

# Multiscale Stochastic Volatility Model for Derivatives on Futures

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April 12, 2013

## Abstract

The goal of this article is to present a new method to extend the singular and regular perturbation techniques developed in the book by Fouque, Papanicolaou, Sircar and Sølna (2011, CUP) to price derivatives on futures when the asset presents mean reversion. We also consider the calibration procedure of the proposed model. We apply it to options on the Henry Hub natural gas futures and to options on stock and its volatility index in a consistent way. The main contribution of our work is a general method to compute the first-order approximation for the price of general compound derivatives such that no additional hypothesis on the regularity of the payoff function must be assumed. The only pre-requisite is the first-order approximation for the underlying derivative. In other words, the method proposed here allows us to derive the first-order approximation for compound derivatives keeping the hypotheses of the original approximation given in Fouque *et al.* (2011, CUP). Furthermore, this method maintains another desirable feature of the perturbation method: the direct calibration of the market group parameters.

## 1 Introduction

In many financial applications the underlying asset of the derivative contract under consideration is a derivative itself. A very important example of this complex and widely traded class of products consists of derivatives on future contracts. We shall study such financial instruments in the context of multiscale stochastic volatility as presented in Fouque et al. [2011].

It is well-known that under the No-Arbitrage Hypothesis, one can find a risk-neutral probability measure such that all *tradable* assets in this market, when properly discounted, are martingales under this measure (see Delbaen and Schachermayer [2008]

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for an extensive exposition on this subject). Here, we assume constant interest rate throughout this paper.

The future contract on the asset  $V$  with maturity  $T$  is a standardized contract traded at a futures exchange for which both parties consent to trade the asset  $V$  at time  $T$  for a price agreed upon the day the contract was written. This previously arranged price is called *strike*. The *future price* at time  $t$  with maturity  $T \geq t$  of the asset  $V$ , which will be denoted by  $F_{t,T}$ , is defined as the strike of the future contract on  $V$  with maturity  $T$  such that no premium is paid when the contract is written. In symbols,

$$(1.1) \quad F_{t,T} = \mathbb{E}_{\mathbb{Q}}[V_T \mid \mathcal{F}_t],$$

where  $\mathbb{Q}$  is a risk-neutral probability. If the asset  $V$  is tradable, then we simply have  $F_{t,T} = e^{r(T-t)}V_t$ , where  $r$  is the constant interest rate, and then derivatives on futures can be treated in the exact same way one handles derivatives on the asset itself.

Future prices are non-trivial when the asset is not tradable and therefore the discounted asset price is not a martingale, see for example [Musielka and Rutkowski, 2008, Chapter 3]. This will be our main assumption: *the asset  $V$  is not tradable*. More precisely, we assume the asset price presents mean reversion. Some examples of such assets are: commodities, currency exchange rates, volatility indices, and interest rates.

Because of the nature of our problem, the future price, which is the underlying asset of the derivative in consideration, has its dynamics explicitly depending on the time scales of the volatility. This creates an important difference from the usual perturbation theory to the derivative pricing problem.

The method presented in this paper can be described as follows:

- (i) Write a stochastic differential equation (SDE) for the future  $F_{t,T}$  with all coefficients depending only on  $F_{t,T}$ . This means we will need to invert the future prices of  $V$  in order to write  $V_t$  as a function of  $F_{t,T}$ .
- (ii) Consider the pricing partial differential equation (PDE) for a European derivative on  $F_{t,T}$ . The coefficients of this PDE will depend on the time scales of the stochastic volatility of the asset in a complicated way. At this point, we use perturbation analysis to treat such PDE by expanding the coefficients.
- (iii) Determine the first-order approximation for the derivatives on  $F_{t,T}$  as it is done in Fouque et al. [2011].

Indeed, this method is not the only way to tackle this problem. Instead, we could have considered this compound derivative as a more elaborate derivative in the asset and then find the first-order approximation proposed in Hikspoors and Jaimungal [2008]. This, in turn, follows the idea designed in Cotton et al. [2004] and is based on the Taylor expansion of the payoff under consideration around the zero-order term of the approximation of the future price  $F_{t,T}$ . Therefore, some smoothness of the payoff function must be assumed. Since the method considered here does not rely on such Taylor expansion, no restriction other than the ones intrinsic to the perturbation method is required. Furthermore, we shall show that although the method presented here is more involved, it allows a cleaner calibration. This is due to the fact we are considering the derivative as a function of the future price, which is the tradable asset and hence a martingale under the pricing risk-neutral measure.

Another important set of examples that can be handled using the method proposed in this paper consists of interest rate derivatives (see Cotton et al. [2004]). Moreover,

in the equity case, the method could be used to tackle the general problem of pricing compound derivatives, as it is done in Fouque and Han [2005] by Taylor expansion of the payoff.

The model presented in this paper was originally developed to consistently calibrate a geometric Brownian motion (GBM)  $S_t$  with instantaneous stochastic variance  $V_t$  to options on  $S$  and options on  $V$ . Such variance in turn would follow an exponential Ornstein-Uhlenbeck process with multiscale stochastic volatility of volatility (see (2.1) below). The prime example we had in mind was the S&P 500 and the VIX. We come back to this problem in Section 5.

The main contribution of our work is a general method to compute the first-order approximation for the price of general compound derivatives such that no additional hypothesis on the regularity of the payoff function must be assumed. The only prerequisite is the first-order approximation for the underlying derivative. In other words, the method proposed here allows us to derive the first-order approximation for compound derivatives keeping the hypotheses of the original approximation given in Fouque et al. [2011]. Furthermore, this method maintains another desirable feature of the perturbation method: the direct calibration of the market group parameters.

This paper is organized as follows: Section 2 describes the dynamics of the underlying asset and then, in Section 3 we follow the method previously outlined to find the first-order approximation for derivatives on future contracts of  $V$ . Section 4 characterizes the calibration procedure to call options and we analyze an example of calibration to options on Henry Hub natural gas futures. In Section 5 we present a model for derivatives of a volatility index on a stock. Finally, we conclude in Section 6 with some suggestions for further research.

## 2 The Model

We assume that the asset value  $V_t$  is described under a risk-neutral probability  $\mathbb{Q}$  by an exponential Ornstein-Uhlenbeck (exp-OU) stochastic process with a multiscale stochastic volatility. Namely,

$$(2.1) \quad \begin{cases} V_t = e^{U_t}, \\ dU_t = \kappa(m - U_t)dt + \eta(Y_t^\varepsilon, Z_t^\delta)dW_t^{(0)}, \\ dY_t^\varepsilon = \frac{1}{\varepsilon}\alpha(Y_t^\varepsilon)dt + \frac{1}{\sqrt{\varepsilon}}\beta(Y_t^\varepsilon)dW_t^{(1)}, \\ dZ_t^\delta = \delta c(Z_t^\delta)dt + \sqrt{\delta}g(Z_t^\delta)dW_t^{(2)}, \end{cases}$$

where  $(W_t^{(0)}, W_t^{(1)}, W_t^{(2)})$  is a correlated  $\mathbb{Q}$ -Brownian motion with

$$dW_t^{(0)}dW_t^{(i)} = \rho_i dt, \quad i = 1, 2, \quad dW_t^{(1)}dW_t^{(2)} = \rho_{12}dt.$$

The main assumptions of this model are:

- There exists a unique solution of the SDE (2.1) for any fixed  $(\varepsilon, \delta)$ .
- The risk-neutral probability  $\mathbb{Q}$  is chosen in order to match the future prices of  $V$  observed in the market to the prices produced by the model (2.1) and the martingale relation (1.1).

- $|\rho_1| < 1$ ,  $|\rho_2| < 1$ ,  $|\rho_{12}| < 1$  and  $1 + 2\rho_1\rho_2\rho_{12} - \rho_1^2 - \rho_2^2 - \rho_{12}^2 > 0$ . These conditions ensure the positive definiteness of the covariance matrix of  $(W_t^{(0)}, W_t^{(1)}, W_t^{(2)})$ .
- The interest rate is constant and equals  $r$ .
- $\alpha$  and  $\beta$  are such the process  $Y^1$  has a unique invariant distribution and is mean-reverting as in [Fouque et al., 2011, Section 3.2].
- $\eta(y, z)$  is a positive function, smooth in  $z$  and such that  $\eta^2(\cdot, z)$  is integrable with respect to the invariant distribution of  $Y^1$ .

It is important to notice that we could have explicitly considered the market prices of volatility risk as it is done in Fouque et al. [2011] and in doing so we have a term of order  $\varepsilon^{-1/2}$  and a term of order  $\delta^{1/2}$  in the drifts of  $Y^\varepsilon$  and  $Z^\delta$  respectively, both depending on  $Y^\varepsilon$  and  $Z^\delta$ , and they could have been handled in the way it is done in the aforesaid reference. For simplicity, we do not consider these market prices of volatility risk here.

Simple generalizations of this model are the addition of a deterministic seasonality factor in  $V$  and a deterministic time-varying long run mean  $m(t)$  in the drift of  $U$ . Both can be easily handled.

We now restate the definition of the future prices of  $V$

$$F_{t,T} = \mathbb{E}_{\mathbb{Q}}[V_T \mid \mathcal{F}_t], \quad 0 \leq t \leq T,$$

and then in next the section we will develop the first-order approximation for derivatives on  $F_{t,T}$ .

**Remark 2.1.** More precisely, we say that a function  $g^{\varepsilon,\delta}$  is a first-order approximation to the function  $f^{\varepsilon,\delta}$  if

$$|g^{\varepsilon,\delta} - f^{\varepsilon,\delta}| \leq C(\varepsilon + \delta),$$

for some constant  $C > 0$  and for sufficiently small  $\varepsilon, \delta > 0$ . We use the notation

$$g^{\varepsilon,\delta} - f^{\varepsilon,\delta} = O(\varepsilon + \delta).$$

## 3 Derivatives on Future Contracts

### 3.1 First-Order Approximation for Future Prices

Here, we present the first-order approximation for future prices on mean-reverting assets. For a fixed maturity  $T > 0$ , we define

$$h^{\varepsilon,\delta}(t, u, y, z, T) = \mathbb{E}_{\mathbb{Q}}[V_T \mid U_t = u, Y_t^\varepsilon = y, Z_t^\delta = z],$$

and note that  $F_{t,T} = h^{\varepsilon,\delta}(t, U_t, Y_t^\varepsilon, Z_t^\delta, T)$ . Consider the formal expansion in powers of  $\sqrt{\varepsilon}$  and  $\sqrt{\delta}$  of  $h^{\varepsilon,\delta}$ :

$$h^{\varepsilon,\delta}(t, u, y, z, T) = \sum_{i,j \geq 0} (\sqrt{\varepsilon})^i (\sqrt{\delta})^j h_{i,j}(t, x, y, z, T).$$

We are interested in the first-order approximation for derivatives on mean-reverting assets, which is presented in Hikspoors and Jaimungal [2008] and Chiu et al. [2011]. We shall denote by  $Y^1$  the process given by the second of the stochastic differential equations in (2.1) when  $\varepsilon = 1$ .

Applying this result to  $h^{\varepsilon, \delta}$ , we choose the first terms of the above formal series to be

$$(3.1) \quad h_0(t, u, z, T) = \exp \left\{ m + (u - m)e^{-\kappa(T-t)} + \frac{\bar{\eta}^2(z)}{4\kappa} \left( 1 - e^{-2\kappa(T-t)} \right) \right\},$$

$$(3.2) \quad h_{1,0}(t, u, z, T) = g(t, T)V_3(z)\frac{\partial^3 h_0}{\partial u^3}(t, u, z, T),$$

$$(3.3) \quad h_{0,1}(t, u, z, T) = f(t, T)V_1(z)\frac{\partial^3 h_0}{\partial u^3}(t, u, z, T),$$

where, denoting the averaging with respect to the invariant distribution of  $Y^1$  by  $\langle \cdot \rangle$ , we have

$$(3.4) \quad \begin{aligned} \bar{\eta}^2(z) &= \langle \eta^2(\cdot, z) \rangle, \\ V_3(z) &= -\frac{\rho_1}{2} \left\langle \eta(\cdot, z) \beta(\cdot) \frac{\partial \phi}{\partial y}(\cdot, z) \right\rangle, \\ V_1(z) &= \rho_2 g(z) \langle \eta(\cdot, z) \bar{\eta}(z) \bar{\eta}'(z) \rangle, \\ f(t, T) &= \frac{e^{3\kappa(T-t)} - e^{2\kappa(T-t)}}{2\kappa^2} - \frac{e^{3\kappa(T-t)} - 1}{6\kappa^2}, \\ g(t, T) &= \frac{e^{-3\kappa(T-t)} - 1}{3\kappa}, \end{aligned}$$

and  $\phi(y, z)$  is the solution of the Poisson equation

$$(3.5) \quad \mathcal{L}_0 \phi(y, z) = \eta^2(y, z) - \bar{\eta}^2(z),$$

with  $\mathcal{L}_0$  being the infinitesimal generator of  $Y^1$ . Moreover, we may assume  $h_{1,1}$  does not depend on  $y$  and choose

$$(3.6) \quad h_{2,0}(t, u, y, z, T) = -\frac{1}{2} \phi(y, z) \frac{\partial^2 h_0}{\partial u^2}(t, u, z, T) + c(t, u, z, T),$$

for some function  $c$  that does not depend on  $y$ . Under all these choices and some regularity conditions similar to the ones presented in Theorem 3.2 at the end of this section, as it was shown in Chiu et al. [2011] and Hikspoors and Jaimungal [2008], we have

$$h^{\varepsilon, \delta}(t, u, y, z) = h_0(t, u, z, T) + \sqrt{\varepsilon} h_{1,0}(t, u, z, T) + \sqrt{\delta} h_{0,1}(t, u, z, T) + O(\varepsilon + \delta).$$

Furthermore, the following simplifications hold:

$$h_{1,0}(t, u, z, T) = g(t, T)V_3(z)e^{-3\kappa(T-t)}h_0(t, u, z, T),$$

and

$$h_{0,1}(t, u, z, T) = f(t, T)V_1(z)e^{-3\kappa(T-t)}h_0(t, u, z, T).$$

## 3.2 The Dynamics of the Future Prices

In this section, we will derive the SDE describing the dynamics of  $F_{t,T}$  and write its coefficients as functions of  $F_{t,T}$ . Since  $F_{t,T}$  is a martingale under  $\mathbb{Q}$ , its dynamics has

no drift and hence, applying Itô's Formula to  $F_{t,T} = h^{\varepsilon,\delta}(t, U_t, Y_t^\varepsilon, Z_t^\delta, T)$ , we get

$$\begin{aligned} dF_{t,T} &= \frac{\partial h^{\varepsilon,\delta}}{\partial u}(t, U_t, Y_t^\varepsilon, Z_t^\delta, T) \eta(Y_t^\varepsilon, Z_t^\delta) dW_t^{(0)} \\ &\quad + \frac{1}{\sqrt{\varepsilon}} \frac{\partial h^{\varepsilon,\delta}}{\partial y}(t, U_t, Y_t^\varepsilon, Z_t^\delta, T) \beta(Y_t^\varepsilon) dW_t^{(1)} \\ &\quad + \sqrt{\delta} \frac{\partial h^{\varepsilon,\delta}}{\partial z}(t, U_t, Y_t^\varepsilon, Z_t^\delta, T) g(Z_t^\delta) dW_t^{(2)}. \end{aligned}$$

We are interested in derivatives on  $F_{t,T}$  and in applying the perturbation method to approximate their prices. Thus, we will rewrite the above SDE with all coefficients depending on  $F_{t,T}$  instead of  $U_t$ . In order to proceed, we need to make sure we can invert  $h^{\varepsilon,\delta}$  with respect to  $u$  for fixed  $\varepsilon, \delta, y, z$  and  $T$ , i.e. there must exist a function  $H^{\varepsilon,\delta}(t, x, y, z, T)$  such that

$$H^{\varepsilon,\delta}(t, \cdot, y, z, T) = (h^{\varepsilon,\delta}(t, \cdot, y, z, T))^{-1}.$$

This result, together with the asymptotics of  $H^{\varepsilon,\delta}$ , is given in the following lemma.

**Lemma 3.1.** *Since  $h_0(t, u, z)$  given by (3.1) is invertible, so is  $h^{\varepsilon,\delta}$  at least for small  $\varepsilon$  and  $\delta$ . Moreover if we choose  $H_0, H_{1,0}, H_{0,1}$  to be*

$$(i) \quad H_0(t, \cdot, z, T) = (h_0(t, \cdot, z, T))^{-1},$$

$$(ii) \quad H_{1,0}(t, x, z, T) = -\frac{h_{1,0}(t, H_0(t, x, z, T), z, T)}{\frac{\partial h_0}{\partial u}(t, H_0(t, x, z, T), z, T)},$$

$$(iii) \quad H_{0,1}(t, x, z, T) = -\frac{h_{0,1}(t, H_0(t, x, z, T), z, T)}{\frac{\partial h_0}{\partial u}(t, H_0(t, x, z, T), z, T)},$$

where  $h_{1,0}$  and  $h_{0,1}$  are given by (3.2) and (3.3) respectively, then, we have

$$H^{\varepsilon,\delta}(t, x, y, z, T) = H_0(t, x, z, T) + \sqrt{\varepsilon} H_{1,0}(t, x, z, T) + \sqrt{\delta} H_{0,1}(t, x, z, T) + O(\varepsilon + \delta).$$

*Proof.* The derivation is straightforward, so we skip the details.  $\square$

Notice that  $H^{\varepsilon,\delta}(t, F_{t,T}, y, z, T) = U_t$  and if we define

$$\begin{aligned} \psi_1^{\varepsilon,\delta}(t, x, y, z, T) &= \frac{\partial h^{\varepsilon,\delta}}{\partial u}(t, H^{\varepsilon,\delta}(t, x, y, z, T), y, z, T), \\ \psi_2^{\varepsilon,\delta}(t, x, y, z, T) &= \frac{\partial h^{\varepsilon,\delta}}{\partial y}(t, H^{\varepsilon,\delta}(t, x, y, z, T), y, z, T), \\ \psi_3^{\varepsilon,\delta}(t, x, y, z, T) &= \frac{\partial h^{\varepsilon,\delta}}{\partial z}(t, H^{\varepsilon,\delta}(t, x, y, z, T), y, z, T), \end{aligned}$$

we obtain the desired SDE for  $F_{t,T}$

$$(3.7) \quad \begin{aligned} dF_{t,T} &= \psi_1^{\varepsilon,\delta}(t, F_{t,T}, Y_t^\varepsilon, Z_t^\delta, T) \eta(Y_t^\varepsilon, Z_t^\delta) dW_t^{(0)} \\ &\quad + \frac{1}{\sqrt{\varepsilon}} \psi_2^{\varepsilon,\delta}(t, F_{t,T}, Y_t^\varepsilon, Z_t^\delta, T) \beta(Y_t^\varepsilon) dW_t^{(1)} \\ &\quad + \sqrt{\delta} \psi_3^{\varepsilon,\delta}(t, F_{t,T}, Y_t^\varepsilon, Z_t^\delta, T) g(Z_t^\delta) dW_t^{(2)}. \end{aligned}$$

### 3.3 A Pricing PDE for Derivatives on Future Contracts

We now fix a future contract on  $V$  with maturity  $T$  and consider a European derivative with maturity  $T_0 < T$  and whose payoff  $\varphi$  depends only on the terminal value  $F_{T_0, T}$ . A no-arbitrage price for this derivative on  $F_{t, T}$  is given by

$$P^{\varepsilon, \delta}(t, x, y, z, T) = \mathbb{E}_{\mathbb{Q}}[e^{-r(T_0-t)}\varphi(F_{T_0, T}) \mid F_{t, T} = x, Y_t^\varepsilon = y, Z_t^\delta = z],$$

where  $\mathbb{Q}$  is the risk-neutral probability discussed in Section 2 and we are using the fact that  $(F_{t, T}, Y_t^\varepsilon, Z_t^\delta)$  is a Markov process. In this section we derive a PDE for  $P^{\varepsilon, \delta}$ . Recall that  $F_{t, T}$  follows Equation (3.7), where  $Y_t^\varepsilon$  and  $Z_t^\delta$  are given in (2.1). Then, we write the infinitesimal generator  $\mathcal{L}^{\varepsilon, \delta}$  of  $(F_{t, T}, Y_t^\varepsilon, Z_t^\delta)$ , where, for simplicity of notation, we will drop the variables  $(t, x, y, z, T)$  of  $\psi_i^{\varepsilon, \delta}$ ,  $i = 1, 2, 3$ ,

$$(3.8) \quad \begin{aligned} \mathcal{L}^{\varepsilon, \delta} = & \frac{1}{\varepsilon} \left( \mathcal{L}_0 + \frac{1}{2}(\psi_2^{\varepsilon, \delta})^2 \beta^2(y) \frac{\partial^2}{\partial x^2} + \psi_2^{\varepsilon, \delta} \beta^2(y) \frac{\partial^2}{\partial x \partial y} \right) \\ & + \frac{1}{\sqrt{\varepsilon}} \left( \rho_1 \psi_1^{\varepsilon, \delta} \psi_2^{\varepsilon, \delta} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x^2} + \rho_1 \psi_1^{\varepsilon, \delta} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x \partial y} \right) \\ & + \frac{\partial}{\partial t} + \frac{1}{2}(\psi_1^{\varepsilon, \delta})^2 \eta^2(y, z) \frac{\partial^2}{\partial x^2} - r. \\ & + \sqrt{\delta} \left( \rho_2 \psi_1^{\varepsilon, \delta} \psi_3^{\varepsilon, \delta} \eta(y, z) g(z) \frac{\partial^2}{\partial x^2} + \rho_2 \psi_1^{\varepsilon, \delta} \eta(y, z) g(z) \frac{\partial^2}{\partial x \partial z} \right) \\ & + \delta \left( \mathcal{M}_2 + \frac{1}{2}(\psi_3^{\varepsilon, \delta})^2 g^2(z) \frac{\partial^2}{\partial x^2} + \psi_3^{\varepsilon, \delta} g^2(z) \frac{\partial^2}{\partial x \partial z} \right) \\ & + \sqrt{\frac{\delta}{\varepsilon}} \left( \rho_{12} \psi_2^{\varepsilon, \delta} \psi_3^{\varepsilon, \delta} \beta(y) g(z) \frac{\partial^2}{\partial x^2} + \rho_{12} \psi_3^{\varepsilon, \delta} \beta(y) g(z) \frac{\partial^2}{\partial x \partial y} \right. \\ & \quad \left. + \rho_{12} \psi_2^{\varepsilon, \delta} \beta(y) g(z) \frac{\partial^2}{\partial x \partial z} + \rho_{12} \beta(y) g(z) \frac{\partial^2}{\partial y \partial z} \right), \end{aligned}$$

where

$$(3.9) \quad \begin{aligned} \mathcal{L}_0 &= \frac{1}{2} \beta^2(y) \frac{\partial^2}{\partial y^2} + \alpha(y) \frac{\partial}{\partial y}, \\ \mathcal{M}_2 &= \frac{1}{2} g^2(z) \frac{\partial^2}{\partial z^2} + c(z) \frac{\partial}{\partial z}. \end{aligned}$$

It is well-known that under some mild conditions, by Feynman-Kac's Formula,  $P^{\varepsilon, \delta}$  satisfies the pricing PDE

$$(3.10) \quad \begin{cases} \mathcal{L}^{\varepsilon, \delta} P^{\varepsilon, \delta}(t, x, y, z, T) = 0, \\ P^{\varepsilon, \delta}(T_0, x, y, z, T) = \varphi(x). \end{cases}$$

### 3.4 Perturbation Framework

We will now develop the formal singular and regular perturbation analysis for European derivatives on  $F_{t, T}$  following the method outlined in Fouque et al. [2011]. However, in our case we have a fundamental difference: the coefficients of the differential operator  $\mathcal{L}^{\varepsilon, \delta}$ , given by Equation (3.8), depend on  $\varepsilon$  and  $\delta$  in an intricate way. In particular, the term corresponding to the factor  $\varepsilon^{-1}$  is not simply of order  $\varepsilon^{-1}$ . To circumvent

this problem, we will expand the coefficients in powers of  $\varepsilon$  and  $\delta$  and then collect the correct terms for each order. Therefore, it will be necessary to compute some terms of the expansion of  $\psi_i^{\varepsilon,\delta}$ . All the details for this expansion are given in the Appendix A and the final result is:

$$\mathcal{L}^{\varepsilon,\delta} = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\varepsilon}\mathcal{L}_3 + \sqrt{\delta}\mathcal{M}_1 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3 + \dots,$$

where  $\mathcal{L}_0$  is given by (3.9) and

$$(3.11) \quad \mathcal{L}_1 = \rho_1 e^{-\kappa(T-t)} \eta(y, z) \beta(y) x \frac{\partial^2}{\partial x \partial y},$$

$$(3.12) \quad \begin{aligned} \mathcal{L}_2 = & \frac{\partial}{\partial t} + \frac{1}{2} e^{-2\kappa(T-t)} \eta^2(y, z) x^2 \frac{\partial^2}{\partial x^2} - r \\ & - \frac{1}{2} e^{-2\kappa(T-t)} \frac{\partial \phi}{\partial y}(y, z) \beta^2(y) x \frac{\partial^2}{\partial x \partial y}, \end{aligned}$$

$$(3.13) \quad \begin{aligned} \mathcal{M}_3 = & \rho_{12} \frac{(1 - e^{-2\kappa(T-t)})}{2\kappa} \beta(y) g(z) \bar{\eta}(z) \bar{\eta}'(z) x \frac{\partial^2}{\partial x \partial y} \\ & + \rho_{12} \beta(y) g(z) \frac{\partial^2}{\partial y \partial z}, \end{aligned}$$

$$(3.14) \quad \begin{aligned} \mathcal{L}_3 = & (\psi_{2,3,0}(t, x, y, z, T) \beta^2(y) \\ & + \rho_1 \psi_{1,2,0}(t, x, y, z, T) \eta(y, z) \beta(y)) \frac{\partial^2}{\partial x \partial y} \\ & - \rho_1 \frac{1}{2} e^{-3\kappa(T-t)} \frac{\partial \phi}{\partial y}(y, z) \eta(y, z) \beta(y) x^2 \frac{\partial^2}{\partial x^2}, \end{aligned}$$

$$(3.15) \quad \begin{aligned} \mathcal{M}_1 = & \rho_2 e^{-\kappa(T-t)} \frac{(1 - e^{-2\kappa(T-t)})}{2\kappa} \eta(y, z) g(z) \bar{\eta}(z) \bar{\eta}'(z) x^2 \frac{\partial^2}{\partial x^2} \\ & + \rho_2 e^{-\kappa(T-t)} \eta(y, z) g(z) x \frac{\partial^2}{\partial x \partial z} + (\psi_{2,2,1}(t, x, y, z, T) \beta^2(y) \\ & + \rho_2 \psi_{1,1,1}(t, x, T) \eta(y, z) \beta(y)) \frac{\partial^2}{\partial x \partial y}. \end{aligned}$$

The fundamental difference with the situation described in Fouque et al. [2011] then materializes in one term: *the differential operator  $\mathcal{L}_3$  which contributes to the order  $\sqrt{\varepsilon}$  in the expansion of  $\mathcal{L}^{\varepsilon,\delta}$* . Also, observe that the coefficients of these operators are time dependent which complicates the asymptotic analysis. This difficulty has also been dealt with in Fouque et al. [2004].

### 3.5 Formal Derivation of the First-Order Approximation

Let us formally write  $P^{\varepsilon,\delta}$  in powers of  $\sqrt{\delta}$  and  $\sqrt{\varepsilon}$ ,

$$P^{\varepsilon,\delta} = \sum_{m,k \geq 0} (\sqrt{\varepsilon})^k (\sqrt{\delta})^m P_{k,m},$$

and denote  $P_{0,0}$  simply by  $P_0$  where we assume that, at maturity  $T_0$ ,  $P_0(T_0, x, y, z, T) = \varphi(x)$ . We are interested in determining  $P_0$ ,  $P_{1,0}$  and  $P_{0,1}$ . We follow the method presented in Fouque et al. [2011] with some minor modifications in order to take into account the new term  $\mathcal{L}_3$ .



In order to compute the leading term  $P_0$  and  $P_{1,0}$ , we set to be zero the following terms of the expansion of  $\mathcal{L}^{\varepsilon,\delta} P^{\varepsilon,\delta}$ :

$$(3.16) \quad (-1, 0) : \mathcal{L}_0 P_0 = 0,$$

$$(3.17) \quad (-1/2, 0) : \mathcal{L}_0 P_{1,0} + \mathcal{L}_1 P_0 = 0,$$

$$(3.18) \quad (0, 0) : \mathcal{L}_0 P_{2,0} + \mathcal{L}_1 P_{1,0} + \mathcal{L}_2 P_0 = 0,$$

$$(3.19) \quad (1/2, 0) : \mathcal{L}_0 P_{3,0} + \mathcal{L}_1 P_{2,0} + \mathcal{L}_2 P_{1,0} + \mathcal{L}_3 P_0 = 0,$$

where we are using the notation  $(i, j)$  to denote the term of  $i$ th order in  $\varepsilon$  and  $j$ th in  $\delta$ .

### 3.5.1 Computing $P_0$

We seek a function  $P_0 = P_0(t, x, z, T)$ , independent of  $y$ , so that the Equation (3.16) is satisfied. Since  $\mathcal{L}_1$  takes derivative with respect to  $y$ ,  $\mathcal{L}_1 P_0 = 0$ . Thus the second (3.17) becomes  $\mathcal{L}_0 P_{1,0} = 0$  and for the same reason as before, we seek a function  $P_{1,0} = P_{1,0}(t, x, z, T)$  independent of  $y$ . The  $(0, 0)$ -order equation (3.18) becomes

$$\mathcal{L}_0 P_{2,0} + \cancel{\mathcal{L}_1 P_{1,0}} + \mathcal{L}_2 P_0 = 0,$$

which is a Poisson equation for  $P_{2,0}$  with solvability condition

$$\langle \mathcal{L}_2 P_0 \rangle = 0,$$

where  $\langle \cdot \rangle$  is the average under the invariant measure of  $\mathcal{L}_0$ . For more details on Poisson equations, see [Fouque et al., 2011, Section 3.2]. Define now

$$(3.20) \quad \mathcal{L}_B(\sigma) = \frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2}{\partial x^2} - r.$$

and

$$\sigma(t, y, z, T) = e^{-\kappa(T-t)} \eta(y, z),$$

where we are using the notation  $\mathcal{L}_B(\sigma)$  for the Black differential operator with volatility  $\sigma$ . Since  $P_0$  does not depend on  $y$  and by the form of  $\mathcal{L}_2$  given in (3.12), the solvability condition becomes

$$\langle \mathcal{L}_2 P_0 \rangle = \langle \mathcal{L}_B(\sigma(t, y, z, T)) \rangle P_0 = 0.$$

Note that

$$\langle \mathcal{L}_B(\sigma(t, y, z, T)) \rangle = \frac{\partial}{\partial t} + \frac{1}{2} x^2 \langle \sigma^2(t, \cdot, z, T) \rangle \frac{\partial^2}{\partial x^2} - r = \mathcal{L}_B(\bar{\sigma}(t, z, T)),$$

where

$$(3.21) \quad \bar{\sigma}^2(t, z, T) = \langle \sigma^2(t, \cdot, z, T) \rangle = e^{-2\kappa(T-t)} \bar{\eta}^2(z),$$

with  $\bar{\eta}(z)$  defined in (3.4). Therefore, we choose  $P_0$  to satisfy the PDE

$$\begin{cases} \mathcal{L}_B(\bar{\sigma}(t, z, T)) P_0(t, x, z, T) = 0, \\ P_0(T_0, x, z, T) = \varphi(x). \end{cases}$$

Note also that  $\mathcal{L}_B(\bar{\sigma}(t, z, T))$  is the Black differential operator with time-varying volatility  $\bar{\sigma}(t, z, T)$  and hence, if we define the time-averaged volatility,  $\bar{\sigma}_{t, T_0}(z, T)$ , by the formula

$$(3.22) \quad \begin{aligned} \bar{\sigma}_{t, T_0}^2(z, T) &= \frac{1}{T_0 - t} \int_t^{T_0} \bar{\sigma}^2(u, z, T) du \\ &= \bar{\eta}^2(z) \left( \frac{e^{-2\kappa(T-T_0)} - e^{-2\kappa(T-t)}}{2\kappa(T_0 - t)} \right), \end{aligned}$$

we can write

$$P_0(t, x, z, T) = P_B(t, x, \bar{\sigma}_{t, T_0}(z, T)),$$

where  $P_B(t, x, \sigma)$  is the price at  $(t, x)$  of the European derivative with maturity  $T_0$  and payoff function  $\varphi$  in the Black model with constant volatility  $\sigma$ .

In order to simplify notation here and in what follows, we define

$$(3.23) \quad \lambda(t, T_0, T, \kappa) = \frac{e^{-\kappa(T-T_0)} - e^{-\kappa(T-t)}}{\kappa(T_0 - t)}.$$

Therefore,

$$\bar{\sigma}_{t, T_0}^2(z, T) = \bar{\eta}^2(z) \lambda_\sigma^2(t, T_0, T, \kappa),$$

where

$$\lambda_\sigma(t, T_0, T, \kappa) = \sqrt{\lambda(t, T_0, T, 2\kappa)}.$$

### 3.5.2 Computing $P_{1,0}^\varepsilon$

By the  $(0, 0)$ -order equation (3.18), we get the formula

$$(3.24) \quad P_{2,0} = -\mathcal{L}_0^{-1}(\mathcal{L}_B(\sigma) - \mathcal{L}_B(\bar{\sigma}))P_0 + c(t, x, z, T),$$

for some function  $c$  which does not depend on  $y$ . Denote by  $\phi(y, z)$  a solution of the Poisson equation

$$\mathcal{L}_0\phi(y, z) = \eta^2(y, z) - \bar{\eta}^2(z).$$

Hence,

$$\begin{aligned} \mathcal{L}_0^{-1}(\mathcal{L}_B(\sigma) - \mathcal{L}_B(\bar{\sigma})) &= \mathcal{L}_0^{-1} \left( \frac{1}{2}(\sigma^2(t, y, z, T) - \bar{\sigma}^2(t, z, T))x^2 \frac{\partial^2}{\partial x^2} \right) \\ &= \frac{1}{2} e^{-2\kappa(T-t)} \mathcal{L}_0^{-1}(\eta^2(y, z) - \bar{\eta}^2(z))x^2 \frac{\partial^2}{\partial x^2} \\ &= \frac{1}{2} e^{-2\kappa(T-t)} \phi(y, z) D_2, \end{aligned}$$

where we use the notation

$$(3.25) \quad D_k = x^k \frac{\partial^k}{\partial x^k}.$$

From the  $(1/2, 0)$ -order equation (3.19), which is a Poisson equation for  $P_{3,0}$ , we get the solvability condition

$$(3.26) \quad \langle \mathcal{L}_1 P_{2,0} + \mathcal{L}_2 P_{1,0} + \mathcal{L}_3 P_0 \rangle = 0.$$

Using formula (3.24) for  $P_{2,0}$  and formula (3.11) for  $\mathcal{L}_1$ , we get

$$\begin{aligned}
\mathcal{L}_1 P_{2,0} &= -\mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_B(\sigma) - \mathcal{L}_B(\bar{\sigma})) P_0 \\
&= -\rho_1 e^{-\kappa(T-t)} \eta(y, z) \beta(y) x \frac{\partial^2}{\partial x \partial y} \left( \frac{1}{2} e^{-2\kappa(T-t)} \phi(y, z) D_2 P_0 \right) \\
&= -\rho_1 e^{-\kappa(T-t)} x \eta(y, z) \beta(y) \frac{\partial}{\partial x} \frac{\partial}{\partial y} \left( \frac{1}{2} e^{-2\kappa(T-t)} \phi(y, z) D_2 P_0 \right) \\
&= -\frac{1}{2} \rho_1 e^{-3\kappa(T-t)} \eta(y, z) \beta(y) \frac{\partial \phi}{\partial y}(y, z) D_1 D_2 P_0.
\end{aligned}$$

We also have by equation (3.12)

$$\mathcal{L}_2 P_{1,0} = \frac{\partial P_{1,0}}{\partial t} + \frac{1}{2} \sigma^2(t, y, z, T) x^2 \frac{\partial^2 P_{1,0}}{\partial x^2} - r P_{1,0},$$

and from (3.14)

$$\mathcal{L}_3 P_0 = -\frac{1}{2} \rho_1 e^{-3\kappa(T-t)} \frac{\partial \phi}{\partial y}(y, z) \eta(y, z) \beta(y) D_2 P_0.$$

Combining these equations, we get

$$\begin{aligned}
\mathcal{L}_1 P_{2,0} + \mathcal{L}_2 P_{1,0} + \mathcal{L}_3 P_0 &= v_3(t, y, z, T) D_2 P_0 \\
&\quad + v_3(t, y, z, T) D_1 D_2 P_0 + \mathcal{L}_B(\sigma(t, y, z, T)) P_{1,0},
\end{aligned}$$

where

$$v_3(t, y, z, T) = -\frac{1}{2} \rho_1 e^{-3\kappa(T-t)} \frac{\partial \phi}{\partial y}(y, z) \eta(y, z) \beta(y).$$

Therefore, averaging with respect to the invariant distribution of  $Y^1$ , we deduce from (3.26) that  $P_{1,0}^\varepsilon = \sqrt{\varepsilon} P_{1,0}$  satisfies the PDE:

$$(3.27) \quad \begin{cases} \mathcal{L}_B(\bar{\sigma}(t, z, T)) P_{1,0}^\varepsilon(t, x, z, T) = -f(t, T) \mathcal{A}^\varepsilon P_0(t, x, z, T), \\ P_{1,0}^\varepsilon(T_0, x, z, T) = 0, \end{cases}$$

where

$$(3.28) \quad \begin{aligned} \mathcal{A}^\varepsilon &= V_3^\varepsilon(z) (D_1 D_2 + D_2), \\ f(t, T) &= e^{-3\kappa(T-t)}, \\ V_3^\varepsilon(z) &= -\sqrt{\varepsilon} \frac{1}{2} \rho_1 \left\langle \frac{\partial \phi}{\partial y}(\cdot, z) \eta(\cdot, z) \beta \right\rangle. \end{aligned}$$

The linear PDE (3.27) is solved explicitly:

$$(3.29) \quad P_{1,0}^\varepsilon(t, x, z, T) = (T_0 - t) \lambda_3(t, T_0, T, \kappa) V_3^\varepsilon(z) (D_1 D_2 + D_2) P_B(t, x, \bar{\sigma}_{t, T_0}(z, T)),$$

where

$$\lambda_3(t, T_0, T, \kappa) = \lambda(t, T_0, T, 3\kappa),$$

and  $\lambda$  is defined by (3.23). To see this, note that the operator  $\mathcal{A}^\varepsilon$  given by (3.28), and the operator  $\mathcal{L}_B(\bar{\sigma}(t, z, T))$  given by (3.20) and (3.21), commute and therefore, the solution of the PDE (3.27) is given by

$$P_{1,0}^\varepsilon(t, x, z, T) = \left( \int_t^{T_0} f(u, T) du \right) \mathcal{A}^\varepsilon P_0(t, x, z, T).$$

Thus, solving the above integral, we get (3.29).

### 3.5.3 Computing $P_{0,1}^\delta$

In order to compute  $P_{0,1}$ , we need to consider terms with order  $1/2$  in  $\delta$ , more explicitly the following ones:

$$(3.30) \quad (-1, 1/2) : \mathcal{L}_0 P_{0,1} = 0,$$

$$(3.31) \quad (-1/2, 1/2) : \mathcal{L}_0 P_{1,1} + \mathcal{L}_1 P_{0,1} + \mathcal{M}_3 P_0 = 0,$$

$$(3.32) \quad (0, 1/2) : \mathcal{L}_0 P_{2,1} + \mathcal{L}_1 P_{1,1} + \mathcal{M}_3 P_{1,0} + \mathcal{L}_2 P_{0,1} + \mathcal{M}_1 P_0 = 0.$$

Recall that  $\mathcal{L}_1$  and  $\mathcal{M}_3$  as defined by Equations (3.11) and (3.13) take derivative with respect to  $y$ . Choosing  $P_{0,1} = P_{0,1}(t, x, z, T)$  and  $P_{1,1} = P_{1,1}(t, x, z, T)$  independent of  $y$ , the first two equations (3.30) and (3.31) are satisfied. The last equation (3.32) becomes

$$\mathcal{L}_0 P_{2,1} + \mathcal{L}_2 P_{0,1} + \mathcal{M}_1 P_0 = 0,$$

and thus the solvability condition for this Poisson equation for  $P_{2,1}$  is

$$\langle \mathcal{L}_2 P_{0,1} + \mathcal{M}_1 P_0 \rangle = 0.$$

From (3.15) we have

$$\begin{aligned} \mathcal{M}_1 P_0 &= \rho_2 \bar{\eta}(z) \bar{\eta}'(z) \frac{(1 - e^{-2\kappa(T-t)})}{2\kappa} e^{-\kappa(T-t)} \eta(y, z) g(z) D_2 P_0 \\ &\quad + \rho_2 e^{-\kappa(T-t)} \eta(y, z) g(z) D_1 \frac{\partial}{\partial z} P_0, \end{aligned}$$

and then, if we write  $P_{0,1}^\delta(t, x, z, T) = \sqrt{\delta} P_{0,1}(t, x, z, T)$ , the solvability condition above can be written as

$$(3.33) \quad \begin{cases} \mathcal{L}_B(\bar{\sigma}(t, z, T)) P_{0,1}^\delta = -f_0(t, T) \mathcal{A}_0^\delta P_0 - f_1(t, T) \mathcal{A}_1^\delta P_0, \\ P_{0,1}^\delta(T_0, x, z, T) = 0, \end{cases}$$

where

$$\begin{aligned} \mathcal{A}_0^\delta &= V_0^\delta(z) D_2, \\ \mathcal{A}_1^\delta &= V_1^\delta(z) D_1 \frac{\partial}{\partial z}, \\ V_0^\delta(z) &= \sqrt{\delta} \frac{1}{2\kappa} \rho_2 \langle \eta(\cdot, z) \rangle g(z) \bar{\eta}(z) \bar{\eta}'(z), \\ f_0(t, T) &= e^{-\kappa(T-t)} - e^{-3\kappa(T-t)}, \\ V_1^\delta(z) &= \sqrt{\delta} \rho_2 \langle \eta(\cdot, z) \rangle g(z), \\ f_1(t, T) &= e^{-\kappa(T-t)}. \end{aligned}$$

The solution for this PDE can be explicitly computed

$$P_{0,1}^\delta(t, x, z, T) = (T_0 - t) V_0^\delta(z) (\lambda_0(t, T_0, T, \kappa) D_2 + \lambda_1(t, T_0, T, \kappa) D_1 D_2) P_B(t, x, \bar{\sigma}_{t, T_0}(z, T)),$$

where

$$\begin{aligned} \lambda_0(t, T_0, T, \kappa) &= \lambda(t, T_0, T, \kappa) - \lambda(t, T_0, T, 3\kappa), \\ \lambda_1(t, T_0, T, \kappa) &= e^{-2\kappa(T-T_0)} \lambda(t, T_0, T, \kappa) - \lambda(t, T_0, T, 3\kappa). \end{aligned}$$

The details of this computation are given in the Appendix C.

### 3.6 Summary and Some Remarks

We now summarize the formulas involved in the first-order asymptotic expansion of the price of European derivative on futures. We recall that, as before,  $D_k = x^k \partial / \partial x_k$ . We have formally derived the first-order approximation for  $P^{\varepsilon, \delta}$ :

$$P^{\varepsilon, \delta} \approx P_0 + P_{1,0}^\varepsilon + P_{0,1}^\delta$$

with

$$P_0(t, x, z, T) = P_B(t, x, \bar{\sigma}_{t, T_0}(z, T)),$$

$$P_{1,0}^\varepsilon(t, x, z, T) = (T_0 - t) \lambda_3(t, T_0, T, \kappa) V_3^\varepsilon(z) (D_2 + D_1 D_2) P_B(t, x, \bar{\sigma}_{t, T_0}(z, T)),$$

$$P_{0,1}^\delta(t, x, z, T) = (T_0 - t) V_0^\delta(z) (\lambda_0(t, T_0, T, \kappa) D_2 + \lambda_1(t, T_0, T, \kappa) D_1 D_2) P_B(t, x, \bar{\sigma}_{t, T_0}(z, T)),$$

where

$$\begin{aligned} \bar{\eta}^2(z) &= \langle \eta^2(\cdot, z) \rangle, \\ V_3^\varepsilon(z) &= -\sqrt{\varepsilon} \frac{1}{2} \rho_1 \left\langle \frac{\partial \phi}{\partial y}(\cdot, z) \eta(\cdot, z) \beta \right\rangle, \\ V_0^\delta(z) &= \sqrt{\delta} \frac{1}{2\kappa} \rho_2 \langle \eta(\cdot, z) \rangle g(z) \bar{\eta}(z) \bar{\eta}'(z), \\ \lambda(t, T_0, T, \kappa) &= \frac{e^{-\kappa(T-T_0)} - e^{-\kappa(T-t)}}{\kappa(T_0 - t)}, \\ \lambda_3(t, T_0, T, \kappa) &= \lambda(t, T_0, T, 3\kappa), \\ \lambda_0(t, T_0, T, \kappa) &= \lambda(t, T_0, T, \kappa) - \lambda(t, T_0, T, 3\kappa), \\ \lambda_1(t, T_0, T, \kappa) &= e^{-2\kappa(T-T_0)} \lambda(t, T_0, T, \kappa) - \lambda(t, T_0, T, 3\kappa), \\ \lambda_\sigma^2(t, T_0, T, \kappa) &= \lambda(t, T_0, T, 2\kappa). \\ \bar{\sigma}_{t, T_0}^2(z, T) &= \bar{\eta}^2(z) \lambda_\sigma^2(t, T_0, T, \kappa). \end{aligned}$$

A valuable feature of the perturbation method is that in order to compute the first-order approximation, we only need the values of the *group market parameters*

$$(\kappa, \bar{\eta}(z), V_0^\delta(z), V_3^\varepsilon(z)).$$

This feature can also be seen as model independence or robustness of this approximation: under some regularity conditions stated in Theorem 3.2, this approximation is independent of the particular form of the coefficients describing the process  $Y^\varepsilon$  and  $Z^\delta$ , i.e. the functions  $\alpha$ ,  $\beta$ ,  $c$  and  $g$  involved in the model (2.1).

From now on we will use the following notation

$$\begin{aligned} \bar{P}(t, x, z, T) &= P_0(t, x, z, T), \\ \bar{P}^{\varepsilon, \delta}(t, x, z, T) &= P_{1,0}^\varepsilon(t, x, z, T) + P_{0,1}^\delta(t, x, z, T), \\ \tilde{P}^{\varepsilon, \delta}(t, x, z, T) &= \bar{P}(t, x, z, T) + \bar{P}^{\varepsilon, \delta}(t, x, z, T). \end{aligned} \tag{3.34}$$

### 3.7 Accuracy of the Approximation

We now state the precise accuracy result for the formal approximation determined in the previous sections. All the reasoning in Section 3.4 is only a formal procedure and a well-thought *choice* for the proposed first-order approximation. The next result establishes the order of accuracy of this approximation and justifies *a posteriori* the choices made earlier.

**Theorem 3.2.** *We assume*

- (i) *Existence and uniqueness of the SDE (2.1) for any fixed  $(\varepsilon, \delta)$ .*
- (ii) *The process  $Y^1$  with infinitesimal generator  $\mathcal{L}_0$  has a unique invariant distribution and is mean-reverting as in [Fouque et al., 2011, Section 3.2].*
- (iii) *The function  $\eta(y, z)$  is smooth in  $z$  and such the solution  $\phi$  to the Poisson equation (3.5) is at most polynomially growing.*
- (iv) *The payoff function  $\varphi(x)$  and its derivatives are smooth and bounded.*

Then,

$$P^{\varepsilon, \delta}(t, x, y, z, T) = \tilde{P}^{\varepsilon, \delta}(t, x, z, T) + O(\varepsilon + \delta).$$

*Proof.* The proof is provided in Appendix B. □

Observe that in the heuristic derivation of the approximation given by Equation (3.34) we did not use the smoothness assumption (iv) of Theorem 3.2. In fact, (iv) is only used in a technical way to derive the accuracy result as shown in Appendix B. Then, assumption (iv) can be relaxed to include the case of *call options* by the regularization argument presented in Fouque et al. [2003], and extended for the addition of the slow factor in Fouque et al. [2011]. Being able to apply the approximation (3.34) to *call options* is essential to the next section.

## 4 Calibration

In this section we will outline a procedure to calibrate the *group market parameters*  $(\kappa, \bar{\eta}(z), V_0^\delta(z), V_3^\varepsilon(z))$  to available prices of call options on  $F_{t,T}$ . As one may conclude from [Fouque et al., 2011, Chapters 6 and 7], or from an application of Functional Itô Calculus (Dupire [2009]) to the perturbation analysis presented in the forthcoming paper Fouque and Saporito, the values of the group market parameters are the only parameters needed to price path-dependent or American options to the same order of accuracy. Therefore, once the group market parameters are calibrated to vanilla options, the *same* parameters are used to price exotic derivatives. This is one of the most important characteristics of the perturbation theory.

### 4.1 Approximate Call Prices on Future Contracts and Implied Volatilities

Assume without loss of generality  $t = 0$  and consider a European call option on  $F_{0,T}$  with maturity  $T_0 \leq T$  and strike  $K$ . As we are interested in the calibration of the market group parameters to call prices at the fixed time  $t = 0$ , we will drop the

variables  $(t, x)$  in the formulas and write the variables  $(T_0, K)$  instead. We will also drop the variable  $z$  since it should be understood as just a parameter. The Black formula for a  $(T_0, K)$ -call option is defined by

$$(4.1) \quad C_B(T_0, K, \sigma) = e^{-rT_0}(F_{0,T}\Phi(d_1(\sigma)) - K\Phi(d_2(\sigma))),$$

where

$$d_{1,2}(\sigma) = \frac{\log(F_{0,T}/K) \pm \frac{\sigma^2}{2}T_0}{\sigma\sqrt{T_0}}.$$

Let us also denote

$$\bar{d}_{1,2} = d_{1,2}(\bar{\sigma}_{0,T_0}),$$

where  $\bar{\sigma}_{0,T_0}$  is the time-averaged volatility defined in (3.22), and notice that

$$(4.2) \quad \bar{P}(0, F_{0,T}, z, T) = C_B(T_0, K, \bar{\sigma}_{0,T_0}).$$

The following relations between Greeks of the Black price are well-known and they will be essential in what follows:

$$\frac{\partial C_B}{\partial \sigma}(T_0, K, \sigma) = T_0\sigma D_2 C_B(T_0, K, \sigma),$$

and

$$(4.3) \quad \begin{aligned} D_1 \frac{\partial C_B}{\partial \sigma}(T_0, K, \sigma) &= \left(1 - \frac{d_1}{\sigma\sqrt{T_0}}\right) \frac{\partial C_B}{\partial \sigma}(T_0, K, \sigma) \\ &= \left(\frac{1}{2} + \frac{\log(K/x)}{\sigma^2 T_0}\right) \frac{\partial C_B}{\partial \sigma}(T_0, K, \sigma), \end{aligned}$$

where the operator  $D_k$  is defined in (3.25). Using the Greeks relations stated above and (4.2), we are able to rewrite (3.34) as

$$\begin{aligned} \tilde{P}^{\varepsilon,\delta} &= \bar{P}_B + \left( \frac{\lambda_3(T_0, T, \kappa)}{\bar{\sigma}_{0,T_0}} V_3^\varepsilon(z) + \frac{\lambda_3(T_0, T, \kappa)}{\bar{\sigma}_{0,T_0}} V_3^\varepsilon(z) \left( \frac{1}{2} + \frac{\log(K/F_{0,T})}{\bar{\sigma}_{0,T_0}^2 T_0} \right) \right. \\ &\quad \left. + \frac{\lambda_0(T_0, T, \kappa)}{\bar{\sigma}_{0,T_0}} V_0^\delta(z) + \frac{\lambda_1(T_0, T, \kappa)}{\bar{\sigma}_{0,T_0}} V_0^\delta(z) \left( \frac{1}{2} + \frac{\log(K/F_{0,T})}{\bar{\sigma}_{0,T_0}^2 T_0} \right) \right) \frac{\partial \bar{P}}{\partial \sigma}. \end{aligned}$$

Now, we convert the price  $P^{\varepsilon,\delta}$  to a Black implied volatility  $I$ :

$$C_B(T_0, K, I(T_0, K, T)) = P^{\varepsilon,\delta} = \tilde{P}^{\varepsilon,\delta} + \dots$$

**Remark.** Since we do not have a spot price readily available for trade in our model we work with futures. Thus, differently from what is done in the equity case, we consider the Black implied volatility instead of the Black–Scholes implied volatility.

Then, expand  $I(T_0, K, T)$  around  $\bar{\sigma}_{0,T_0}$ :

$$I(T_0, K, T) - \bar{\sigma}_{0,T_0} = \sqrt{\varepsilon} I_{1,0}(T_0, K, T) + \sqrt{\delta} I_{0,1}(T_0, K, T) + \dots$$

Hence, matching both expansions gives us

$$\begin{aligned}\sqrt{\varepsilon}I_{1,0}(T_0, K, T) &= \frac{3}{2} \frac{\lambda_3(T_0, T, \kappa)}{\lambda_\sigma(T_0, T, \kappa)} \frac{V_3^\varepsilon(z)}{\bar{\eta}(z)} + \frac{\lambda_3(T_0, T, \kappa)}{\lambda_\sigma^3(T_0, T, \kappa)} \frac{V_3^\varepsilon(z)}{\bar{\eta}^3(z)} \frac{\log(K/F_{0,T})}{T_0}, \\ \sqrt{\delta}I_{1,0}(T_0, K, T) &= \left( \frac{\lambda_0(T_0, T, \kappa)}{\lambda_\sigma(T_0, T, \kappa)} + \frac{1}{2} \frac{\lambda_1(T_0, T, \kappa)}{\lambda_\sigma(T_0, T, \kappa)} \right) \frac{V_0^\delta(z)}{\bar{\eta}(z)} \\ &\quad + \frac{\lambda_1(T_0, T, \kappa)}{\lambda_\sigma^3(T_0, T, \kappa)} \frac{V_0^\delta(z)}{\bar{\eta}^3(z)} \frac{\log(K/F_{0,T})}{T_0}.\end{aligned}$$

So, in terms of the reduced variable LMMR, the log-moneyness to maturity ratio defined by

$$\text{LMMR} = \frac{\log(K/F_{0,T})}{T_0},$$

the first-order approximation of the implied volatility  $I(T_0, K, T)$  can be written as

$$(4.4) \quad \begin{aligned}I(T_0, K, T) &\approx \bar{\eta}(z) \bar{b}(T_0, T, \kappa) + \frac{V_3^\varepsilon(z)}{\bar{\eta}(z)} b^\varepsilon(T_0, T, \kappa) + \frac{V_0^\delta(z)}{\bar{\eta}(z)} b^\delta(T_0, T, \kappa) \\ &\quad + \left( \frac{V_3^\varepsilon(z)}{\bar{\eta}^3(z)} a^\varepsilon(T_0, T, \kappa) + \frac{V_0^\delta(z)}{\bar{\eta}^3(z)} a^\delta(T_0, T, \kappa) \right) \text{LMMR},\end{aligned}$$

where

$$\begin{aligned}\bar{b}(T_0, T, \kappa) &= \lambda_\sigma(T_0, T, \kappa), \\ b^\varepsilon(T_0, T, \kappa) &= \frac{3}{2} \frac{\lambda_3(T_0, T, \kappa)}{\lambda_\sigma(T_0, T, \kappa)}, \\ b^\delta(T_0, T, \kappa) &= \frac{\lambda_0(T_0, T, \kappa)}{\lambda_\sigma(T_0, T, \kappa)} + \frac{1}{2} \frac{\lambda_1(T_0, T, \kappa)}{\lambda_\sigma(T_0, T, \kappa)}, \\ a^\varepsilon(T_0, T, \kappa) &= \frac{\lambda_3(T_0, T, \kappa)}{\lambda_\sigma^3(T_0, T, \kappa)}, \\ a^\delta(T_0, T, \kappa) &= \frac{\lambda_1(T_0, T, \kappa)}{\lambda_\sigma^3(T_0, T, \kappa)}.\end{aligned}$$

Therefore, the model predicts at first-order accuracy and for fixed maturities  $T_0$  and  $T$  that the implied volatility is affine in the LMMR variable.

## 4.2 Calibration Procedure

Suppose that at the present time  $t = 0$ , there is available the finite set of Black implied volatilities  $\{I(T_{0ij}, K_{ijl}, T_i)\}$ , which we understand in the following way: for each  $i$  (i.e. for each future price  $F_{0,T_i}$ ) there are available call option prices with maturities  $T_{0ij}$  and, for each of these maturities, strikes  $K_{ijl}$ .

Since the data is more abundant in the  $K$  direction, we will first linearly regress the implied volatilities against the variable  $\text{LMMR}_{ijl}$ ,

$$\text{LMMR}_{ijl} = \frac{\log(K_{ijl}/F_{0,T_i})}{T_{0ij}},$$



for fixed  $i$  and  $j$ . More precisely, we will use the least-squares criterion to perform this regression, namely

$$(\hat{a}_{ij}, \hat{b}_{ij}) = \operatorname{argmin}_{(a_{ij}, b_{ij})} \sum_l (I(T_{0ij}, K_{ijl}, T_i) - (a_{ij} \text{LMMR}_{ijl} + b_{ij}))^2.$$

Now, using Equation (4.4), we first regress the estimate  $\hat{a}$  against  $a^\varepsilon$  and  $a^\delta$ :

$$(4.5) \quad (\hat{a}_0, \hat{a}_1, \hat{\kappa}) = \operatorname{argmin}_{(a_0, a_1, \kappa)} \sum_{i,j} \left( \hat{a}_{ij} - \left( a_0 a^\varepsilon(T_{0ij}, T_i, \kappa) + a_1 a^\delta(T_{0ij}, T_i, \kappa) \right) \right)^2$$

and then knowing  $(\hat{a}_0, \hat{a}_1, \hat{\kappa})$  we regress  $\hat{b}$  against  $\bar{b}$ ,  $b^\varepsilon$  and  $b^\delta$ :

$$\hat{b}_0 = \operatorname{argmin}_{b_0} \sum_{i,j} \left( \hat{b}_{ij} - \left( b_0 \bar{b}(T_{0ij}, T_i, \hat{\kappa}) + b_0^2 \left( \hat{a}_0 b^\varepsilon(T_{0ij}, T_i, \hat{\kappa}) + \hat{a}_1 b^\delta(T_{0ij}, T_i, \hat{\kappa}) \right) \right) \right)^2.$$

Therefore, we find the following estimates for the market group parameters:

$$\begin{aligned} \widehat{\bar{\eta}}(z) &= \hat{b}_0, \\ \widehat{V_3^\varepsilon}(z) &= \hat{a}_0 \hat{b}_0^3, \\ \widehat{V_0^\delta}(z) &= \hat{a}_1 \hat{b}_0^3, \end{aligned}$$

and  $\hat{\kappa}$  is given by Equation (4.5). In order to perform the above minimizations, the initial guesses are of utmost importance. Since we expect terms of order  $\sqrt{\varepsilon}$  to be small, we set the initial guesses of  $a_0$  and  $a_1$  to be 0. On the other hand, we can construct initial guesses of  $\kappa$  and  $b_0$  by estimating  $\kappa$  and  $\bar{\eta}(z)$  from historical data of  $F_{t,T}$ .

### 4.3 Calibration Example

For this numerical example we will use the prices of call options on the Henry Hub future contracts on May 9th, 2007. For each future contract (i.e. for each maturity  $T_i$ ), the option data is available just for the same maturity as the underlying future (i.e.  $T_{0ij} = T_i$ ) and various strikes  $K_{ijl}$  between US\$ 3.00 and US\$ 20.00. The future prices are shown in Figure 1 and one can clearly notice that seasonality is a main factor in the data. However, we have not included seasonality in this work, although it can be done without additional mathematical difficulty, see Eydeland and Wolyniec [2002] for instance, but doing so would be outside the scope of the paper. Henceforth, we will restrict ourselves to options maturing only in the same month of the year, for example we choose November. In the following calibration, we assume  $r = 0$  here. If one wants to capture the convexity of the implied volatility, one would have to use the second-order approximation as it is done for the Equity markets in Fouque et al. [2012].

We show in Figure 2 the implied volatility fit, where the solid line is the model implied volatility and the circles are the implied volatilities observed in the market. On the title of each subplot it is shown the time-to-maturity (TTM) of each option in years. The calibrated group market parameters are given in Table 1. It is important to notice that  $V_3^\varepsilon(z)$  and  $V_0^\delta(z)$  are indeed small and hence these parameters are compatible with our model.

Parameter	Value
$\hat{\kappa}$	1.1602
$\widehat{\bar{\eta}}(z)$	0.5298
$\widehat{V}_3^\varepsilon(z)$	0.0050
$\widehat{V}_0^\delta(z)$	0.0180

Table 1: Calibrated Parameters

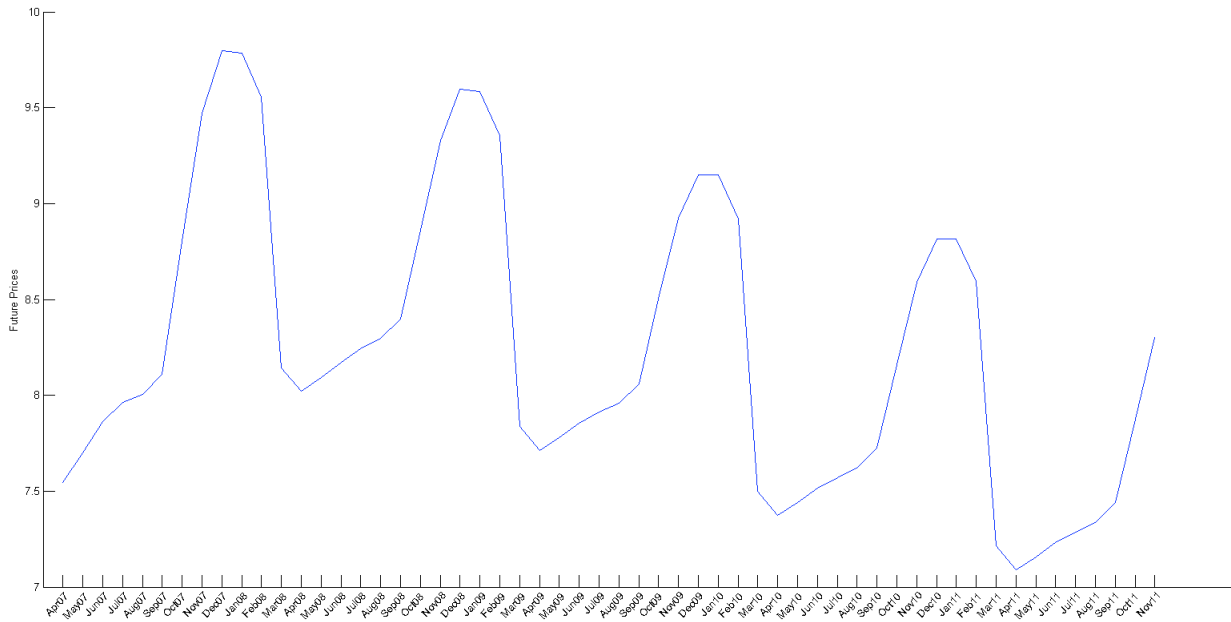


Figure 1: Future Prices on May 9th, 2007

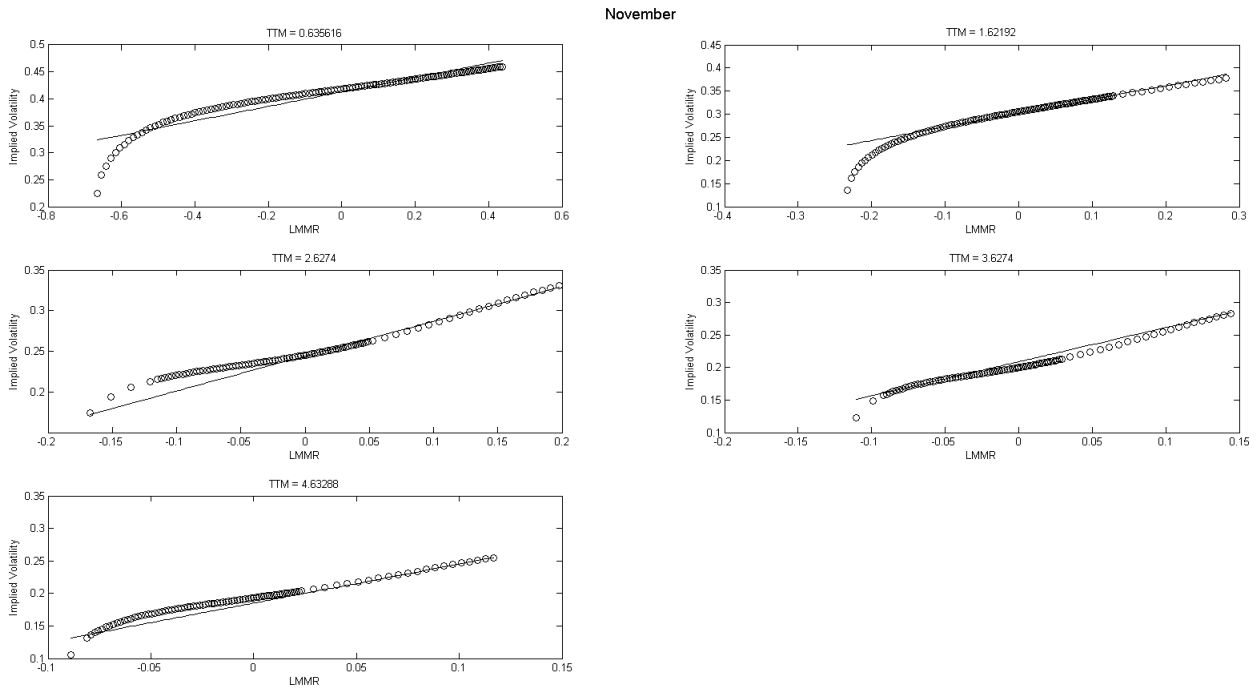


Figure 2: Implied Volatility for Call Options on the Henry Hub Future Contracts with Maturity in November

## 5 Joint Calibration to Options on a Stock and its Volatility Index

The method detailed in Section 3 was initially developed to jointly calibrate the model described by Equation (5.2) to data from options of a stock and its volatility index. The main desired property is the tractability of the model in which the calibration procedure is direct at least for the volatility index market and that is the reason behind the choice of the exp-OU model for the volatility, which is a log-normal model under constant volatility of volatility. The crucial example to have in mind is the S&P 500 and the VIX. For the precise definition of the CBOE Volatility Index (VIX), see CBOE [2003]. Loosely, the VIX, also known as the “fear index”, represents the market’s expectation of the volatility for the next 30 days and it is computed using options on the S&P 500 index. If we assume that the S&P 500 index is described by a geometric Brownian motion (GBM) with instantaneous stochastic variance  $V_t$ , then

$$(5.1) \quad \text{VIX}_t^2 = \mathbb{E} \left[ \frac{1}{\tau_0} \int_t^{t+\tau_0} V_\gamma d\gamma \mid \mathcal{F}_t \right],$$

where  $\tau_0$  is 30 days.

We now state precisely the problem we aim to solve. Denote the value of the stock at time  $t$  by  $S_t$  and assume the following multiscale stochastic volatility dynamics:

$$(5.2) \quad \left\{ \begin{array}{l} dS_t = rS_t dt + \sqrt{V_t} S_t dW_t^S, \\ V_t = e^{2U_t}, \\ dU_t = \kappa(m - U_t) dt + \eta(Y_t^\varepsilon, Z_t^\delta) dW_t^{(0)}, \\ dY_t^\varepsilon = \frac{1}{\varepsilon} \alpha(Y_t^\varepsilon) dt + \frac{1}{\sqrt{\varepsilon}} \beta(Y_t^\varepsilon) dW_t^{(1)}, \\ dZ_t^\delta = \delta c(Z_t^\delta) dt + \sqrt{\delta} g(Z_t^\delta) dW_t^{(2)}, \end{array} \right.$$

where  $dW_t^S dW_t^{(j)} = \rho_{Sj} dt$  and with all notation and hypotheses stated in Section 2 also holding true. In this model,  $V_t$  denotes the instantaneous variance of  $S_t$ . Notice the difference between the definitions of  $V_t$  in the SDE systems (5.2) and (2.1). While  $V_t = e^{U_t}$  in (2.1),  $V_t = e^{2U_t}$  in (5.2). This will be justified in what follows. The financial instruments to which the model will be calibrated are:

$$(5.3) \quad \begin{aligned} C_S(T_S, K_S) &= \mathbb{E}_{\mathbb{Q}} \left[ e^{-r(T_S-t)} (S_{T_S} - K_S)^+ \mid \mathcal{F}_t \right], \\ C_V(T_V, K_V) &= \mathbb{E}_{\mathbb{Q}} \left[ (\sqrt{V_{T_V}} - K_V)^+ \mid \mathcal{F}_t \right], \end{aligned}$$

for various  $T_S$ ,  $K_S$ ,  $T_V$  and  $K_V$ .

Clearly, this is only a toy model to understand the joint calibration problem, because VIX options are actually options on the square root of integrated future variance, as one can clearly realize by (5.1). Nevertheless, the same equation shows  $\text{VIX}_t^2 \approx V_t$ , if we assume  $V_t$  does not change much from  $t$  to  $t + \tau_0$ , and hence the joint calibration

being solved is an approximation of the original problem of joint calibration to options on the S&P 500 and the VIX.

Although it is not shown explicitly by Equation (5.3), options on VIX are actually options on future contracts of VIX. Anyway, since the option and the underlying future have the same maturity, it is mathematically equivalent to understanding the price of this option as a function of the future price or the VIX. However, since the future price is a martingale (as opposed to the VIX), better formulas for the Greeks of the 0-order term are available. Mainly, Equation (4.3) holds true. Hence, the calibration procedure will be easier to describe as it is done in Section 4.2. Nevertheless, the full strength of the method described in Section 3.4 cannot be realized in this example because the maturities of the option and the underlying future contract coincide.

Furthermore, in order to compute the first-order approximation for  $C_V$ , we would like to use the results derived in Section 3. Note now  $\sqrt{V_T} = e^{U_T}$  and then the formulas of the first-order approximation for options on  $\sqrt{V_T}$  are given in Section 3.6 and no change needs to be made concerning the square root in Equation (5.3).

The idea is to first approximate the price of options on  $S$  to the 0-order and then approximate options on  $\sqrt{V}$  to the first-order using the method described in Section 3. The reason of not going further in the approximation for the price of options on  $S$  is twofold. Firstly, the 0-order term will be the price of the option still under a stochastic volatility model, then we expect to be able to fit the observed implied volatility. Secondly, since the 0-order term will not be given in closed form, it will have to be computed by Monte Carlo methods, and it will be harder to calculate the Greeks of the 0-order needed in the first-order correction.

One should also notice that it is possible to compute the first-order approximation for VIX options, but no closed formula could be found. We will not pursue this computations here, we refer to [Fouque et al., 2011, Chapter 6] where the first-order approximation of Asian options is derived.

## 5.1 Options on the Stock

Here, we describe the 0-order term of the price of a European option on  $S$ .

Fix a maturity  $T_S$  and a call payoff  $\varphi_S(s) = (s - K_S)^+$ . A no-arbitrage price of this vanilla European option is given by the conditional expectation

$$P_S^{\varepsilon, \delta}(t, s, u, y, z) = \mathbb{E}_{\mathbb{Q}}[e^{-r(T_S-t)} \varphi_S(S_{T_S}) \mid S_t = s, U_t = u, Y_t^\varepsilon = y, Z_t^\delta = z].$$

Let us write  $f(u) = e^u$  and define the differential operator

$$\mathcal{L}_S^{\varepsilon, \delta} = \frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_1^S + \mathcal{L}_2^S + \sqrt{\delta} \mathcal{M}_1^S + \delta \mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}} \mathcal{M}_3^S,$$

where

$$\begin{aligned}
\mathcal{L}_0 &= \alpha(y) \frac{\partial}{\partial y} + \frac{1}{2} \beta^2(y) \frac{\partial^2}{\partial y^2}, \\
\mathcal{L}_1^S &= \rho_{S1} f(u) s \beta(y) \frac{\partial^2}{\partial s \partial y} + \rho_{01} \eta(y, z) \beta(y) \frac{\partial^2}{\partial u \partial y}, \\
\mathcal{L}_2^S &= \frac{\partial}{\partial t} + \frac{1}{2} f^2(u) s^2 \frac{\partial^2}{\partial s^2} + r s \frac{\partial}{\partial s} - r \\
&\quad + \kappa(m - u) \frac{\partial}{\partial u} + \frac{1}{2} \eta^2(y, z) \frac{\partial^2}{\partial u^2} + \rho_{S0} f(u) s \eta(y, z) \frac{\partial^2}{\partial u \partial s}, \\
\mathcal{M}_1^S &= \rho_{S2} f(u) s g(z) \frac{\partial^2}{\partial s \partial z} + \rho_{02} \eta(y, z) g(z) \frac{\partial^2}{\partial u \partial z}, \\
\mathcal{M}_2 &= c(z) \frac{\partial}{\partial z} + \frac{1}{2} g^2(z) \frac{\partial^2}{\partial z^2}, \\
\mathcal{M}_3^S &= \rho_{12} \beta(y) g(z) \frac{\partial^2}{\partial y \partial z}.
\end{aligned}$$

Hence, Feynman-Kac's Formula tells us  $P_S^{\varepsilon, \delta}$  satisfies the following pricing PDE

$$\begin{cases} \mathcal{L}_S^{\varepsilon, \delta} P_S^{\varepsilon, \delta}(t, s, u, y, z) = 0, \\ P_S^{\varepsilon, \delta}(T_S, s, u, y, z) = \varphi_S(s). \end{cases}$$

### 5.1.1 Formal Derivation of the Zero Order Approximation

If we formally write  $P_S^{\varepsilon, \delta}$  in powers of  $\sqrt{\delta}$ ,

$$P_S^{\varepsilon, \delta} = P_{S_0}^\varepsilon + \sqrt{\delta} P_{S_1}^\varepsilon + \dots,$$

we then choose  $P_{S_0}^\varepsilon$  to satisfy

$$(5.4) \quad \begin{cases} \left( \frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_1^S + \mathcal{L}_2^S \right) P_{S_0}^\varepsilon(t, s, u, y, z) = 0, \\ P_{S_0}^\varepsilon(T_S, s, u, y, z) = \varphi_S(x). \end{cases}$$

Now, expand  $P_{S_0}^\varepsilon$  in powers of  $\sqrt{\varepsilon}$ ,

$$P_0^\varepsilon = \sum_{m \geq 0} (\sqrt{\varepsilon})^m P_{S_{m,0}},$$

and substitute into Equation (5.4) to get the following PDEs

$$(5.5) \quad (-1, 0) : \mathcal{L}_0 P_{S_0} = 0,$$

$$(5.6) \quad (-1/2, 0) : \mathcal{L}_0 P_{S_{1,0}} + \mathcal{L}_1^S P_{S_0} = 0,$$

$$(5.7) \quad (0, 0) : \mathcal{L}_0 P_{S_{2,0}} + \mathcal{L}_1^S P_{S_{1,0}} + \mathcal{L}_2^S P_{S_0} = 0.$$

Therefore, we choose  $P_{S_0} = P_{S_0}(t, s, u, z)$  in order to satisfy Equation (5.5). Since  $\mathcal{L}_1^S$  differentiates with respect to  $y$  in all its terms,  $\mathcal{L}_1^S P_{S_0} = 0$  and then, we also choose

$P_{S_{1,0}} = P_{S_{1,0}}(t, s, u, z)$  independent of  $y$  to satisfy (5.6). Now the 0-order PDE (5.7) gives us

$$\mathcal{L}_0 P_{S_{2,0}} + \mathcal{L}_1^S P_{1,0} + \mathcal{L}_2^S P_{S_0} = 0,$$

which is a Poisson equation for  $P_{S_{2,0}}$  with solvability condition:

$$\langle \mathcal{L}_2^S P_{S_0} \rangle = 0,$$

where  $\langle \cdot \rangle$  is the average under the invariant measure of  $\mathcal{L}_0$ . Since  $P_{S_0}$  does not depend on  $y$ , the solvability condition becomes:

$$\langle \mathcal{L}_2^S \rangle P_0 = 0.$$

Now, we define

$$\begin{aligned} \mathcal{L}_{SV}(\eta, \rho) &= \frac{\partial}{\partial t} + \frac{1}{2} f^2(u) s^2 \frac{\partial^2}{\partial s^2} + r s \frac{\partial}{\partial s} - r \cdot \\ &+ \kappa(m-u) \frac{\partial}{\partial u} + \frac{1}{2} \eta^2 \frac{\partial^2}{\partial u^2} + \rho f(u) s \eta \frac{\partial^2}{\partial u \partial s} \end{aligned}$$

and notice that

$$\langle \mathcal{L}_2^S \rangle = \mathcal{L}_{SV}(\bar{\eta}(z), \bar{\rho}(z)),$$

with  $\bar{\eta}(z) = \sqrt{\langle \eta^2(\cdot, z) \rangle}$  and

$$\bar{\rho}(z) = \rho_{S_0} \frac{\langle \eta(\cdot, z) \rangle}{\bar{\eta}(z)}.$$

Therefore, we choose  $P_{S_0}$  to satisfy the PDE

$$\begin{cases} \mathcal{L}_{SV}(\bar{\eta}(z), \bar{\rho}(z)) P_{S_0}(t, s, u, z) = 0 \\ P_{S_0}(T_S, s, u, z) = \varphi_S(s). \end{cases}$$

## 5.2 Options on Volatility

Here we show again the results summarized in Section 3.6 with  $T_0 = T$ , payoff  $\varphi(x) = (\sqrt{x} - K_V)^+$  and  $r = 0$ . Remember that  $V_t = e^{2U_t}$  and then no change in the formulas below is needed to take into account the square root in  $\varphi$ .

$$\begin{aligned} P_{V_0}(t, x, z) &= C_B(t, x, \bar{\sigma}_{t,T}(z)), \\ P_{V_{1,0}}^\varepsilon(t, x, z) &= (T-t) \lambda_3(t, T, \kappa) V_3^\varepsilon(z) (D_1 D_2 + D_2) C_B(t, x, \bar{\sigma}_{t,T}(z)), \\ P_{V_{0,1}}^\delta(t, x, z) &= (T-t) \lambda_0(t, T, \kappa) V_0^\delta(z) (D_1 D_2 + D_2) C_B(t, x, \bar{\sigma}_{t,T}(z)), \end{aligned}$$

where

$$\begin{aligned}
\bar{\eta}^2(z) &= \langle \eta^2(\cdot, z) \rangle, \\
V_3^\varepsilon(z) &= -\sqrt{\varepsilon} \frac{1}{2} \rho_1 \left\langle \frac{\partial \phi}{\partial y}(\cdot, z) \eta(\cdot, z) \beta \right\rangle, \\
V_0^\delta(z) &= \sqrt{\delta} \frac{1}{2\kappa} \rho_2 \langle \eta(\cdot, z) \rangle g(z) \bar{\eta}(z) \bar{\eta}'(z), \\
\lambda(t, T, \kappa) &= \frac{1 - e^{-\kappa(T-t)}}{\kappa(T-t)}, \\
\lambda_3(t, T, \kappa) &= \lambda(t, T, 3\kappa), \\
\lambda_0(t, T, \kappa) &= \lambda(t, T, \kappa) - \lambda(t, T, 3\kappa), \\
\lambda_\sigma^2(t, T, \kappa) &= \lambda(t, T, 2\kappa), \\
\bar{\sigma}_{t,T}^2(z) &= \bar{\eta}^2(z) \lambda_\sigma^2(t, T, \kappa).
\end{aligned}$$

We have dropped the variable  $T_0$ , because it is equal to  $T$ , and we have also used the function  $C_B$  defined by Equation (4.1) instead of  $P_B$  because of the specific payoff  $\varphi$  we are using. Notice that since  $T_0 = T$ ,  $\lambda_0 = \lambda_1$  and the formula for  $P_{V_0,1}^\delta$  simplifies.

### 5.3 Joint Calibration

Considering there is much more data available for options on the stock than for options on its volatility, the strategy we will follow for the calibration is:

- Calibrate  $(\kappa, m, \bar{\eta}(z), \bar{\rho}(z))$  to call options on the stock;
- Then, calibrate  $(V_3^\varepsilon(z), V_0^\delta(z))$  to call options on volatility using the values for  $\kappa$  and  $\bar{\eta}(z)$  found in the previous step.

A desirable feature is to have a negative skew in the implied volatility for options on the stock and a positive skew for options on volatility. This toy model presents such feature and in order to exemplify it, consider the parameter values given in Table 2 and the implied volatilities given in the Figures 3 and 4.

Table 2: Parameter Values

Parameter	Value	Parameter	Value
$r$	0.05	$\kappa$	1
$T$	1 year	$\bar{\eta}(z)$	0.5
$F_{0,T}$	20	$m$	0.3
$S_0$	100	$\rho$	-0.7
$V_0$	0.4	$V_3^\varepsilon(z)$	-0.001
$K_S$	$\in [70, 130]$	$V_0^\delta(z)$	0.0008
$K_V$	$\in [5, 25]$		



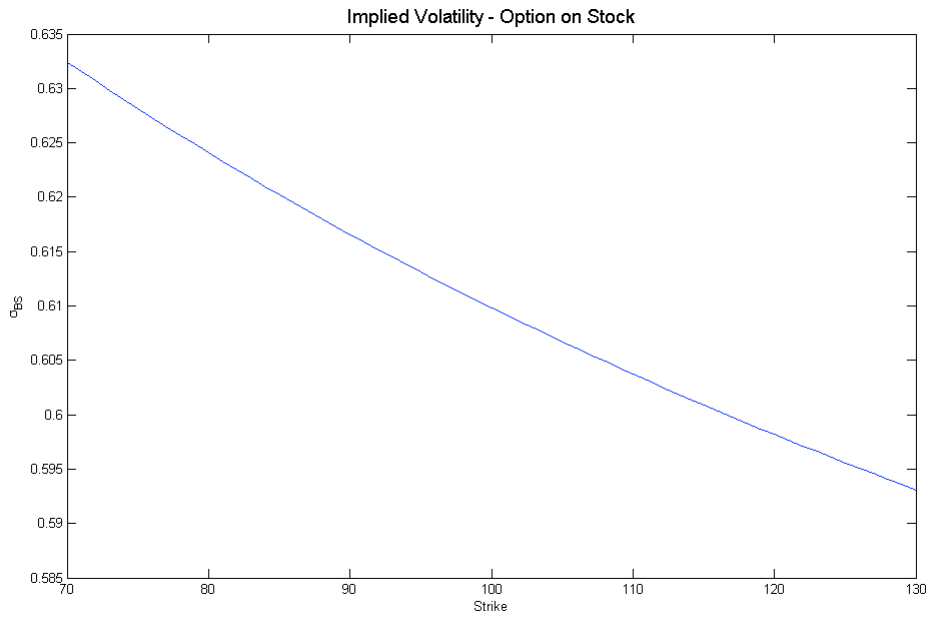


Figure 3: Black–Scholes Implied Volatility for Options on the Stock

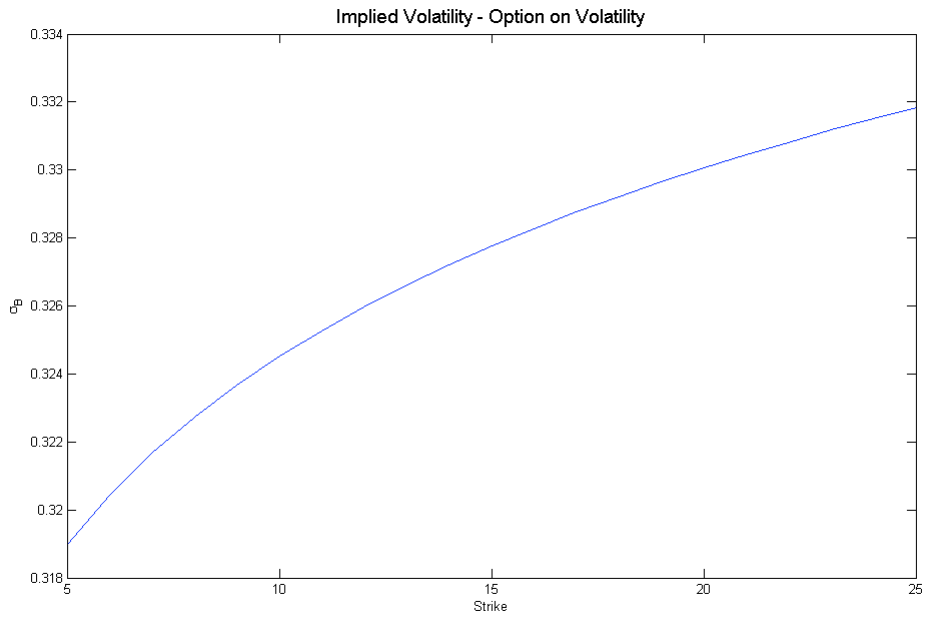


Figure 4: Black Implied Volatility for Options on Volatility

## 6 Concluding Remarks and Future Directions

We have presented a general method to derive the first-order approximation for compound derivatives and develop it thoroughly in the case of derivatives on future contracts. Although the method may seem to be involved, it does not require any additional hypothesis on the regularity of the payoff function other than the ones inherent to the perturbation theory. Moreover, we presented a calibration procedure associated with the method, for which we derive formulas for the market group parameters. A practical numerical example of the calibration procedure is given using data of call options on Henry Hub futures. Furthermore, we show that the method can also be used to study the joint calibration of a multiscale stochastic volatility model to option prices on the stock and its volatility index.

One direction for further study would be to connect the model and asymptotic expansions proposed in this work with the celebrated Schwartz-Smith Schwartz and Smith [2000a,b] and Gibson-Schwartz Gibson and Schwartz [1990] models for commodity prices. In particular, an important question would be the computation of the risk premium as in Gibson and Schwartz [1990].

In a different direction, one could compute the first-order correction to the price of options on the stock studied in Section 5 using Monte Carlo to compute the Greeks of the 0-order approximation. The methods developed by Fournié et al. [1999, 2001] might be efficient in this context, but they would have to be adapted to the stochastic volatility context.

## A PDE Expansion

Formally write, for  $k = 1, 2, 3$ ,

$$\psi_k^{\varepsilon, \delta}(t, x, y, z, T) = \sum_{i, j \geq 0} (\sqrt{\varepsilon})^i (\sqrt{\delta})^j \psi_{k, i, j}(t, x, y, z, T).$$

In what follows we will compute only the terms of the above expansion that are necessary for the computation of the first-order approximation of derivatives on  $F_{t, T}$ .

### A.1 Expanding $\psi_1^{\varepsilon, \delta}$

By the chain rule

$$\psi_1^{\varepsilon, \delta}(t, x, y, z, T) = \frac{1}{\frac{\partial H^{\varepsilon, \delta}}{\partial x}(t, x, y, z, T)},$$

and then one can easily see that

$$\psi_{1,0,0}(t, x, y, z, T) = \frac{1}{\frac{\partial H_0}{\partial x}(t, x, z, T)} = \frac{\partial h_0}{\partial u}(t, H_0(t, x, z, T), z, T).$$

Since

$$\frac{\partial h_0}{\partial u}(t, u, z, T) = e^{-\kappa(T-t)} h_0(t, u, z, T),$$

we have the following formula for  $\psi_{1,0,0}$ :

$$\psi_{1,0,0}(t, x, T) = e^{-\kappa(T-t)} h_0(t, H_0(t, x, z, T), z, T) = e^{-\kappa(T-t)} x.$$

Moreover, by Lemma 3.1,

$$\begin{aligned} H_{1,0}(t, x, z, T) &= -\frac{h_{1,0}(t, H_0(t, x, z, T), z, T)}{\frac{\partial h_0}{\partial u}(t, H_0(t, x, z, T), z, T)} \\ &= -\frac{g(t, T)V_3(z)e^{-3\kappa(T-t)}x}{e^{-\kappa(T-t)}x} = -g(t, T)V_3(z)e^{-2\kappa(T-t)}, \end{aligned}$$

which is independent of  $x$ , and therefore,

$$\psi_{1,1,0}(t, x, z, T) = -\frac{1}{\left(\frac{\partial H_0}{\partial x}(t, x, z, T)\right)^2} \frac{\partial H_{1,0}}{\partial x}(t, x, z, T) = 0.$$

We also have

$$\psi_{1,0,1}(t, x, z, T) = 0.$$

## A.2 Expanding $\psi_2^{\varepsilon, \delta}$

Recall that the first four terms of the expansion of  $h^{\varepsilon, \delta}$  do not depend on  $y$ . Thus

$$\psi_{2,0,0} = \psi_{2,0,1} = \psi_{2,1,0} = \psi_{2,1,1} = 0.$$

Furthermore, again by the chain rule, we have

$$\psi_2^{\varepsilon, \delta}(t, x, y, z, T) = -\psi_1^{\varepsilon, \delta}(t, x, y, z, T) \frac{\partial H^{\varepsilon, \delta}}{\partial y}(t, x, y, z, T).$$

From this, we get

$$\begin{aligned} \psi_{2,2,0}(t, x, y, z, T) &= -\psi_{1,0,0}(t, x, T) \frac{\partial H_{2,0}}{\partial y}(t, x, y, z, T) \\ &= -e^{-\kappa(T-t)}x \frac{\partial H_{2,0}}{\partial y}(t, x, y, z, T). \end{aligned}$$

In order to compute  $H_{2,0}$  we need to go further in the expansions of  $h^{\varepsilon, \delta}$  and  $H^{\varepsilon, \delta}$ . One can compute the term of order  $(2, 0)$  in  $H^{\varepsilon, \delta}$  and then conclude

$$\begin{aligned} H_{2,0}(t, x, y, z, T) &= -\frac{h_{2,0}(t, H_0(t, x, z, T), y, z, T)}{\frac{\partial h_0}{\partial u}(t, H_0(t, x, z, T), z, T)} \\ &\quad - \frac{1}{2} \frac{\frac{\partial^2 h_0}{\partial u^2}(t, H_0(t, x, z, T), z, T)}{\frac{\partial h_0}{\partial u}(t, H_0(t, x, z, T), z, T)} H_{1,0}^2(t, x, z, T) \\ &\quad - \frac{\frac{\partial h_{1,0}}{\partial u}(t, H_0(t, x, z, T), z, T)}{\frac{\partial h_0}{\partial u}(t, H_0(t, x, z, T), z, T)} H_{1,0}(t, x, z, T), \end{aligned}$$

which implies

$$\frac{\partial H_{2,0}}{\partial y}(t, x, y, z, T) = -\frac{\frac{\partial h_{2,0}}{\partial y}(t, H_0(t, x, z, T), y, z, T)}{\frac{\partial h_0}{\partial u}(t, H_0(t, x, z, T), z, T)}.$$

From (3.6), we know that

$$\begin{aligned}\frac{\partial h_{2,0}}{\partial y}(t, u, y, z, T) &= -\frac{1}{2} \frac{\partial \phi}{\partial y}(y, z) \frac{\partial^2 h_0}{\partial u^2}(t, u, z, T) \\ &= -\frac{1}{2} e^{-2\kappa(T-t)} \frac{\partial \phi}{\partial y}(y, z) h_0(t, u, z, T),\end{aligned}$$

and hence

$$\frac{\partial H_{2,0}}{\partial y}(t, x, y, z, T) = \frac{\frac{1}{2} e^{-2\kappa(T-t)} \frac{\partial \phi}{\partial y}(y, z) x}{e^{-\kappa(T-t)} x} = \frac{1}{2} e^{-\kappa(T-t)} \frac{\partial \phi}{\partial y}(y, z).$$

Finally, this gives the formula

$$\psi_{2,2,0}(t, x, y, z, T) = -\frac{1}{2} e^{-2\kappa(T-t)} \frac{\partial \phi}{\partial y}(y, z) x.$$

### A.3 Expanding $\psi_3^{\varepsilon, \delta}$

By the chain rule, we have

$$\psi_3^{\varepsilon, \delta}(t, x, y, z, T) = -\psi_1^{\varepsilon, \delta}(t, x, y, z, T) \frac{\partial H^{\varepsilon, \delta}}{\partial z}(t, x, y, z, T),$$

and then

$$\begin{aligned}\psi_{3,0,0}(t, x, z) &= -\psi_{1,0,0}(t, x, T) \frac{\partial H_0}{\partial z}(t, x, y, z, T) \\ &= \frac{\partial h_0}{\partial z}(t, H_0(t, x, z, T), z, T) \\ &= \frac{1}{2\kappa} \bar{\eta}(z) \bar{\eta}'(z) (1 - e^{-2\kappa(T-t)}) x.\end{aligned}$$

### A.4 An Expansion of the Pricing PDE

Define

$$\Psi_{n,p}^{k,m}(t, x, y, z, T) = \sum_{i=0}^n \sum_{j=0}^p \psi_{k,i,j}(t, x, y, z, T) \psi_{m,n-i,p-j}(t, x, y, z, T),$$

which is the  $(n, p)$ -order coefficient of the formal power series of  $\psi_k^{\varepsilon, \delta} \psi_m^{\varepsilon, \delta}$ :

$$\begin{aligned}&\psi_k^{\varepsilon, \delta}(t, x, y, z, T) \psi_m^{\varepsilon, \delta}(t, x, y, z, T) \\ &= \sum_{i,j,l,r \geq 0} (\sqrt{\varepsilon})^{l+i} (\sqrt{\delta})^{j+r} \psi_{k,i,j}(t, x, y, z, T) \psi_{m,l,r}(t, x, y, z, T) \\ &= \sum_{n,p \geq 0} (\sqrt{\varepsilon})^n (\sqrt{\delta})^p \left( \sum_{i=0}^n \sum_{j=0}^p \psi_{k,i,j}(t, x, y, z, T) \psi_{m,n-i,p-j}(t, x, y, z, T) \right).\end{aligned}$$

Therefore, we have the following formal expansion for the operator  $\mathcal{L}^{\varepsilon, \delta}$  where we drop the variables  $(t, x, y, z, T)$  for simplicity,

$$\begin{aligned}
\mathcal{L}^{\varepsilon,\delta} = & \frac{1}{\varepsilon} \left( \mathcal{L}_0 + \frac{1}{2} \Psi_0^{2,2} \beta^2(y) \frac{\partial^2}{\partial x^2} + \psi_{2,0,0} \beta^2(y) \frac{\partial^2}{\partial x \partial y} \right) \\
& + \frac{1}{\sqrt{\varepsilon}} \left( \rho_1 \Psi_0^{1,2} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x^2} + \rho_1 \psi_{1,0,0} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x \partial y} \right. \\
& + \left. \frac{1}{2} \Psi_{1,0}^{2,2} \beta^2(y) \frac{\partial^2}{\partial x^2} + \psi_{2,1,0} \beta^2(y) \frac{\partial^2}{\partial x \partial y} \right) \\
& + \left( \frac{\partial}{\partial t} + \frac{1}{2} \Psi_0^{1,1} \eta^2(y, z) \frac{\partial^2}{\partial x^2} - r \cdot + \rho_1 \Psi_{1,0}^{1,2} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x^2} \right. \\
& + \rho_1 \psi_{1,1,0} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x \partial y} + \frac{1}{2} \Psi_{2,0}^{2,2} \beta^2(y) \frac{\partial^2}{\partial x^2} + \psi_{2,2,0} \beta^2(y) \frac{\partial^2}{\partial x \partial y} \left. \right) \\
& + \sqrt{\varepsilon} \left( \frac{1}{2} \Psi_{3,0}^{2,2} \beta^2(y) \frac{\partial^2}{\partial x^2} + \psi_{2,3,0} \beta^2(y) \frac{\partial^2}{\partial x \partial y} + \rho_1 \Psi_{2,0}^{1,2} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x^2} \right. \\
& + \left. \rho_1 \psi_{1,2,0} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x \partial y} + \frac{1}{2} \Psi_{1,0}^{1,1} \beta^2(y) \frac{\partial^2}{\partial x^2} \right) \\
& + \sqrt{\delta} \left( \rho_2 \Psi_0^{1,3} \eta(y, z) g(z) \frac{\partial^2}{\partial x^2} + \rho_2 \psi_{1,0,0} \eta(y, z) g(z) \frac{\partial^2}{\partial x \partial z} \right. \\
& + \left. \frac{1}{2} \Psi_{2,1}^{2,2} \beta^2(y) \frac{\partial^2}{\partial x^2} + \psi_{2,2,1} \beta^2(y) \frac{\partial^2}{\partial x \partial y} \right) \\
& + \rho_2 \Psi_{1,1}^{1,2} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x^2} + \rho_2 \psi_{1,1,1} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x \partial y} + \frac{1}{2} \Psi_{0,1}^{1,1} \eta^2(y, z) \frac{\partial^2}{\partial x^2} \left. \right) \\
& + \sqrt{\frac{\delta}{\varepsilon}} \left( \rho_{12} \Psi_0^{2,3} \beta(y) g(z) \frac{\partial^2}{\partial x^2} + \rho_{12} \psi_{3,0,0} \beta(y) g(z) \frac{\partial^2}{\partial x \partial y} \right. \\
& + \rho_{12} \psi_{2,0,0} \beta(y) g(z) \frac{\partial^2}{\partial x \partial z} + \rho_{12} \beta(y) g(z) \frac{\partial^2}{\partial y \partial z} + \frac{1}{2} \Psi_{1,1}^{2,2} \beta^2(y) \frac{\partial^2}{\partial x^2} \\
& + \left. \psi_{2,1,1} \beta^2(y) \frac{\partial^2}{\partial x \partial y} + \rho_1 \Psi_{0,1}^{1,2} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x^2} + \rho_1 \psi_{1,0,1} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x \partial y} \right). \\
& + \mathcal{L}_R^{\varepsilon,\delta}.
\end{aligned}$$

Then we write, with obvious notation,

$$\mathcal{L}^{\varepsilon,\delta} = \frac{1}{\varepsilon} \mathcal{D}_{-2,0} + \frac{1}{\sqrt{\varepsilon}} \mathcal{D}_{-1,0} + \mathcal{D}_{0,0} + \sqrt{\varepsilon} \mathcal{D}_{1,0} + \sqrt{\delta} \mathcal{D}_{0,1} + \sqrt{\frac{\delta}{\varepsilon}} \mathcal{D}_{-1,1} + \mathcal{L}_R^{\varepsilon,\delta}.$$

In order to simplify the operators  $\mathcal{D}_{i,j}$ , note that  $\Psi_0^{2,k} = 0$ , for  $k = 1, 2, 3$ ,

$$\Psi_{1,0}^{2,2} = \Psi_{0,1}^{2,2} = \Psi_{1,0}^{1,2} = \Psi_{2,0}^{2,2} = \Psi_{3,0}^{2,2} = \Psi_{2,1}^{2,2} = \Psi_{1,1}^{1,2} = 0,$$

and  $\Psi_{1,0}^{1,1} = \Psi_{0,1}^{1,1} = 0$ . Hence,

$$\begin{aligned}
\bullet \mathcal{D}_{-2,0} &= \mathcal{L}_0 + \frac{1}{2} \cancel{\Psi_0^{2,2}} \beta^2(y) \frac{\partial^2}{\partial x^2} + \cancel{\psi_{2,0,0}} \beta^2(y) \frac{\partial^2}{\partial x \partial y} = \mathcal{L}_0 \\
\bullet \mathcal{D}_{-1,0} &= \rho_1 \cancel{\Psi_0^{1,2}} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x^2} + \rho_1 \psi_{1,0,0} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x \partial y} \\
&+ \frac{1}{2} \cancel{\Psi_{1,0}^{2,2}} \beta^2(y) \frac{\partial^2}{\partial x^2} + \cancel{\psi_{2,1,0}} \beta^2(y) \frac{\partial^2}{\partial x \partial y} = \rho_1 \psi_{1,0,0} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x \partial y} \\
\bullet \mathcal{D}_{0,0} &= \frac{\partial}{\partial t} + \frac{1}{2} \Psi_0^{1,1} \eta^2(y, z) \frac{\partial^2}{\partial x^2} - r \cdot + \rho_1 \cancel{\Psi_{1,0}^{1,2}} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x^2} \\
&+ \rho_1 \cancel{\psi_{1,1,0}} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x \partial y} + \frac{1}{2} \cancel{\Psi_{2,0}^{2,2}} \beta^2(y) \frac{\partial^2}{\partial x^2} + \psi_{2,2,0} \beta^2(y) \frac{\partial^2}{\partial x \partial y} \\
&= \frac{\partial}{\partial t} + \frac{1}{2} \psi_{1,0,0}^2 \eta^2(y, z) \frac{\partial^2}{\partial x^2} - r \cdot + \psi_{2,2,0} \beta^2(y) \frac{\partial^2}{\partial x \partial y} \\
\bullet \mathcal{D}_{1,0} &= \frac{1}{2} \cancel{\Psi_{3,0}^{2,2}} \beta^2(y) \frac{\partial^2}{\partial x^2} + \psi_{2,3,0} \beta^2(y) \frac{\partial^2}{\partial x \partial y} + \rho_1 \cancel{\Psi_{2,0}^{1,2}} \psi_{1,0,0} \psi_{2,2,0} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x^2} \\
&+ \rho_1 \psi_{1,2,0} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x \partial y} + \frac{1}{2} \cancel{\Psi_{1,0}^{1,2}} \beta^2(y) \frac{\partial^2}{\partial x^2} \\
&= (\psi_{2,3,0} \beta^2(y) + \rho_1 \psi_{1,2,0} \eta(y, z) \beta(y)) \frac{\partial^2}{\partial x \partial y} + \rho_1 \psi_{1,0,0} \psi_{2,2,0} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x^2} \\
\bullet \mathcal{D}_{0,1} &= \rho_2 \Psi_0^{1,3} \eta(y, z) g(z) \frac{\partial^2}{\partial x^2} + \rho_2 \psi_{1,0,0} \eta(y, z) g(z) \frac{\partial^2}{\partial x \partial z} + \frac{1}{2} \cancel{\Psi_{2,1}^{2,2}} \beta^2(y) \frac{\partial^2}{\partial x^2} \\
&+ \psi_{2,2,1} \beta^2(y) \frac{\partial^2}{\partial x \partial y} + \rho_2 \cancel{\Psi_{1,1}^{1,2}} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x^2} + \rho_2 \psi_{1,1,1} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x \partial y} \\
&+ \frac{1}{2} \cancel{\Psi_{0,1}^{1,2}} \eta^2(y, z) \frac{\partial^2}{\partial x^2} = (\rho_2 \psi_{1,0,0} \psi_{3,0,0} \eta(y, z) g(z)) \frac{\partial^2}{\partial x^2} \\
&+ \rho_2 \psi_{1,0,0} \eta(y, z) g(z) \frac{\partial^2}{\partial x \partial z} + (\psi_{2,2,1} \beta^2(y) + \rho_2 \psi_{1,1,1} \eta(y, z) \beta(y)) \frac{\partial^2}{\partial x \partial y} \\
\bullet \mathcal{D}_{-1,1} &= \rho_{12} \cancel{\Psi_0^{2,3}} \beta(y) g(z) \frac{\partial^2}{\partial x^2} + \rho_{23} \psi_{3,0,0} \beta(y) g(z) \frac{\partial^2}{\partial x \partial y} \\
&+ \rho_{12} \cancel{\psi_{2,0,0}} \beta(y) g(z) \frac{\partial^2}{\partial x \partial z} + \rho_{12} \beta(y) g(z) \frac{\partial^2}{\partial y \partial z} + \frac{1}{2} \cancel{\Psi_{1,1}^{2,2}} \beta^2(y) \frac{\partial^2}{\partial x^2} \\
&+ \cancel{\psi_{2,1,1}} \beta^2(y) \frac{\partial^2}{\partial x \partial y} + \rho_1 \cancel{\Psi_{0,1}^{1,2}} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x^2} \\
&+ \rho_1 \cancel{\psi_{1,0,1}} \eta(y, z) \beta(y) \frac{\partial^2}{\partial x \partial y} = \rho_{12} \psi_{3,0,0} \beta(y) g(z) \frac{\partial^2}{\partial x \partial y} + \rho_{12} \beta(y) g(z) \frac{\partial^2}{\partial y \partial z}
\end{aligned}$$

It is clear from the above choices that the coefficients of the operator  $\mathcal{L}_R^{\varepsilon, \delta}$  are of

order  $(\varepsilon + \delta)$ . Hence, if  $f$  is smooth and bounded with all derivatives bounded,

$$(A.1) \quad \mathcal{L}_R^{\varepsilon, \delta} f(t, x, y, z) = O(\varepsilon + \delta).$$

## B Proof of Theorem 3.2

Following the proof of Theorem 4.10 given in Fouque et al. [2011] for the Equity case, we go further in the approximation of  $P^{\varepsilon, \delta}$  and define the higher-order approximation:

$$(B.1) \quad \widehat{P}^{\varepsilon, \delta} = \widetilde{P}^{\varepsilon, \delta} + \varepsilon(P_{2,0} + \sqrt{\varepsilon}P_{3,0}) + \sqrt{\delta}(\sqrt{\varepsilon}P_{1,1} + \varepsilon P_{2,1}),$$

and moreover, we introduce

$$\widehat{\mathcal{L}}^{\varepsilon, \delta} = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\varepsilon}\mathcal{L}_3 + \sqrt{\delta}\mathcal{M}_1 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3.$$

Necessary properties of the additional terms in the expansion (B.1) can be derived in the same way as it is done in Fouque et al. [2011] and thus we skip the details here. Next, we define the residual

$$R^{\varepsilon, \delta} = P^{\varepsilon, \delta} - \widehat{P}^{\varepsilon, \delta}$$

and use the pricing PDE (3.10) to conclude that

$$\begin{aligned} \mathcal{L}^{\varepsilon, \delta} R^{\varepsilon, \delta} &= \mathcal{L}^{\varepsilon, \delta}(P^{\varepsilon, \delta} - \widehat{P}^{\varepsilon, \delta}) = -\mathcal{L}^{\varepsilon, \delta} \widehat{P}^{\varepsilon, \delta} \\ &= -(\widehat{\mathcal{L}}^{\varepsilon, \delta} + \mathcal{L}_R^{\varepsilon, \delta})\widehat{P}^{\varepsilon, \delta} = -\widehat{\mathcal{L}}^{\varepsilon, \delta} \widehat{P}^{\varepsilon, \delta} - \mathcal{L}_R^{\varepsilon, \delta} \widehat{P}^{\varepsilon, \delta}. \end{aligned}$$

Hence, mimicking the computations from Fouque et al. [2011], we can write

$$\widehat{\mathcal{L}}^{\varepsilon, \delta} \widehat{P}^{\varepsilon, \delta} = \varepsilon R_1^\varepsilon + \sqrt{\varepsilon\delta}R_2^\varepsilon + \delta R_3^\varepsilon,$$

where  $R_1^\varepsilon$ ,  $R_2^\varepsilon$  and  $R_3^\varepsilon$  can be exactly computed as linear combinations of  $\mathcal{L}_k P_{i,j}^\varepsilon$  and  $\mathcal{M}_l P_{i,j}^\varepsilon$ , for some  $k, l, i, j$ . However, the important fact about  $R_i^\varepsilon$  is that they are smooth functions of  $t, x, y, z$ , for small  $\varepsilon$  and  $\delta$ , bounded by smooth functions of  $t, x, y, z$  independent of  $\varepsilon$  and  $\delta$ , uniformly bounded in  $t, x, z$  and at most linearly growing in  $y$ . Additionally, from (A.1), we can define  $R_4^\varepsilon$  and conclude that

$$R_4^\varepsilon = \mathcal{L}_R^{\varepsilon, \delta} \widehat{P}^{\varepsilon, \delta} = O(\varepsilon + \delta),$$

since  $\widehat{P}^{\varepsilon, \delta}$  is bounded and smooth with all derivatives bounded. This follows from the same arguments presented in proof of Theorem 4.10 of Fouque et al. [2011] with some minor differences to include  $\mathcal{L}_3$ . Therefore, the residual  $R^{\varepsilon, \delta}$  solves the PDE

$$\begin{cases} \mathcal{L}^{\varepsilon, \delta} R^{\varepsilon, \delta} + \varepsilon R_1^\varepsilon + \sqrt{\varepsilon\delta}R_2^\varepsilon + \delta R_3^\varepsilon + R_4^\varepsilon = 0, \\ R^{\varepsilon, \delta}(T_0, x, y, z, T) = -\varepsilon(P_{2,0} + \sqrt{\varepsilon}P_{3,0})(T_0, x, y, z, T) \\ \quad - \sqrt{\varepsilon\delta}(P_{1,1} + \sqrt{\varepsilon}P_{2,1})(T_0, x, y, z, T), \end{cases}$$

and then all the computations regarding the Feynman-Kac probabilistic representation of  $R^{\varepsilon, \delta}$  and the growth control of the source and final condition can be carried out just as in Fouque et al. [2011] in order to conclude that  $R^{\varepsilon, \delta} = O(\varepsilon + \delta)$ . Lastly, the desired result follows because

$$|P^{\varepsilon, \delta} - \widetilde{P}^{\varepsilon, \delta}| \leq |R^{\varepsilon, \delta}| + |\widehat{P}^{\varepsilon, \delta} - \widetilde{P}^{\varepsilon, \delta}|.$$

□

## C Computing $P_{0,1}^\delta$ explicitly for European derivatives

Let us write  $P_{0,1}^\delta$  in terms of Greeks of  $P_B$  by solving the PDE (3.33). Note first that

$$\frac{\partial P_0}{\partial z}(t, x, z, T) = \frac{\partial P_B}{\partial \sigma}(t, x, \bar{\sigma}_{t, T_0}(z, T)) \frac{\partial \bar{\sigma}_{t, T_0}}{\partial z}(z, T).$$

Since  $P_B$  satisfies the Black equation and the European derivative has maturity  $T_0$ , we have the relation between the Vega and the Gamma:

$$\frac{\partial P_B}{\partial \sigma}(t, x, \sigma) = (T_0 - t)\sigma D_2 P_B(t, x, \sigma).$$

Moreover,

$$\frac{\partial \bar{\sigma}_{t, T_0}}{\partial z}(z, T) = \bar{\eta}'(z)\lambda_\sigma(t, T_0, T, \kappa).$$

Combining the above equations, we obtain

$$\begin{aligned} & -f_0(t, T)\mathcal{A}_0^\delta P_0 - f_1(t, T)\mathcal{A}_1^\delta P_0 \\ &= -f_0(t, T)\mathcal{A}_0^\delta P_0 - f_1(t, T)V_1^\delta(z)(T_0 - t)\bar{\eta}(z)\bar{\eta}'(z)\lambda_\sigma^2(t, T_0, T, \kappa)D_1 D_2 P_0 \\ &= -f_0(t, T)\mathcal{A}_0^\delta P_0 - \tilde{f}_1(t, T)\mathcal{A}_0^\delta D_1 P_0, \end{aligned}$$

where

$$\tilde{f}_1(t, T) = f_1(t, T)(e^{-2\kappa(T-T_0)} - e^{-2\kappa(T-t)}).$$

Hence, we deduce the following PDE for  $P_{0,1}^\delta$ :

$$\begin{cases} \mathcal{L}_B(\bar{\sigma}(t, z, T))P_{0,1}^\delta = -f_0(t, T)\mathcal{A}_0^\delta P_0 - \tilde{f}_1(t, T)\mathcal{A}_0^\delta D_1 P_0, \\ P_{0,1}^\delta(T_0, x, z, T) = 0. \end{cases}$$

By the same argument used to deduce Equation (3.29), and using the linearity of the differential operators involved, we can easily conclude

$$P_{0,1}^\delta(t, x, z, T) = \left( \int_t^{T_0} f_0(u, T) du \right) \mathcal{A}_0^\delta P_0 + \left( \int_t^{T_0} \tilde{f}_1(u, T) du \right) \mathcal{A}_0^\delta D_1 P_0.$$

Computing these integrals gives us the formulas

$$\begin{aligned} \lambda_0(t, T_0, T, \kappa) &= \frac{1}{T_0 - t} \int_t^{T_0} f_0(u, T) du \\ &= \frac{e^{-\kappa(T-T_0)} - e^{-\kappa(T-t)}}{\kappa(T_0 - t)} - \frac{e^{-3\kappa(T-T_0)} - e^{-3\kappa(T-t)}}{3\kappa(T_0 - t)} \\ &= \lambda(t, T_0, T, \kappa) - \lambda(t, T_0, T, 3\kappa), \\ \lambda_1(t, T_0, T, \kappa) &= \frac{1}{T_0 - t} \int_t^{T_0} \tilde{f}_1(u, T) du \\ &= e^{-2\kappa(T-T_0)} \frac{e^{-\kappa(T-T_0)} - e^{-\kappa(T-t)}}{\kappa(T_0 - t)} - \frac{e^{-3\kappa(T-T_0)} - e^{-3\kappa(T-t)}}{3\kappa(T_0 - t)} \\ &= e^{-2\kappa(T-T_0)} \lambda(t, T_0, T, \kappa) - \lambda(t, T_0, T, 3\kappa). \end{aligned}$$



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