x) = \dfrac{m}{k}\left[k_1\left(\dfrac{x}{x^{(1)}_{m,1}}\right)^{k_2} + k_2\left(\dfrac{x}{x^{(1)}_{m,1}}\right)^{-k_1}\right], & x^{(1)}_{m,1} \le x < x^{(1)}_{m,2}, \\ x, & x \ge x^{(1)}_{m,2}. \end{cases}$$

This proves part (i) for $n = 1$. Suppose that $V^{(n)}(x, m)$ and $W^{(n)}_m(y)$ are given by (20) and (35), respectively, for some $n \ge 1$. Let us show that they also hold for $n + 1$. It is easy to check that

$$\begin{aligned} V^{(n)}(m,m) = \ell^{(n)}_m(m) = m\Bigg\{&\frac{k_1}{k}\left[\left(\frac{k_1}{1+k_1}\right)^{1+k_1}\left(\frac{k_2}{k_2-1}\right)^{k_2-1}\right]^{-k_2/k}(a_n)^{k_2} \\ &+\frac{k_2}{k}\left[\left(\frac{k_1}{1+k_1}\right)^{1+k_1}\left(\frac{k_2}{k_2-1}\right)^{k_2-1}\right]^{k_1/k}(a_n)^{-k_1}\Bigg\} = a_{n+1}m \end{aligned}$$

by the definition of a_{n+1} in (19). Therefore,

$$g^{(n+1)}_m(x) = V^{(n)}(x, m \vee x) = \begin{cases} V^{(n)}(x,m), & 0 \le x < m, \\ a_{n+1}x, & x \ge m, \end{cases}$$

and since $V^{(n)}(F^{-1}(y), m)/\varphi(F^{-1}(y)) = W^{(n)}_m(y)$ by Proposition 7(ii), we have

$$G^{(n+1)}_m(y) = \left\{\begin{matrix} W^{(n)}_m(y), & 0 \le y < m^k \\ a_{n+1}y^{(1+k_1)/k}, & y \ge m^k \end{matrix}\right\} = \left\{\begin{matrix} my^{k_1/k}, & 0 \le y < y^{(n)}_{m,1} \\ L^{(n)}_m(y), & y^{(n)}_{m,1} \le y < m^k \\ a_{n+1}y^{(1+k_1)/k}, & y \ge m^k \end{matrix}\right\}$$

by the induction hypothesis. The function $y \mapsto G^{(n+1)}_m$ is the maximum of the concave and increasing curves $y \mapsto W^{(n)}_m(y)$ and $y \mapsto a_{n+1}y^{(1+k_1)/k}$, which meet at $y = m^k$; see Figure 6(c). The valley centered at $y = m^k$ in the graph of $y \mapsto G^{(n+1)}_m$ can be bridged by a straight line $L^{(n+1)}_m(y)$, which majorizes $G^{(n+1)}_m$ everywhere and is tangent to the curves $y \mapsto W^{(n)}_m(y)$ and $y \mapsto a_{n+1}y^{(1+k_1)/k}$ at some points $y = y^{(n+1)}_{m,1}$ and $y = y^{(n+1)}_{m,2}$, respectively. Since $y \mapsto a_{n+1}y^{(1+k_1)/k}$ is above the line $L^{(n)}_m(y)$ on $y \in [m^k, \infty)$, the points $y^{(n+1)}_{m,1}$ and $y^{(n+1)}_{m,2}$ are the unique solutions $u < y^{(n)}_{m,1} < m^k < v$ of

$$\frac{d}{dy}\left(my^{k_1/k}\right)\Big|_{y=u} = \frac{a_{n+1}v^{(1+k_1)/k} - mu^{k_1/k}}{v-u} = \frac{d}{dy}\left(a_{n+1}y^{(1+y_1)/k}\right)\Big|_{y=v}.$$

After straight-forward algebra, we obtain

$$\begin{aligned} y_{m,1}^{(n+1)} &= \left(\frac{k_1}{1+k_1}\right)^{1+k_1}\left(\frac{k_2}{k_2-1}\right)^{k_2-1}\left(\frac{m}{a_{n+1}}\right)^{k} < y_{m,1}^{(n)} < m^k, \\ m^k < y_{m,2}^{(n+1)} &= \left(\frac{k_1}{1+k_1}\right)^{k_1}\left(\frac{k_2}{k_2-1}\right)^{k_2}\left(\frac{m}{a_{n+1}}\right)^{k} < y_{m,2}^{(n)}. \end{aligned}$$

Last inequality follows from that $a_{n+1} > a_n$. The equation of the line $L_m^{(n+1)}(y)$ becomes

$$W_m^{(n)}\left(y_{m,1}^{(n+1)}\right) + \left(y - y_{m,1}^{(n+1)}\right)\cdot \frac{d}{dy}W_m^{(n)}(y)\big|_{y=y_{m,1}^{(n+1)}} = m\frac{k_1}{k}\left[y_{m,1}^{(n+1)}\right]^{-k_2/k} y + m\frac{k_2}{k}\left[y_{m,1}^{(n+1)}\right]^{k_1/k},$$

and the smallest nonnegative concave majorant of $G_m^{(n+1)}$ is given by

$$W_m^{(n+1)}(y) = \begin{cases} m y^{k_1/k}, & 0 \le y < y_{m,1}^{(n+1)} < y_{m,1}^{(n)} < m^k, \\ L_m^{(n+1)}(y), & y_{m,1}^{(n+1)} \le y < y_{m,2}^{(n+1)} < y_{m,2}^{(n)}, \\ y^{(1+k_1)/k}, & y \ge y_{m,2}^{(n+1)} > m^k, \end{cases}$$

which is the same as (35) with $n+1$ instead of n. Finally, let us define $x_{m,j}^{(n+1)} = F^{-1}(y_{m,j}^{(n+1)}) = (y_{m,j}^{(n+1)})^{1/k}$ for $j = 1, 2$. Then $x_{m,1}^{(n)} < x_{m,1}^{(n+1)} < m^k < x_{m,2}^{(n+1)} < x_{m,2}^{(n)}$ proves (ii), and by Proposition 7(ii) we have that $V^{(n+1)}(x, m) = \varphi(x) W_m^{(n+1)}(F(x))$ equals

$$V^{(n+1)}(x,m) = \begin{cases} m, & 0 \le x < x_{m,1}^{(n+1)}, \\ \ell_m^{(n+1)}(x) = \dfrac{m}{k}\left[k_1\left(\dfrac{x}{x_{m,1}^{(n+1)}}\right)^{k_2} + k_2\left(\dfrac{x}{x_{m,1}^{(n+1)}}\right)^{-k_1}\right], & x_{m,1}^{(n+1)} \le x < x_{m,1}^{(n+1)}, \\ a_{n+1}x, & x \ge x_{m,2}^{(n+1)} \end{cases}$$

in terms of $\ell_m^{(n+1)}(x) \triangleq x^{-k_1} L_m^{(n+1)}(x^k)$. This completes the proof of Proposition 5 (i) and (ii). Finally, (iii) follows from Proposition 7 (iii). □

4.3. **Proof of Proposition 6.** We need to prove only the first equality; the rest follows immediately from it. The sequence $(a_n)_{n\ge 1}$ in (19) is increasing. Therefore, the limit $a \triangleq \lim_{n\to\infty} a_n \ge 1$ exists and satisfies the equation

$$x = \frac{k_1}{k}\left[\left(\frac{k_1}{1+k_1}\right)^{1+k_1}\left(\frac{k_2}{k_2-1}\right)^{k_2-1}\right]^{-k_2/k} x^{k_2} + \frac{k_2}{k}\left[\left(\frac{k_1}{1+k_1}\right)^{1+k_1}\left(\frac{k_2}{k_2-1}\right)^{k_2-1}\right]^{k_1/k} x^{-k_1}, \tag{36}$$

obtained by passing to limit as $n \to \infty$ in (19). In order to find a, we will guess its value, verify that it satisfies the above equation, and that the equation has exactly one solution.

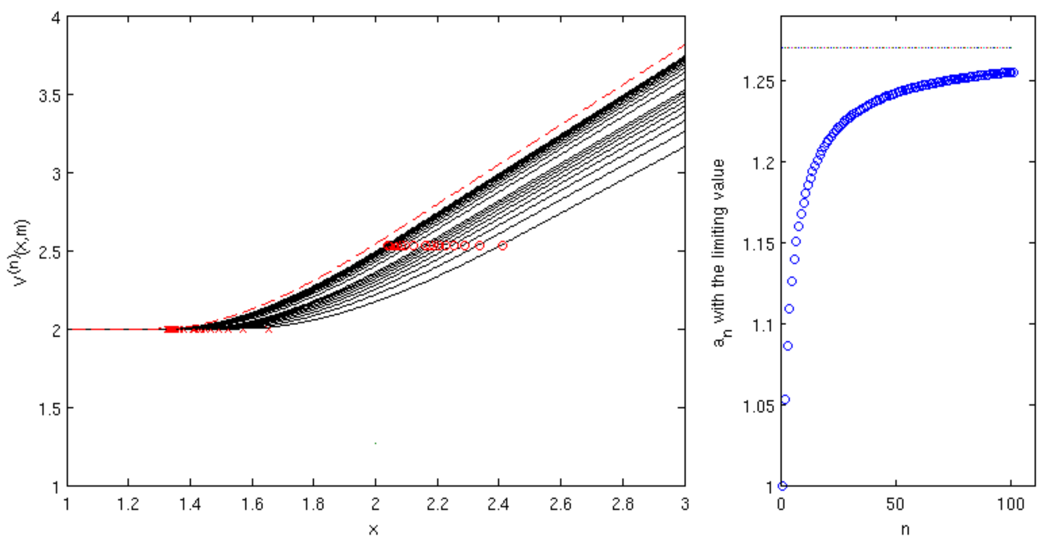

FIGURE 7. Behavior of $V^{(n)}(x,m)$ as a function of n in Proposition 5. We take $\sigma = 0.2, \mu = 0.03, r = 0.06$, $m = 2$. The left panel shows $V^{(n)}(x,m)$ for a range of x around m with $n = 1, 2, \ldots, 10, 15, 20, \ldots, 60$. The thresholds $x_1^{(n)}(m)$ are marked with a red cross and are seen to be decreasing and approaching the limit $\hat{x}$. The thresholds $x_2^{(n)}(m)$ are shown with a red circle and are also decreasing and approaching $m = 2$. Recall that $V^{(n)}(x,m)$ is affine on $[x_2^{(n)}(m), \infty)$. The Russian option value $V^{(\infty)}(x,m)$ is shown as the dashed red curve. The right panel shows the sequence (a_n) from (19). As proven the sequence converges to $\hat{a}$ indicated with a dashed line.

From Proposition 5 (i) and (ii) we know that

$$x_{m,2}^{(n)} \triangleq \left(\frac{k_1}{1+k_1}\right)^{k_1/k} \left(\frac{k_2}{k_2-1}\right)^{k_2/k} \frac{m}{a_n} > m, \qquad n \geq 1.$$

Since $a_n \nearrow a$, this inequality implies that a is finite. Recall that $x_{m,2}^{(n)}$ gives the upper exercise threshold when there are n exercise rights. Intuitively, as the number of exercise rights increases, the optimal waiting time before marking a new record of the process X should get shorter. In the limit, this waiting time should reduce to zero if the process starts at m. Therefore, we expect $\lim_{n\to\infty} x_{m,2}^{(n)} = m$, which implies that a equals

$$\widehat{a} = \left(\frac{k_1}{1+k_1}\right)^{k_1/k} \left(\frac{k_2}{k_2-1}\right)^{k_2/k}.$$

It is easy to verify that $\widehat{a}$ satisfies (36).

To show that $a = \widehat{a}$, we shall prove that $\widehat{a}$ is the unique solution of (36). Note that every x satisfying (36) must be nonzero. If we divide (36) by $\varphi(x) = x^{-k_1}$ and replace in the

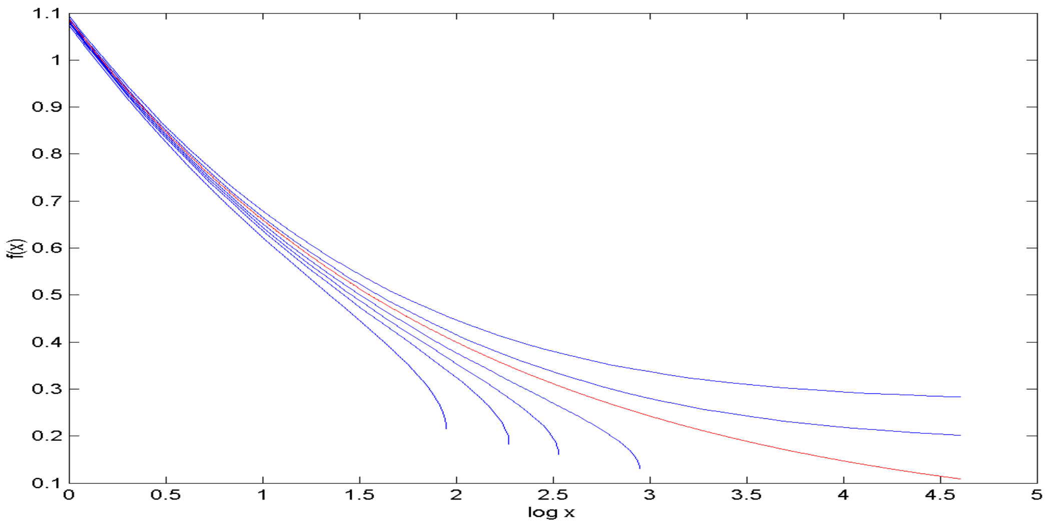

FIGURE 8. Solutions of the ordinary differential equation (38) for different initial conditions. The true solution $f(m)$ from (35) is shown in red. As can be seen, the true solution is the smallest non-exploding one. All solutions were obtained using a `ode15s` Matlab solver.

resulting equation every x with $F^{-1}(x) = x^{1/k}$, we obtain

$$x^{(1+k_1)/k} = \frac{k_1}{k}\left[\left(\frac{k_1}{1+k_1}\right)^{1+k_1}\left(\frac{k_2}{k_2-1}\right)^{k_2-1}\right]^{-k_2/k} x + \frac{k_2}{k}\left[\left(\frac{k_1}{1+k_1}\right)^{1+k_1}\left(\frac{k_2}{k_2-1}\right)^{k_2-1}\right]^{k_1/k}. \tag{37}$$

Note that x solves (36) if and only if $F(x) = x^k$ solves (37). Therefore, it is enough to show that $F(\widehat{a}) = \widehat{a}^k$ is the unique solution of (37). However, its easy to check that the straight line on the right is tangent to the strictly concave curve on the left exactly at $x = \widehat{a}$; because of the properties of strictly concave functions, they cannot meet at anywhere else. $\square$

Remark 4. *For the geometric BM, the fixed point method gives that the slope $f(m)$ of the affine segment of $W^{(\infty)}(x,m)$ must solve*

$$f'(m) = \frac{\hat{y}^{\frac{k_1}{k}}}{\hat{y} - m^k} \quad \textit{where} \quad \hat{y} = \left(\frac{f(m)}{m}\frac{k}{k_1}\right)^{-\frac{k}{k_2}}. \tag{38}$$

It can verified that this ODE is indeed satisfied by $f(m) = m\frac{k_1}{k}[\hat{y}]^{-k_2/k}$ from (35) with $\hat{y} = \left(\frac{k_1}{1+k_1}\right)^{1+k_1}\left(\frac{k_2}{k_2-1}\right)^{k_2-1}\left(\frac{m}{\hat{a}}\right)^k$ and $\hat{a}$ as above. From numerical experiments, we conjecture that $f(m)$ is the smallest non-exploding solution of (38), as illustrated in Figure 8.

5. Acknowledgment

This work was initiated while both authors were attending the *Symposium on Optimal Stopping with Applications* in Manchester, United Kingdom. They are very grateful to the organizers for this stimulating event.

References

[1] René Carmona and Savas Dayanik, *Optimal multiple-stopping of linear diffusions and swing options*, Preprint. (http://www.princeton.edu/~sdayanik/papers/swing.pdf), 2003.

[2] Rene Carmona and Nizar Touzi, *Optimal multiple stopping and valuation of swing options*, to appear in *Mathematical Finance*, 2003.

[3] Savas Dayanik and Ioannis Karatzas, *On the optimal stopping problem for one-dimensional diffusions*, Stochastic Process. Appl. **107** (2003), no. 2, 173–212. MR MR1999788 (2004d:60104)

[4] J. Darrell Duffie and J. Michael Harrison, *Arbitrage pricing of Russian options and perpetual lookback options*, Ann. Appl. Probab. **3** (1993), no. 3, 641–651. MR MR1233618 (94i:90028)

[5] Stephen M. Samuels, *Secretary problems*, Handbook of sequential analysis, Statist. Textbooks Monogr., vol. 118, Dekker, New York, 1991, pp. 381–405. MR MR1174312

[6] L. A. Shepp and A. N. Shiryaev, *A new look at the "Russian option"*, Teor. Veroyatnost. i Primenen. **39** (1994), no. 1, 130–149. MR MR1348192 (97e:90007)

[7] Larry Shepp and A. N. Shiryaev, *The Russian option: reduced regret*, Ann. Appl. Probab. **3** (1993), no. 3, 631–640. MR MR1233617 (94i:90027)

(S. Dayanik) Department of Operations Research and Financial Engineering, and Bendheim Center for Finance, Princeton University, Princeton, NJ 08544

E-mail address: `sdayanik@princeton.edu`

(M. Ludkovski) Department of Mathematics, University of Michigan, Ann Arbor, MI 48109

E-mail address: `mludkov@umich.edu`