Motivation.

1. Questions and Motivation

Let us consider a linear torus directed graph (or directed network) of vertices \( \{1, \ldots, n\} \) in the sense that each node \( i \) in the network connects only with its neighboring vertex \( i+1 \) for \( i = 1, \ldots, n-1 \), and the boundary vertex \( n \) connects with vertex 1.

On some probability space, based on this torus graph let us consider the simple Ornstein-Uhlenbeck type system (or a Gaussian cascade)

\[
dX_{t,i} = (X_{t,i+1} - X_{t,i})dt + dW_{t,i}; \quad t \geq 0, \quad i = 1, \ldots, n-1, \quad (1)
\]

with initial and independently and identically distributed random variables \( X_{0,i} \), independent of standard Brownian motions \( (W_{t,i}) \), \( 1 \leq i \leq n \).

For comparison, on the same probability space, we consider a typical mean-field interacting system where each particle is attracted towards the mean \( \sum_{j=1}^{n} X_{t,j} / n \) of the system.

Questions.

Q1. What is the essential difference between the system (1) and (2) for large \( n \rightarrow \infty \)?

Q2. Can we detect the type of interaction from the particle behavior at the boundary vertex with initial independent and identically distributed random variables on some probability space, based on this torus graph let us consider a non-linear diffusion pair \( (X_{t,i}, \hat{X}_{t,i}) \), \( t \geq 0 \), described by the stochastic differential equation of McKean-Vlasov type:

\[
dX_{t}^{i} = b(t, X_{t}^{i}, F_{t}^{i})dt + dB_{t}; \quad t \geq 0, \quad (4)
\]

driven by a BM \( (B_{t}, t \geq 0) \), where \( F_{t}^{i} \) is the weighted p.m.

\[
F_{t}^{i}(\cdot) := u \cdot \delta_{\hat{X}_{t}^{i}}(\cdot) + (1 - u) \cdot \mathcal{L}_{X_{t}^{i}}(\cdot) \quad (5)
\]

of the Dirac mass \( \delta_{\hat{X}_{t}^{i}}(\cdot) \) of \( \hat{X}_{t}^{i} \) and law \( \mathcal{L}_{X_{t}^{i}} \) of \( X_{t}^{i} \). We shall assume that the law of \( X_{t}^{i} \) is identical to that of \( \hat{X}_{t}^{i} \), and \( \hat{X}_{t}^{i} \) is independent of the Brownian motion, i.e.

\[
\text{Law}(\{X_{t}^{i}, t \geq 0\}) \equiv \text{Law}(\{\hat{X}_{t}^{i}, t \geq 0\}) \quad (6)
\]

Let us also assume that the Brownian motion \( B \) is independent of the initial value \( (X_{0}^{i}, \hat{X}_{0}^{i}) \).

Proposition. Suppose that \( b(\cdot) \) is Lipschitz and is at most linear growth. Then, for each \( u \in [0,1] \) there exists a weak solution \((\Omega, \mathcal{F}, (\mathcal{F}_{t}), \mathbb{P})\), \((X_{t}^{i}, \hat{X}_{t}^{i}, B)\) to the infinite-dimensional McKean-Vlasov equation (4) with (5)-(6). This solution is unique in law.

Let us take a linear functional \( b(t,x,u) := - \int_{0}^{U} (x - y) \mu(dy) \) for \( t \geq 0 \), \( x \in \mathbb{R} \), \( \mu \in \mathcal{M}(\mathbb{R}) \) of mean-reverting type. (4) is reduced to

\[
dX_{t}^{i} = -(u(X_{t}^{i} - \hat{X}_{t}^{i}) + (1 - u)(X_{t}^{i} - \mathbb{E}[X_{T}^{i}]))dt + dB_{t}; \quad (7)
\]

As \( n \rightarrow \infty \), the first two components \((X_{t,j}, X_{t,j+1})\) of (3) converges weakly to \((X_{0}^{i}, \hat{X}_{0}^{i})\) in (7). Following the martingale method [O], one can show that the joint distribution and marginal distribution of \((X_{t}^{i}, \hat{X}_{t}^{i})\) satisfies an integral equation. The details, variants and fluctuation results are found in [DFI]. This answers the question Q1.

3. Detecting presence of mean-field

Assume \( X_{t}^{i} \equiv 0 \equiv \hat{X}_{t}^{i} \). Let us consider the following problem of a single observer: The observer only observes \( X_{t} := X_{t}^{i} \), \( t \geq 0 \) but does neither know the value \( u \in [0,1] \) nor \( \hat{X}_{t}^{i}, t \geq 0 \) in (7).

The value \( u \) in (7) indicates how much \( X_{t}^{i} \) is attracted towards the neighborhood \( \hat{X}_{t}^{i} \) and \((1-u)\) indicates how much it is attracted towards the average \( \mathbb{E}[X_{T}^{i}] \). In this context, Q2 is reformulated as Q2'. Only given the filtration \( \mathcal{F}^{X} := \{\sigma(X_{s}^{i}, 0 \leq s \leq t), t \geq 0\} \), can the observer detect the value \( u \in [0,1] \)?

Yes, after observing for a sufficiently long time! (7) with (6) is solvable explicitly, and a method-of-moments estimator is consistent:

\[
\lim_{T \to \infty} \hat{u}_{T} = u \quad \text{a.s.},
\]

where

\[
\hat{u}_{T} := \left[ 1 - \left( \frac{2}{T} \int_{0}^{T} X_{t}^{i} dt \right)^{-1/2} \right].
\]

A modified version

\[
\hat{u}_{T} := \left( \int_{0}^{T} X_{t}^{i} dt \right)^{-1} \left( \int_{0}^{T} X_{t}^{i} dt + \int_{0}^{T} \hat{X}_{t}^{i} dt \right) \equiv 1 - \left( \frac{2}{T} \int_{0}^{T} X_{t}^{i} dt \right)^{-1} (T - X_{T}^{i})
\]

of the conditional maximum likelihood estimator

\[
\hat{u} := \left( \int_{0}^{T} \mathbb{E}[\hat{X}_{T}^{i} X_{T}^{i}] dt \right)^{-1} \left( \int_{0}^{T} \hat{X}_{t}^{i} dt + \int_{0}^{T} \hat{X}_{t}^{i} dt \right)
\]

underestimates the value, i.e., \( \lim_{T \to \infty} \hat{u}_{T} = 1 - \sqrt{1 - u^{2}} \leq u \in [0,1] \).

The detailed study of \( \hat{u} \) and filtering problem is an ongoing research.

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References

