Mathematical Logic

“The point of philosophy is to start with something so simple as not to seem worth stating and to end with something so paradoxical that no one will believe it.”

Bertrand Russell
Informal questions for Mathematicians

- What is left to study in mathematics? A new way to add or multiply?
- What exactly do mathematicians do?

Mathematics is...

- The Science of Numbers
- Problem Solving
- Theorem Proving
- The Science of Reasoning
- The Science of Patterns
Mathematics is like...

- A language
- A science
- An art
- A process

Mathematics is...

- Vast
- Performed in a wide variety of ways
- By a wide variety of people
What do mathematicians do?

- Add, Multiply, Subtract, Divide, etc.
- Do Algebra, Make Geometry T-Proofs
- Solve Problems, Model Nature
- Experiment, Conjecture, Prove
- Precisely identify assumptions (axioms)
- Precisely define terms
- Categorize, Classify, Generalize, Reason

Is New Mathematics ...

- Discovered, Invented, or Created?
Keith Devlin on the Nature of Mathematics

- Mathematics is the Science of Patterns
- Not only the patterns of numbers (arithmetic)...
- But also the patterns of shapes (geometry), reasoning (logic), motion (calculus), surfaces and knots (topology), etc.

Reference:
Mathematics--The Science of Patterns: The Search for Order in Life, Mind and the Universe
(Scientific American Paperback Library)

Mathematics is like...Music

- Both appreciated by many professional scientists and mathematicians
- Similar tasks in learning: practice, drill, learn a language, learn to sight-read, learn aesthetics
- In Tasks and Roles:
  - Teach/Study
  - Compose (Experiment-Conjecture-Prove, Invent new mathematical ideas)
  - Conduct (Seminar Presentation at a Conference)
  - Perform (trained student of mathematics)
  - Improvise (problem solve: do all of the above...)
Einstein

- Perhaps mathematics is communicated via its special language...but new mathematical concepts do not always originate from “a language.”

Georg Cantor (1845-1918)

- Developed a systematic study of the “infinite” and transfinite numbers.
- Developed new concepts: ordinals, cardinals, and topological connectivity.
- His highly original views were vigorously attacked by contemporaries.
First Thoughts

Aristotle creates potential infinite and actual infinite
Only potential infinite is permissible to thought
Actual infinite is not “sensible”

Cantor

Hierarchy of Infinity
Transfinite numbers
The First Circle

~ Cantor shows that rational numbers are countable by explanation through a table
~ Although some numbers are repeated, all rational numbers are present
~ There is some way to enumerate all numbers on the real line
Cantor's Set Theory

~ Being able to count infinity in sets

~ Ratio of 1:1 between sets
Transfinite Numbers

A number that is greater than any finite number
Googol, googolplex, etc.
Symbolized by last letter of Greek alphabet, omega
Therefore, Cantor had an infinity of infinite numbers

Cardinality

A measure of the number of elements in a set

E.g. a set of 106 people in a movie theatre has cardinal number of 106.

Symbolized by the Hebrew letter aleph
Conclusion

~ There are as many rational numbers \(1/2, 1/3, \text{etc.}\) as there are counting numbers \(1, 2, 3, \text{etc}\)
~ Aleph null is lowest infinity
~ Always larger number with sub-intervals of number
  ex: \(\{1, 2, 3\} = 2^3 = 8\)

Axiom of Set Theory

“From any clearly defined property \(P\), we may specify the set of all sets that have that property.”

Examples:
\(E = \text{Empty set} = \{ x \mid x \text{ is not equal to } x\}\)
(Read: The set of all \(x\) such that \(x\) is not equal to \(x\).)
\(U = \text{Universal set} = \{x \mid x \text{ is equal to } x\}\)

Note:
\(E\) is not an element of \(E\).
\(U\) is an element of \(U\).
This looked fine but ...
Bertrand Russell sent a letter to Frege

- Russell’s set \( R = \{ x \mid x \text{ is not an element of } x \} \)
- Question: Is \( R \) in \( R \)? Is \( R \) not in \( R \)?
- Neither can be true...(Check it!)
- This problem in foundations became known as “Russell’s Paradox.”

Related Semantic Paradoxes

- Consider:
  “I am now lying to you.”
  “This statement is false.”

- Question:
  Are these statements true or false?
Paradox and Mystery...

- “The most beautiful thing we can experience is the mysterious. It is the source of all true art and science.”
  --Albert Einstein

Major Philosophical Views

1. Rationalism (we have already met)
2. Empiricism (we have already met)
3. Logicism
4. Formalism
5. Intuitionism