A CONTROL VARIATE METHOD TO EVALUATE OPTION PRICES UNDER MULTI-FACTOR STOCHASTIC VOLATILITY MODELS

JEAN-PIERRE FOUQUE* AND CHUAN-HSIANG HAN†

Abstract. We propose a control variate method with multiple controls to effectively reduce variances of Monte Carlo simulations for pricing European options under multi-factor stochastic volatility models. Based on an application of Ito's formula, the option price is decomposed by its discounted payoff and stochastic integrals with the appearance of gradients of the unknown option price with respect to state variables (the risky asset price and driving volatility levels). Taking advantage of the closed-form option price approximations obtained by Fouque et al. (SIAM Journal on Multiscale Modeling and Simulation 2(1), 2003), we are able to build controls by substituting approximate option prices into the stochastic integrals. This setup leads to an unbiased control variate and naturally suggests estimates for control parameters. Several numerical experiments are provided to demonstrate the performance of variance reductions by this control variate method. In comparison with variance reduction ratios obtained from importance sampling, we generally find that the control variate method is numerically more stable and efficient than importance sampling for the European option pricing problems under stochastic volatility models.

Key words. Multi-factor stochastic volatility models, Control variates, Importance Sampling.

AMS(MOS) subject classifications. Primary 65C05, 65C50, 91B70.

1. Introduction. Stochastic volatility, SV in short, models have become a very convenient framework to (1) model the randomness of volatility of the risky asset price and incorporate the leverage between returns of the price and volatility, and (2) reproduce the “skew” of implied volatility. See for example [5] and the references therein, where one-factor SV models such as Heston model [10] are considered. Recent empirical studies reveal that one-factor SV models fail to match either the high conditional kurtosis of returns [2] or the full term structure of implied volatility surface [3]. Adding jump components in returns and/or volatility process, or considering multi-factor SV models are two primary generalizations of one-factor SV models.

From the viewpoint of modeling historical asset returns, Chernov et al. [2] use the efficient method of moments [8] to obtain comparable empirical goodness-of-fit from affine jump-diffusion models and two-factor SV family models. Molina et al. [13] use a Markov Chain Monte Carlo method to find strong evidence of two-factor SV models with well-separated time scales in foreign exchange data. From the viewpoint of calibration to op-

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tion prices, Cont and Tankov [3] find that jump-diffusion models have a fairly good fit to the implied volatility surface. Fouque et al. find that two-factor SV models provide a better fit to the term structure of implied volatility than one factor SV models by capturing the behavior at short and long maturities. From the viewpoint of derivative pricing and hedging, a class of jump-diffusion models, such as in [1], enjoys closed-form solutions for option prices but the complexity of hedging strategies increases due to jumps. On the contrary, multi-factor SV models do not admit in general explicit solutions for option prices but have direct implications on hedges. To our best knowledge, there is no strong empirical evidence to judge the overwhelming position between jump-diffusion family models and multi-factor SV family models.

In this paper, multi-factor SV models are considered and our goal is to explore efficient Monte Carlo simulations for option prices. Among a large collection of techniques, presented for instance in [9], we focus on two widely applied variance reduction techniques known as importance sampling and control variates (see also [11]). In our previous work [4] to evaluate European and Asian option prices, the importance sampling was employed, in which approximate option prices were used to drive the change of measure in order to reduce the variance. Here we develop a control variate method with multiple controls for which a general background can be found in [9]. By virtue of Ito’s formula, the option price can be decomposed by its discounted payoff and several stochastic integrals with the appearance of gradients of the unknown option price with respect to state variables, namely the risky asset price and driving volatility levels. As in importance sampling in [4], our proposed unbiased controls consist of approximating the unknown gradient of option prices in the stochastic integrals. Option price approximations are derived from the two-scale perturbation analysis by Fouque et al. [6, 4]. We recall their results in Section 3.1. Furthermore, this setup naturally suggests multiple estimates of optimal control parameters. The details of this method is stated in Section 3. In Section 4, the performance of variance reduction from importance sampling and control variate is illustrated. We find a significant superiority of control variates against importance sampling for pricing European options under two-factor SV models. Conclusion and future directions are given in Section 5.

2. Multi-factor Stochastic Volatility Models and Option Price Approximations. Under the physical probability measure, a family of multi-factor stochastic volatility models evolves as

\[ dS_t = \mu S_t dt + \sigma_t S_t d\tilde{W}_t^{(0)}, \]
\[ \sigma_t = f(Y_t, Z_t), \]
\[ dY_t = \alpha c_1(Y_t) dt + \sqrt{\alpha g_1(Z_t)} d\tilde{W}_t^{(1)}, \]
\[ dZ_t = \delta c_2(Z_t) dt + \sqrt{\delta g_2(Z_t)} d\tilde{W}_t^{(2)}, \]
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where $S_t$ is the underlying asset price with a constant rate of return $\mu$ and a random volatility $\sigma_t$ driven by the stochastic processes $Y_t$ and $Z_t$ varying on the respective time scales $1/\alpha$ and $1/\delta$ (in the regime studied in the following section, typically $1/\alpha$ is small and $1/\delta$ is large). The standard Brownian motions $(\tilde{W}_t^{(0)}, \tilde{W}_t^{(1)}, \tilde{W}_t^{(2)})$ are correlated according to (2.1) below. The volatility function $f$ is assumed to be bounded and bounded away from 0. The coefficient functions of $Y_t$, namely $c_1$ and $g_1$, are assumed to be chosen such that $Y_t$ is an ergodic diffusion. The Ornstein-Uhlenbeck (OU) process is a typical example by defining the rate of mean-reversion $\alpha$, $c_1(y) = m_1 - y$, and $g_1(y) = \nu_1 \sqrt{2}$, where $m_1$ is the long run mean and $\nu_1$ is the long run standard deviation, such that $\Phi = N(m_1, \nu_1^2)$ is the invariant distribution. The coefficient functions of $Z_t$, namely $c_2$ and $g_2$ are assumed to satisfy existence and uniqueness conditions for diffusions as given in [12] for instance. For simplicity, we set the process $Z_t$ to be another OU by choosing $c_2(z) = m_2 - z$, and $g_2(z) = \nu_2 \sqrt{2}$, where $m_2$ is the long run mean and $\nu_2$ is the long run standard deviation. In order to incorporate a correlation between the Brownian motions $(\tilde{W}_t^{(0)}, \tilde{W}_t^{(1)}, \tilde{W}_t^{(2)})$, we set

\begin{align}
\tilde{W}_t^{(0)} &= W_t^{(0)}, \\
\tilde{W}_t^{(1)} &= \rho_1 W_t^{(0)} + \sqrt{1 - \rho_1^2} W_t^{(1)}, \\
\tilde{W}_t^{(2)} &= \rho_2 W_t^{(0)} + \rho_{12} W_t^{(1)} + \sqrt{1 - \rho_2^2 - \rho_{12}^2} W_t^{(2)},
\end{align}

where $(W_t^{(0)}, W_t^{(1)}, W_t^{(2)})$ are independent standard Brownian motions, and the correlation coefficients $\rho_1$, $\rho_2$, and $\rho_{12}$ satisfy $|\rho_1| < 1$ and $|\rho_2^2 + \rho_{12}^2| < 1$.

Under the pricing risk-neutral probability measure $\mathbb{P}^*$, a family of multi-factor SV models can be described as follows

\begin{align}
(2.2) dS_t &= rS_t dt + \sigma_t S_t dW_t^{(0)*}, \\
\sigma_t &= f(Y_t, Z_t), \\
dY_t &= \left(\alpha(m_1 - Y_t) - \nu_1 \sqrt{2\alpha} \Lambda_1(Y_t, Z_t)\right) dt \\
&\quad + \nu_1 \sqrt{2\alpha} \left(\rho_1 dW_t^{(0)*} + \sqrt{1 - \rho_1^2} dW_t^{(1)*}\right), \\
dZ_t &= \left(\delta(m_2 - Z_t) - \nu_2 \sqrt{2\delta} \Lambda_2(Y_t, Z_t)\right) dt \\
&\quad + \nu_2 \sqrt{2\delta} \left(\rho_2 dW_t^{(0)*} + \rho_{12} dW_t^{(1)*} + \sqrt{1 - \rho_2^2 - \rho_{12}^2} dW_t^{(2)*}\right),
\end{align}

where $(W_t^{(0)*}, W_t^{(1)*}, W_t^{(2)*})$ are independent standard Brownian motions.

The constant risk-free interest rate of return is denoted by $r$. The functions $\Lambda_1$ and $\Lambda_2$ are the combined market prices of volatility risk and they are
assumed to be bounded and dependent on the variables \( y \) and \( z \) only. Under this model the joint process \((S_t, Y_t, Z_t)\) is Markovian. The payoff of an European-style option is an integrable function, say \( H \), of the stock price \( S_T \) at the maturity date \( T \). The price of this option is given by the expectation of the discounted payoff conditioning on the current stock price and driving volatility levels due to the Markov property of the joint dynamics (2.2). By introducing the notation \( \varepsilon = 1/\alpha \), the European option price at time \( t \) is given by \( P^{\varepsilon, \delta}(t, S_t, Y_t, Z_t) \) where

\[
P^{\varepsilon, \delta}(t, x, y, z) = \mathbb{E}^\pi \left\{ e^{-r(T-t)} H(S_T) \mid S_t = x, Y_t = y, Z_t = z \right\}.
\]

2.1. Vanilla European Option Price Approximations. By an application of Feynman-Kac formula, \( P^{\varepsilon, \delta}(t, x, y, z) \) defined in (2.3) is the solution of the following three-dimensional partial differential equation

\[
\frac{\partial P^{\varepsilon, \delta}}{\partial t} + \mathcal{L}^{\varepsilon, \delta}_{(S, Y, Z)} P^{\varepsilon, \delta} - r P^{\varepsilon, \delta} = 0,
\]

\[
P^{\varepsilon, \delta}(T, x, y, z) = H(x),
\]

where \( \mathcal{L}^{\varepsilon, \delta}_{(S, Y, Z)} \) denotes the infinitesimal generator of the Markovian process \((S_t, Y_t, Z_t)\) given by (2.2). Assuming that the parameters \( \varepsilon \) and \( \delta \) are relatively small, that is

\[
\varepsilon \ll 1 \ll 1/\delta,
\]

Fouque et al. in [6] utilize a combination of regular and singular perturbation analysis to derive the following pointwise option price approximation

\[
P^{\varepsilon, \delta}(t, x, y, z) \approx \tilde{P}(t, x, z),
\]

where

\[
\tilde{P} = P_{BS} + (T-t) \left( V_0 \frac{\partial}{\partial \sigma} + V_1 x \frac{\partial^2}{\partial x \partial \sigma} + V_2 x^2 \frac{\partial^2}{\partial x^2} + V_3 x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2}{\partial x^2} \right) \right) P_{BS},
\]

with an accuracy of order \( (\varepsilon |\log \varepsilon| + \delta) \) for call options. The leading order price \( P_{BS}(t, x; \bar{\sigma}(z)) \) is independent of the \( y \) variable and is the homogenized price which solves the Black-Scholes equation

\[
\mathcal{L}_{BS}(\bar{\sigma}(z)) P_{BS} = 0,
\]

\[
P_{BS}(T, x; \bar{\sigma}(z)) = H(x),
\]

where \( \mathcal{L}_{BS}(\sigma) \) denotes the usual Black-Scholes operator with constant volatility \( \sigma \). Here the \( z \)-dependent effective volatility \( \bar{\sigma}(z) \) is defined by

\[
\bar{\sigma}^2(z) = \langle f^2(\cdot, z) \rangle.
\]
where the brackets denote the average with respect to the invariant distribution \( \mathcal{N}(m_1, \nu_1^2) \) of the fast factor \((Y_t)\). The \(z\)-dependent parameters \((V_0, V_1, V_2, V_3)\) are given by

\begin{align*}
V_0 &= -\frac{\nu_2 \sqrt{\delta}}{\sqrt{2}} \langle \Lambda_2 \rangle \bar{\sigma}, \\
V_1 &= \frac{\rho_2 \nu_3 \sqrt{\delta}}{\sqrt{2}} \langle f \rangle \bar{\sigma}, \\
V_2 &= \frac{\nu_1 \sqrt{2}}{\sqrt{2}} \langle \Lambda_1 \frac{\partial \phi}{\partial y} \rangle, \\
V_3 &= -\frac{\rho_1 \nu_1 \sqrt{2}}{\sqrt{2}} \langle f \frac{\partial \phi}{\partial y} \rangle,
\end{align*}

where \(\bar{\sigma}\) denotes the derivative of \(\bar{\sigma}\) with respect to \(z\), and the function \(\phi(y, z)\) is a solution of the Poisson equation

\[
\left( \nu_2 \frac{\partial^2}{\partial y^2} + (m_1 - y) \frac{\partial}{\partial y} \right) \phi(y, z) = f^2(y, z) - \bar{\sigma}^2(z),
\]

associated to the OU infinitesimal generator. The parameters \(V_0\) and \(V_1\) (resp. \(V_2\) and \(V_3\)) are small of order \(\sqrt{\delta}\) (resp. \(\sqrt{\epsilon}\)). The parameters \(V_0\) and \(V_2\) reflect the effect of the market prices of volatility risk. The parameters \(V_1\) and \(V_3\) are proportional to the correlation coefficients \(\rho_2\) and \(\rho_1\) respectively. In [6], these parameters are calibrated using the observed implied volatilities. In the present work, the model (2.2) will be fully specified, and these parameters are computed using the formulas (2.7, 2.8, 2.9, 2.10) above.

3. Two Variance Reduction Methods. In this section we formulate two variance reduction methods, namely importance sampling and control variates, to evaluate European option prices by Monte Carlo simulations. The technique of importance sampling using the approximations described in the previous section has been introduced to evaluate European options in [7], and Asian options in [4]. We briefly review this methodology in Section 3.1, and then we develop our control variate method in Section 3.2.

To simplify notations, we present the stochastic volatility model (2.2) in the vector form

\[
dV_t = b(t, V_t)dt + a(t, V_t)d\eta_t,
\]

where we set

\[
v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad V_t = \begin{pmatrix} S_t \\ Y_t \\ Z_t \end{pmatrix}, \quad \eta_t = \begin{pmatrix} W_t^{(0)*} \\ W_t^{(1)*} \\ W_t^{(2)*} \end{pmatrix},
\]
we define the drift
\[ b(t, v) = \begin{pmatrix} rx \\ \alpha (m_1 - y) - \nu_1 \sqrt{2\alpha} \Lambda_1(y, z) \\ \delta (m_2 - z) - \nu_2 \sqrt{2\delta} \Lambda_2(y, z) \end{pmatrix}, \]
and the diffusion matrix
\[ a(t, v) = \begin{pmatrix} f(y, z) & 0 & 0 \\ \nu_1 \sqrt{2\alpha} \rho_1 & \nu_1 \sqrt{2\alpha} \sqrt{1 - \rho_1^2} & 0 \\ \nu_2 \sqrt{2\delta} \rho_2 & \nu_2 \sqrt{2\delta} \sqrt{1 - \rho_2^2 - \rho_2^2} \rho_1^2 & \nu_2 \sqrt{2\delta} \sqrt{1 - \rho_2^2 - \rho_2^2 \rho_1^2} \end{pmatrix}. \]

The pricing function \( P(t, x, y, z) \) of an European option at time \( t \) is given by
\[ P(t, v) = \mathbb{E}^* \{ e^{-r(T-t)} H(S_T) | V_t = v \}, \tag{3.2} \]
and a direct Monte Carlo simulation consists in approximating \( P(0, v) \) by the unbiased estimator
\[ P_{MC} = \frac{1}{N} \sum_{k=1}^{N} e^{-rT} H(S_T^k), \]
where \( S^k \), for \( k = 1, \ldots, N \), are independent realizations of the process \( S \) under the dynamics (2.2).

3.1. Importance Sampling. A change of drift in the model dynamics (3.1) can be obtained by
\[ dV_t = (b(t, V_t) - a(t, V_t) h(t, V_t)) dt + a(t, V_t) d\tilde{\eta}_t, \tag{3.3} \]
where
\[ \tilde{\eta}_t = \eta_t + \int_0^t h(s, V_s) ds. \]

The instantaneous shift \( h(s, V_s) \) is assumed to satisfy the Novikov’s condition
\[ \mathbb{E}^* \left\{ \exp \left( \frac{1}{2} \int_0^T h^2(s, V_s) ds \right) \right\} < \infty. \]

By Girsanov Theorem, one can construct the new probability measure \( \tilde{\mathbb{P}}^\nu \) given by
\[ \frac{d\tilde{\mathbb{P}}^\nu}{d\mathbb{P}} = Q_T, \]
where the Radon-Nikodym derivative is defined as

\begin{equation}
Q_T = \exp \left( \int_0^T h(s, V_s) d\tilde{\eta}_s - \frac{1}{2} \int_0^T \|h(s, V_s)\|^2 ds \right),
\end{equation}

such that \( \tilde{\eta}_t \) is a Brownian motion under \( \tilde{\mathbb{I}} \mathbb{P} \). The option price \( P \) at time 0 can be written as

\begin{equation}
P(0, v) = \tilde{\mathbb{E}} \{ e^{-rT} H(S_T)Q_T \mid V_0 = v \}.
\end{equation}

By an application of Ito’s formula to \( P(t, V_t) Q_t \), we obtain the following variance

\[ \text{Var}_P (H(S_T)Q_T) = \tilde{\mathbb{E}} \left\{ \int_0^T Q_t^2 \|a^T \nabla P + Ph\|^2 dt \right\}, \]

and therefore the following optimal choice for \( h \)

\begin{equation}
h = -\frac{1}{P} (a^T \nabla P).
\end{equation}

We refer to \([4, 11]\) for the details. The superscript in \( a^T \) denotes the transpose of \( a \), and \( \nabla \) denotes the gradient with respect to \( v \). However, neither the price \( P \) nor its gradient \( \nabla P \) are known.

The idea of the importance sampling technique introduced in \([4]\) is to approximate unknown option price \( P = P_{\varepsilon, \delta} \) by \( \tilde{P} \) as in (2.5). Then the Monte Carlo simulations are done under the new measure \( \tilde{\mathbb{I}} \mathbb{P} \):

\begin{equation}
P(0, x, y, z) \approx \frac{1}{N} \sum_{k=1}^N e^{-rT} H(S_T^{(k)})Q_T^{(k)},
\end{equation}

where \( N \) is the total number of simulations, and \( S_T^{(k)} \) and \( Q_T^{(k)} \) denote the final value of the \( k \)-th realized trajectory (3.3) and weight (3.4) respectively.

3.2. Control Variates. The setup of a conventional control variate with \( m \) multiple controls to estimate the expectation \( P \) in (3.2) is the following:

\begin{equation}
P^{CV} \triangleq P^{MC} + \sum_{i=1}^m \lambda_i (\hat{P}_C^i - P_C^i),
\end{equation}

where we denote by \( P^{MC} \) the unbiased estimator of \( P \) obtained by the sample mean of outputs from \( N \) i.i.d. realizations. Each \( \hat{P}_C^i \) represents a sample mean obtained from the same realizations than those used in \( P^{MC} \). In addition, we assume that \( P_C^i \) is an unbiased estimator of \( P_C^i \), which admits a closed-form expression. The control variate \( P^{CV} \) is thus
an unbiased estimator of $P$. The control parameters $\lambda_i$’s need to be chosen to minimize the variance of $\hat{P}^{CV}$. A detailed discussion on control variates can be found in [9].

We now introduce a constructive way to build control variate estimators under the diffusion model (2.2). Based on Ito’s formula and (2.4), the discounted option price satisfies

$$d\left(e^{-rs}P(s, S_s, Y_s, Z_s)\right) = e^{-rs}(a^T \nabla P)(s, S_s, Y_s, Z_s) \cdot d\eta_s.$$ \hspace{1cm} (3.9)

Integrating between the initial time 0 and the maturity time $T$, and using the terminal condition $P(T, S_T, Y_T, Z_T) = H(S_T)$, we deduce that

$$P(0, S_0, Y_0, Z_0) = e^{-rT}H(S_T) - \int_0^T e^{-rs}(a^T \nabla P)(s, S_s, Y_s, Z_s) \cdot d\eta_s.$$ \hspace{1cm} (3.9)

This suggests that the martingale term in (3.9) is the “perfect” control variate. However, the unknown price process $(P(s, X_s, Y_s, Z_s))_{0 \leq s \leq T}$ appears through its gradient in the stochastic integrals in (3.9). The use of the price approximation (2.5)

$$P^{\varepsilon \delta}(t, x, z) \approx \tilde{P}(t, x, z),$$

suggests a constructive way to build the multiple control variates given by the following centered martingales

$$M_1(\tilde{P}) = \int_0^T e^{-rs} \frac{\partial \tilde{P}}{\partial x}(s, S_s, Z_s) f(Y_s, Z_s) S_s dW_s^{(0)*},$$ \hspace{1cm} (3.10)

$$M_2(\tilde{P}) = \nu_2 \sqrt{2} \int_0^T e^{-rs} \frac{\partial \tilde{P}}{\partial z}(s, S_s, Z_s) dW_s^*, $$ \hspace{1cm} (3.11)

where we have defined the Brownian motion

$$W_s^* = \rho_2 W_s^{(0)*} + \rho_{12} W_s^{(1)*} + \sqrt{1 - \rho_2^2 - \rho_{12}^2} W_s^{(2)*},$$

and we used that $\tilde{P}$ does not depend on the variable $y$. To fit in the control variate setup with multiple controls (3.8), we choose $m = 2$ and:

$$P_{MC} = \frac{1}{N} \sum_{k=1}^N e^{-rT}H(S_T^{(k)}),$$

$$\hat{P}_C^i = \frac{1}{N} \sum_{k=1}^N M_i^{(k)}(\tilde{P}), \hspace{1cm} i = 1, 2,$$

$$\lambda_i = -1, \hspace{1cm} i = 1, 2,$$
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Table 1
Parameters used in the one-factor stochastic volatility model (2.2).

<table>
<thead>
<tr>
<th>$r$</th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$\nu_1$</th>
<th>$\nu_2$</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$f(y, z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>-2.6</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-0.3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0 exp(y)</td>
</tr>
</tbody>
</table>

Table 2
Initial conditions and call option parameters.

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>$Y_0$</th>
<th>$Z_0$</th>
<th>$K$</th>
<th>$T$ years</th>
</tr>
</thead>
<tbody>
<tr>
<td>110</td>
<td>-2.32</td>
<td>0</td>
<td>100</td>
<td>1</td>
</tr>
</tbody>
</table>

where the superscript corresponds to the kth realization. Since $P_C^1 = P_C^2 = 0$, we deduce

$$P_{CV} = P_{MC} - \hat{P}_1 - \hat{P}_2.$$  

The variance of this estimator is given by the sum of the quadratic variations of the martingales $M_1(P - \tilde{P})$ and $M_2(P - \tilde{P})$. It involves only the gradient of $P - \tilde{P}$ in contrast with the importance sampling method where division by $P$ is required as can be seen in (3.6).

4. Numerical Simulations. Two sets of numerical experiments are provided in order to illustrate the relative performance of importance sampling and control variate as described in Section 3. The first set of experiments is for one-factor SV models and the second set is for two-factor SV models. These experiments are done only for vanilla European call options.

4.1. One-Factor SV Models. Under the framework of the two-factor SV model (2.2), an one-factor SV model is considered by setting all parameters as well as the initial condition used to describe the second factor $Z_t$ in (2.2) being zeros. Our test model is chosen the same as in Fouque and Tullie [7], in which they used an Euler scheme to discretize the diffusion process $V_t$ to run the Monte Carlo simulations. The time step is $10^{-3}$ and the number of realizations is 10000.

The one-factor stochastic volatility model is specified in Tables 1 and 2. In [7] the authors proposed an importance sampling technique by using an approximate option price obtained by a fast mean-reversion expansion. This approach is described in Section 3.1. Since only one-factor SV model is considered, the zero-order price approximation reduces to

$$P_{BS}(t, x; \sigma) = x \mathcal{N}(d_1(x)) - Ke^{-r(T-t)} \mathcal{N}(d_2(x)),$$

where

$$d_1(x) = \frac{\ln(x/K) + (r + \frac{1}{2} \sigma^2)(T - t)}{\sigma \sqrt{T - t}},$$

$$d_2(x) = \frac{\ln(x/K) + (r - \frac{1}{2} \sigma^2)(T - t)}{\sigma \sqrt{T - t}}.$$
\[d_2(x) = d_1(x) - \sigma \sqrt{T-t},\]
\[N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-u^2/2} \, du.\]

The constant volatility \(\bar{\sigma} = \bar{\sigma}(0)\) is defined in (2.6), and the first-order price approximation reduces to
\[\tilde{P} = P_{BS} + (T-t) \left( V_0 \frac{\partial}{\partial \sigma} + V_1 x \frac{\partial^2}{\partial x \partial \sigma} \right) P_{BS}.\]

It is found in [7] that the importance sampling technique performs best by employing the first-order price approximation \(\tilde{P}\). According to different levels of the rate of mean-reversion \(\alpha\), we summarize the results from Table 1 in [7] into the second column of Table 3, in which we illustrate variance ratios between the variance in the basic Monte Carlo simulations denoted by \(V^{MC}\), and the variance in the Monte Carlo simulations with importance sampling denoted by \(V^{IS}(P)\).

Our construction of the control variate is described in Section 3.2. Since only one-factor model is considered here, the control variate given in (3.10) reduces to the martingale
\[M_1(P_{BS}) = \int_0^T e^{-r_s} \frac{\partial P_{BS}}{\partial x}(s,S_s)f(Y_s)S_s dW_s^{(0)*}.\]

Notice that we choose the zero-order option price approximation \(P_{BS}\) instead of the first-order price approximation \(\tilde{P}\) since, from our numerical results, we have not found any apparent improvement when using \(\tilde{P}\) versus \(P_{BS}\). In the third column of Table 3, we list the sample variance ratios obtained from the basic Monte Carlo and the Monte Carlo with our control variate, namely \(V^{CV}(P_{BS})/V^{MC}\). From this test example, the control variate given in (4.2) apparently dominates the importance sampling for the range of values of \(\alpha\).

4.2. **Two-Factor SV models.** We continue to investigate the performance of variance reduction for the two-factor SV model (2.2) defined in Table 4 and 5. Fouque and Han [4] present an importance sampling technique as described in Section 3.1 to evaluate European option prices. Their numerical results extracted from Table 3 in [4] are summarized as variance ratios in the third column of Table 6. According to different rates of mean-reversion \(\alpha\)'s and \(\delta\)'s for each factor, we illustrate ratios of sample variances computed from the basic Monte Carlo, denoted by \(V^{MC}\) and the Monte Carlo simulations with importance sampling, denoted by \(V^{IS}(P)\). Among these Monte Carlo simulations, there are a total of 5000 sample paths in (3.7), simulated based on the discretization of the diffusion process \(V_t\) using an Euler scheme with time step \(\Delta t = 0.005\).

As in the case of one-factor SV models, we do not find an apparent advantage of variance reduction by choosing the first-order approximate option
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Table 3
Comparison of estimated variance reduction ratios for European call options with various \( \alpha \)'s. Notation \( V^{MC} \) is the sample variance from basic Monte Carlo simulation, and \( V^{IS}(\tilde{P}) \) is the sample variance computed from the important sampling with \( \tilde{P} \) defined in (2.5) as an approximate option price. \( V^{CV}(P_{BS}) \) is the sample variance computed from the control variate with \( P_{BS} \) defined in (4.1) as an approximate option price.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( V^{IS}(\tilde{P})/V^{MC} )</th>
<th>( V^{CV}(P_{BS})/V^{MC} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.128</td>
<td>0.020</td>
</tr>
<tr>
<td>1</td>
<td>0.063</td>
<td>0.033</td>
</tr>
<tr>
<td>5</td>
<td>0.061</td>
<td>0.031</td>
</tr>
<tr>
<td>10</td>
<td>0.034</td>
<td>0.019</td>
</tr>
<tr>
<td>25</td>
<td>0.027</td>
<td>0.010</td>
</tr>
<tr>
<td>50</td>
<td>0.021</td>
<td>0.005</td>
</tr>
<tr>
<td>100</td>
<td>0.009</td>
<td>0.004</td>
</tr>
</tbody>
</table>

Table 4
Parameters used in the two-factor stochastic volatility model (2.2).

<table>
<thead>
<tr>
<th>( r )</th>
<th>( m_1 )</th>
<th>( m_2 )</th>
<th>( \nu_1 )</th>
<th>( \nu_2 )</th>
<th>( \rho_1 )</th>
<th>( \rho_2 )</th>
<th>( \rho_{12} )</th>
<th>( \Lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>( f(y, z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>-0.8</td>
<td>-0.8</td>
<td>0.5</td>
<td>0.8</td>
<td>-0.2</td>
<td>-0.2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \exp(y + z) )</td>
</tr>
</tbody>
</table>

price \( \tilde{P} \) compared to using the zero-order approximation \( P_{BS} \) only. Hence the control variates implemented in this numerical experiment are given as in (3.10) and (3.11) by

\[
M_1(P_{BS}) = \int_0^T e^{-rs} \left( \frac{\partial P_{BS}}{\partial x}(s, S_s)f(Y_s, Z_s)S_s dW_s^{(0)*} \right),
\]

\[
M_2(P_{BS}) = \nu_2 \sqrt{2\delta} \int_0^T e^{-rs} \left( \frac{\partial P_{BS}}{\partial z}(s, S_s)dW_s^* \right),
\]

where

\[
(4.3) \quad P_{BS}(t, x; \sigma(z)) = x\mathcal{N}(d_1(x, z)) - Ke^{-r(T-t)}\mathcal{N}(d_2(x, z)),
\]

\[
d_1(x, z) = \ln(x/K) + (r + \frac{1}{2}\sigma^2(z))(T - t) \over \sigma(z)\sqrt{T - t},
\]

\[
d_2(x, z) = d_1(x, z) - \sigma(z)\sqrt{T - t}.
\]

In the fourth column of Table 6, we list the sample variance ratios obtained from the basic Monte Carlo and the Monte Carlo with our control variate, namely \( V^{CV}(P_{BS})/V^{MC} \). Comparing the third and fourth columns in Table 6, a significant variance reduction is readily observed. From this test example and the other extensive numerical experiments that we have
Table 5
Initial conditions and call option parameters.

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>$Y_0$</th>
<th>$Z_0$</th>
<th>$K$</th>
<th>$T$ years</th>
</tr>
</thead>
<tbody>
<tr>
<td>55</td>
<td>-1</td>
<td>-1</td>
<td>50</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 6
Comparison of sample variances for various values of $\alpha$ and $\delta$. Notation $V^{MC}$ is the sample variance from basic Monte Carlo simulation, $V^{IS}(\tilde{P})$ is the sample variance computed from the important sampling with $\tilde{P}$ defined in (2.5) as an approximate option price. $V^{CV}(P_{BS})$ is the sample variance computed from the control variate with $P_{BS}$ defined in (4.3) as an approximate option price.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\delta$</th>
<th>$V^{IS}(P)/V^{MC}$</th>
<th>$V^{CV}(P_{BS})/V^{MC}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
<td>0.074</td>
<td>0.064</td>
</tr>
<tr>
<td>20</td>
<td>0.1</td>
<td>0.057</td>
<td>0.010</td>
</tr>
<tr>
<td>50</td>
<td>0.05</td>
<td>0.030</td>
<td>0.006</td>
</tr>
<tr>
<td>100</td>
<td>0.01</td>
<td>0.041</td>
<td>0.004</td>
</tr>
</tbody>
</table>

performed, we conclude that the control variate given above is superior to the importance sampling for the range of values of $\alpha$ and $\delta$.

5. Conclusion. In the context of pricing European option prices under stochastic volatility models, we find that the control variate method is more efficient to reduce variance than importance sampling via Monte Carlo simulations. In the case of the two-factor SV model varying with well-separated time scales, the reduction power is significant. Under the framework of multi-factor SV models, this computational advantage not only benefits option pricing problems but also provides a suitable tool to allow some parameter estimation procedures such as the efficient method of moments or Markov Chain Monte Carlo to explore volatility premium. In addition, the control variate variance reduction method for path dependent options such as Asian, Barrier, and American options will be further investigated and compared with importance sampling already implemented for Asian options in [4].

REFERENCES

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