A Simpler Form of the Russell Paradox

- A barber claims he shaves those people, and only those people, who do not shave themselves. Does he shave himself?
  - If he does, he can’t; if he doesn’t, he must.
  - Some attempts have been made to disallow such definitions (the barber, or the set of sets that have themselves as members), calling them impredicative definitions (meaning self-referential) and claiming they should be disallowed.
  - Disallowing impredicative definitions eliminates some important mathematical results.
Russell’s Paradox - Why is this a Crisis?

- If a logical system contains any statement that is both true and false no proof can be trusted.

If $p$ is both true and false any proposition $q$ follows from $p$:

$$p \Rightarrow (p \cup q)$$

[because $p$ is true $p \cup$ any proposition is true]

$$ (p \cup q) \Rightarrow q$$

[p is false so $q$ must be true]

The Ramifications of the Crisis

- If set theory is in doubt, all of mathematics is in doubt because all propositions follow from a contradiction.
Attempts to Resolve the Crisis

- 1910 - Russell and Whitehead started to build mathematics from logic.
- On page 360 of Russell and Whitehead’s *Principia Mathematica* they reach almost far enough to establish that $1+1 = 2$

Principia Mathematica

- Russell said that the efforts on *Principia Mathematica* broke him and he was never as sharp afterwards. He went on to receive the Nobel Prize for Literature in 1951.
There are at least 3 approaches to resolving the third crisis, Logicism, Intuitionism, and Formalism.

**Logicism**
- Leibniz (1666)
- Dedekind (1888)
- Frege (1903)
- Peano (1909)
- Russell (1913)

Mathematics is a branch of logic. Whitehead and Russell are of this school and Principia Mathematica was a step toward building mathematics from logic.

**Intuitionism (~1909)**
- Brouwer (1908)
- Poincare (1906)
- Kronecker (1880s)

Build mathematics solely by finite constructive methods on the intuitively given sequence of the natural numbers.
Brouwer's Mathematics

- Mathematics is synthetic.
- Truth is a priori.
- Mathematical knowledge is mind dependent.
  - Anti Realist in Ontology
  - Anti Realist in Truth Value
  - But: non-empiricist
- Found mathematics on Kantian view of time.
  - Sequence of moments gives natural numbers
  - Infinite divisibility of moments gives linear continuum
- Found geometry on reals using Cartesian techniques.

Brouwer's Mathematical Method

- Mathematics is about idealized mental constructions
- Existential proofs require a construction procedure.
- No unknowable truths.
  - Knowledge is provability. Truths are provable statements.
- Impredicative definitions are circular.
- Impossible to consider infinite collections of mathematical objects as completed totalities.
  - Construction of the elements can never be completed.
  - Example of a meaningless sentence:
    The set of continuous univariate functions has property P.
### Example: reals

<table>
<thead>
<tr>
<th>Classical reals</th>
<th>Intuitionistic reals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Any decimal expansion</td>
<td>Early Brouwer</td>
</tr>
<tr>
<td>Digit n is fixed from the beginning for each n</td>
<td>rule-governed decimal expansions (Cauchy sequences)</td>
</tr>
<tr>
<td>Actual infinity</td>
<td>Later Brouwer</td>
</tr>
<tr>
<td></td>
<td>Free-choice sequences, Digit are iterated on demand by a \textit{creative subject}.</td>
</tr>
<tr>
<td></td>
<td>Potential infinity</td>
</tr>
</tbody>
</table>

### Intuitionistic logic

- \textit{Classical logic} is about truth
- \textit{Intuitionistic logic} is about constructibility
- Intuitionistic logic is therefore also called \textit{constructive logic}
- All statements are classically regarded as either true or false: \( p \lor \neg p \) holds (\textit{tertium non datur})
  - "there is 7 seven's somewhere in the decimal representation of the number"
  - "there are irrational \( p \) and \( q \) such that \( p^q \) is rational"
- Intuitionistic logic, in contrast, requires that we can \textit{construct} the objects whose existence we assert and hence rejects the \textit{tertium non datur} principle
Intuitionistic logic

- L.E.J. Brouwer (1881-1966)
- Arend Heyting (1898-1980)

Brouwer's Mathematics

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Brouwer's Mathematical Method

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Heyting

- Logic
  - Codification of the rules of communicating mathematics via language
  - Dependent on mathematics. New methods of reasoning require extensions of logical system
  - Not truth conditions, proof conditions.
Formalism

Formalism (Hilbert 1904)
Collaborators (1920):
• Bernays
• Ackermann
• von Neumann

Developed to eliminate the problems in set theory and save classical mathematics from contradictions.

Idea is that mathematics is concerned with formal symbolic systems devoid of meaning. It was believed (until the 1930s) that a system could be shown to be consistent through this approach.

The original vision fell after Gödel’s breakthroughs.

Is Mathematics Complete?

- Can every question within geometry be answered?
  By “answered” we mean proved either true or false.
Mathematics Can Never be Complete
Kurt Gödel

In 1931 Kurt Gödel, a 25 year-old mathematician at the University of Vienna, published a paper that changed the nature of mathematics:

- On Formally Undecidable Propositions of Principia Mathematica and Related Systems
- This was the death knell for any hope to “complete” mathematics or any branch of mathematics that encompasses arithmetic.

The Dream is Denied

- Gödel showed that even ordinary arithmetic of the integers cannot be fully axiomatized
- Worse yet, it is impossible to establish the internal logical consistency of even elementary arithmetic
- The hope for consistency is lost, the hope for a complete formalization of arithmetic, and hence virtually all of mathematics, is lost.
How Did Gödel Do It?

- Gödel’s work is based in meta-mathematics; he proved theorems about the nature of mathematics.
- Gödel’s proof is very difficult but we can get some idea of what it is about.

The Analogy with Chess

- In pure mathematics, we follow rules to carry out proofs.
- In chess the pieces and board are not interpreted as real people
- The starting position is analogous to the axioms
- The rules are the rules of inference,
- The legal positions are the theorems.
Breaking Away from Interpretations

- Bertrand Russell: pure mathematics is the subject in which we do not know what we are talking about, or whether what we are saying is true.
- Technical terms are implicitly defined by their use in axioms.

Gödel’s Arguments: the bridge between Mathematics and Meta-Mathematics

<table>
<thead>
<tr>
<th>Mathematical Statements</th>
<th>Meta-Mathematical Statements</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 + 3$</td>
<td>‘$2 + 3$’ is a formula</td>
</tr>
<tr>
<td>$X = 5$</td>
<td>$X$ is a variable</td>
</tr>
<tr>
<td>$5 = y + 1$</td>
<td>‘$5 = y + 1$’ has only one solution, $y = 4$</td>
</tr>
<tr>
<td></td>
<td>Arithmetic is consistent.</td>
</tr>
</tbody>
</table>
Meta-mathematical statements contain the names of variables, not the variables themselves.
We write: Orlando is south of Atlanta
but to talk about the words themselves we use quotes:
“Orlando” is not often misspelled or “Atlanta” is written with 7 letters.

Similarly we do not write $x = 5$ is an equation
we write “$x = 5$” is an equation.

Bridging the Gap

- By mapping Meta - Mathematical Statements to Mathematical Statements
- Assign a unique number to each symbol to be used.

Let $A$ be 1, $B$ be 2, $C$ be 3, ... $Z$ be 26, space be 27, and period be 28.
To a written document (novel, newspaper, this slide, mathematical proof) form a number as follows:
- for the $n^{th}$ character in the document raise the $n^{th}$ prime to the power of the code for the character in that position.
- Assign the product of the values so formed

A Sample Mapping

The word “HELLO” is mapped to $2^8 \times 3^5 \times 5^{12} \times 7^{12} \times 11^{15}$
This is a huge number (15 digits) for a single word, but it does not matter.
- Formal mathematics is not written in English, this is just to illustrate the mapping technique.
Gödel’s Method

Once Gödel developed a way to map mathematical proofs to natural numbers, he used proofs in the realm of the natural numbers to establish theorems about meta-mathematical statements.

Bibliography

- Gödel, Escher, Bach: An Eternal Golden Braid; Douglas R. Hofstadter, 1979
- Principia Mathematica Alfred North Whitehead & Bertrand Russell (1910)
- The Basic Writings of Bertrand Russell (1961)