A Test of Goodness of Fit based on Gini’s Index of Spacings

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Abstract

This paper introduces a new test for goodness of fit based on the Gini index which is the sum over all pairs, of the absolute differences of the observed spacings. We derive its exact and asymptotic distributions under the null hypothesis, after showing that it is distributionally equivalent to the sum of uniform observations on the unit interval. After a discussion of local powers of this and related tests, we provide simulated power comparisons, which demonstrate that the Gini test is better than all the other competitors considered, against a wide variety of alternatives.

Keywords: Goodness of fit tests, spacings, Gini Index, Order statistics, Directional data.

1 Introduction

Consider a random sample $X_1, \ldots, X_n$ from a continuous distribution function (d.f) $F^*$ on the real line and let $(X_{(1)}, \ldots, X_{(n)})$ denote the order statistics from this sample. Considerable attention has been devoted in the literature to the problem of goodness of fit i.e., testing the hypothesis

$$H : F^*(x) = F_0^*(x) \quad \text{against the alternative} \quad K : F^*(x) \neq F_0^*(x),$$

where $F_0^*$ is a completely specified continuous d.f.. Apart from various ad hoc tests, there are 3 general classes of tests for this problem, viz.

1. the $\chi^2$ tests which depend on groupings
2. tests based on empirical d.f.s, and
3. tests based on spacings i.e. the differences between order statistics.

There have been suggestions in the literature that tests based on empirical d.f.s such as the Kolmogorov-Smirnov and Cramer-von Mises statistics perform better in detecting differences between d.f.s, whereas tests based on spacings are particularly effective in revealing the differences between densities. See for instance the discussion in Pyke (1965), which also provides an excellent survey on the topic of spacings. Jammalamadaka and Tiwari (1987) show that comparable spacings tests are better than chi-squared tests, in terms of local power. It should be
noted that, outside of the goodness-of-fit framework, spacings based tests arise in many other contexts including reliability. See eg. Soofi, Ebrahimi, and Habibullah (1995), and Jammalamadaka and Taufer (2003). In a survey paper Birnbaum (1953) discusses general methods of chi-square, Kolmogorov-Smirnov, Cramer-von Mises and spacings. The reader is also referred to Shorack and Wellner (1986) for a thorough treatment of distribution-free goodness of fit tests based on empirical d.f.’s.

Without loss of any generality, one can reduce the above problem of goodness of fit, to testing the hypothesis of uniformity on the unit interval, by means of the probability integral transformation

$$U_i = F^*(X_i), \quad i = 1, 2, \ldots, n$$

that is, on the basis of the transformed sample, the problem becomes one of testing uniformity i.e.,

$$H : F(u) = u, \quad 0 < u < 1 \quad \text{against the alternative} \quad K : F(u) \neq u, \quad 0 < u < 1.$$  

From here on, we shall assume that such a transformation has been made and define

$$D_i = U_{(i)} - U_{(i-1)}, \quad i = 1, 2, \ldots, n + 1,$$

with the notation $U_{(0)} \equiv 0, U_{(n+1)} \equiv 1$.

Although there has been significant work prior to it, it is fair to say that the impetus for work on the theory of spacings came from the paper by Greenwood (1946), who proposed the statistic

$$\sum_{i=1}^{n+1} D_i^2$$

to test whether certain events such as the spread of disease occur at random (or follow a Poisson process with fixed rate) on the time axis. Since then, many tests based on spacings have been proposed in literature; see Pyke (1965) for a good review. Sethuraman and Rao (1970) and Rao and Sethuraman (1975) provide a unified treatment of the asymptotic distribution theory for symmetric spacings statistics, by expressing such statistics in terms of the empirical spacings process.

The $\{D_i\}$ defined above are called one-step or simple spacings. One can generalize this idea of spacings by defining, for any fixed integer $m$, ($1 \leq m < n$), the $m$-step (or higher-order) spacings. The overlapping $m$-step spacings are defined by

$$D_{(m)}^i = \begin{cases} 
U_{(i+m)} - U_{(i)}, & i = 0, 1, \ldots, n + 1 - m \\
1 + U_{(i+m-n-1)} - U_{(i)}, & i = n + 2 - m, \ldots, n 
\end{cases}$$

whereas the non-overlapping $m$-step spacings are defined by

$$D_{(m)}^i = U_{(i+1),m} - U_{(i),m}, \quad i = 0, 1, \ldots, [n/m] - 1.$$  

Jammalamadaka and Kuo (1984) consider tests based on these higher order spacings and show that tests based on $m$-step spacings have higher efficiency compared to their counterparts based on simple or 1-step spacings.

The spacings $\{D_i, i = 1, 2, \ldots, n + 1\}$, under the null hypothesis of uniformity, form an exchangeable set of random variables with an expected value of $1/(n+1)$. Since they add up to one, their arithmetic mean is also $\bar{D} = 1/(n+1)$. Thus tests of uniformity can be constructed by measuring how different $\{D_i, i = 1, 2, \ldots, n + 1\}$ are from their average value of $1/(n+1)$. In Section 2 we show that many of the spacings tests considered in the literature can be viewed in terms of various choices of measures of dispersion of these $\{D_i\}$.

An alternate way to construct a general class of such tests is to use the Csiszár divergence measure between two probability distributions $F_1(\cdot)$ and $F_2(\cdot)$:

$$I_h(F_1, F_2) = \int_{\mathbb{R}} h \left( \frac{dF_1(x)}{dF_2(x)} \right) dF_2(x)$$
where \( h : (0, \infty) \to \mathbb{R} \) is a convex function with \( h(1) = 0 \). See eg. Ekstrom (1997) and Ghosh and Jammalamadaka (2001). A special case of this measure is the Kullback-Liebler distance, which takes the simple form \( \sum_{i=1}^{n+1} \log D_i \) in this particular instance. Darling (1953) has studied this statistic and its first order approximation \( \sum_{i=1}^{n+1} 1/D_i \).

In Section 2, we propose a new test for goodness-of-fit based on the Gini’s measure of dispersion and in Section 3, we discuss its exact and asymptotic distributions under the null hypothesis. Some comments on the local powers of this and related statistics are provided in Section 4 and numerical comparison of powers is provided in Section 5. Finally, in Section 6, we provide conclusions and discuss possible future research.

2 Spacings tests as measures of dispersion

As stated earlier, a test of uniformity based on spacings corresponds to a dispersion measure for \( D_i \) and one can use any dispersion measure to serve that purpose.

Common among the dispersion measures are the variance, mean absolute deviation, range and interquartile distance. While measures such as the mean absolute deviation and Gini’s index are less frequently used, perhaps due to the computational difficulties involved, these are known to be more robust to the outliers than the others mentioned.

We remark that the test based on the variance

\[
V_n = \sum_{i=1}^{n+1} (D_i - 1/(n + 1))^2
\]

(1)
corresponds to the Greenwood statistic (see also Kimball (1947)). A generalized Greenwood statistic based on \( m \)-step spacings (overlapping, as well as non-overlapping) and its asymptotic optimality, has been studied in detail by Rao and Kuo (1984).

An alternate dispersion measure, namely the mean absolute deviation leads to the statistic

\[
R_n = \sum_{i=1}^{n+1} |D_i - 1/(n + 1)|
\]

(2)
which was suggested by Sherman (1950). This latter statistic was independently introduced in the context of circular data by Rao (1969) with Russell and Levitin (1995) providing an extensive table of critical values for it. More generally, it is clear that any convex function of the spacings or of higher order spacings can be used for this purpose.

The range, the interquartile range and the Gini’s index of the observed spacings can also be used for goodness of fit testing. This has not been discussed in literature, although Rao and Sobel (1980) provide a pertinent and useful discussion of tests based on ordered spacings. Our goal here is to study the test statistic based on the Gini’s index, while the others are deferred to a later study.

Gini, in introducing his measure of dispersion, raised objection to using the variance or the mean absolute deviation, since they measure the deviation of individual observations from the “center,” thus interlinking the concept of location with variability. According to him, these are two distinct properties, needing distinct measures which do not depend on each other. Therefore he proposed the sum of pairwise distances between the observations, as a measure.
of dispersion. Gini measure can be related to the measure of concentration and has thus found considerable appeal in economics and received renewed attention.

Applying this idea to \( \{D_i\} \) in our context, we may define the Gini index of dispersion for spacings, as

\[
G_n = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} |D_i - D_j|.
\] (3)

Our goal in this paper is to study this Gini statistic \( G_n \), its distributional properties and how it compares with other competing tests of uniformity.

**Remark 1:**

One can consider a generalized Gini statistic defined by

\[
G_n(r) = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} |D_i - D_j|^r, \quad r > 0.
\]

Clearly the special case \( r = 1 \) corresponds to the Gini statistic \( G_n \) that we discuss here, while the special case \( r = 2 \) corresponds to the Greenwood statistic.

**Remark 2:**

If we consider the vector of spacings, \( \mathbf{D} = (D_1, \ldots, D_{n+1}) \), the Greenwood statistic \( V = \mathbf{D}'\mathbf{D} \) is a quadratic form which gives equal weights to all the spacings. Since \( \{D_i\} \) are correlated, it is tempting to ask whether one gets different (and possibly better) test by taking into account the covariance-structure between these spacings. It is known (see e.g. Pyke (1965)) that

\[
\operatorname{Cov}(\mathbf{D}) = \Sigma = (\sigma_{ij}) \quad i, j = 1, 2, \ldots, n + 1
\]

where

\[
\sigma_{ij} = -\frac{1}{(n+1)^2(n+2)}, \quad \sigma_{ii} = \frac{n}{(n+1)^2(n+2)}.
\]

Since the covariance matrix is singular, we may take its generalized inverse \( \Sigma^{-} \). Consider the statistic

\[
V^* = \mathbf{D}'\Sigma^{-}\mathbf{D}
\]

which introduces appropriate weights into the quadratic form. It can be checked that the matrix

\[
\Sigma^{-} = ((\sigma^{ij}))
\]

with

\[
\sigma^{ij} = (n+1)(n+2), \quad \sigma^{ii} = 2(n+1)(n+2)
\]

is indeed the Penrose inverse of \( \Sigma \). The resulting statistic \( V^* \) is exactly identical to the un-weighted Greenwood statistic, \( V \). Thus there is no gain in such a weighting scheme.

Our objective in the next few sections is to discuss the test statistic \( G_n \) based on Gini’s index and compare its performance with competitors like the Greenwood statistic.
3 Exact and asymptotic distribution of $G_n$ under the hypothesis.

In this section, we derive the exact as well as the asymptotic distribution of $G_n$ under the null hypothesis of uniformity. The Gini’s measure of dispersion, $G_n$ can be rewritten in terms ordered spacings as follows:

$$ G_n = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} |D_{(i)} - D_{(j)}| $$

$$ = 2 \sum_{i=1}^{n+1} \sum_{j>i} (D_{(j)} - D_{(i)}) $$

$$ = 2 \sum_{i=1}^{n+1} [( \sum_{j=i+1}^{n+1} D_{(j)}) - (n - i + 1)D_{(i)}]. \tag{4} $$

On writing the above explicitly and simplifying, we obtain

$$ G_n = 2 \sum_{i=1}^{n+1} (2i - n - 2)iD_{(i)} = 4 \sum_{i=1}^{n+1} iD_{(i)} - 2(n + 2) = 4L_n - 2(n + 2) \tag{5} $$

where

$$ L_n = \sum_{i=1}^{n+1} iD_{(i)}. \tag{6} $$

Now define

$$ E_i = (n - i + 2)[D_{(i)} - D_{(i-1)}], \quad i = 1, 2, \ldots, n + 1 $$

or, conversely

$$ D_{(i)} = \sum_{j=1}^{i} E_j/(n - j + 2). $$

Then $L_n$ defined in Equation (6) can be expressed as

$$ L_n = \sum_{i=1}^{n+1} i \sum_{j=1}^{i} E_j/(n - j + 2). $$

On explicitly writing the above double sum, we get

$$ L_n = \sum_{i=1}^{n+1} \left[ \frac{(n + 1)(n + 2)}{2} - \frac{(i - 1)i}{2} \right] E_i/(n - i + 2). $$

On further simplification and using the fact that $\sum_{i=1}^{n+1} E_i = 1$, we get

$$ L_n = 1/2 \sum_{i=1}^{n+1} (n + i + 1)E_i = (n + 1)/2 + 1/2 \sum_{i=1}^{n+1} iE_i. $$

Starting with the joint density of $U_{(i)}$, Durbin (1961) shows by repeated transformation that $(E_1, \ldots, E_{n+1})$ are distributionally equivalent to $(D_1, \ldots, D_{n+1})$ under the null hypothesis.
of uniformity. This result can also be checked directly by representing uniform spacings in terms of scaled exponential random variables and using a very similar property that holds for exponential order statistics. Thus, under the null hypothesis, the distribution of $L_n$ is the same as that of

$$L_n \sim (n + 1)/2 + 1/2 \sum_{i=1}^{n+1} i D_i = (n + 1)/2 + L_n^*/2$$  

(7)

where $L_n^* = \sum_{i=1}^{n+1} i D_i$ is related to the sum $S_n = \sum_{i=1}^n U_i$ of $n$ uniform random variables on the unit interval. Indeed

$$S_n = \sum_{i=1}^n U_i = \sum_{i=1}^n U_{(i)}$$
$$= \sum_{i=1}^n \sum_{j=1}^i D_j = \sum_{i=1}^{n+1} (n - i + 1) D_i$$
$$= (n + 1) - \sum_{i=1}^{n+1} i D_i = (n + 1) - L_n^*.$$  

(8)

Combining Equations (5), (7) and (8), we get

$$G_n \sim 2(n - S_n).$$  

(9)

under the null hypothesis. Note that this is a distributional equivalence that holds under the hypothesis of uniformity and should not be interpreted any more broadly. Again, a uniform observation $U_i$ has the same distribution as that of $(1 - U_i)$, it clear that the distribution of $n - S_n$ is the same as that of $S_n$ which has the following well-known density (see e.g. Wilks (1962, p.204))

$$f_n(x) = \frac{1}{(n-1)!} \sum_{k=0}^{\infty} (-1)^k \left(\frac{n}{k}\right) < x - n >^{n-1}, 0 < x < n.$$  

From this, one can write the exact null density of the Gini statistic $G_n$, in the range $0 < y < 2n$, as

$$g_n(y) = \frac{1}{(n-1)!2} \sum_{k=0}^{\infty} (-1)^k \left(\frac{n}{k}\right) < \frac{y}{2} - n >^{n-1}.$$  

As for the large sample distribution under the null hypothesis, note that by the Central Limit Theorem for iid variables,

$$S_n \simeq N(n/2, n/12)$$

under the null hypothesis of uniformity. Therefore by using the relationship (9) of $G_n$ to $S_n$, we can claim the asymptotic normality

$$G_n \simeq N(n, n/3)$$

under the null. More precisely, as $n$ goes to infinity,

$$(3/n)^{1/2}(G_n - n)$$
converges to a standard Normal distribution.

Note that the exact as well as the asymptotic distributions of both $L^*_n$ and $L_n$ can be easily written down from that of $S_n$ using equations (8) and (7) respectively.

Since $E(D_{(i)}) \approx i/(n + 1)^2$, it is reasonable to consider yet another test statistic for goodness of fit, viz.

$$T_n = \sum_{i=1}^{n+1} [D_{(i)} - i/(n + 1)^2]^2. \quad (10)$$

When expanded, this can be written as a linear combination of the Gini Statistic $G_n$ (or its equivalent $L_n$) and $V_n$, which is equivalent to the Greenwood statistic i.e.,

$$T_n = V_n + 1/(n + 1) - 2L_n/(n + 1)^2 + (n + 2)(2n + 3)/\{6(n + 1)^3\}$$

$$= V_n - 2L_n/(n + 1)^2 + (8n^2 + 19n + 12)/\{6(n + 1)^3\} \quad (11)$$

or

$$(n + 1)^{3/2}T_n - 5/6(n + 1)^{1/2} = [(n + 1)^{3/2}V_n - 2(n + 1)^{1/2}] - 2(n + 1)^{-1/2}[L_n - 3/4(n + 1)].$$

The exact null distribution of Greenwood statistic has been tabulated (see e.g. Burrows (1979)) and for large samples,

$$(n + 1)^{3/2}V_n - 2(n + 1)^{1/2}$$

is asymptotically normal with zero mean and variance 4 (see e.g. Darling (1953)). However since $G_n$ and $V_n$ are not necessarily independent, the exact and asymptotic null distribution of $T_n$ needs further investigation.

4 Some comments on power for specific alternatives

Here we offer some comments on the powers of the tests $S_n$, $L_n$ and $L^*_n$ under various alternatives. It is well known (see eg. Section 3 of Rao and Sethuraman (1975) and references therein) that tests which are symmetric in spacings, have lower asymptotic powers in the Pitman sense. However, their small sample powers are reasonable (see the small simulation in the last section) and they are especially relevant in the field of directional data, where spacings form the maximal invariant. We should also remark that although the Gini test is equivalent to $S_n$ under the null hypothesis, its behavior and performance can be very different under alternatives. We consider a close sequence of the so-called “smooth” alternatives of the form

$$H_n : F_n(x) = x + n^{-1/2}M_n(x), \quad M_n(0) = 0 \text{ and } M_n(1) = 0. \quad (12)$$

Here we assume that $M_n(x)$ is continuously differentiable with derivative $m_n(x)$ and that it converges uniformly to $M(x)$ on $[0,1]$, a twice continuously differentiable function with first and second derivatives $m(x), m'(x)$ respectively, so that

$$\sup_{0 \leq x \leq 1} |m_n(x) - m(x)| = o(1).$$
Against such a sequence of alternatives, Holst and Rao (1981) show that (see their Theorem 3.1 and Equation (3.6)), the test based on

\[ T_n = \sum_{i=1}^{n+1} m(i/(n+1))D_i \]

is Locally Most Powerful (LMP) for testing uniformity. The following remark is also taken from there:

**Remark 3:**

Consider testing the hypothesis that a random sample is from a logistic distribution with

\[ G(t) = \frac{1}{1 + e^{-t}} \]

against local translation alternatives, \( G(t - \theta/\sqrt{n}) \). After transforming the data to the interval \([0, 1]\) through \( x = G(t) \), the null hypothesis translates to testing uniformity, while \( M_n(x) \) described in Equation (12) converges to

\[ M(x) = G'(G^{-1}(x)) \]

with derivative

\[ m(x) = (2x - 1), \quad 0 \leq x \leq 1 \]

(cf. Holst and Rao (1981)). Thus the LMP test for this context reduces to one based on \( L_n^* \).

Things are not so clear when one is dealing with tests based on ordered spacings like \( L_n \) or its equivalent \( G_n \). This is mainly because of the fact that although spacings under general alternatives can be expressed in terms of uniform spacings for close alternatives (see for example Equation (3.8) of Rao and Sethuraman (1975)), their ordering depends very much on what the alternative density is. Hence one has to resort to simulations.

**Remark 4:** When considering alternatives at a distance of \( n^{-1/4} \) (i.e., Equation (12) with \( n^{-1/2} \) replaced by \( n^{-1/4} \)), it is known that the Greenwood test \( V_n \) is locally optimal among tests based symmetrically on the spacings. However it has no power against alternatives at a distance of \( n^{-1/2} \) i.e., its power is the same as the level and is not a competitor for \( L_n \) or \( L_n^* \) (see Sethuraman and Rao (1969)).

**5 Simulated power comparisons of tests**

In this section, we report on a small simulation study that was carried out to compare the Monte Carlo powers of the four tests of goodness of fit, namely the test based on unordered spacings \( L_n^* \), the Gini test based on ordered spacings \( G_n \), the Greenwood statistic \( V_n \), and the hybrid statistic \( T_n \) mentioned in Equation (11). For this purpose we consider testing the null hypothesis of a standard Normal distribution against several alternatives, at 0.05 level. These alternatives include heavy-tailed distributions such as the Student \( t(\nu) \), \( \nu = 1, 2, 3 \), thin long-tailed, Laplace, and short tailed, Logistic. Samples of sizes 10, 20, 50, 100 with replications of 1000 were generated using the MATLAB package. The resulting simulated powers are reported in the table below.
Simulated powers of the tests at various alternatives. $\alpha = 0.05$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Test</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$t_3$</th>
<th>Logistic</th>
<th>Laplace</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$L^*_n$</td>
<td>0.093</td>
<td>0.078</td>
<td>0.065</td>
<td>0.089</td>
<td>0.068</td>
</tr>
<tr>
<td></td>
<td>$G_n$</td>
<td>0.462</td>
<td>0.273</td>
<td>0.172</td>
<td>0.398</td>
<td>0.196</td>
</tr>
<tr>
<td></td>
<td>$V_n$</td>
<td>0.291</td>
<td>0.190</td>
<td>0.122</td>
<td>0.288</td>
<td>0.127</td>
</tr>
<tr>
<td></td>
<td>$T_n$</td>
<td>0.260</td>
<td>0.175</td>
<td>0.107</td>
<td>0.269</td>
<td>0.112</td>
</tr>
<tr>
<td>20</td>
<td>$L^*_n$</td>
<td>0.074</td>
<td>0.064</td>
<td>0.058</td>
<td>0.096</td>
<td>0.071</td>
</tr>
<tr>
<td></td>
<td>$G_n$</td>
<td>0.610</td>
<td>0.300</td>
<td>0.159</td>
<td>0.519</td>
<td>0.209</td>
</tr>
<tr>
<td></td>
<td>$V_n$</td>
<td>0.356</td>
<td>0.184</td>
<td>0.112</td>
<td>0.374</td>
<td>0.145</td>
</tr>
<tr>
<td></td>
<td>$T_n$</td>
<td>0.321</td>
<td>0.165</td>
<td>0.099</td>
<td>0.345</td>
<td>0.136</td>
</tr>
<tr>
<td>50</td>
<td>$L^*_n$</td>
<td>0.077</td>
<td>0.079</td>
<td>0.065</td>
<td>0.082</td>
<td>0.052</td>
</tr>
<tr>
<td></td>
<td>$G_n$</td>
<td>0.890</td>
<td>0.502</td>
<td>0.281</td>
<td>0.762</td>
<td>0.262</td>
</tr>
<tr>
<td></td>
<td>$V_n$</td>
<td>0.624</td>
<td>0.288</td>
<td>0.164</td>
<td>0.550</td>
<td>0.178</td>
</tr>
<tr>
<td></td>
<td>$T_n$</td>
<td>0.589</td>
<td>0.275</td>
<td>0.163</td>
<td>0.537</td>
<td>0.175</td>
</tr>
<tr>
<td>100</td>
<td>$L^*_n$</td>
<td>0.097</td>
<td>0.071</td>
<td>0.064</td>
<td>0.092</td>
<td>0.073</td>
</tr>
<tr>
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<td>$G_n$</td>
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<td>0.645</td>
<td>0.364</td>
<td>0.928</td>
<td>0.408</td>
</tr>
<tr>
<td></td>
<td>$V_n$</td>
<td>0.837</td>
<td>0.383</td>
<td>0.202</td>
<td>0.741</td>
<td>0.260</td>
</tr>
<tr>
<td></td>
<td>$T_n$</td>
<td>0.782</td>
<td>0.318</td>
<td>0.183</td>
<td>0.689</td>
<td>0.240</td>
</tr>
</tbody>
</table>

From an examination of this table, the following statements can be asserted:

1. It is clear that the Gini test $G_n$ dominates all the other competing tests, against each of the alternatives considered. $T_n$ and $V_n$ have comparable powers and this is to be expected in view of the high weight given to $V_n$ in Equation (11). Put differently, $T_n$ ignores the better statistic $G_n$ in its weighting, at its own peril!

2. It may be observed that for the Student’s $t(\nu)$, $\nu = 1, 2, 3$ alternatives, the power decreases with increasing degrees of freedom. Again this is to be expected since the larger degrees of freedom represent closeness to normality (although with d.f. this small, we are quite far from normality in an absolute sense).

3. The power performance of the new test we introduced, the Gini test $G_n$ is quite decent even for moderate sample sizes like 20 and is nearly twice that of the other competitors considered. We have done more extensive simulations with several other alternatives with similar ordering in relative performance.

### 6 Conclusion

This paper introduces a new test for goodness of fit based on the Gini index of spacings. Its exact and asymptotic distributions are provided under the null hypothesis and a simulation study shows that this test has higher powers than others at certain common alternatives. There are several interesting open questions that are worth further investigation. For instance, one could study the range and the inter-quartile range of spacings, the more general statistic $G_n(r)$ mentioned in Remark 1, or the statistic $T_n$ defined in Equation (10). One could also investigate each of these statistics using non-overlapping or overlapping $m$-step spacings. Finally, it
would be of interest to theoretically derive the Pitman efficiency of such tests under smooth alternatives, as in Sethuraman and Rao (1969).

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