PARABOLIC AND GAUSSIAN WHITE NOISE APPROXIMATION FOR WAVE PROPAGATION IN RANDOM MEDIA

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Abstract. The parabolic or forward scattering approximation has been used extensively in the study of wave propagation. This approximation is combined with a gaussian white noise approximation for waves propagating in a random medium. The validity of this approximation is proved for stratified weakly fluctuating random media in the high frequencies regime. The limiting distribution of the wave field is characterized as the unique solution of a random Schrödinger equation studied by Dawson and Papanicolaou [4].
The proofs are based on various generalizations of the perturbed test function method developped by Kushner [9].

Key words. waves in random media, parabolic approximation, diffusion-approximation

AMS subject classifications. 60H15, 35R60, 73D25

1. Introduction. We consider the reduced wave equation for the scalar field $\Psi(r), r \in \mathbb{R}^3$:

$$\nabla^2\Psi + k^2 n^2(r)\Psi = 0 \tag{1.1}$$

where k denotes the free space wave number and $n(r)$ a random index of refraction given by:

$$\begin{cases} n^2(r) = 1 + \beta\eta(\alpha r_1, \alpha r_2, r_3) & \text{for } 0 \le r_3 \le L \\ n^2(r) = 1 & \text{elsewhere} \end{cases} \tag{1.2}$$

The random field η is assumed to be real bounded and centered with mixing properties in the r_3-direction; β is a positive parameter measuring the size of the fluctuations such that $\beta|\eta|$ is bounded by a constant less than 1; α is a positive scale ratio between the r_3-direction and the (r_1, r_2) orthogonal directions and L is a length scale corresponding to a distance of propagation in the r_3-direction.
Equation (1.1) is to be solved on the half-space $\{r_3 \ge 0\}$ with the following boundary conditions:

$$\begin{cases} \Psi(r_1, r_2, 0) = \Psi_0(\alpha r_1, \alpha r_2) \\ \text{radiation condition on } r_3 > L \end{cases} \tag{1.3}$$

where Ψ_0 is a smooth function which has a bounded Fourier transform $\hat{\Psi}_0$ with compact support included in a ball centered at 0 with radius R; our radiation condition corresponds to the absence of left going wave on $r_3 > L$ (this condition will be made precise later on).

In this introduction, we shall exhibit various regimes for which the elliptic problem (1.1) with boundary conditions (1.3) is expected to converge to a stochastic parabolic problem with an initial condition at $r_3 = 0$. We are not concerned by the wellposedness of problem (1.1),(1.3): this will be guaranteed in the stratified model described in the next section.

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We are interested in the solution of (1.1) which propagates mainly in the r_3-direction. In order to center our problem, we define the field $u(r)$ by:

$$\Psi(r) = e^{\mathrm{i}kr_3}u(r) \tag{1.4}$$

which satisfies:

$$\left(\frac{\partial^2}{\partial r_1^2} + \frac{\partial^2}{\partial r_2^2}\right) u + \frac{\partial^2 u}{\partial r_3^2} + 2\mathrm{i}k\frac{\partial u}{\partial r_3} + k^2(n^2(r) - 1)u = 0 \tag{1.5}$$

with the boundary conditions corresponding to (1.3).
Introducing a new positive parameter ϵ, we perform the following rescaling:

$$\left\{ \begin{array}{l} (r_1, r_2) = \left(\dfrac{x_1}{\alpha}, \dfrac{x_2}{\alpha}\right) \\ r_3 = \dfrac{t}{\epsilon^2} \end{array} \right. \tag{1.6}$$

so that, using (1.2) and denoting (x_1, x_2) by x and $\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ by Δ, (1.5) becomes:

$$\mathrm{i}\frac{\partial u}{\partial t} + \frac{\epsilon^2}{2k}\frac{\partial^2 u}{\partial t^2} + \frac{\alpha^2}{2k\epsilon^2}\Delta u + \frac{k\beta}{2\epsilon^2}\eta(\frac{t}{\epsilon^2}, x)u = 0 \tag{1.7}$$

with the boundary conditions: $u(0, x) = \Psi_0(x_1, x_2)$ also denoted $u_0(x)$ and a radiation condition on $t > \epsilon^2 L$.
Let us assume that L is of order ϵ^{-2} such that $L = T/\epsilon^2$. Equation (1.7) is to be solved on $[0, T]$.
We are interested in the parabolic and white noise approximation of this equation: the parabolic or forward scattering approximation consists in neglecting the backscattered waves, namely the second derivative with respect to t, keeping of order 1 the second derivative in the orthogonal direction x. This approximation is commonly made in the litterature (Ishimaru[7],Sobszyk[14],Dawson-Papanicolaou[4],Nair-White[12]).
The second approximation is the white noise approximation: it consists in approximating $\dfrac{k\beta}{2\epsilon^2}\eta(\dfrac{t}{\epsilon^2}, x)$ by a white noise in the t-variable.
The parabolic approximation is of great interest because solving directly (1.5) requires a large amount of computations. It has been justified when the angle of propagation is small, far from the source and for small variations in the range of the sound speed (see for example Bamberger and al [2] or Bernardi-Pelissier[3]). On the other hand, the white noise approximation enables us to use stochastic calculus on the limiting field in order, for instance in our linear case, to close the equations for the moments of the solution.
The aim of this paper is to show that it can be made simultaneously for regimes corresponding to high frequencies.
We now reformulate our problem (1.7) in the regimes described above. For the parabolic approximation we impose $\frac{\alpha^2}{2k\epsilon^2}$ to be of order 1 and $\frac{\epsilon^2}{2k}$ to be small. For the white noise approximation we impose $\frac{k\beta}{2\epsilon^2}$ to be of order ϵ^{-1}.
Setting $\frac{\epsilon^2}{2k} = \epsilon^p$ for some parameter p, one gets:

$$\left\{ \begin{array}{l} k = \dfrac{1}{2}\epsilon^{2-p} \\ \beta = 4\epsilon^{p-1} \\ \alpha = \epsilon^{\frac{4-p}{2}} \end{array} \right. \tag{1.8}$$

The condition $1 \leq p \leq 4$ will ensure that β and α are small or at most of order 1. It will turn out that these approximations will be valid only in the high frequencies regimes which correspond to $p > 2$. So we assume:

$$2 < p \leq 4 \tag{1.9}$$

With p satisfying (1.9), our problem is now only ϵ dependent and it can be rewritten as:

$$\mathrm{i}\frac{\partial u^\epsilon}{\partial t} + \epsilon^p \frac{\partial^2 u^\epsilon}{\partial t^2} + \Delta u^\epsilon + \frac{1}{\epsilon}\eta(\frac{t}{\epsilon^2}, x)u^\epsilon = 0 \tag{1.10}$$

with the boundary conditions:

$$\begin{cases} u^\epsilon(0,x) = u_0(x) \\ \text{radiation condition on } t > T \end{cases} \tag{1.11}$$

REMARK 1.1. In the applications, β and α will be given; we are interested in the cases where β is small and α at most of order 1 so that by (1.8) our parameter ϵ is of order $(\alpha^2\beta)^{1/3}$, k is of order $(\frac{\alpha}{\beta})^{2/3}$. For instance, the case $\alpha = 1$, β small $(p = 4)$ corresponds to the model used in [12],[13].

Denoting by $W(t,x)$ the infinite dimensional brownian motion obtained as the limit in law of the process $W^\epsilon(t,x) = \frac{1}{\epsilon}\int_0^t \eta(\frac{s}{\epsilon^2}, x)ds$ and assuming η to be stationary in the x-direction, $u^\epsilon(x,t)$ should converge to a solution of the following stochastic equation:

$$u(t,x) = u_0(x) + \int_0^t \left(\mathrm{i}\Delta - \frac{1}{2}\sigma^2\right) u(s,x)ds + \mathrm{i}\int_0^t u(s,x)dW(s,x) \tag{1.12}$$

where the stochastic integral is an Itô integral and $\frac{1}{2}\sigma^2 u$ is a Stratonovich correction with $\sigma^2 = \int_0^\infty \mathbb{E}\eta(s,0)\eta(0,0)ds$.
This random Schrödinger equation has been studied by Dawson-Papanicolaou: they proved the uniqueness of the solution. One can observe that such a limiting result requires to take a limit in an elliptic problem (1.10) with boundary conditions (1.11) and to obtain a parabolic problem with an initial condition; in particular the radiation condition in (1.10) disappears in the limit.
In order to explain how we shall derive such a result let us introduce the following parabolic problem:

$$\mathrm{i}\frac{\partial v^\epsilon}{\partial t} + \Delta v^\epsilon + \frac{1}{\epsilon}\eta(\frac{t}{\epsilon^2}, x)v^\epsilon = 0 \tag{1.13}$$

with the initial condition:

$$v^\epsilon(0,x) = u_0(x) \tag{1.14}$$

We first prove that $v^\epsilon(t,x)$ converges in law, as ϵ goes to zero, to the solution $u(t,x)$ of (1.12). This is done on the space of $\mathbb{L}^2$-valued continuous processes. We then expand $u^\epsilon(t,x)$ as an infinite series involving $u_0(x)$, $\frac{\partial u^\epsilon}{\partial t}(0,x)$ and their derivatives. This gives a strong solution of equation (1.10) and requires strong uniform estimates on $\frac{\partial u^\epsilon}{\partial t}(0,x)$ and its derivatives. We then show that $u^\epsilon(t,x)$ has the same limit as $v^\epsilon(t,x)$. In order to derive rigorously the strong uniform estimates needed, we shall restrict ourself to a particular model, namely a stratified random medium, that we

describe in the next section.
To end this introduction we show the relation between $\frac{\partial u^\epsilon}{\partial t}(0,x)$ denoted by $u_1^\epsilon(x)$ and $\frac{\partial \Psi}{\partial r_3}(r_1,r_2,0)$ denoted by $\Psi_1^\epsilon(\alpha r_1,\alpha r_2)$ in the α-scale: we already have that $u_0^\epsilon(x) = u_0(x) = \Psi_0(x)$ and by using (1.4) we get that $u_1^\epsilon(x) = \frac{1}{\epsilon^2}(\Psi_1^\epsilon(x) - \mathrm{i}k\Psi_0(x))$; note that $\Psi_1^\epsilon(x)$ depends on $\Psi_0(x)$ itself and the entire equation through the radiation condition at infinity. It will be easier to work out estimates on the field Ψ and deduce the corresponding estimates on $u_1^\epsilon(x)$ thanks to the previous relation. This is the content of the next section. In the third section we give the mixing condition assumed on the noise η and we state our results. The last section will be devoted to the proofs.

2. The stratified model. Let $\{\eta_n, n \geq 1\}$ be a sequence of random variables defined on the same probability space $(\Omega_\eta, \mathcal{A}_\eta, \mathbb{P}_\eta)$. We assume that these random variables are uniformly bounded and centered. We do not assume the independence of these random variables but a mixing property will be needed in the next section. For a given integer N we define the following pure jump process:

$$\eta(r_3) = \sum_{n=1}^{N} \eta_n \mathbb{1}_{[n-1,n)}(r_3) \tag{2.1}$$

This will be our noise η in (1.2) with N of order L. We could have considered a more general model with varying distances between jumps. In each interval $(n-1,n)$ equation (1.1), in the α-scale and in the Fourier domain in the orthogonal direction, becomes:

$$\frac{\partial^2 \hat{\Psi}_n}{\partial r_3^2} + (k^2(1+\beta\eta_n) - \alpha^2|\omega|^2)\hat{\Psi}_n = 0 \tag{2.2}$$

The continuity condition of the solution and its derivative at the interfaces reads:

$$\left\{ \begin{array}{l} \hat{\Psi}_n(n) = \hat{\Psi}_{n+1}(n) \\ \dfrac{\partial \hat{\Psi}_n}{\partial r_3}(n) = \dfrac{\partial \hat{\Psi}_{n+1}}{\partial r_3}(n) \end{array} \right. \tag{2.3}$$

The solution is zero for frequencies $|\omega|^2$ greater than R^2 since we have supposed that $\hat{\Psi}_0$ has a support included in the ball centered at zero with radius R. k being of order ϵ^{2-p} which goes to infinity under the hypothesis (1.9), we may always suppose that the coefficients $k_n^2 = k^2(1+\beta\eta_n)$ are greater than $\alpha^2|\omega|^2$ for frequencies in the support of $\hat{\Psi}_0$. With this remark we can decompose, on each interval $(n-1,n)$, the solution $\hat{\Psi}_n$ into right and left going waves:

$$\hat{\Psi}_n(r_3,\omega) = a_n(\omega)e^{\mathrm{i}\sqrt{k_n^2-|\alpha\omega|^2}r_3} + b_n(\omega)e^{-\mathrm{i}\sqrt{k_n^2-|\alpha\omega|^2}r_3} \tag{2.4}$$

The relations (2.3) give $2N$ equations; the boundary conditions (1.3) give the following two equations:

$$\left\{ \begin{array}{l} \Psi(0,x) = \Psi_0(x) \\ \text{radiation condition} \end{array} \right. \Rightarrow \left\{ \begin{array}{l} a_1(\omega) + b_1(\omega) = \hat{\psi}_0(\omega) \\ b_{N+1} = 0 \end{array} \right. \tag{2.5}$$

b_{N+1} corresponding to the left going wave in the domain $(N,+\infty)$ with $k_{N+1} = k$. Our $2(N+1)$ unknowns $(a_n(\omega), b_n(\omega))$, $1 \leq n \leq N+1$ are uniquely determined by

(2.3) and (2.5).
Using the following notation:

$$h_n = \sqrt{k_n^2 - |\alpha\omega|^2} = \sqrt{k^2(1+\beta\eta_n) - |\alpha\omega|^2} \tag{2.6}$$

we easily obtain:

$$\left[\begin{array}{c} a_n \\ b_n \end{array}\right] = M_n \left[\begin{array}{c} a_{n+1} \\ b_{n+1} \end{array}\right]$$

where:

$$M_n = \begin{pmatrix} (\frac{1}{2} + \frac{h_{n+1}}{h_n})e^{\mathrm{i}(h_{n+1}-h_n)} & (\frac{1}{2} - \frac{h_{n+1}}{h_n})e^{-\mathrm{i}(h_{n+1}-h_n)} \\ (\frac{1}{2} - \frac{h_{n+1}}{h_n})e^{\mathrm{i}(h_{n+1}-h_n)} & (\frac{1}{2} + \frac{h_{n+1}}{h_n})e^{-\mathrm{i}(h_{n+1}-h_n)} \end{pmatrix}$$

Denoting the product $M_1 M_2 \dots M_q \dots M_N$ by Π_N, we have:

$$\left[\begin{array}{c} a_1 \\ b_1 \end{array}\right] = \Pi_N \left[\begin{array}{c} a_{N+1} \\ 0 \end{array}\right] \tag{2.7}$$

Using the matrix norm $|M| = \sqrt{\frac{1}{2}\mathrm{trace} MM^*}$ which satisfies $|MN| \leq |M|.|N|$, one easily obtains:

$$|M_n|^2 = \frac{1}{2}\left(1 + \frac{h_{n+1}^2}{h_n^2}\right) \quad \text{and} \quad \det M_n = \frac{h_{n+1}}{h_n}$$

each matrix M_n can be written as:

$$M_n = \begin{pmatrix} f_n & \bar{g}_n \\ g_n & \bar{f}_n \end{pmatrix}$$

the set of these matrices being a multiplicative group, Π_N can also be written as:

$$\Pi_N = \begin{pmatrix} f & \bar{g} \\ g & \bar{f} \end{pmatrix}$$

The relation (2.7) implies that:

$$a_1 = f a_{N+1} \quad \text{and} \quad b_1 = g a_{N+1}$$

which combined with (2.5) gives:

$$\begin{cases} a_1 = \dfrac{f}{f+g}\hat{\psi}_0 \\ b_1 = \dfrac{g}{f+g}\hat{\psi}_0 \end{cases} \tag{2.8}$$

where f and g depend on $\alpha\omega$, k, β and $\{\eta_n, 1 \leq n \leq N\}$. The following result is the estimate needed:

PROPOSITION 2.1. *There exists a constant C and $\nu(\epsilon)$ going to zero as ϵ goes to zero such that:*

$$|\hat{u}_1^\epsilon(\omega)| \leq \frac{C|\hat{u}_0(\omega)|}{\epsilon^p}\nu(\epsilon) \tag{2.9}$$

Proof. Computing the derivative with respect to r_3 at the origin, we deduce:

$$\hat{\Psi}_1 = \mathrm{i}h_1a_1 - \mathrm{i}h_1b_1$$

Using the relations between Ψ and u given at the end of the introduction, we get:

$$\begin{aligned}
\hat{u}_1^\epsilon &= \frac{1}{\epsilon^2}(\hat{\Psi}_1^\epsilon - \mathrm{i}k\hat{\Psi}_0)\\
&= \frac{\mathrm{i}}{\epsilon^2}(h_1a_1 - h_1b_1 - k\hat{\Psi}_0)\\
&= \frac{\mathrm{i}}{\epsilon^2}(\frac{h_1f}{f+g} - \frac{h_1g}{f+g} - k)\hat{\Psi}_0
\end{aligned}$$

therefore:

$$\hat{u}_1^\epsilon = \frac{\mathrm{i}}{\epsilon^2}(\frac{f}{f+g}(h_1-k) - \frac{g}{f+g}(h_1+k))\hat{\Psi}_0$$

Since $\hat{\Psi}_0 = \hat{u}_0$ we get:

$$|\hat{u}_1^\epsilon(\omega)| \leq \frac{1}{\epsilon^2}(|f|.|h_1-k| + |g|.|h_1+k|)|f+g|^{-1}|\hat{u}_0| \tag{2.10}$$

using (2.6) with $n = 1$ and the renormalizations (1.8) we easily deduce the existence of constants C_1 and C_2 and $\nu_1(\epsilon)$ going to 0 as ϵ goes to 0 such that:

$$|h_1 - k| \leq \frac{C_1}{\epsilon^{p-2}}\nu_1(\epsilon) \quad \text{and} \quad |h_1 + k| \leq \frac{C_2}{\epsilon^{p-2}} \tag{2.11}$$

we have:

$$\det \Pi_N = |f|^2 - |g|^2 = \prod_{n=1}^{N} \det M_n = \frac{h_{N+1}}{h_1}$$

which implies that there exists $\nu_2(\epsilon)$ going to 0 as ϵ goes to 0 such that:

$$|f|^2 - |g|^2 = 1 + \nu_2(\epsilon) \tag{2.12}$$

Also:

$$\begin{aligned}
|\Pi_N|^2 &= |f|^2+|g|^2 \leq \prod_{n=1}^{N} |M_n|^2 = \prod_{n=1}^{N} \frac{1}{2}\left(1+\frac{h_{n+1}^2}{h_n^2}\right) \\
&= \prod_{n=1}^{N} \frac{2k^2+k^2\beta(\eta_n+\eta_{n+1})-2\alpha^2|\omega|^2}{2k^2(1+\beta\eta_n)-2\alpha^2|\omega|^2} \\
&= \prod_{n=1}^{N}\left(1+\frac{k^2\beta(\eta_n-\eta_{n+1})}{2k^2(1+\beta\eta_n)-2\alpha^2|\omega|^2}\right)
\end{aligned}$$

Using the uniform boundedness of the (η_n) one can deduce the existence of a constant C_3 such that:

$$\log|\Pi_N|^2 \leq C_3(\beta+\beta^2 N)$$

N being of order ϵ^{-2} and p being strictly greater then 2, $\beta^2 N$ goes to 0 as ϵ goes to 0 and therefore there exists $\nu_3(\epsilon)$ going to 0 as ϵ goes to 0 such that:

$$|\Pi_N|^2 = |f|^2+|g|^2 \leq 1+\nu_3(\epsilon) \tag{2.13}$$

¿From (2.12) and (2.13) we deduce that $\lim_{\epsilon\to 0}|f|=1$, $\lim_{\epsilon\to 0}|g|=0$ and $\lim_{\epsilon\to 0}|f+g|=1$ which combined with (2.10) and (2.11) give the desired estimate (2.9). □

To conclude this section we may observe that the estimate (2.9) is a purely deterministic result due to the smallness of the stratified noise and the high frequencies regime. However $u_1^\epsilon(x)$ is measurable with respect to the $\sigma-$field generated by the (η_n) up to n of order ϵ^{-2}.

3. Statement of the result. Let us first rewrite our problem (1.10) in the stratified model described in the previous section. T_0 and T being fixed such that $0<T_0<T$, we define $\eta(t)$, $\eta^\epsilon(t)$ and $\eta_T^\epsilon(t)$ for $t\geq 0$ as follows:

$$\begin{cases}
\eta(t) = \sum_{n=1}^{\infty} \eta_n \mathbb{1}_{[n-1,n)}(t) \\
\eta^\epsilon(t) = \frac{1}{\epsilon}\eta(\frac{t}{\epsilon^2}) \\
\eta_T^\epsilon(t) = \eta^\epsilon(t)\mathbb{1}_{\{t\leq \epsilon^2[\frac{T}{\epsilon^2}]\}}
\end{cases} \tag{3.1}$$

where $N=[\frac{T}{\epsilon^2}]$ denotes the integer part of $\frac{T}{\epsilon^2}$, the number of layers in our model. The random process $\{u^\epsilon(t,x),\, t\geq 0,\, x\in\mathbb{R}^2\}$ is defined as the unique solution of:

$$\mathrm{i}\frac{\partial u^\epsilon}{\partial t}+\epsilon^p\frac{\partial^2 u^\epsilon}{\partial t^2}+\Delta u^\epsilon+\eta_T^\epsilon(t)u^\epsilon = 0 \tag{3.2}$$

with the boundary conditions (1.3).

We consider u^ϵ as a stochastic process with continuous trajectories from $[0,T_0]$ in $\mathbb{L}^2_{\mathbb{C}}(\mathbb{R}^2)$ equipped with its weak topology and denoted by $\mathbb{L}^2_w$. We shall denote by $<.>$ and $\|.\|$ the scalar product and the hilbertian norm in the space $\mathbb{L}^2$. To (3.2) we associate the following parabolic problem:

$$\mathrm{i}\frac{\partial v^\epsilon}{\partial t}+\Delta v^\epsilon+\eta_T^\epsilon(t)v^\epsilon = 0 \tag{3.3}$$

with the initial value $v^\epsilon(0,x)=u_0(x)$.
Replacing η_T^ϵ by η^ϵ in (3.3) does not affect $v^\epsilon(t,x)$ for $t\le T_0$.
We give now the precise hypothesis on the noise $(\eta_n)_{n\ge1}$. We suppose that the sequence of uniformly bounded centered random variables $(\eta_n)_{n\ge1}$ is stationary; denoting by $\mathcal{F}_m^n$ the $\sigma-$field generated by $\{\eta_k,\ m\le k\le n\}$, we define the mixing coefficient $\phi(n)$ by:

$$\phi(n)=\sup_{A\in\mathcal{F}_1^m,B\in\mathcal{F}_{m+n}^\infty}|P(B/A)-P(B)|$$

and we assume that:

$$\sum_{n=1}^\infty \phi(n)^{1/2}<\infty \tag{3.4}$$

In terms of the process $\eta(t)$ we define $\mathcal{F}_0^t$ as $\mathcal{F}_1^{[t]+1}$ and we denote $\mathcal{F}_0^{\frac{t}{\epsilon^2}}$ by $\mathcal{F}_t^\epsilon$ and the conditional expectation with respect to $\mathcal{F}_t^\epsilon$ by $\mathbb{E}_t^\epsilon$. The following estimates are the basic ingredients in the perturbed test function method, they are consequences of (3.4) (see Kushner[9]):

$$\begin{cases}\left|\int_t^T \mathbb{E}_t^\epsilon\eta^\epsilon(s)ds\right|\le C\epsilon\\ \left|\eta^\epsilon(t)\int_t^T \mathbb{E}_t^\epsilon\eta^\epsilon(s)ds\right|\le C\\ \lim_{\epsilon\to0}\mathbb{E}\left|\int_0^t\int_s^T(\mathbb{E}_t^\epsilon\eta^\epsilon(s)\eta^\epsilon(u)-\mathbb{E}\eta^\epsilon(s)\eta^\epsilon(u))dsdu\right|=0\\ \lim_{\epsilon\to0}\int_0^t\int_s^T\mathbb{E}\eta^\epsilon(s)\eta^\epsilon(u)dsdu=t\int_0^\infty\lim_{A\to\infty}\frac{1}{A}\int_0^A\mathbb{E}\eta(s)\eta(s+u)dsdu\end{cases}\tag{3.5}$$

(for a constant C independent from ϵ). We shall denote:

$$\sigma^2=2\int_0^\infty\lim_{A\to\infty}\frac{1}{A}\int_0^A\mathbb{E}\eta(s)\eta(s+u)dsdu=\mathbb{E}\eta_1^2+2\sum_{n=2}^\infty\mathbb{E}(\eta_1\eta_n)\tag{3.6}$$

Under these hypothesis, the process $\int_0^t\eta^\epsilon(t)ds$ converges in distribution, as ϵ goes to 0, on $C[0,T_0;\mathbb{R}]$ to the brownian motion σB_t where B_t is a standard brownian motion (cf Ethier-Kurtz[6]).
Our first result concerns the convergence in distribution on $C[0,T_0;\mathbb{L}_w^2]$ of v^ϵ, solution of (3.3), to the unique solution of the following random Schrödinger equation:

$$u(t,x)=u_0(x)+\int_0^t\left(\mathrm{i}\Delta-\frac{1}{2}\sigma^2\right)u(s,x)ds+\mathrm{i}\sigma\int_0^t u(s,x)dB_s\tag{3.7}$$

We point out that although $C[0,T_0;\mathbb{L}_w^2]$ is not a Polish space, Prohorov criterium for weak relative compactness of sets of probability measures holds (cf Métivier[11]).

THEOREM 3.1. *v^ϵ converges in distribution, as ϵ goes to 0, on $C[0,T_0;\mathbb{L}_w^2]$ to the solution u of equation (3.7).*

Proof. The following two properties imply the tightness in $C[0,T_0;\mathbb{L}_w^2]$:

i/ $\sup_{\epsilon} \mathbb{E}\left(\sup_{t\in[0,T_0]} \|v^\epsilon(t,.)\|^2\right) < \infty.$

ii/ For all θ in a dense subset of $\mathbb{L}^2$, the sequence $< v^\epsilon, \theta >$ is tight in $C[0, T_0; \mathbb{C}]$.
¿From (3.3) we deduce that:

$$2\frac{\partial}{\partial t}|v^\epsilon(t,x)|^2 = \mathrm{i}\bar{v}^\epsilon(t,x)\Delta v^\epsilon(t,x) - \mathrm{i}v^\epsilon(t,x)\Delta\bar{v}^\epsilon(t,x)$$

which implies that $\frac{\partial}{\partial t}\|v^\epsilon(t,.)\|^2 = 0$ and therefore:

$$\|v^\epsilon(t,.)\| = \|v^\epsilon(0,.)\| = \|u_0\|, \ \forall t$$

On the other hand $u_0 = \Psi_0$, described in the introduction; it has a bounded Fourier trasnform with compact support and hence is in $\mathbb{L}^2$. This proves $i/$.
Denoting by $X^\epsilon(t,\omega)$ the Fourier transform of $v^\epsilon(t,x)$, we have:

$$\frac{dX^\epsilon_t}{dt} = -\mathrm{i}|\omega|^2 X^\epsilon_t + \mathrm{i}\eta^\epsilon(t)X^\epsilon_t, \quad X^\epsilon(0,\omega) = \hat{u}_0(\omega) \tag{3.8}$$

X^ϵ is explicitely given by:

$$X^\epsilon(t,\omega) = \hat{u}_0(\omega)e^{-\mathrm{i}|\omega|^2 t + \mathrm{i}\int_0^t \eta^\epsilon(s)ds} \tag{3.9}$$

θ being a test function in $\mathbb{L}^2$, we get:

$$< v^\epsilon(t,.), \theta > = < X^\epsilon(t,.), \hat{\theta} > = e^{\mathrm{i}\int_0^t \eta^\epsilon(s)ds} \int_{\mathbb{R}^2} \hat{\theta}(\omega)\hat{u}_0(\omega)e^{-\mathrm{i}|\omega|^2 t}d\omega$$

which obviously converges in distribution to:

$$e^{\mathrm{i}\sigma B_t}\int_{\mathbb{R}^2} \hat{\theta}\hat{u}_0(\omega)e^{-\mathrm{i}|\omega|^2 t}d\omega$$

and shows that $ii/$ is satisfied. Denoting by $X(t,\omega)$ the limit in distribution of $X^\epsilon(t,\omega)$, we have:

$$X(t,\omega) = \hat{u}_0(\omega)e^{-\mathrm{i}|\omega|^2 t + \sigma B_t} \tag{3.10}$$

which is the unique solution of:

$$\left\{\begin{array}{ll} dX_t & = -\mathrm{i}|\omega|^2 X_t dt + \mathrm{i}\sigma X_t \circ dB_t \\ & = -(\mathrm{i}|\omega|^2 + \frac{1}{2}\sigma^2)X_t dt + \mathrm{i}\sigma X_t dB_t \\ X_0 & = \hat{u}_0 \end{array}\right. \tag{3.11}$$

The first stochastic integral being a stratonovich integral and the second one an Itô integral.

By Fourier inverse, we identify the limit in distribution of v^ϵ as the unique solution of (3.4).
□

REMARK 3.1. The proof of this result is very simple because we have an explicit representation of the solution as a continuous functional of the process $\int_0^t \eta^\epsilon(s)ds$ which converges in distribution to σB_t. This is obviously due to the fact that our noise does not depend on the orthogonal direction x in the stratified model.
We would like to point out that the theorem 3.1 remains true for a general class of noise $\eta(t,x)$, stationary in the x variable and mixing in the t direction as we assumed above for $\eta(t)$. The proof is much more involved and requires to carry out the perturbed test function method through the dual martingale problem characterizing the solution of (1.12) as in Dawson and Papanicolaou[4]. Details can be found in Bailly[1].

We turn now to the main difficulty of this paper, namely the convergence of u^ϵ solution of (3.2) to the unique solution u of (3.7). The main result of this paper is:
THEOREM 3.2.
a/ $(u^\epsilon)_{\epsilon>0}$ *is tight in* $C[0,T_0;\mathbb{L}^2_w]$.
b/ $\lim_{\epsilon\to 0}\sup_{t\leq T_0} \mathbb{E}\|u^\epsilon(t,x)-v^\epsilon(t,x)\|^2 = 0$.
c/ u^ϵ *converges in distribution, as* ϵ *goes to 0, on* $C[0,T_0;\mathbb{L}^2_w]$ *to the unique solution of equation (3.7).*

REMARK 3.2. $c/$ is a direct consequence of $a/$, $b/$ and theorem 3.1 since $b/$ and theorem 3.1 imply the convergence of the finite dimensional distributions of u^ϵ to the one's of u and $a/$ insures the tightness of (u^ϵ).

REMARK 3.3. El-Ansary and Khalil (cf [5]) have studied the following problem:

$$\frac{\partial \xi^\epsilon}{\partial t} + \epsilon^p \frac{\partial^2 \xi^\epsilon}{\partial t^2} + \frac{1}{\epsilon} F(\eta(\frac{t}{\epsilon^2}), \xi^\epsilon) = 0,\ t>0 \tag{3.12}$$

with initial conditions ξ^ϵ and its derivatives at $t=0$.

Using directly the perturbed test function method of Kushner, they proved a convergence result for the solution of (1.10) under regularity assumptions on F.

This case is quite different from ours. Indeed ξ^ϵ is the solution of a Cauchy problem instead of a Dirichlet problem. On the other hand, it is not possible to carry out the perturbation method directly in the complex case. We succeeded in solving these difficulties only when F is a linear mapping.

We shall now derive a series expansion for $u^\epsilon(t,x)$ and decompose the proof of theorem 3.2 into lemmas. The proof of these lemmas will be given in the last section. From now on we shall not make any difference between $\eta^\epsilon(t)$ and $\eta^\epsilon_T(t)$ for notational simplicity.

3.1. Expansion of the solution. Let us first consider the following equation:

$$\epsilon^p \frac{\partial^2 u^\epsilon}{\partial t^2} + \mathrm{i}\frac{\partial u^\epsilon}{\partial t} = 0 \tag{3.13}$$

Its general solution can be written as:

$$Ae^{\mathrm{i}\lambda_1^\epsilon t} + Be^{\mathrm{i}\lambda_2^\epsilon t}$$

where $\lambda_1^\epsilon = -\frac{1}{\epsilon^p}$ and $\lambda_2^\epsilon = 0$.
Writing the solution of (3.2) as:

$$u^\epsilon(t,x) = f^\epsilon(t,x)e^{\mathrm{i}\lambda_1^\epsilon t} + g^\epsilon(t,x)e^{\mathrm{i}\lambda_2^\epsilon t}$$

we apply the variation of constant method which means that f^ϵ and g^ϵ should satisfy:

$$(f^\epsilon)'(t,x)e^{\mathrm{i}\lambda_1^\epsilon t} + (g^\epsilon)'(t,x)e^{\mathrm{i}\lambda_2^\epsilon t} = 0$$

which combined with (3.2) and the notation $u_1^\epsilon(x) = \frac{\partial u^\epsilon}{\partial t}(0,x)$ give:

$$(3.14)\begin{cases} f^\epsilon(t,x) = \mathrm{i}\epsilon^p u_1^\epsilon(x) - \mathrm{i}\displaystyle\int_0^t e^{-\mathrm{i}\lambda_1^\epsilon s}\left(\eta^\epsilon(s)u^\epsilon(s,x) + \Delta u^\epsilon(s,x)\right)ds \\ \\ g^\epsilon(t,x) = u_0(x) - \mathrm{i}\epsilon^p u_1^\epsilon(x) + \mathrm{i}\displaystyle\int_0^t e^{-\mathrm{i}\lambda_2^\epsilon s}\left(\eta^\epsilon(s)u^\epsilon(s,x) + \Delta u^\epsilon(s,x)\right)ds \end{cases}$$

Therefore $u^\epsilon(t,x)$ satisfies the implicit equation:

$$(3.15)\quad \begin{aligned} u^\epsilon(t,x) &= \mathrm{i}\epsilon^p u_1^\epsilon(x)e^{\mathrm{i}\lambda_1^\epsilon t} + (u_0(x) - \mathrm{i}\epsilon^p u_1^\epsilon(x))e^{\mathrm{i}\lambda_2^\epsilon t} \\ &\quad + \mathrm{i}\int_0^t \left[e^{\mathrm{i}\lambda_2^\epsilon(t-s)} - e^{\mathrm{i}\lambda_1^\epsilon(t-s)}\right]\left(\eta^\epsilon(s)u^\epsilon(s,x) + \Delta u^\epsilon(s,x)\right)ds \end{aligned}$$

Iterating (3.15) gives a series expansion for $u^\epsilon(t,x)$: in order to write this series let us introduce some notations:

- For $n \geq 1$, A_n denotes the set $\{0,1\}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ its elements.
- For $n \geq 0$, B_n denotes the set $\{\lambda_1^\epsilon, \lambda_2^\epsilon\}^{n+1}$ and $\beta = (\beta_1, \ldots, \beta_{n+1})$ its elements.
- $|\beta|$ denotes the number of times that λ_1^ϵ appears in $(\beta_1, \ldots, \beta_n)$ for an element β in B_n.
- $f^\epsilon(0,x) = \mathrm{i}\epsilon^p u_1^\epsilon(x)$ (resp. $g^\epsilon(0,x) = u_0(x) - \mathrm{i}\epsilon^p u_1^\epsilon(x)$) is denoted by $H^{\lambda_1^\epsilon}(x)$ (resp. $H^{\lambda_2^\epsilon}(x)$).
- Δ^1 and Δ^0 will denote respectively the laplace operator Δ and the identity operator.

For $(\alpha,\beta) \in A_n \mathrm{x} B_n$, $n \geq 1$, we set:

$$(3.16)\quad \begin{aligned} S^\epsilon_{\alpha,\beta}(t,x) &= (-1)^{|\beta|}(\mathrm{i})^n e^{\mathrm{i}\beta_1 t}\int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n e^{\mathrm{i}\left(\sum_{j=2}^{n+1}(\beta_j - \beta_{j-1})\, t_{j-1}\right)} \\ &\prod_{i=1}^n (\alpha_1\eta^\epsilon(t_i) + (1-\alpha_i))\Delta^{1-\alpha_1}\circ\ldots\circ\Delta^{1-\alpha_n}H^{\beta_{n+1}}(x) \end{aligned}$$

and for $n = 0$, $B_0 = \{\lambda_1^\epsilon, \lambda_2^\epsilon\}$ and:

$$\sum_{\alpha \in A_0} S^\epsilon_{\alpha,\beta}(t,x) = S^\epsilon_\beta(t,x) = e^{\mathrm{i}\beta t} H^\beta(x) \tag{3.17}$$

For ϵ fixed, if it converges, the following series is the unique solution of (3.2):

$$u^\epsilon(t,x) = \sum_{n=0}^{\infty} \sum_{\alpha \in A_n, \beta \in B_n} S^\epsilon_{\alpha,\beta}(t,x) \tag{3.18}$$

The convergence of this series will be obtained later on, which will give a strong solution to (3.2) and requires strong regularity assumptions made on $u_0(x)$ as well as on $u_1^\epsilon(x)$ obtained in section 2 (2.9).
The solution v^ϵ of equation (3.3) is given formally by:

$$v^\epsilon(t,x) = e^{\mathrm{i}\int_0^t (\Delta + \eta^\epsilon(s))ds} u_0(x)$$

and can also be expanded:

$$v^\epsilon(t,x) = \sum_{n=0}^{\infty} \sum_{\alpha \in A_n} \tilde{S}^\epsilon_{\alpha,\{\lambda_2^\epsilon\}^{n+1}}(t,x) \tag{3.19}$$

where $\tilde{S}^\epsilon_{\alpha,\beta}$ is defined as $S^\epsilon_{\alpha,\beta}$ in (3.16) with $g^\epsilon(0,x)$ replaced by $u_0(x)$.

3.2. Main steps of the proof. The proof of theorem 3.2 will be divided into four lemmas. We first give a definition which will be useful when dealing with series:

DEFINITION 3.3. *A sequence $(e_n)_{n>0}$ of non-negative real numbers belongs to the class S if and only if:*

$$\forall\, a > 0,\ \sum_{n \geq 0} a^n e_n < \infty$$

LEMMA 3.4. *The sequence (e_n^ϵ) defined by:*

$$e_n^\epsilon = \sum_{\alpha \in A_n, \beta \in B_n} \mathbb{E} \sup_{t \leq T_0} \|S^\epsilon_{\alpha,\beta}(t,x)\|^2$$

is in S uniformly with respect to ϵ, i.e.:

$$\forall\, a > 0,\ \exists C(a) < \infty,\ \sup_\epsilon \sum_{n=0}^{\infty} a^n e_n^\epsilon \leq C(a)$$

LEMMA 3.5. *For every test function θ in $\mathbb{L}^2$ the family of continuous processes $(< u^\epsilon(t,x), \theta(x) >, t \geq 0)_{\epsilon>0}$ is tight in $C[0,T_0;\mathbb{C}]$.*

These two lemmas imply the tightness of u^ϵ (part $a/$ of theorem 3.2). Indeed:

$$\begin{aligned}
|u^\epsilon(t,x)|^2 &\leq \left|\sum_{n=0}^{\infty} \sum_{\alpha\in A_n \beta\in B_n} S^\epsilon_{\alpha,\beta}(t,x)\right|^2 \\
&\leq \left|\sum_{n=0}^{\infty} 2^{-n}2^n \sum_{\alpha\in A_n,\beta\in B_n} S^\epsilon_{\alpha,\beta}(t,x)\right|^2 \\
&\leq \left(\sum_{n=0}^{\infty} 4^{-n}\right)\left(\sum_{n=0}^{\infty} 4^n \left|\sum_{\alpha\in A_n,\beta\in B_n} S^\epsilon_{\alpha,\beta}(t,x)\right|^2\right) \\
&\leq C\left(\sum_{n=0}^{\infty} 16^n \sum_{\alpha\in A_n,\beta\in B_n} \left|S^\epsilon_{\alpha,\beta}(t,x)\right|^2\right)
\end{aligned}$$

for a constant C independent from ϵ. This implies that:

$$\|u^\epsilon(t,x)\|^2 \leq C\sum_{n=0}^{\infty} 16^n \sum_{\alpha\in A_n,\beta\in B_n} \|S^\epsilon_{\alpha,\beta}(t,x)\|^2$$

Applying lemma 3.4 with $a = 16$, we get:

$$\sup_\epsilon \mathbb{E} \sup_{t\leq T_0} \|u^\epsilon(t,x)\|^2 \leq C \sup_\epsilon \sum_{n=0}^{\infty} a^n e^\epsilon_n < \infty$$

which gives the first part of the tightness conditions; combined with lemma 3.5, we obtain part $a/$ of theorem 3.2. This shows also that the series expansion converges and is the solution of (3.2).

LEMMA 3.6. *For $\beta \neq (\lambda^\epsilon_2, \ldots, \lambda^\epsilon_2)$,*

$$\lim_{\epsilon\to 0}\left(\sup_{t\leq T_0} \mathbb{E}\|S^\epsilon_{\alpha,\beta}(t,x)\|^2\right) = 0$$

LEMMA 3.7. *For every $n \geq 0$:*

$$\lim_{\epsilon\to 0}\left(\sup_{t\leq T_0} \mathbb{E}\|S^\epsilon_{\alpha,\{\lambda^\epsilon_2\}^{n+1}}(t,x) - \tilde{S}^\epsilon_{\alpha,\{\lambda^\epsilon_2\}^{n+1}}(t,x)\|^2\right) = 0$$

These two lemmas imply part $b/$ of theorem 3.2; indeed

$$
\begin{aligned}
\mathbb{E}\left\|u^{\epsilon}(t,x)-v^{\epsilon}(t,x)\right\|^2 &\leq 2\mathbb{E}\left\|\sum_{n=0}^{\infty}\sum_{\alpha\in A_n,\beta\neq\{\lambda_2^{\epsilon}\}^{n+1}} S^{\epsilon}_{\alpha,\beta}(t,x)\right\|^2 \\
&+2\mathbb{E}\left\|\sum_{n=0}^{\infty}\sum_{\alpha\in A_n}\|S^{\epsilon}_{\alpha,\{\lambda_2^{\epsilon}\}^{n+1}}(t,x)-\tilde{S}^{\epsilon}_{\alpha,\{\lambda_2^{\epsilon}\}^{n+1}}(t,x)\right\|^2
\end{aligned}
$$

A similar computation as the one following lemma 3.5 shows that:

$$
\begin{aligned}
\mathbb{E}\left\|u^{\epsilon}(t,x)-v^{\epsilon}(t,x)\right\|^2 &\leq C\sum_{n=0}^{\infty}a^n\sum_{\alpha\in A_n,\beta\neq\{\lambda_2^{\epsilon}\}^{n+1}}\mathbb{E}\left\|S^{\epsilon}_{\alpha,\beta}(t,x)\right\|^2 \\
&+C\sum_{n=0}^{\infty}a^n\sum_{\alpha\in A_n}\mathbb{E}\left\|S^{\epsilon}_{\alpha,\{\lambda_2^{\epsilon}\}^{n+1}}(t,x)-\tilde{S}^{\epsilon}_{\alpha,\{\lambda_2^{\epsilon}\}^{n+1}}(t,x)\right\|^2
\end{aligned}
$$

Taking a supremum in $t\leq T_0$, by lemma 3.4, these two series converge uniformly in ϵ and lemma 3.6 and 3.7 imply that each term of these series converges to 0 as ϵ goes to 0. This gives part $b/$ of theorem 3.2 and consequently theorem 3.2

4. Proof of the lemmas. These proofs involve a detailed study of $S^{\epsilon}_{\alpha,\beta}(t,x)$ given in (3.16) and (3.17). It will be convenient to separate t and x variables by writing $S^{\epsilon}_{\alpha,\beta}(t,x)=R^{\epsilon}_{\alpha,\beta}(t)Q^{\epsilon}_{\alpha,\beta}(x)$ where:

$$
\begin{aligned}
R^{\epsilon}_{\alpha,\beta}(t) &= (-1)^{|\beta|}(\mathrm{i})^n e^{\mathrm{i}\beta_1 t}\int_0^t dt_1\ldots\int_0^{t_{n-1}}dt_n\exp\mathrm{i}\left(\sum_{j=2}^{n+1}(\beta_j-\beta_{j-1})\,t_{j-1}\right). \\
&\quad .\prod_{i=1}^{n}(\alpha_i\eta^{\epsilon}(t_i)+(1-\alpha_i)) \qquad (4.1)\\
Q^{\epsilon}_{\alpha,\beta}(x) &= \Delta^{1-\alpha_1}\circ\ldots\circ\Delta^{1-\alpha_n}H^{\beta_{n+1}}(x)
\end{aligned}
$$

A similar decomposition will be made on $\tilde{S}^{\epsilon}_{\alpha,\beta}(t,x)$ written as $R^{\epsilon}_{\alpha,\beta}(t)\tilde{Q}^{\epsilon}_{\alpha,\beta}(x)$ where $\tilde{Q}^{\epsilon}_{\alpha,\beta}(x)$ is defined as $Q^{\epsilon}_{\alpha,\beta}(x)$ with $g^{\epsilon}(0,x)$ replaced by $u_0(x)$.
The space dependency is handled in the following result:

Lemma 4.1.

i/ There exists a constant M such that for every $(\alpha,\beta)\in A_n\mathrm{x}B_n$, one has $\|Q^{\epsilon}_{\alpha,\beta}\|^2\leq M^n$.

ii/ If $\beta_{n+1}=\lambda_1^{\epsilon}$, for every α, $\lim_{\epsilon\to 0}\|Q^{\epsilon}_{\alpha,\beta}\|^2=0$.

iii/ If $\beta_{n+1}=\lambda_2^{\epsilon}$, for every α, $\lim_{\epsilon\to 0}\|Q^{\epsilon}_{\alpha,\beta}-\tilde{Q}^{\epsilon}_{\alpha,\beta}\|^2=0$.

Proof. We have supposed that u_0 has a bounded Fourier transform with a compact support contained in the ball centered at 0 with radius R. Therefore there exists a constant C_0 such that:

$$
\left\|\frac{\partial^{k+l}u_0(x)}{\partial x_1^k\partial x_2^l}\right\|^2_{\mathbf{L}^2}=\|\omega_1^k\omega_2^l\hat{u}_0(\omega)\|^2_{\mathbf{L}^2}\leq C_0^{k+l} \tag{4.2}
$$

Using the estimate (2.9) obtained in section 2, there exists also a constant C_1 such that:

$$
\left\|\frac{\partial^{k+l}u_1^{\epsilon}(x)}{\partial x_1^k\partial x_2^l}\right\|^2_{\mathbf{L}^2}=\|\omega_1^k\omega_2^l\hat{u}_1^{\epsilon}(\omega)\|^2_{\mathbf{L}^2}\leq\frac{C_1^{k+l}}{\epsilon^p}\nu(\epsilon) \tag{4.3}
$$

By a simple rescaling of the space variable x we can always suppose that $C_0 = C_1 = 1$. Using the relation $\Delta(\phi\psi) = \phi\Delta\psi + 2\nabla\phi.\nabla\psi + \psi\Delta\phi$, $Q^\epsilon_{\alpha,\beta}(x)$ is a sum of at most 8^n terms as $\left(\frac{\partial^{k+l}u_0(x)}{\partial x_1^k \partial x_2^l}\right)$ or $\left(\frac{\partial^{k+l}u_1^\epsilon(x)}{\partial x_1^k \partial x_2^l}\right)$ with $k+l \leq n$. In these terms u_1^ϵ is always multiplied by ϵ^p as can be seen in the definition of $H^{\beta_n}(x)$. Therefore $\|Q^\epsilon_{\alpha,\beta}\|^2$ is less than 8^n which proves $i/$ with $M = 8$.
$ii/$ and $iii/$ are obvious since $H^{\lambda_1^\epsilon}(x) = f^\epsilon(0,x)$ and $H^{\lambda_2^\epsilon}(x) = g^\epsilon(0,x)$ converge respectively to 0 and $u_0(x)$ by (2.9). □
The proofs of the lemmas require now a detailed study of the complex-valued processes $R^\epsilon_{\alpha,\beta}(t)$. For each β in B_n we have 2^{n+1} real multiple integrals corresponding to the choice of real or imaginary parts in the complex exponentials. To avoid more notations we shall not detail this decomposition but simply write:

$$R^\epsilon_{\alpha,\beta}(t) = \sum_{\gamma \in C_n} (-1)^{|\beta|} \mathrm{i}^{n+\|\gamma\|} R^\epsilon_{\alpha,\beta,\gamma}(t) \tag{4.4}$$

where γ is a parameter varying in a set C_n with $\mathrm{card}(C_n) = 2^n$, $\|\gamma\|$ is an integer associated to γ and $R^\epsilon_{\alpha,\beta,\gamma}(t)$ are real multiple integrals.
All the proofs will be given for the particular $R^\epsilon_{\alpha,\beta,\gamma}(t)$ denoted by $R^\epsilon_{\alpha,\beta,0}(t)$ and obtained by replacing the complex exponentials by cosinus:

$$\begin{aligned} R^\epsilon_{\alpha,\beta,0}(t) &= \cos(\beta_1 t) \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \left(\prod_{j=2}^{n+1} \cos((\beta_j - \beta_{j-1})\, t_{j-1}) \right) . \\ &\quad . \prod_{i=1}^{n} (\alpha_i \eta^\epsilon(t_i) + (1-\alpha_i)) \end{aligned} \tag{4.5}$$

The various estimates that we shall obtain do not depend on this particular choice; estimates on $R^\epsilon_{\alpha,\beta}(t)$ itself will be deduced easily.

4.1. Proof of lemma 3.7. With the notation (4.1), lemma 4.1 and the equality (4.4), we have, for every $a > 0$:

$$\begin{aligned} \sum_{n=0}^{\infty} a^n \sum_{(\alpha,\beta)\in A_n \mathrm{x} B_n} \mathbb{E} \sup_{t\leq T_0} \|S^\epsilon_{\alpha,\beta}(t,x)\|^2 &\leq \sum_n (Ma)^n \sum_{(\alpha,\beta)\in A_n \mathrm{x} B_n} \mathbb{E} \sup_{t\leq T_0} |R^\epsilon_{\alpha,\beta}(t)|^2 \\ &\leq \sum_{n=0}^{\infty} (2Ma)^n \sum_{(\alpha,\beta)\in A_n \mathrm{x} B_n} \sup_\gamma \mathbb{E} \sup_{t\leq T_0} (R^\epsilon_{\alpha,\beta,\gamma}(t))^2 \\ &\leq \sum_{n=0}^{\infty} (8Ma)^n \sup_{\alpha,\beta,\gamma} \mathbb{E} \sup_{t\leq T_0} (R^\epsilon_{\alpha,\beta,\gamma}(t))^2 \end{aligned} \tag{4.6}$$

We now consider the particular term $R^\epsilon_{\alpha,\beta,0}(t)$ defined in (4.5).
Using $|\eta^\epsilon(t_i)| \leq \frac{C}{\epsilon}$ and $|\cos(t_i)| \leq 1$, we get:

$$\mathbb{E} \sup_{t\leq T_0} |R^\epsilon_{\alpha,\beta,0}(t)|^2 \leq \mathbb{E} \sup_{t\leq T_0} \left[\int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \left(\prod_{i=1}^{n} (1 + \frac{C}{\epsilon} \right) \right]^2 \leq \left[(1+\frac{C}{\epsilon})^n \frac{T_0^n}{n!} \right]^2$$

This shows with (4.6) that $\sum_{n=0}^{\infty} a^n e^\epsilon_n < \infty$ for ϵ fixed. As ϵ goes to 0, we shall prove that there exists $\epsilon_a > 0$ such that:

$$\sup_{\epsilon\leq\epsilon_a}\sum_{n=0}^{\infty}(8Ma)^n \sup_{\alpha,\beta,\gamma}\mathbb{E}\sup_{t\leq T_0}(R^\epsilon_{\alpha,\beta,\gamma}(t))^2<\infty \tag{4.7}$$

which will achieve the proof of the lemma since:

$$\begin{aligned}\sup_\epsilon\sum_n a^n \sum_{(\alpha,\beta)\in A_n\text{x}B_n}\mathbb{E}\sup_{t\leq T_0}\|S^\epsilon_{\alpha,\beta}(t,x)\|^2 &\leq \sup_{\epsilon\leq\epsilon_a}\sum_{n=0}^{\infty}(8Ma)^n\sup_{\alpha,\beta,\gamma}\mathbb{E}\sup_{t\leq T_0}(R^\epsilon_{\alpha,\beta,\gamma}(t))^2\\ &\quad+\sup_{\epsilon>\epsilon_a}\sum_{n=0}^{\infty}(8Ma)^n\left(\left(1+\frac{C}{\epsilon}\right)^n\frac{T_0^n}{n!}\right)^2\\ &<\infty\end{aligned}$$

We shall restrict the summation to $n\geq 1$ because $\|S^\epsilon_\beta(t,x)\|^2$ is uniformly bounded. In order to prove (4.7), we introduce the following system of differential equations:

$$\left\{\begin{array}{ll}\frac{dY_1^\epsilon}{dt}(t) & =(\alpha_1\eta^\epsilon(t)+(1-\alpha_1))\cos((\beta_2-\beta_1)t)\\ \quad\vdots & \\ \frac{dY_m^\epsilon}{dt}(t) & =(\alpha_m\eta^\epsilon(t)+(1-\alpha_m))\cos((\beta_{m+1}-\beta_m)t)Y^\epsilon_{m-1}(t)\\ \quad\vdots & \\ \frac{dY_n^\epsilon}{dt}(t) & =(\alpha_n\eta^\epsilon(t)+(1-\alpha_n))\cos((\beta_{n+1}-\beta_n)t)Y^\epsilon_{n-1}(t)\\ Y_1^\epsilon(0) & =\ldots=Y_n^\epsilon(0)=0\end{array}\right. \tag{4.8}$$

Reindexing $(\alpha_1,\ldots,\alpha_n)$ and $(\beta_1,\ldots,\beta_{n+1})$ as $(\alpha_n,\ldots,\alpha_1)$ and $(\beta_{n+1},\ldots,\beta_1)$, the integral $R^\epsilon_{\alpha,\beta,0}(t)$ is now given by $\cos(\beta_1 t)Y_n^\epsilon(t)$ and any $R^\epsilon_{\alpha,\beta,\gamma}(t)$ will be given by such a system (4.8). We have $\sup_{t\leq T_0}|R^\epsilon_{\alpha,\beta,0}(t)|\leq\sup_{t\leq T_0}|Y_n^\epsilon(t)|$. This remark enables us to work on $Y_n^\epsilon(t)$ directly by applying the perturbed test function method to the system (4.8).

The proof itself is divided into two parts. Let $b=8Ma$

1/ There exists ϵ_1^a such that $\sup_{\epsilon\leq\epsilon_1^a}\sum_{n=1}^{\infty}b^n\sup_{s\leq T_0}\mathbb{E}(Y_n^\epsilon(s))^2<\infty$.

2/ There exists ϵ_a such that $\sup_{\epsilon\leq\epsilon_a}\sum_{n=1}^{\infty}b^n\mathbb{E}\sup_{s\leq T_0}(Y_n^\epsilon(s))^2<\infty$

4.1.1. first part.

¿From (4.8), we get:

$$(Y_n^\epsilon(t))^2=2\int_0^t(\alpha_n\eta^\epsilon(s)+(1-\alpha_n))\cos((\beta_{n+1}-\beta_n)s)Y_n^\epsilon(s)Y^\epsilon_{n-1}(s)ds \tag{4.9}$$

If $\alpha_n=0$:

$$\sup_{s\leq t}\mathbb{E}(Y_n(s))^2\leq\int_0^t\left(\sup_{s\leq u}\mathbb{E}(Y_n^\epsilon(s))^2+\sup_{s\leq u}\mathbb{E}(Y^\epsilon_{n-1}(s))^2\right)du \tag{4.10}$$

Otherwise, we introduce the perturbation:

$$V_n^\epsilon(t)=2\left(\int_t^{T_0}\mathbb{E}^\epsilon_t\eta^\epsilon(u)\cos((\beta_{n+1}-\beta_n)u)du\right)Y_n^\epsilon(t)Y^\epsilon_{n-1}(t)$$

As in Kushner[9], we define the pseudo-generator $\mathcal{A}$ acting on $V_n^\epsilon(t)$ as the L^1-limit when δ goes to 0 of:

$$\frac{1}{\delta}(\mathbb{E}_t^\epsilon V_n^\epsilon(t+\delta) - V_n^\epsilon(t))$$

We easily obtain:

$$\mathcal{A}V_n^\epsilon(t) = -2\eta^\epsilon(t)\cos((\beta_{n+1}-\beta_n)t)Y_n^\epsilon(t)Y_{n-1}^\epsilon(t) + L_{n,1}^\epsilon(t) + L_{n,2}^\epsilon(t)$$

with:

$$\begin{aligned}
L_{n,1}^\epsilon(t) &= 2\left(\int_t^{T_0}\mathbb{E}_t^\epsilon\eta^\epsilon(u)\cos((\beta_{n+1}-\beta_n)u)du\right)\eta^\epsilon(t)\cos((\beta_{n+1}-\beta_n)t)(Y_{n-1}^\epsilon(t))^2\\
L_{n,2}^\epsilon(t) &= 2\left(\int_t^{T_0}\mathbb{E}_t^\epsilon\eta^\epsilon(u)\cos((\beta_{n+1}-\beta_n)u)du\right)(\alpha_{n-1}\eta^\epsilon(t)+(1-\alpha_{n-1}))\\
&\quad\cos((\beta_n-\beta_{n-1})t)Y_n^\epsilon(t)Y_{n-2}^\epsilon(t)
\end{aligned}$$

(4.11)

For ϵ fixed, all these integrals are well defined. As in Kushner[9], we have the following essential property:

$$M_n^\epsilon(t) = V_n^\epsilon(t) - \int_0^t \mathcal{A}V_n^\epsilon(s)ds \tag{4.12}$$

is a $\mathcal{F}_t^\epsilon$-martingale. Adding (4.12) to (4.9), one obtains for $n \geq 1$:

$$(Y_n^\epsilon(t))^2 = M_n^\epsilon(t) - V_n^\epsilon(t) + \int_0^t (L_{n,1}^\epsilon(s) + L_{n,2}^\epsilon(s))ds \tag{4.13}$$

with the convention $L_{1,2}^\epsilon(s) = 0$.

Our mixing properties (3.5) imply the following bounds:

$$\begin{aligned}
\mathbb{E}|V_n^\epsilon(t)| &\leq \epsilon C_0\mathbb{E}|Y_n^\epsilon(t)|.|Y_{n-1}^\epsilon(t)|\\
\mathbb{E}|L_{n,1}^\epsilon(t)| &\leq C_1\mathbb{E}(Y_{n-1}^\epsilon(t))^2\\
\mathbb{E}|L_{n,2}^\epsilon(t)| &\leq C_2\mathbb{E}|Y_n^\epsilon(t)|.|Y_{n-2}^\epsilon(t)|
\end{aligned}$$

Taking the expectation in (4.13), we get:

$$\begin{aligned}
\sup_{s\leq t}\mathbb{E}(Y_n^\epsilon(s))^2 \quad &\leq \epsilon C_0\sup_{s\leq t}\mathbb{E}|Y_n^\epsilon(s)|.|Y_{n-1}^\epsilon(s)|\\
&\quad +(C_1+C_2)\int_0^t\left(\sup_{s\leq u}\mathbb{E}(Y_{n-1}^\epsilon(s))^2 + \sup_{s\leq u}\mathbb{E}|Y_n^\epsilon(s)|.|Y_{n-2}^\epsilon(s)|\right)du
\end{aligned}$$

(4.14)

Mulitplying (4.14) by b^n and adding all these inequalities for n between 1 and N, one obtains:

$$\begin{aligned}\sum_{n=1}^{N} b^n \sup_{s\le t} \mathbb{E}(Y_n^\epsilon(s))^2 \quad &\le \epsilon C_0 \sum_{n=1}^{N} b^n \sup_{s\le t} \mathbb{E}|Y_n^\epsilon(s)|.|Y_{n-1}^\epsilon(s)| \\ &+(C_1+C_2)\int_0^t \sum_{n=1}^{N} b^n \sup_{s\le u} \mathbb{E}(Y_{n-1}^\epsilon(s))^2 du \\ &+(C_1+C_2)\int_0^t \sum_{n=1}^{N} b^n \sup_{s\le u} \mathbb{E}|Y_n^\epsilon(s)|.|Y_{n-2}^\epsilon(s)| du\end{aligned}$$

Using the basic inequality $ab \le a^2+b^2$, we get:

$$\begin{aligned}\sum_{n=1}^{N} b^n \sup_{s\le t} \mathbb{E}(Y_n^\epsilon(s))^2 \quad &\le \epsilon C_0(1+b) \sum_{n=1}^{N} b^n \sup_{s\le t} \mathbb{E}(Y_n^\epsilon(s))^2 \\ &+(C_1+C_2)(1+b+b^2)\int_0^t \left(\sum_{n=1}^{N} b^n (\sup_{s\le u} \mathbb{E}(Y_n^\epsilon(s))^2\right) du\end{aligned}$$

Applying the same technics on (4.10) one can see that this inequality is also valid in the case $\alpha_n = 0$ provided that $C_1 + C_2 > 1$ which one can always assume. It is also independent from the choice of γ in C_n.
For $\epsilon \le \epsilon_1^a$ such that $C_0(1+b)\epsilon_1^a < 1$, we let N go to infinity and we apply Gronwall's lemma to end the proof.

4.1.2. Second part. We have:

$$Y_n^\epsilon(t) = \int_0^t (\alpha_n \eta^\epsilon(s) + (1-\alpha_n)) \cos((\beta_{n+1}-\beta_n)s) Y_{n-1}^\epsilon(s) ds \tag{4.15}$$

If $\alpha_n = 0$ then, by Cauchy-Schwartz inequality:

$$\mathbb{E}\sup_{s\le t}(Y_n^\epsilon(s))^2 \le T_0 \int_0^t \left(\mathbb{E}\sup_{s\le u}(Y_{n-1}^\epsilon(s))^2\right) du \tag{4.16}$$

otherwise, we use the perturbation:

$$W_n^\epsilon(t) = \left(\int_t^{T_0} \mathbb{E}_t^\epsilon \eta^\epsilon(u) \cos((\beta_{n+1}-\beta_n)u) du\right) Y_{n-1}^\epsilon(t)$$

Using the same pseudo-generator as previously, we get:

$$\mathcal{A}W_n^\epsilon(t) = -\eta^\epsilon(t)\cos(\beta_n t) Y_{n-1}^\epsilon(t) + K_n^\epsilon(t)$$

where K_n^ϵ is defined by:

$$\begin{aligned}K_n^\epsilon(t) = \quad &\left(\int_t^{T_0} \mathbb{E}_t^\epsilon \eta^\epsilon(u)\cos((\beta_{n+1}-\beta_n)u) du\right) \\ &(\alpha_{n-1}\eta^\epsilon(t) + (1-\alpha_{n-1})\cos((\beta_n-\beta_{n-1})t) Y_{n-2}^\epsilon(t)\end{aligned} \tag{4.17}$$

When ϵ is fixed, we obtain the following martingale:

$$N_n^\epsilon(t) = W_n^\epsilon(t) - \int_0^t \mathcal{A}W_n^\epsilon(s)ds \tag{4.18}$$

Adding (4.18) to equality (4.15), we get for $n \geq 1$:

$$Y_n^\epsilon(t) = N_n^\epsilon(t) - W_n^\epsilon(t) + \int_0^t K_n^\epsilon(s)ds \tag{4.19}$$

with $K_1^\epsilon(s) = 0$.
¿From the bounds $|W_n^\epsilon(t)| \leq C|Y_{n-1}^\epsilon(t)|$ and $|K_n^\epsilon(t)| \leq C|Y_{n-2}^\epsilon(t)|$, we get:

$$\mathbb{E}(N_n^\epsilon(t))^2 \leq C\left(\mathbb{E}(Y_n^\epsilon(t))^2 + \mathbb{E}(Y_{n-1}^\epsilon(t))^2 + T_0\int_0^t \mathbb{E}(Y_{n-2}^\epsilon(s))^2 ds\right)$$

So the first part of the proof, shows that $N_n^\epsilon(t)$ is a square integrable martingale and moreover, owing to Burkholder's inequality, we have, for $b = 8Ma$:

$$\sup_{\epsilon\leq\epsilon_1^a}\sum_{n=0}^\infty b^n\mathbb{E}\sup_{s\leq T_0}(N_n^\epsilon(s))^2 < \sup_{\epsilon\leq\epsilon_1^a}\sum_{n=0}^\infty b^n\mathbb{E}(N_n^\epsilon(T_0))^2 < \infty$$

On the other hand (4.19) leads to:

$$(Y_n^\epsilon(t))^2 \leq 2\left((N_n^\epsilon(t))^2 + (W_n^\epsilon(t))^2 + T_0\int_0^t(\sup_{u\leq s}K_n^\epsilon(u))^2 ds\right) \tag{4.20}$$

Using the bounds on K_n^ϵ and W_n^ϵ and including the $\alpha_n = 0$ case, we finally obtain the following inequality:

$$\begin{aligned}\mathbb{E}\sup_{s\leq t}(Y_n^\epsilon(s))^2 \quad &\leq C_0\epsilon^2\mathbb{E}\sup_{s\leq t}(Y_{n-1}^\epsilon(s))^2 + 2\mathbb{E}\sup_{s\leq t}(N_n^\epsilon(s))^2 \\ &+C_1\int_0^t\left(\mathbb{E}\sup_{s\leq u}(Y_{n-1}^\epsilon(s))^2 + \mathbb{E}\sup_{s\leq u}(Y_{n-2}^\epsilon(s))^2\right)du\end{aligned} \tag{4.21}$$

Multiplying (4.21) by b^n and adding the inequalities, we obtain for an arbitrary N:

$$\begin{aligned}\sum_{n=1}^N b^n\mathbb{E}\sup_{s\leq t}(Y_n^\epsilon(s))^2 \quad &\leq \sum_{n=1}^N b^n\mathbb{E}\sup_{s\leq T_0}(N_n^\epsilon(s))^2 + C_0 b\epsilon^2\sum_{n=1}^N b^n\mathbb{E}\sup_{s\leq t}(Y_n^\epsilon(s))^2 \\ &+C_1(b+b^2)\int_0^t\sum_{n=1}^N b^n\mathbb{E}\sup_{s\leq u}(Y_n^\epsilon(s))^2 du\end{aligned}$$

Taking N to ∞, we get for $\epsilon < \epsilon_a^2$ small enough:

$$\sum_{n=1}^\infty b^n\mathbb{E}\sup_{s\leq t}(Y_n^\epsilon(s))^2 \leq C\sum_{n=1}^\infty b^n\mathbb{E}\sup_{s\leq T_0}(N_n^\epsilon(s))^2 + \int_0^t\sum_{n=1}^\infty b^n\mathbb{E}\sup_{s\leq u}(Y_n^\epsilon(s))^2 du$$

Let $\epsilon_a = \min(\epsilon_a^1, \epsilon_a^2)$; thanks to Gronwall's lemma:

$$\sup_{\epsilon \le \epsilon_a} \sum_{n=1}^{\infty} b^n \mathbb{E} \sup_{s \le T_0} (Y_n^\epsilon(s))^2 < \infty$$

This ends the proof of the lemma.

4.2. Proof of lemma 3.9. Before giving the proof, we shall state an averaging lemma on which is based all the proof: it enables us to show that processes such that:

$$X_t^\epsilon = \frac{1}{\epsilon} \int_0^t \eta(\frac{s}{\epsilon^2}) \cos(\frac{s}{\epsilon^p}) ds$$

go to 0 in the law sense as ϵ goes to 0 under the assumption $p > 2$.

LEMMA 4.2. *Let $(F_n(t))_{n>0}$ be a sequence of continuous functions such that:*

i/ $\sup_n \int_0^{T_0} |F_n(t)| dt < \infty$.

ii/ There exists two sequences of real number $(a_n)_{n>0}$ and $(b_n)_{n>0}$ going to $+\infty$ such that: $\lim_{n\to\infty} \frac{a_n}{b_n} = +\infty$.

iii/ $\lim_{n\to\infty} \int_0^1 \left(\int_0^{T_0} \left| F_n(t + \frac{s}{b_n}) - F_n(t) \right| dt \right) ds = 0$. *We shall call this property the "continuity in the $\frac{1}{b_n}$ scale ".*

Then $\lim_{n\to\infty} \sup_{s \le T_0} \int_0^s F_n(t) \cos(a_n t) dt = 0$.

Proof. Let $(t_i^n)_{i \in \{0,1,\ldots,[b_n T_0]\}}$ be a subdivision of $[0, T_0]$ in b_n intervals of length $\frac{1}{b_n}$ (it is always possible to choose the sequence (b_n) such that this subdivision exists). Let s in $[0, T_0]$, we write then:

$$\begin{aligned} \int_0^s F_n(t) \cos(a_n t) dt &= \sum_{i, t_{i+1} \le s} \int_{t_i}^{t_{i+1}} (F_n(t) - F_n(t_i)) \cos(a_n t) dt \\ &+ \sum_{i, t_{i+1} \le s} \int_{t_i}^{t_{i+1}} F_n(t_i) \cos(a_n t) dt + \int_{t_m}^s F_n(t) \cos(a_n t) dt \end{aligned}$$

where m is the last index such that $t_m \le s$. Owing to assumption i/, the last integral goes to 0 when n goes to infinity, uniformly with respect to s. Making now the change of variable $t' = b_n(t - t_i)$ leads to the following bound:

$$\begin{aligned} \sup_{s \le T_0} \left| \int_0^s F_n(t) \cos(a_n t) dt \right| &\le \frac{1}{b_n} \sum_i \left| \int_0^1 (F_n(t_i + \frac{t}{b_n}) - F_n(t_i)) \cos(\frac{a_n}{b_n} t + a_n t_i) dt \right| \\ &+ \frac{1}{b_n} \sum_i \left| F_n(t_i) \int_0^1 \cos(\frac{a_n}{b_n} t + a_n t_i) dt \right| + O(1/b_n) \end{aligned}$$

$$
\begin{aligned}
\sup_{s\le T_0}\left|\int_0^s F_n(t)\cos(a_n t)dt\right| \quad &\le \frac{1}{b_n}\sum_i \int_0^1 \left|F_n(t_i+\frac{t}{b_n})-F_n(t_i)\right| dt \\
&\quad +2\frac{b_n}{a_n}\left(\frac{1}{b_n}\sum_i |F_n(t_i)|\right)+O(1/b_n)
\end{aligned}
$$

The first sum can be approximated by $\int_0^1\left(\int_0^{T_0}\left|F_n(t+\frac{s}{b_n})-F_n(t)\right|dt\right)ds$ and so goes to 0 owing to assumption iii/; the second one goes also to zero owing to assumptions i/ and ii/. □
To apply this lemma, instead of assumption iii/, we shall check that:

$$
\lim_{n\to\infty}\sup_{0\le s\le T_0}\int_0^{T_0}\left|F_n(t+\frac{s}{b_n})-F_n(t)\right|dt=0
$$

which obviously implies iii/. Note also that this result remains true if we replace cos by sin.

As in the proof of lemma 3.4 we shall only consider the real process $R^\epsilon_{\alpha,\beta,0}(t)$.
We show the following proposition:

If $\beta\neq(\lambda_2^\epsilon,\ldots,\lambda_2^\epsilon)$, for all (α,γ), $\mathbb{E}(R^\epsilon_{\alpha,\beta,\gamma}(t))^2$ goes to 0 when ϵ goes to 0, uniformly with respect to t in $[0,T_0]$.

If $\beta_{n+1}=\lambda_1^\epsilon$ then $Q^\epsilon_{\alpha,\beta}(x)$ goes to 0 when ϵ goes to 0 (cf lemma 4.1). Hence, we shall restrict ourself to the case $\beta_{n+1}=\lambda_2^\epsilon$.
There exists an index $m\le n$ such that $\beta_{m+1}=\lambda_2^\epsilon$ and $\beta_m=\lambda_1^\epsilon$ (otherwise β would be equal to $(\lambda_2^\epsilon,\ldots,\lambda_2^\epsilon)$).
Under these conditions, system (4.8) writes as:

$$
\left\{
\begin{aligned}
\frac{dY_1^\epsilon}{dt}(t) \quad &= (\alpha_1\eta^\epsilon(t)+(1-\alpha_1))\cos((\beta_2-\beta_1)t)\\
&\vdots\\
\frac{dY_m^\epsilon}{dt}(t) \quad &= (\alpha_m\eta^\epsilon(t)+(1-\alpha_m))\cos(\frac{t}{\epsilon^p})Y^\epsilon_{m-1}(t)\\
&\vdots\\
\frac{dY_n^\epsilon}{dt}(t) \quad &= (\alpha_n\eta^\epsilon(t)+(1-\alpha_n))\cos((\beta_{n+1}-\beta_n)t)Y^\epsilon_{n-1}(t)
\end{aligned}
\right.
$$

Coming back to the proof of lemma 3.4, we recall that (cf (4.14)):

$$
\mathbb{E}(Y_n^\epsilon(t))^2\le C_0\epsilon\mathbb{E}|Y^\epsilon_{n-1}(t)Y_n^\epsilon(t)|+C_1\int_0^t(\mathbb{E}(Y^\epsilon_{n-2}(s)^2+\mathbb{E}(Y^\epsilon_{n-1}(s))^2+\mathbb{E}(Y_n^\epsilon(s)^2)ds
$$

Hence, it is enough to show that $\sup_{t\le T_0}\mathbb{E}(Y_m^\epsilon(t))^2$ and $\sup_{t\le T_0}\mathbb{E}(Y^\epsilon_{m+1}(t))^2$ go to 0 when ϵ goes to 0 to prove that $\sup_{t\le T_0}\mathbb{E}(Y_n^\epsilon(t))^2$ goes also to 0.

Let us begin with $\sup_{t\le T_0}\mathbb{E}(Y_m^\epsilon(t))^2$.
If $\alpha_m=0$ then:

$$\mathbb{E}(Y_m^\epsilon(t))^2 = 2\int_0^t \cos(\frac{s}{\epsilon^p})\mathbb{E}Y_m^\epsilon(s)Y_{m-1}^\epsilon(s)ds$$

We apply the averaging lemma 4.2 to the function $\mathbb{E}Y_m^\epsilon(s)Y_{m-1}^\epsilon(s)$. Thanks to lemma 3.4, assumption i/ is satisfied and it remains to check the continuity hypothesis (assumption iii/). Let q such that $2<q<p$,we compute:

$$\begin{aligned}\int_0^{T_0}|\mathbb{E}Y_m^\epsilon(t+\epsilon^q s)Y_{m-1}^\epsilon(t+\epsilon^q s)-\mathbb{E}Y_m^\epsilon(t)Y_{m-1}^\epsilon(t)|dt &\le C\epsilon^q\sup_{t\le T_0}\mathbb{E}\left|\frac{d}{dt}Y_m^\epsilon(t)Y_{m-1}^\epsilon(t)\right|\\ &\le C\epsilon^q(1+\frac{C}{\epsilon})\sup_{t\le T_0}(\mathbb{E}(Y_{m-1}^\epsilon(t))^2+\mathbb{E}|Y_m^\epsilon(t)Y_{m-2}^\epsilon(t)|)\end{aligned}$$

which goes to 0 uniformly with respect to $s\le T_0$. So applying lemma 4.2 shows that $\sup\limits_{t\le T_0}\mathbb{E}(Y_m^\epsilon(t))^2$ goes to 0.
If $\alpha_m=1$, we must use the same perturbation method as in the proof of lemma 3.4. This leads to:

$$\mathbb{E}(Y_m^\epsilon(t))^2\le C\epsilon+\left|\int_0^t\mathbb{E}L_{m,1}^\epsilon(s)ds\right|+\left|\int_0^t\mathbb{E}L_{m,2}^\epsilon(s)ds\right|$$

Recall now what are $L_{m,1}^\epsilon$ et $L_{m,2}^\epsilon$ (equation (4.11)):

$$\begin{aligned}L_{m,1}^\epsilon(t)&=2\left(\int_t^{T_0}\mathbb{E}_t^\epsilon\eta^\epsilon(u)\cos(\frac{u}{\epsilon^p})du\right)\eta^\epsilon(t)\cos(\frac{t}{\epsilon^p})(Y_{m-1}^\epsilon(t))^2\\ L_{m,2}^\epsilon(t)&=2\left(\int_t^{T_0}\mathbb{E}_t^\epsilon\eta^\epsilon(u)\cos(\frac{u}{\epsilon^p})du\right)\\ &\quad(\alpha_{m-1}\eta^\epsilon(t)+(1-\alpha_{m-1}))\cos((\beta_m-\beta_{m-1})t)Y_m^\epsilon(t)Y_{m-2}^\epsilon(t)\end{aligned}$$

We shall prove that, for t fixed:

$$\tau^\epsilon(t)=\frac{1}{\epsilon}\int_t^{T_0}\mathbb{E}_t^\epsilon\eta^\epsilon(u)\cos(\frac{u}{\epsilon^p})du=\int_0^{T_0}\frac{1}{\epsilon^2}\mathbb{I}_{\{u>t\}}\mathbb{E}_t^\epsilon\eta(\frac{u}{\epsilon^2})\cos(\frac{u}{\epsilon^p})du$$

goes to 0 when ϵ goes to 0. As $\mathbb{E}(Y_n^\epsilon(t))^2$ is uniformly bounded, this proves that $\sup_{t\le T_0}\mathbb{E}(Y_m^\epsilon(t))^2$ goes to 0.
We use again the averaging lemma. Assumption i/ is a direct consequence of the mixing properties. Let us now compute:

$$
\begin{aligned}
\delta\tau^\epsilon(s,t) &= \frac{1}{\epsilon^2}\int_0^{T_0} |\mathbb{1}_{\{u>t\}}\mathbb{E}_t^\epsilon\eta(\frac{u}{\epsilon^2}) - \mathbb{1}_{\{u+\epsilon^q s>t\}}\mathbb{E}_t^\epsilon\eta(\frac{u+\epsilon^q s}{\epsilon^2})|du \\
&= \int_0^{\frac{T_0}{\epsilon^2}} |\mathbb{1}_{\{u>\frac{t}{\epsilon^2}\}}\mathbb{E}_t^\epsilon\eta(u) - \mathbb{1}_{\{u+\epsilon^{q-2}s>\frac{t}{\epsilon^2}\}}\mathbb{E}_t^\epsilon\eta(u+\epsilon^{q-2}s)|du
\end{aligned}
$$

Last quantity goes to 0 uniformly with respect to s by Lebesgue's theorem. This proves the continuity hypothesis and enables us to apply the averaging lemma.

We are now concerned in $\sup_{t\le T_0}\mathbb{E}(Y_{m+1}^\epsilon(t))^2$.
When $\beta_{m+2}=\lambda_1^\epsilon$, we have $\beta_{m+2}-\beta_{m+1}=\frac{1}{\epsilon^p}$ which leads to exactly the same computations as previously and we shall not do it again. So let us suppose that $\beta_{m+2}=\lambda_2^\epsilon$.
If $\alpha_{m+1}=0$, we have:

$$
\mathbb{E}(Y_{m+1}^\epsilon(t))^2 = 2\int_0^t \mathbb{E}Y_{m+1}^\epsilon(s)Y_m^\epsilon(s)ds \le \int_0^t (\mathbb{E}(Y_m^\epsilon(s))^2+\mathbb{E}(Y_{m+1}^\epsilon(s))^2)ds
$$

Applying Gronwall's lemma, we find that $\sup_{t\le T_0}\mathbb{E}(Y_{m+1}^\epsilon(t))^2$ goes to 0.
If $\alpha_{m+1}=1$, we use the perturbation method which leads to:

$$
\mathbb{E}(Y_{m+1}^\epsilon(t))^2 \le C\epsilon + \left|\int_0^t \mathbb{E}L_{m+1,1}^\epsilon(s)ds\right| + \left|\int_0^t \mathbb{E}L_{m+2,2}^\epsilon(s)ds\right|
$$

with:

$$
\begin{aligned}
L_{m+1,1}^\epsilon(t) &= 2\left(\int_t^{T_0}\mathbb{E}_t^\epsilon\eta^\epsilon(u)\right)\eta^\epsilon(t)(Y_m^\epsilon(t))^2 \\
L_{m+1,2}^\epsilon(t) &= 2\left(\int_t^{T_0}\mathbb{E}_t^\epsilon\eta^\epsilon(u)\right)(\alpha_m\eta^\epsilon(t)+(1-\alpha_m))\cos(\frac{t}{\epsilon^p})Y_{m+1}^\epsilon(t)Y_{m-1}^\epsilon(t)
\end{aligned}
$$

As $\sup_{t\le T_0}\mathbb{E}(Y_m^\epsilon(t))^2$ goes to 0, $\left|\mathbb{E}\int_0^t L_{m+1,1}^\epsilon(s)ds\right|$ goes also uniformly to 0. It remains to prove that $\left|\mathbb{E}\int_0^t L_{m+1,2}^\epsilon(s)ds\right|$ goes to 0.
If $\alpha_m=0$, it is not necessary to use the averaging lemma because the mixing property immediatly gives:

$$
|\mathbb{E}L_{m+1,2}^\epsilon(t)| \le C\epsilon\mathbb{E}|Y_{m+1}^\epsilon(t)Y_{m-1}^\epsilon(t)| \le C\epsilon(\mathbb{E}(Y_{m+1}^\epsilon(t))^2+\mathbb{E}(Y_{m-1}^\epsilon(t))^2)
$$

If $\alpha_m=1$, we use the averaging lemma. We write:

$$
\mathbb{E}\int_0^t L_{m+1,2}^\epsilon(s)ds = \int_0^t \tilde{\tau}^\epsilon(s)\cos(\frac{s}{\epsilon^p})ds
$$

with $\tilde{\tau}^\epsilon(t) = 2\mathbb{E}\left(\int_t^{T_0} \mathbb{E}_t^\epsilon \eta^\epsilon(u)du\right)\eta^\epsilon(t)Y_{m+1}^\epsilon(t)Y_{m-1}^\epsilon(t)$.
Assumption i/ is once again a direct consequence of the mixing properties. To check the continuity hypothesis we compute:

$$\begin{aligned}\int_0^{T_0}|\tilde{\tau}^\epsilon(t+\epsilon^q s)-\tilde{\tau}^\epsilon(t)|dt &\leq C\frac{1}{\epsilon^2}\int_0^{T_0}\left|\int_{t+\epsilon^q s}^{T_0}\mathbb{E}_{t+\epsilon^q s}^\epsilon\eta(\frac{u}{\epsilon^2})du-\int_t^{T_0}\mathbb{E}_t^\epsilon\eta(\frac{u}{\epsilon^2})du\right|dt\\ &+C\int_0^{T_0}|\eta(\frac{t+\epsilon^q s}{\epsilon^2})-\eta(\frac{t}{\epsilon^2})|dt\\ &+C\int_0^{T_0}|\mathbb{E}Y_{m+1}^\epsilon(t+\epsilon^q s)Y_{m-1}^\epsilon(t+\epsilon^q s)-\mathbb{E}Y_{m+1}^\epsilon(t)Y_{m-1}^\epsilon(t)|dt\end{aligned}$$

The first two terms go to 0 owing to Lebesgue's theorem. For the last one it is enough to bound the derivative of $\mathbb{E}Y_{m+1}^\epsilon(t)Y_{m-1}^\epsilon(t)$ by $\frac{C}{\epsilon^2}$.
A last application of the averaging lemma ends the proof.

4.3. Proof of lemma 3.8. We want to show that the sequence of complex processes $(< u^\epsilon(t,x),\theta(x) >)_{\epsilon>0}$ is tight in $C[0,T_0;\mathbb{C}\,]$ when $\theta(x)$ is a test function in $\mathbb{L}^2$.

We divide into two parts the expansion of $u^\epsilon(t,x)$ according to the value of β_1:

$$u^\epsilon(t,x)=\sum_{n=0}^{\infty}\sum_{\alpha\in A_n,\beta\in B_n,\beta_1=\lambda_1^\epsilon}S_{\alpha,\beta}^\epsilon(t,x)+\sum_{n=0}^{\infty}\sum_{\alpha\in A_n,\beta\in B_n,\beta_1=\lambda_2^\epsilon}S_{\alpha,\beta}^\epsilon(t,x)$$

Each term of the first sum has a common phase $e^{\mathrm{i}\beta_1 t}=e^{\mathrm{i}\lambda_1^\epsilon t}=e^{-\mathrm{i}t/\epsilon^p}$; in the second sum, there is no such phase since $e^{\mathrm{i}\beta_1 t}=e^{\mathrm{i}\lambda_2^\epsilon t}=1$. Therefore, with obvious notations, $u^\epsilon(t,x)$ can be written as:

$$u^\epsilon(t,x)=e^{-\frac{\mathrm{i}}{\epsilon^p}t}u_{\beta_1\neq 0}^\epsilon(t,x)+u_{\beta_1=0}^\epsilon(t,x)$$

It is enough to prove the tightness of the two families of processes $< u_{\beta_1\neq 0}^\epsilon(t,x),\theta(x) >$ and $< u_{\beta_1=0}^\epsilon(t,x),\theta(x) >$. Indeed, as a consequence of lemmas 3.4 and 3.6, we know that:

$$\lim_{\epsilon\to 0}\sup_{t\leq T_0}\mathbb{E}\|u_{\beta_1\neq 0}^\epsilon(t,x)\|^2=0$$

Therefore $< u_{\beta_1\neq 0}^\epsilon(t,x),\theta(x) >$ will converge to 0 in the law sense on the space $C[0,T_0]$. This implies that $< e^{-\mathrm{i}t/\epsilon^p}u_{\beta_1\neq 0}^\epsilon(t,x),\theta(x) >$ converges also to 0 in the law sense on $C[0,t_0]$ and therefore is tight. Combined with the tightness of $< u_{\beta_1=0}^\epsilon(t,x),\theta(x) >$, this gives the tightness of $< u^\epsilon(t,x),\theta(x) >$.
Separating x and t variables and real and imaginary part in $u_{\beta_1\neq 0}^\epsilon(t,x)$ and $u_{\beta_1=0}^\epsilon(t,x)$ as we did previously for $u^\epsilon(t,x)$, it is easy to see that we have to check the tightness

of the sequence:

$$\mathcal{R}^\epsilon(t,x) = \sum_{n=0}^{\infty} \sum_{\alpha\in\tilde{A}_n,\beta\in\tilde{B}_n,\gamma\in\tilde{C}_n} \tilde{R}^\epsilon_{\alpha,\beta,\gamma}(t) Q^\epsilon_{\alpha,\beta}(x)$$

Where $\tilde{A}_n$, $\tilde{B}_n$ and $\tilde{C}_n$ are subset of A_n, B_n and C_n and $\tilde{R}^\epsilon_{\alpha,\beta,\gamma}(t)$ is a real multiple integral obtained as $R^\epsilon_{\alpha,\beta,\gamma}(t)$ after removing the phase $e^{\mathrm{i}\beta_1 t}$. As we did previously, we shall work only on a specific term and we choose:

$$\tilde{R}^\epsilon_{\alpha,\beta,0}(t) = \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \left(\prod_{j=2}^{n+1} \cos((\beta_j - \beta_{j-1})\, t_{j-1}) \right) \prod_{i=1}^{n} (\alpha_i \eta^\epsilon(t_i) + (1-\alpha_i)) \tag{4.22}$$

Each of the processes $\tilde{R}^\epsilon_{\alpha,\beta,\gamma}(t)$ can be defined with the help of a system of differential equations such as system (4.8) and we have exactly $\tilde{R}^\epsilon_{\alpha,\beta,0}(t) = Y^\epsilon_n(t)$ for $n \geq 1$ (we take $Y^\epsilon_0(t) = 1$). Let:

$$\mathcal{R}^\epsilon(t,x) = \sum_{n=0}^{\infty} \sum_{\alpha\in\tilde{A}_n,\beta\in\tilde{B}_n,\gamma\in\tilde{C}_n} \tilde{R}^\epsilon_{\alpha,\beta,\gamma}(t) Q^\epsilon_{\alpha,\beta}(x)$$

We shall prove the following bound, for a constant C:

$$\mathbb{P}^\epsilon(\mu,\delta) = \mathbb{P}(\sup_{|t-s|\leq\mu} \|\mathcal{R}^\epsilon(t,x) - \mathcal{R}^\epsilon(s,x)\|^2 > \delta) \leq C\frac{\mu+\epsilon}{\delta} \tag{4.23}$$

This will show the tightness of $< \mathcal{R}^\epsilon(t,x), \theta(x) >$. We have:

$$\begin{aligned}
\|\mathcal{R}^\epsilon(t,x) - \mathcal{R}^\epsilon(s,x)\|^2 &\leq \left\| \sum_{n=0}^{\infty} \sum_{\alpha\in\tilde{A}_n,\beta\in\tilde{B}_n,\gamma\in\tilde{C}_n} (\tilde{R}^\epsilon_{\alpha,\beta,\gamma}(t) - \tilde{R}^\epsilon_{\alpha,\beta,\gamma}(s)) Q^\epsilon_{\alpha,\beta}(x) \right\|^2 \\
&\leq C \sum_{n=0}^{\infty} (32)^n \sum_{\alpha\in\tilde{A}_n,\beta\in\tilde{B}_n,\gamma\in\tilde{C}_n} \|(\tilde{R}^\epsilon_{\alpha,\beta,\gamma}(t) - \tilde{R}^\epsilon_{\alpha,\beta,\gamma}(s)) Q^\epsilon_{\alpha,\beta}(x)\|^2 \\
&\leq C \sum_{n=0}^{\infty} (32M)^n \sum_{\alpha\in\tilde{A}_n,\beta\in\tilde{B}_n,\gamma\in\tilde{C}_n} (\tilde{R}^\epsilon_{\alpha,\beta,\gamma}(t) - \tilde{R}^\epsilon_{\alpha,\beta,\gamma}(s))^2
\end{aligned}$$

where we have used lemma 4.1 in the last inequality. Let (c_n) be a sequence of non-negative real numbers such that $\sum_{n=0}^{\infty} c_n = 1$ and $c_n > 2^{-n}$ (at least for n big enough). From the classic inequality:

$$\mathbb{P}(X+Y > Z) \leq \mathbb{P}(X > Z/2) + \mathbb{P}(Y > Z/2)$$

and $\mathrm{card}(\tilde{A}_n \mathrm{x} \tilde{B}_n \mathrm{x} \tilde{C}_n) \leq 8^n$, we obtain:

$$\mathbb{P}^\epsilon(\mu,\delta) \leq \sum_{n=0}^{\infty} \sum_{\alpha\in\tilde{A}_n,\beta\in\tilde{B}_n,\gamma\in\tilde{C}_n} \mathbb{P}\left(\sup_{|t-s|\leq\mu}(\tilde{R}^\epsilon_{\alpha,\beta,\gamma}(t)-\tilde{R}^\epsilon_{\alpha,\beta,\gamma}(s))^2 > \frac{c_n\delta}{C8^n(32M)^n}\right) \tag{4.24}$$

Each of the probabilities $\mathbb{P}(\sup_{|t-s|\leq\mu}(\tilde{R}^\epsilon_{\alpha,\beta,\gamma}(t)-\tilde{R}^\epsilon_{\alpha,\beta,\gamma}(s))^2 > A)$ can be bounded by the same perturbation method no matter what is the exact value of (α,β,γ) so we shall use the variable $\tilde{R}^\epsilon_{\alpha,\beta,0}(t)$ defined by (4.22). We have:

$$|Y^\epsilon_n(t)-Y^\epsilon_n(s)|^2 = 2\int_s^t (\alpha_n\eta^\epsilon(u)+(1-\alpha_n))\cos((\beta_{n+1}-\beta_n)u)Y^\epsilon_{n-1}(u)(Y^\epsilon_n(u)-Y^\epsilon_n(s))du$$

We shall denote by T^ϵ_n the bounded quantity $\mathbb{E}\sup_{t\leq T_0}(Y^\epsilon_n(t))^2$. If $\alpha_n=0$, we have:

$$\sup_{|t-s|\leq\mu}\mathbb{E}(Y^\epsilon_n(t)-Y^\epsilon_n(s))^2 \leq \mu(T^\epsilon_{n-1}+T^\epsilon_n)$$

which leads to:

$$\mathbb{P}(\sup_{|t-s|\leq\mu}(Y^\epsilon_n(t)-Y^\epsilon_n(s))^2 > A) \leq \frac{\mu(T^\epsilon_{n-1}+T^\epsilon_n)}{A} \tag{4.25}$$

If $\alpha_n=1$, we use the perturbation:

$$\tilde{V}^\epsilon_n(t) = 2\left(\int_t^{T_0}\mathbb{E}^\epsilon_t\eta^\epsilon(u)\cos((\beta_{n+1}-\beta_n)u)du\right)(Y^\epsilon_n(t)-Y^\epsilon_n(s))Y^\epsilon_{n-1}(t)$$

Applying the pseudo-generator gives:

$$\begin{aligned}\mathcal{A}\tilde{V}^\epsilon_n(t) = \quad & 2\eta^\epsilon(t)\cos((\beta_{n+1}-\beta_n)t)Y^\epsilon_{n-1}(t)(Y^\epsilon_n(t)-Y^\epsilon_n(s)) \\ & +\tilde{L}^\epsilon_{n,1}(t)+\tilde{L}^\epsilon_{n,2}(t)\end{aligned}$$

with:

$$\begin{aligned}\tilde{L}^\epsilon_{n,1}(t) &= 2\left(\int_t^{T_0}\mathbb{E}^\epsilon_t\eta^\epsilon(u)\cos((\beta_{n+1}-\beta_n)u)du\right)\eta^\epsilon(t)\cos((\beta_{n+1}-\beta_n)t)(Y^\epsilon_{n-1}(t))^2 \\ \tilde{L}^\epsilon_{n,2}(t) &= 2\left(\int_t^{T_0}\mathbb{E}^\epsilon_t\eta^\epsilon(u)\cos((\beta_{n+1}-\beta_n)u)du\right)(Y^\epsilon_n(t)-Y^\epsilon_n(s)). \\ &\quad .(\alpha_{n-1}\eta^\epsilon(t)+(1-\alpha_{n-1}))\cos((\beta_n-\beta_{n-1})t)Y^\epsilon_{n-2}(t)\end{aligned}$$

We have the following bounds:

$$\begin{aligned}&\mathbb{E}\sup_{t\leq T_0}\|\tilde{V}^\epsilon_n(t)\|^2 \leq C\epsilon(T^\epsilon_{n-1}+T^\epsilon_n) \\ &\mathbb{E}\sup_{t\leq T_0}L^\epsilon_{n,1}(t) \leq CT^\epsilon_{n-1} \\ &\mathbb{E}\sup_{t\leq T_0}L^\epsilon_{n,2}(t) \leq C(T^\epsilon_{n-1}+T^\epsilon_n)\end{aligned} \tag{4.26}$$

This enables us to construct a centered $\mathcal{F}_t^\epsilon$-martingale $\tilde{M}_n^{\epsilon,s}(t)$ which equals 0 for $t = s$ and such that:

$$(Y_n^\epsilon(t) - Y_n^\epsilon(s))^2 = \tilde{M}_n^{\epsilon,s}(t) - \tilde{V}_n^\epsilon(t) + \int_s^t (\tilde{L}_{n,1}^\epsilon(s) + \tilde{L}_{n,2}^\epsilon(s))ds \tag{4.27}$$

with $\tilde{L}_{1,2}^\epsilon(s) = 0$.
Taking the expectation in (4.27) we get, thanks to (4.26):

$$\sup_{|t-s|\leq\mu} \mathbb{E}(Y_n^\epsilon(t) - Y_n^\epsilon(s))^2 \leq C(\mu+\epsilon)(T_{n-2}^\epsilon + T_{n-1}^\epsilon + T_n^\epsilon)$$

thus:

$$\sup_{|t-s|\leq\mu} \mathbb{E}\tilde{M}_n^{\epsilon,s}(t) \leq C(\mu+\epsilon)(T_{n-2}^\epsilon + T_{n-1}^\epsilon + T_n^\epsilon)$$

We apply now Doob's maximal inequality to the martingale $\tilde{M}_n^{\epsilon,s}(t)$ in (4.27) to get:

$$\mathbb{P}(\sup_{|t-s|\leq\mu} (Y_n^\epsilon(t) - Y_n^\epsilon(s))^2 > A) \leq \frac{C(\mu+\epsilon)}{A}(T_{n-2}^\epsilon + T_{n-1}^\epsilon + T_n^\epsilon)$$

This is also true if $\alpha_n = 0$ (cf (4.25)). Coming back to (4.24), we have obtained:

$$\mathbb{P}^\epsilon(\mu,\gamma) \leq C\frac{\mu+\epsilon}{\gamma}\sum_{n=0}^{\infty}\frac{8^n(32M)^n}{c_n}(T_{n-2}^\epsilon + T_{n-1}^\epsilon + T_n^\epsilon) \tag{4.28}$$

As $c_n \geq 2^{-n}$ we can use lemma 3.4 which shows that the series in the right-hand-side converges uniformly with respect to ϵ. This leads to (4.23) and proves the tightness.

4.4. Proof of lemma 3.10. The proof is obvious. Indeed, $S_{\alpha,\{\lambda_2^\epsilon\}^{n+1}}^\epsilon(t,x) - \tilde{S}_{\alpha,\{\lambda_2^\epsilon\}^{n+1}}^\epsilon(t,x)$ writes as:

$$S_{\alpha,\{\lambda_2^\epsilon\}^{n+1}}^\epsilon(t,x) - \tilde{S}_{\alpha,\{\lambda_2^\epsilon\}^{n+1}}^\epsilon(t,x) = R_{\alpha,\{\lambda_2^\epsilon\}^{n+1}}^\epsilon(t)(Q_{\alpha,\{\lambda_2^\epsilon\}^{n+1}}^\epsilon(x) - \tilde{Q}_{\alpha,\{\lambda_2^\epsilon\}^{n+1}}(x))$$

According to lemma 3.4, $\mathbb{E}(R_{\alpha,\{\lambda_2^\epsilon\}^{n+1}}^\epsilon(t))^2$ is uniformly bounded with respect to t and ϵ and, according to lemma 4.1, $(Q_{\alpha,\{\lambda_2^\epsilon\}^{n+1}}^\epsilon(x) - \tilde{Q}_{\alpha,\{\lambda_2^\epsilon\}^{n+1}}(x))$ goes to zero as ϵ goes to 0.

REFERENCES

[1] F. Bailly, Phd dissertation, in preparation.
[2] A. Bamberger, B. Engquist, L. Halpern and P. Joly, *Parabolic wave equation approximations in heterogeneous media* , SIAM J.applied math. , 48(1) (1988) , pp. 99-128.
[3] C. Bernardi and M.C. Pelissier, *Spectral approximation of a Schrödinger type equation* , Mathematical models and methods in applied sciences , 4(1) (1994) , pp. 49-88.
[4] D.A. Dawson and G.C. Papanicolaou, *A random wave process*, Applied Mathematics and Optimization, 12 (1984), pp. 97-114.

[5] M. El-Ansary and H. Kahlil, *On the interplay of singular perturbations and wide-band stochastic fluctuations* , SIAM J.Control and Optimization, 24 (1986), pp. 83-94.
[6] S. Ethier and T. Kurtz, *Markov processes*, Wiley , 1985.
[7] A. Ishimaru, *Wave propagation and scattering in random media*, Academic press , 1977.
[8] V. Klyatskin, *Ondes et équations stochastiques dans les milieux aléatoirement non homogènes*, Traduction française: Editions de physique, 1980.
[9] H. Kushner, *Approximation and weak convergence methods for random processes* , M.I.T. press, 1984.
[10] H. Kushner and H. Huang, *Limits for parabolic partial differential equations with wide-band stochastic coefficients and an application to filtering theory* , Stochastics , 14 (1985), pp. 115-148.
[11] M. Metivier, *Stochastic partial differential equation in infinite dimensional spaces* , Scuola Normale Superiore Pisa , 1988.
[12] B. Nair and B. White, *High-frequency wave propagation in random media -a unified approach*, SIAM J. Appl. Math., 51 (1991), pp. 374-411.
[13] C. Noel, *Propagation des signaux sonores en milieu marin aleatoire* , Phd dissertation, Aix-Marseille, 1993.
[14] K. Sobczyk, *Stochastic wave propagation*, Elsevier , 1985.