Approximation for Option Prices under Uncertain Volatility

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Abstract

In this paper, we study the asymptotic behavior of the worst case scenario option prices as the volatility interval in an uncertain volatility model (UVM) degenerates to a single point, and then provide an approximation procedure for the worst case scenario prices in a UVM with small volatility interval. Numerical experiments show that this approximation procedure performs well even as the size of the volatility band is not so small.

1 Introduction to UVM

Since their introduction in [3] and [13], uncertain volatility models have received intensive attention in mathematical finance. In a simplest UVM, it is assumed that the market has two assets: one riskless asset and one risky asset. Their price processes are denoted as $(B_t)$ and $(X_t)$. It is also assumed that the price process of the riskless asset $(B_t)$ has dynamics

$$dB_t = rB_t dt,$$

where $r$ is a constant.

The price process of the risky asset $(X_t)$ solves the following stochastic differential equation (SDE)

$$dX_t = rX_t dt + \alpha_t X_t dW_t,$$

where $(W_t)$ is a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{Q})$ and the volatility process $(\alpha_t) \in \mathcal{A}$ which is a family of progressively measurable and $[\sigma, \mathcal{F}]$-valued processes. For each stochastic volatility process $\alpha \in \mathcal{A}$, one has a general stochastic volatility model for $(X_t)$. In a UVM, we only know that the true model lies in the above family of general stochastic volatility models. Note that we do not have a prior belief (a probability distribution) over the family of general stochastic volatility models. Therefore, we use “ambiguity” to distinguish this type of uncertainty. Intuitively, we can consider the size of the volatility interval as the degree of model ambiguity.

Due to the presence of model ambiguity (or absence of a prior distribution), the worst case scenario analysis is applied in derivatives pricing under a UVM. Suppose that $\chi$ is a European derivative written on the risky asset with maturity $T$ and payoff $\varphi(X_T)$. It is known that its worst case scenario price at time $t < T$ is given by

$$V(t, X_t) := \sup_{\alpha \in \mathcal{A}} \mathbb{E}_t [\varphi(X_T)],$$

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where $E_t[\cdot]$ is the conditional expectation given $\mathcal{F}_t$ with respect to the measure $Q$, see [3] [13]. It is proved in [8] that the seller of the derivative $\chi$ can super-rePLICATE $\chi$ with initial wealth $\sup_{\alpha \in \mathcal{A}} E[\varphi(X_T)]$ whatever the true volatility process is. The importance of the worst case scenario price is not only because of its super-replication property, but also due to its relationship with coherent risk measures [2] [5]. Following the arguments in stochastic control theory, $V(t, x)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation (in math-finance it is called Black-Scholes-Barenblatt (BSB) equation)

$$\partial_t V + r(x \partial_x V - V) + \sup_{\alpha \in [\sigma, \overline{\sigma}]} \left[ \frac{1}{2} \sigma^2 \alpha^2 \partial_x^2 V \right] = 0, \quad (1.1)$$

$$V(T) = \varphi.$$

If $\varphi$ is convex (like the payoff of a European call), it is known that the worst case scenario price of $\chi$ is equal to its Black-Scholes price with constant volatility $\overline{\sigma}$. For concave $\varphi$, we have a similar result, see [17] for details. However, the fully nonlinear PDE (1.1) does not have a closed form solution, like Black-Scholes formula, for a general terminal payoff function $\varphi$. In order to evaluate the worst case scenario price, we have to resort to numerical methods [3] and [15]. Similarly, the best case scenario price of $\chi$ can be defined as $\inf_{\alpha \in \mathcal{A}} E[\chi]$. Moreover, it is shown in [17] that given any price between $\inf_{\alpha \in \mathcal{A}} E[\chi]$ and $\sup_{\alpha \in \mathcal{A}} E[\chi]$, the market is arbitrage free.

It is clear that the worst case scenario price is larger than any Black-Scholes price with a constant volatility $\sigma \in [\underline{\sigma}, \overline{\sigma}]$. In this paper, we shall consider how the worst case scenario price behaves as the volatility interval $[\underline{\sigma}, \overline{\sigma}]$ degenerates to a single point $\sigma \in [\underline{\sigma}, \overline{\sigma}]$. Intuitively, if the model ambiguity is reduced, then the extra price (which is included in the worst case scenario price) paid for that should be less. As it is shown in the sequel, the worst case scenario price of $\chi$ will converges to its Black-Scholes price with constant volatility $\sigma$. Indeed, Cont suggested in [5] a measure of impact of model uncertainty on the worst case scenario price for any derivative $\chi$:

$$\mu(\chi) = \sup_{\alpha \in \mathcal{A}} E[\chi] - \inf_{\alpha \in \mathcal{A}} E[\chi],$$

which vanishes as the volatility interval shrinks to a single point.

In addition, our study partially answers the sensitivity problem of the worst case scenario to the degree of model ambiguity which is proposed in [5]. In fact, we obtain the rate of convergence of the worst case scenario prices as volatility interval shrinks to a single point. Therefore, this result gives us an approximation of the worst case scenario price when the interval is sufficiently small. Along the paper we denote the Black-Scholes price as $V_0$ and the rate of convergence as $V_1$, which are the solutions to linear partial differential equations. Consequently, the first order approximation $V_0 + (\sigma - \underline{\sigma}) V_1$ of the worst case scenario price is achieved. Of course, the approximated price $V_0 + (\overline{\sigma} - \sigma) V_1$ does not have the property of super-replication. What did we gain in the approximation procedure? First, the problem of solving a fully nonlinear BSB equation is reduced to solving two Black-Scholes like PDEs. The numerical examples also show that the approximation procedure is stable even with reasonably large volatility interval. Second, we are able to see how a linear expectation turn into a sublinear expectation.

In order to study the asymptotic behavior of worst case scenario prices, we re-parameterize our uncertain volatility model and assume that the risky asset price process $(X_t^{\alpha, \varepsilon})$ has a dynamic

$$dX_t^{\alpha, \varepsilon} = r X_t^{\alpha, \varepsilon} dt + \alpha t X_t^{\alpha, \varepsilon} dW_t, \quad (1.2)$$

where $\alpha := (\alpha_t) \in \mathcal{A}$, the family of progressively measurable, $[\sigma, \sigma + \varepsilon]$-valued processes and $(W_t)$ is a Brownian motion on the probability space $(\Omega, \mathcal{F}, F, Q)$. If $\varepsilon = 0$ and no danger of confusion, we shall use $(X_t)$ to denote $(X^{0,0})$ which is indeed a geometric Brownian motion with constant volatility $\sigma$,

$$dX_t = r X_t dt + \sigma X_t dW_t.$$
We define the worst case scenario price as a value function of a stochastic control problem

\[ J^\varepsilon(t, x, \alpha) := \mathbb{E}_{t,x}[\varphi(X_T^{\alpha,\varepsilon})], \]

\[ V^\varepsilon(t, x) = \sup_{\alpha \in A^\varepsilon} [J^\varepsilon(t, x, \alpha)], \]

where the conditional expectation \( \mathbb{E}_{t,x}[\cdot] \) is taken with respect to the law of \( X_T^{\alpha,\varepsilon} \) given \( X_t^{\alpha,\varepsilon} = x \). The worst case scenario option price when \( \varepsilon = 0 \) is a Black-Scholes price. We also represent it as a value function of a trivial stochastic control problem

\[ V_0(t, x) = J(t, x, \sigma) := \mathbb{E}_{t,x} [\varphi(X_T)], \]

where the subscripts in \( \mathbb{E}_{t,x}[\cdot] \) also mean that \( X_t = x \).

This paper is structured as follows. In Section 2, we briefly recall some well-known results from stochastic control theory. The continuity of the worst case scenario price with respect to the parameter \( \varepsilon \) is discussed in Section 3. In Section 4, we heuristically derive the equation for the convergence rate of \( V^\varepsilon \) as \( \varepsilon \) vanishes. Section 5 is devoted to studying the worst case scenario asset price process for the claim \( \chi \). In Section 6, the analysis of error term is developed. In Sections 7, 8 we perform a numerical experiment and conclude the paper.

## 2 Preliminary results from stochastic control

Because we are only interested in the case where \( \varepsilon \) is close to 0, it is legitimate to assume that all \( \varepsilon \leq 1 \). In particular, \( |\alpha_t| \leq \sigma + 1 \). In order to introduce the results that are needed in this paper, we borrow the notations from [12].

Given a SDE with random coefficients:

\[ dX_t = b_t(X_t)dt + \sigma_t(X_t)dW_t, \quad (2.1) \]

if there exists a constant \( K > 0 \) such that

\[ |b_t(x) - b_t(y)| \leq K |x - y|, \quad |\sigma_t(x) - \sigma_t(y)| \leq K |x - y| \]

for all \( t \in [0, T], \omega \in \Omega \), and \( x, y \in \mathbb{R} \), then we say that \((\mathcal{L})\) condition is satisfied. If for all \( t \in [0, T], \omega \in \Omega \), and \( x \in \mathbb{R} \) there exist some \( K > 0 \) such that

\[ |b_t(x)| \leq K |x| + h_t, \quad |\sigma_t(x)|^2 \leq K |x|^2 + r_t^2 \]

for some stochastic processes \((h_t)\) and \((r_t)\), we say that \((R)\) condition is satisfied. It can be seen that the condition \((\mathcal{L})\) implies the condition \((R)\), by noticing that

\[ |b_t(x)| \leq |b_t(x) - b_t(0)| + |b_t(0)| \]

\[ |\sigma_t(x)| \leq |\sigma_t(x) - \sigma_t(0)| + |\sigma_t(0)|. \]

Note that

\[ |x - y| \leq r |x - y|, \quad |\alpha_t x - \alpha_t y| \leq |\alpha_t| |x - y| \leq (\sigma + 1) |x - y|. \]

Therefore, it is clear that the SDE of \((X_t^{\alpha,\varepsilon})\) satisfies the condition \((\mathcal{L})\). According to Corollary 12 in the section 2.5 [12], we have the following universal estimates of the moments of \((X_t^{\alpha,\varepsilon})\)

\[ \mathbb{E} \left[ \sup_{x \in [0, t]} |X_t^{\alpha,\varepsilon}|^q \right] \leq Ne^{Nt} (1 + |x_0|)^q, \quad (2.2) \]

for all \( \alpha \in A^\varepsilon \), \( t \in [0, T] \) and \( q > 0 \), where \( N = N(q, \sigma, r) \) (we assumed that \( \varepsilon < 1 \)) and \( X_0^{\alpha,\varepsilon} = x_0 \).
For another $\varepsilon' \in (0, 1]$, it is assumed that $\varepsilon' < \varepsilon$ without losing generality. We consider the process $(X_t^{\alpha_0(\sigma + \varepsilon'), \varepsilon})$ which satisfies the SDE (1.2) with volatility process $(\alpha_t \wedge (\sigma + \varepsilon'))$ for some $(\alpha_t) \in \mathcal{A}^\varepsilon$. It is also assumed that $X_0^{\alpha, \varepsilon'} = x_0$. By Theorem 9 in the section 2.9 [12] and the estimates of the moments, we can conclude that

$$\mathbb{E}\left[\sup_{s \in [0,t]} |X_s^{\alpha, \varepsilon} - X_s^{\alpha_0(\sigma + \varepsilon'), \varepsilon}|^{2q}\right]$$

$$\leq N't^{q-1}e^{N't}\int_0^t |X_s^{\varepsilon}|^{2q} \cdot |\alpha_s - \alpha_s \wedge (\sigma + \varepsilon')|^{2q} ds$$

$$\leq N't^{q-1}e^{N't}N''e^{N''t}(1 + |x_0|^{2q})(\varepsilon - \varepsilon')$$

$$= N't^{q}e^{N't}N''e^{N''t}(1 + |x_0|^{2q})(\varepsilon - \varepsilon'),$$

for any $q \geq 1$, where $N' = N'(q, \sigma, r)$, $N'' = N''(q, \sigma, r)$, and $N = \max\{N', N'' \}$.

### 3 Continuity of $V^\varepsilon$ in $\varepsilon$

In this section, we mainly analyze the continuity of $V^\varepsilon$ with respect to $\varepsilon$ and the main result is:

**Theorem 3.1.** Given $\varphi$ which is Lipschitz continuous with Lipschitz constant $K_1$ and for any $\varepsilon_0 \in [0, 1)$

$$\lim_{\varepsilon \to \varepsilon_0} V^\varepsilon(t, x) = V^{\varepsilon_0}(t, x)$$

for all $(t, x) \in [0, T] \times \mathbb{R}$.

**Proof.** First, recall that $\mathcal{A}^\varepsilon$ is the family of progressively measurable and $[\sigma, \sigma + \varepsilon]$-valued processes. If $0 \leq \varepsilon_0 < \varepsilon < 1$, we have that

$$V^{\varepsilon_0}(t, x) = \sup_{\alpha \in \mathcal{A}^{\varepsilon_0}} \mathbb{E}_{tx}[\varphi(X_T^{\alpha, \varepsilon_0})] = \sup_{\alpha \in \mathcal{A}^\varepsilon} \mathbb{E}_{tx}\left[\varphi(X_T^{\alpha_0(\sigma + \varepsilon_0), \varepsilon})\right].$$

Therefore, by the estimate (2.3)

$$|V^\varepsilon(t, x) - V^{\varepsilon_0}(t, x)| \leq \sup_{\alpha \in \mathcal{A}^\varepsilon} \left|\mathbb{E}_{tx}[\varphi(X_T^{\alpha, \varepsilon})] - \mathbb{E}_{tx}[\varphi(X_T^{\alpha_0(\sigma + \varepsilon_0), \varepsilon})]\right|$$

$$\leq K_1 \sup_{\alpha \in \mathcal{A}^\varepsilon} \left(\mathbb{E}_{tx}\left[X_T^{\alpha, \varepsilon} - X_T^{\alpha_0(\sigma + \varepsilon_0), \varepsilon}\right]^2\right)^{1/2}$$

$$\leq K_1 \left[N(T-t)e^{N(T-t)}(1 + x^2)(\varepsilon - \varepsilon_0)\right]^{1/2}.$$ 

It can be seen that for any fixed point $(t, x) \in [0, T] \times \mathbb{R}$, $|V^\varepsilon(t, x) - V^{\varepsilon_0}(t, x)| \to 0$ as $\varepsilon$ approaches $\varepsilon_0$ from above.

It can be proved similarly that $\lim_{\varepsilon \to \varepsilon_0} |V^\varepsilon(t, x) - V^{\varepsilon_0}(t, x)| = 0$ for $\varepsilon > 0$.

In particular, when $\varepsilon_0 = 0$ we have one-sided convergence of $\{V^\varepsilon(t, x)\}_{\varepsilon > 0}$ which is stated in the following corollary.

**Corollary 3.1.** When the conditions in Theorem 3.1 are satisfied,

$$\lim_{\varepsilon \downarrow 0} V^\varepsilon(t, x) = V_0(t, x),$$

for all $(t, x) \in [0, T] \times \mathbb{R}$. 

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As the volatility interval $[\sigma, \sigma + \varepsilon]$ becomes smaller, the above corollary tells us that the Black-Scholes price $V_0$ of $\chi$ with constant volatility $\sigma$ is the main contributor to its worst case scenario price relative to the extra price paid for the model ambiguity, i.e. $V_0$ would be the leading term in the approximation of $V^\varepsilon$. We now investigate the first order correction term in this approximation, which will help us understand the sensitivity of the worst case scenario price to the degree of model ambiguity.

4 Heuristic derivation of the first order correction term

To simplify the notations, we assume $r = 0$ in the sequel, but all the results still hold when $r \neq 0$. Recall that $V^\varepsilon$ solves the following BSB equation

$$\partial_t V^\varepsilon + \sup_{\alpha \in [\sigma, \sigma + \varepsilon]} \frac{1}{2} \sigma^2 x^2 \partial_x^2 V^\varepsilon = 0,$$

$$V^\varepsilon(T) = \varphi.$$

To study the asymptotic behavior of $V^\varepsilon$, we also re-parameterize the BSB equation as follows

$$\partial_t V^\varepsilon + \sup_{g \in [0,1]} \frac{1}{2} (\sigma + \varepsilon g)^2 x^2 \partial_x^2 V^\varepsilon = 0, \quad (4.1)$$

$$V^\varepsilon(T) = \varphi.$$

Note that $V_0$ is the solution of the following Black-Scholes equation

$$\partial_t V_0 + \frac{1}{2} \sigma^2 x^2 \partial_x^2 V_0 = 0, \quad (4.2)$$

$$V_0(T) = \varphi.$$

In this heuristic derivation, we assume the differentiability of $V^\varepsilon$ with respect to $\varepsilon$ and the interchangeability of partial differential operators $\partial_\varepsilon$, $\partial_t$ and $\partial_x^2$. We differentiate the equation (4.1) with respect to $\varepsilon$

$$\partial_\varepsilon \left\{ \partial_t V^\varepsilon + \frac{1}{2} \sigma^2 x^2 \partial_x^2 V^\varepsilon + \varepsilon^2 \sup_{g \in [0,1]} \frac{1}{2} g^2 x^2 \partial_x^2 V^\varepsilon + \varepsilon \sup_{g \in [0,1]} g \sigma x^2 \partial_x^2 V^\varepsilon \right\} = 0.$$

Let $V_1 = \partial_\varepsilon V^\varepsilon|_{\varepsilon = 0}$, and according to Corollary 3.1 we also note that $V^\varepsilon|_{\varepsilon = 0} = V_0$. Therefore, $V_1$ is the unique solution of the following linear PDE with source and zero terminal condition

$$\partial_t V_1 + \frac{1}{2} \sigma^2 x^2 \partial_x^2 V_1 + \sup_{g \in [0,1]} g \sigma x^2 \partial_x^2 V_1 = 0, \quad (4.3)$$

$$V_1(T) = 0.$$

Note that the source term in the above equation is known from $V_0$ and is in fact equal to $\sigma x^2 \partial_x^2 V_0 I_{(\partial_x^2 V_0 > 0)}$. It is nonlinear in $V_0$ and can be seen as the first manifestation of the nonlinearity of the full problem (4.1).

In the above heuristic argument, we obtained the equation characterizing $V_1$. It will be verified that $V_1$ which solves (4.3) is the first order derivative of $V^\varepsilon$ with respect to $\varepsilon$ at $\varepsilon = 0$. That is, we shall prove that the error term $V^\varepsilon - (V_0 + \varepsilon V_1)$ is of order $o(\varepsilon)$. More precisely, the following theorem is achieved with more technical conditions imposed on $\varphi$.

**Theorem 4.1.** Assume that the payoff function $\varphi \in C^4_p(\mathbb{R}^+)$ ($p$ for polynomial growth), $\varphi$ is Lipschitz and its derivatives up to order 4 have polynomial growth. Moreover, we also assume that the second derivative of $\varphi$ has a finite number of zero points. Then, pointwise

$$\lim_{\varepsilon \to 0} \frac{V^\varepsilon - (V_0 + \varepsilon V_1)}{\varepsilon} = 0.$$
5 Asset price process in the worst case scenario and estimates of its moments

It is known from [18] and [6] that if $\varphi$ is locally Lipschitz continuous, $\varphi$ and $\varphi'$ have polynomial growth, then the viscosity solution $V^\varepsilon$ of (4.1) belongs to $C^{1,2}_p(0, T) \times \mathbb{R}$ and there exists $\kappa \in (0, 1]$ such that $\partial^2 V^\varepsilon$ is Hölder-\(\kappa\) continuous.

5.1 Existence and uniqueness of $(X_t^{*, \varepsilon})$

The equation (4.1) would produce the worst case scenario volatility process $\alpha^{*, \varepsilon} = \sigma + \varepsilon g^{*, \varepsilon}$ for the claim $\chi$, where

$$g^{*, \varepsilon}(t, x) = \begin{cases} 1 & \partial^2_x V^\varepsilon(t, x) \geq 0, \\ 0 & \partial^2_x V^\varepsilon(t, x) < 0. \end{cases}$$

From the equation (4.3) of $V_1$, we would have another choice of the volatility process for the claim $\chi$: $\bar{\alpha} = \sigma + \varepsilon \bar{g}$, where

$$\bar{g}(t, x) = \begin{cases} 1 & \partial^2_x V_0(t, x) \geq 0, \\ 0 & \partial^2_x V_0(t, x) < 0. \end{cases}$$

Therefore, the asset price process in the worst case scenario for the claim $\chi$ is a stochastic process which satisfies the SDE (1.2) with $\alpha = \alpha^{*, \varepsilon}$ and $r = 0$, i.e.

$$dX^{*, \varepsilon}_t = \alpha^{*, \varepsilon}_t X^{*, \varepsilon}_t dW_t. \tag{5.1}$$

Define a transformation

$$Y^{*, \varepsilon}_t := \log X^{*, \varepsilon}_t,$$

which is well-defined for any $t < \tau^\delta$ where

$$\tau^\delta = \inf \left\{ t > 0 \mid X^{*, \varepsilon}_t = \delta \text{ or } X^{*, \varepsilon}_t = 1/\delta \right\} = \inf \left\{ t > 0 \mid Y^{*, \varepsilon}_t = \log \delta \text{ or } Y^{*, \varepsilon}_t = -\log \delta \right\},$$

for any $\delta > 0$. By Itô’s formula, the process $(Y_t^{*, \varepsilon})$ satisfies the following SDE

$$dY^{*, \varepsilon}_t = \frac{1}{2} \left(\alpha^{*, \varepsilon}_t\right)^2 dt + \alpha^{*, \varepsilon}_t dW_t. \tag{5.2}$$

It is noted that the coefficients in (5.2) are bounded and progressively measurable. Moreover, the diffusion coefficient is bounded away from zero: $\alpha^{*, \varepsilon}_t \geq \sigma > 0$. Therefore, thanks to Theorem 1 in section 2.6 in [12] or the result 7.3.3 in [16], the SDE (5.2) has a unique weak solution. That is, we have a unique solution to the SDE (5.1) until $\tau^\delta$ for any $\delta > 0$. In order to prove that the SDE (5.1) has a unique solution for all $t \in (0, \infty)$, it suffices to show that for any $T > 0$

$$\lim_{\delta \downarrow 0} Q\left( \tau^\delta < T \right) = 0. \tag{5.3}$$

Indeed, by Chebyshev inequality, it holds that

$$\lim_{\delta \downarrow 0} Q\left( \tau^\delta < T \right) \leq \lim_{\delta \downarrow 0} Q \left( \sup_{t \in [0, T]} |Y^{*, \varepsilon}_t| > |\log \delta| \right) \leq \lim_{\delta \downarrow 0} E \left[ \sup_{t \in [0, T]} |Y^{*, \varepsilon}_t| \right] |\log \delta| = 0$$

which implies (5.3).
5.2 Estimates of moments and exit probability of \((X_t^{*,\varepsilon})\)

As a special case of (2.2), we have that
\[
E_{t,x} \left[ \sup_{s \in [t,T]} |X_s^{*,\varepsilon}|^q \right] \leq N e^{N(T-t)} (1 + |x|^q),
\]
(5.4)

for all \(q > 0\), \(N = N(q, \sigma)\) and \(X_T^{*,\varepsilon} = x\).

Given \(\rho > 0\), define a stopping time
\[
\tau_\rho := \inf \{ s \in [t,T], \text{such that } |X_s^{*,\varepsilon}| \geq \rho \}.
\]

Conventionally, \(\inf 0 = \infty\).

By using the estimates of moments and Chebyshev inequality,
\[
Q_{t,x}(\tau_\rho < T) \leq Q_{t,x}(\sup_{s \in [t,T]} |X_s^{*,\varepsilon}| \geq \rho) \leq \frac{Ne^{N(T-t)}(1 + |x|)}{\rho},
\]
(5.5)

where \(N = N(\sigma)\). This control on the exit probability enables us to use localization arguments in the sequel.

Remark 5.1. It is important to notice that the estimates in this section are independent of \(\varepsilon\), due to our assumption \(\varepsilon \leq 1\).

6 Analysis of the error term

Define the error term of the suggested approximation
\[
Z^\varepsilon = V^\varepsilon - (V_0 + \varepsilon V_1).
\]

Let
\[
\mathcal{L}(\sigma) := \partial_t + \frac{1}{2} \sigma^2 x^2 \partial_x^2.
\]

Next, we apply the operator \(\mathcal{L}(\alpha^{*,\varepsilon})\) to the error term:
\[
\mathcal{L}(\alpha^{*,\varepsilon}) Z^\varepsilon = -\mathcal{L}(\alpha^{*,\varepsilon})(V_0 + \varepsilon V_1)
\]
\[
= -(\mathcal{L}(\alpha^{*,\varepsilon}) - \mathcal{L}(\sigma)) V_0 - \varepsilon (\mathcal{L}(\alpha^{*,\varepsilon}) - \mathcal{L}(\sigma)) V_1 - \varepsilon \mathcal{L}(\sigma)V_1
\]
\[
= \frac{1}{2} \left( \sigma^2 - (\sigma + \varepsilon g^{*,\varepsilon})^2 \right) x^2 \partial_x^2 V_0 + \varepsilon \frac{1}{2} \left( \sigma^2 - (\sigma + \varepsilon g^{*,\varepsilon})^2 \right) x^2 \partial_x^2 V_1
\]
\[
+ \varepsilon \bar{g} \sigma x^2 \partial_x^2 V_0
\]
\[
= -\varepsilon (g^{*,\varepsilon} - \bar{g}) \sigma x^2 \partial_x^2 V_0 - \varepsilon^2 \left( \frac{1}{2} (g^{*,\varepsilon})^2 x^2 \partial_x^2 V_0 + g^{*,\varepsilon} \sigma x^2 \partial_x^2 V_1 \right)
\]
\[
- \varepsilon^3 \left( \frac{1}{2} (g^{*,\varepsilon})^2 x^2 \partial_x^2 V_1 \right).
\]
(6.1)

Note that the terminal condition of \(Z^\varepsilon\) is
\[
Z^\varepsilon(T) = V^\varepsilon(T) - V_0(T) - \varepsilon V_1(T) = 0.
\]

From now on, we impose more regularity conditions on the terminal data \(\varphi\), i.e. polynomial growth condition on the first four derivatives of \(\varphi(x)\):
\[
\begin{cases}
\varphi'(x) \leq K_1, \\
\varphi''(x) \leq K_2 (1 + |x|^m), \\
\varphi'''(x) \leq K_3 (1 + |x|^n), \\
\varphi^{(4)}(x) \leq K_4 (1 + |x|^l),
\end{cases}
\]
(6.2)

where \(K_i\) for \(i \in \{1, 2, 3, 4\}\), and \(m, n,\) and \(l\) are positive constants.
6.1 Feynman-Kac representation of the error term $Z^\varepsilon$

Given the equation (6.1) for $Z^\varepsilon$ together with the existence and uniqueness of $(X_t^t)$, we have the following probabilistic representation of $Z^\varepsilon$:

\[
Z^\varepsilon = -\varepsilon\mathbb{E}_{tx} \left[ \int_t^T \left\{ (g_s^\varepsilon - \bar{g}_s) \sigma (X_s^\varepsilon)^2 \partial_x^2 V_0 (s, X_s^\varepsilon) \right\} ds \right]
\]

\[
-\varepsilon^2 \mathbb{E}_{tx} \left[ \int_t^T \left\{ \frac{1}{2} (g_s^\varepsilon)^2 (X_s^\varepsilon)^2 \partial_x^2 V_0 (s, X_s^\varepsilon) + g_s^\varepsilon \sigma (X_s^\varepsilon)^2 \partial_x^2 V_1 (s, X_s^\varepsilon) \right\} ds \right]
\]

\[
-\varepsilon^3 \mathbb{E}_{tx} \left[ \int_t^T \left\{ \frac{1}{2} (g_s^\varepsilon)^2 (X_s^\varepsilon)^2 \partial_x^2 V_1 (s, X_s^\varepsilon) \right\} ds \right]
\]

\[
= -\varepsilon I_1 - \varepsilon^2 I_2 - \varepsilon^3 I_3,
\]  

(6.3)

where

\[
I_1 = \mathbb{E}_{tx} \left[ \int_t^T \left\{ (g_s^\varepsilon - \bar{g}_s) \sigma (X_s^\varepsilon)^2 \partial_x^2 V_0 (s, X_s^\varepsilon) \right\} ds \right],
\]

\[
I_2 = \mathbb{E}_{tx} \left[ \int_t^T \left\{ \frac{1}{2} (g_s^\varepsilon)^2 (X_s^\varepsilon)^2 \partial_x^2 V_0 (s, X_s^\varepsilon) + g_s^\varepsilon \sigma (X_s^\varepsilon)^2 \partial_x^2 V_1 (s, X_s^\varepsilon) \right\} ds \right],
\]

\[
I_3 = \mathbb{E}_{tx} \left[ \int_t^T \left\{ \frac{1}{2} (g_s^\varepsilon)^2 (X_s^\varepsilon)^2 \partial_x^2 V_1 (s, X_s^\varepsilon) \right\} ds \right].
\]

We shall derive the bounds of $I_2$ and $I_3$ in the next section. The term $I_1$ will be dealt with in Section 6.3.

6.2 Controls of the terms $I_2$ and $I_3$

As a special case of (2.2), the process $(X_t)$ which solves

\[
dX_t = \sigma X_t dW_t
\]

(6.4)

has the following estimates of moments

\[
\mathbb{E}_{tx} \left[ \sup_{s \in [t,T]} |X_s|^q \right] \leq N e^{N(T-t)} (1 + |x|^q),
\]

(6.5)

for all $q > 0$, where $N = N(q, \sigma)$.

Because $V_0$ is the solution of the Black-Scholes equation (4.2), the following lemma holds.

**Lemma 6.1.** Given $\varphi(x)$ which satisfies the condition (6.2), there exist constants $M_1$, $M_2$, and $M_3$ which only depend on $\sigma$, $T$, $m$, $n$, and $l$, such that $|\partial_x^2 V_0| \leq M_1 (1 + |x|^m)$, $|\partial_x^2 V_0| \leq M_2 (1 + |x|^n)$, and $|\partial_t \partial_x^2 V_0| \leq M_3 (1 + |x|^{p_1})$, where $p_1 = \max \{m, n + 1, l + 2\}$.

**Proof.** We shall use the probabilistic representation of $V_0$

\[
V_0(t, x) = \mathbb{E}_{tx}[\varphi(X_T)],
\]

(6.6)

where the process $(X_t)$ is the geometric Brownian motion (6.4). Then, starting at $x$ at time $t$, we have $X_T = xe^{-\frac{\sigma^2}{2}t+\sigma \sqrt{t} \xi}$. Therefore, using the notation $\tau = T - t$, we obtain

\[
|\partial_x^2 V_0| = \int_{\mathbb{R}} e^{-\sigma^2 \tau + 2 \sigma \sqrt{\tau} y + \sigma \sqrt{\tau} y} \varphi''(xe^{-\frac{\sigma^2}{2}t+\sigma \sqrt{\tau} y}) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy
\]

\[
\leq K_2 \int_{\mathbb{R}} e^{-\sigma^2 \tau + 2 \sigma \sqrt{\tau} y} \left( 1 + |x|^m e^{-\frac{\sigma^2}{2}t+\sigma \sqrt{\tau} y} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy
\]

\[
\leq M_1 (1 + |x|^m),
\]
with \( M_1 = M_1(K_2, m, T, \sigma). \)

Similarly, we obtain controls of \( \partial_x^2 V_0 \) and \( \partial_x^4 V_0 \):

\[
|\partial_x^2 V_0| = \left| \int \frac{e^{-\frac{1}{2} \sigma^2 \tau + 3\alpha \sqrt{\tau} y} \varphi''(x e^{-\frac{1}{2} \alpha^2 \tau + \alpha \sqrt{\tau} y})}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \right| \\
\leq K_3 \int e^{-\frac{1}{2} \sigma^2 \tau + 3\alpha \sqrt{\tau} y} (1 + |x|^m) e^{-\frac{\alpha^2}{2} \tau + \alpha \sqrt{\tau} y} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
\leq M_2 (1 + |x|^n),
\]

\[
|\partial_x^4 V_0| = \left| \int e^{-2\sigma^2 \tau + 4\alpha \sqrt{\tau} y} \varphi^{(4)}(x e^{-\frac{1}{2} \alpha^2 \tau + \alpha \sqrt{\tau} y}) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \right| \\
\leq K_4 \int e^{-2\sigma^2 \tau + 4\alpha \sqrt{\tau} y} (1 + |x|^n) e^{-\frac{\alpha^2}{2} \tau + \alpha \sqrt{\tau} y} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
\leq M_4 (1 + |x|^l),
\]

with \( M_2 = M_2(K_3, n, T, \sigma) \) and \( M_4 = M_4(K_4, l, T, \sigma). \)

From the PDE of \( V_0 \), it can be seen that \( \partial_x \partial_x^2 V_0 = -\partial_x^2 \left( \frac{1}{2} \sigma^2 x^2 \partial_x^2 V_0 \right) = -\frac{1}{2} \sigma^2 \left( 2\partial_x^2 V_0 + 4x \partial_x^3 V_0 + x^2 \partial_x^4 V_0 \right) \).

Therefore, the control of the mixed third partial derivative \( \partial_x \partial_x^2 V_0 \) is obtained through the controls on the partial derivatives only with respect to \( x \). Consequently, there exist two constants \( M_3 = M_3(M_1, M_2, M_4, m, n, l) \) and \( p_1 = \max\{m, n + 1, l + 2\} \) such that

\[
|\partial_x \partial_x^2 V_0| \leq \frac{\sigma^2}{2} \left\{ 2M_1 (1 + |x|^m) + 4|x| M_2 (1 + |x|^n) + x^2 M_4 (1 + |x|^l) \right\} \\
\leq M_3(1 + |x|^{p_1}).
\]

\[ \square \]

Given \( V_0 \) which solves \((4.2)\), we define

\[
f(t, x) := \frac{1}{2} \sigma x^2 \partial_x^2 V_0(t, x).
\]

The first order partial derivatives of \( f \) are

\[
f_t(t, x) = \frac{1}{2} \sigma x^2 \partial_t^2 V_0(t, x),
\]

\[
f_x(t, x) = \frac{1}{2} \sigma \left[ 2x \partial_x^2 V_0(t, x) + x^2 \partial_x^3 V_0(t, x) \right].
\]

Due to the lemma \( 6.1 \), it can be observed that

\[
|f_t(t, x)| \leq \frac{1}{2} \sigma x^2 M_1 (1 + |x|^m) \leq K_5(1 + |x|^{m+2}) \tag{6.7}
\]

for a constant \( K_5 = K_5(M_1, \sigma) \) and that

\[
|f_t(t, x)| + |f_x(t, x)| \leq \frac{\sigma}{2} \left( x^2 M_4 (1 + |x|^{p_1}) + 2|x| M_2 (1 + |x|^n) \right) \\
+ x^2 M_4 (1 + |x|^n) \leq K_6(1 + |x|^{p_1}), \tag{6.8}
\]

where \( K_6 = K_6(M_1, M_2, M_4, p_1, m, n, \sigma) \) and \( p = \max\{p_1 + 2, m + 1, n + 2\} \).

In order to analyze the partial derivatives of \( V_1 \), we use \((X^{t,x}_s)^{i,j} \) to denote the solution of the SDE \((6.4)\) with the initial condition \( X_0^{t,x} = x \). Similarly, \((X^{t+\sigma,x}_s)^{i,j} \) denotes the solution of the same SDE with the initial condition \( X_0^{t+\sigma,x} = x \). We also define

\[
\Delta X_s = \frac{X_s^{t,x+\sigma} - X_s^{t,x}}{h}. \tag{6.9}
\]
From D.10 in [9], we have that
\[ E_t \left| \Delta X_s \right|^2 \leq N, \quad \text{(6.10)} \]
for any \( s \in [0, T - t] \), where \( N = N(\sigma) \).

Due to (6.8), we can derive that
\[
\left| \frac{f(s, X_s^{t,x+h}) - f(s, X_s^t)}{h} \right| \leq \int_0^1 \left| f_x(s, X_s^\lambda) \right| |\Delta X_s| \, d\lambda \\
\leq K_6 \left( 1 + \left| X_s^{t,x} \right|^p + \left| X_s^{t,x+h} \right|^p \right) |\Delta X_s|, 
\]
where \( X_s^\lambda = (1 - \lambda)X_s^t + \lambda X_s^{t,x+h} \) for \( \lambda \in (0, 1) \). Moreover, by the mean value theorem, it holds that
\[
\left| \frac{f(s + h, X_s^{t+h,x}) - f(s, X_s^{t+h,x})}{h} \right| \leq K_6 \left( 1 + \left| X_s^{t+h,x} \right|^p \right). 
\]

If we denote the source term in the equation of (4.3) as \( f^+(t, x) \), then it is noted that
\[
f^+(t, x) := - \sup_{g \in [0, 1]} \left\{ \frac{1}{2} g \sigma x^2 \partial_x^2 V_0 \right\} = - \max \left\{ \frac{1}{2} \sigma x^2 \partial_x^2 V_0, 0 \right\} = - \max \{ f, 0 \}. 
\]

Therefore, it is also clear that
\[
\left| f^+(t, x) \right| \leq |f(t, x)|, \\
\left| f^+(t, x) - f^+(s, y) \right| \leq |f(t, x) - f(s, y)|, \quad \text{(6.13)}
\]
for any \( (t, x), (s, y) \in [0, T] \times \mathbb{R} \).

Note that the first order difference quotient of \( V_1 \) with respect to the time variable \( t \) can be represented as follows
\[
\frac{V_1(t + h, x) - V_1(t, x)}{h} = E_t \int_0^{T - t - h} \frac{f^+(s + h, X_s^{t+h,x}) - f^+(s, X_s^{t+h,x})}{h} \, ds \\
+ E_t \int_0^{T - t - h} \frac{f^+(s, X_s^{t+h,x}) - f^+(s, X_s^{t,x})}{h} \, ds \\
- \frac{1}{h} E_t \int_{T - t - h}^{T - t} f^+(s, X_s^{t,x}) \, ds. \quad \text{(6.14)}
\]

**Lemma 6.2.** Given \( \varphi(x) \) which satisfies the condition (6.2), there exists a constant \( M_5 \) which depends on \( K_5, K_6, T, p, m, \) and \( \sigma \) such that \( |\partial_t V_1| \leq M_5(1 + |x|^p) \).

**Proof.** Due to (6.13) and (6.12), the first term in (6.14) can be estimated as follows
\[
\left| E_t \int_0^{T - t - h} \frac{f^+(s + h, X_s^{t+h,x}) - f^+(s, X_s^{t+h,x})}{h} \, ds \right| \\
\leq E_t \int_0^{T - t - h} \left| f(s + h, X_s^{t+h,x}) - f(s, X_s^{t+h,x}) \right| \, ds \\
\leq K_6 E_t \int_0^{T - t - h} \left( 1 + \left| X_s^{t+h,x} \right|^p \right) \, ds \\
\leq A_1(1 + |x|^p),
\]
where \( A_1 = A_1(K_6, T, p, \sigma) \). For the second term in (6.14), by noticing that the shifted process \( X_s^{t,x} \) is identical to \( X_s^{t+h,x} \) up to time \( T - t - h \), we therefore can conclude that
\[
\left| E_t \int_0^{T - t - h} \frac{f^+(s, X_s^{t+h,x}) - f^+(s, X_s^{t,x})}{h} \, ds \right| = 0.
\]
Due to (6.13) and (6.7), the third term in (6.14) can be controlled as follows,

\[
\frac{1}{h} E_t x \left| \int_{T-t-h}^{T-t} f^+(s, X_s^x) ds \right| \leq \frac{1}{h} E_t x \int_{T-t-h}^{T-t} \left| f^+(s, X_s^x) \right| ds
\]

\[
\leq \frac{1}{h} K_5 E_t x \int_{T-t-h}^{T-t} (1 + |X_s^x|^{m+2}) ds
\]

\[
\leq A_2 (1 + |x|^{m+2}),
\]

where \( A_2 = A_2(K_5, T, m, \sigma) \). Then, after summarizing the above three estimates the lemma follows.

**Proposition 6.1.** Given \( \varphi(x) \) which satisfies the condition (6.2), there exist two constants \( p' = \max\{p, m+2\} \) and \( M_6 \) which depends on \( K_5, M_5, T, m, p, \) and \( \sigma \), such that \( |x^2 \partial_x^2 V_1| \leq M_6 (1 + |x|^{p'}) \).

**Proof.** By noticing that

\[ \partial_t V_1 + \frac{1}{2} \sigma^2 x^2 \partial_x^2 V_1 = f^+, \]

we have

\[ |x^2 \partial_x^2 V_1| \leq \frac{2}{\sigma^2} (|\partial_t V_1| + |f^+|). \]

The above fact together with (6.7), (6.13) and the lemma 6.2 results in the proposition.

**Theorem 6.1.** Given \( \varphi(x) \) which satisfies the condition (6.2), there exists a constant \( D_1 \) which depends on \( M_1, M_6, T - t, m, p', \) and \( \sigma \), such that \( I_2 \) and \( I_3 \) in (6.3) satisfy

\[ |I_2| + |I_3| \leq D_1 (1 + |x|^p). \]

**Proof.** Due to the proposition 6.1 and (5.4), it follows that

\[
|I_3| = E_t x \left[ \int_t^T \frac{1}{2} (g_s^x)^2 (X_s^x)^2 \left| \partial_x^2 V_1 (s, X_s^x) \right| ds \right]
\]

\[
\leq \frac{M_6}{2} E_t x \int_t^T \left( 1 + |X_s^x|^{p'} \right) ds
\]

\[
\leq \frac{M_6}{2} \left[ \int_t^T + N(\sigma, p') e^{N(\sigma, p')(T-t)} (1 + |x|^{p'}) ds \right]
\]

\[
= \frac{M_6}{2} \left[ T - t + N(\sigma, p') e^{N(\sigma, p')(T-t)} (1 + |x|^{p'}) (T - t) \right].
\]

Similarly for \( I_2 \), we have

\[
|I_2| \leq E_t x \left[ \int_t^T \frac{1}{2} (X_s^x)^2 \left| \partial_x^2 V_0 (s, X_s^x) \right| ds \right]
\]

\[
+ E_t x \left[ \int_t^T \sigma (X_s^x)^2 \left| \partial_x V_1 (s, X_s^x) \right| ds \right]
\]

\[
\leq \frac{M_1}{2} E_t x \int_t^T |X_s^x|^2 (1 + |X_s^x|^m) ds + M_7 \sigma E_t x \int_t^T (1 + |X_s^x|^{p'}) ds
\]

\[
\leq \frac{M_1}{2} (T - t) N(\sigma) e^{N(\sigma)(T-t)} (1 + |x|^2)
\]

\[
+ \frac{M_1}{2} (T - t) N(m + 2, \sigma) e^{N(m+2, \sigma)(T-t)} (1 + |x|^{m+2})
\]

\[
+ M_6 (T - t) N(p', \sigma) e^{N(p', \sigma)(T-t)} (1 + |x|^{p'}). \]

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By summarizing the above controls on $I_2$ and $I_3$, there exists $D_1 = D_1(M_1, M_6, T - t, m, p', \sigma)$ such that $|I_2| + |I_3| < D_1(1 + |x|^p)$.

### 6.3 Convergence of the term $I_1$

Given the controls of $I_2$ and $I_3$ in the theorem 6.1, to prove

$$Z^\varepsilon(t, x) \sim o(\varepsilon)$$

for a fixed point $(t, x) \in [0, T] \times \mathbb{R}$, it suffices to prove that

$$\lim_{\varepsilon \to 0} I_1 = 0.$$

Let $K_\rho := [-\rho, \rho]$ and $g^{d, \varepsilon} := g^{*, \varepsilon} - \bar{g}$ which depends on $\varepsilon$ through $g^{*, \varepsilon}$ as follows

$$|g^{d, \varepsilon}(t, x)| = \begin{cases} 1 & \frac{\partial^2 V^\varepsilon(t, x)}{\partial x^2} \frac{\partial^2 V_0(t, x)}{\partial x^2} < 0, \\ 0 & \frac{\partial^2 V^\varepsilon(t, x)}{\partial x^2} \frac{\partial^2 V_0(t, x)}{\partial x^2} \geq 0. \end{cases}$$

Intuitively, as $\varepsilon$ approaches 0, $V^\varepsilon$ and its derivatives will get closer and closer to $V_0$ and its corresponding derivatives. In order to lay out the analysis of this intuition, we decompose the range of $X_s^{*, \varepsilon}$ for each $s \in [t, T]$ into two parts: a compact set $K_\rho$ and its complement. Therefore, $I_1$ can be written as the expectation of a sum of (i) the compact part (when $X_s^{*, \varepsilon}$ lies in the compact set) and (ii) the tail part (when $X_s^{*, \varepsilon}$ lies outside of the compact set):

$$I_1 = \mathbb{E}_{tx} \left[ \int_t^T g^{d, \varepsilon}(X_s^{*, \varepsilon})^2 \frac{\partial^2 V_0(s, X_s^{*, \varepsilon})}{\partial x^2} I_{K_\rho}(X_s^{*, \varepsilon}) \, ds \right] + \mathbb{E}_{tx} \left[ \int_t^T g^{d, \varepsilon}(X_s^{*, \varepsilon})^2 \frac{\partial^2 V_0(s, X_s^{*, \varepsilon})}{\partial x^2} I_{K_\rho^c}(X_s^{*, \varepsilon}) \, ds \right]$$

$$= \mathbb{E}_{tx}[i] + \mathbb{E}_{tx}[ii]. \quad (6.15)$$

In order to achieve $\lim_{\varepsilon \to 0} I_1 = 0$, we shall use localization arguments to deal with $\mathbb{E}_{tx}[ii]$ and the problem is reduced onto a compact set. On the compact set, it will be proved that $V^\varepsilon$ and its partial derivatives converge to $V_0$ and its corresponding derivatives. Then, it is followed by the convergence of $\mathbb{E}_{tx}[i]$ to 0.

#### 6.3.1 Control of the tail part

We apply Hölder’s inequality to the second term of (6.15)

$$\mathbb{E}_{tx}[ii] \leq \sigma M_1 \left[ \mathbb{E}_{tx} \left[ \int_t^T (X_s^{*, \varepsilon})^4 (1 + |X_s^{*, \varepsilon}|^m)^2 \, ds \right] \right]^{1/2} \mathbb{E}_{tx}(\tau_\rho < T)^{1/2}$$

where

$$\mathbb{E}_{tx} \left[ \int_t^T (X_s^{*, \varepsilon})^4 (1 + |X_s^{*, \varepsilon}|^m)^2 \, ds \right]$$

$$\leq \int_t^T N(4, \sigma)e^{N(4, \sigma)(T-t)}(1 + |x|^4) + N(m + 4, \sigma)e^{N(m + 4, \sigma)(T-t)}(1 + |x|^{m+4}) + N(2m + 4, \sigma)e^{N(2m + 4, \sigma)(T-t)}(1 + |x|^{2m+4}) \, ds$$

$$\leq M_7(1 + |x|^{2m+4}), \quad (6.16)$$

for some constant $M_7 = M_7(T - t, m, \sigma)$. 

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From (6.16) and the exit probability estimate of the process \((X_t^{\varepsilon})\) in (5.5), it is concluded that
\[
\mathbb{E}_{t,x}[\|ii\|] \leq \sigma M_1 \sqrt{M_T(1 + |x|^{2m+4})} \frac{N(\sigma) e^{N(\sigma)(T-t)(1+|x|)(T-t)}}{\rho} \\
\leq D_2 \frac{(1 + |x|^{m+3/2})}{\sqrt{\rho}},
\]
for some constant \(D_2 = D_2(M_1, T-t, m, \sigma).

As \(\rho\) increases, i.e. the compact set \(K_\rho\) becomes larger and larger, the process will be less and less likely to deviate outside of the set \(K_\rho\). Then, we would expect the tail part is small enough for sufficiently large \(\rho\). This result is summarized in the following proposition.

**Proposition 6.2.** Given \(\varphi(x)\) which satisfies the condition (6.2), \(\mathbb{E}_{t,x}[\|ii\|] \to 0\) as \(\rho \to \infty\).

### 6.3.2 Control of the compact part

In order to prove that the compact term is negligible when \(\varepsilon\) is sufficiently small, we need the convergence of \(\partial_x^2 V^\varepsilon\) to \(\partial_x^2 V_0\) which would imply that \(g^\varepsilon\) gradually vanishes as \(\varepsilon\) tends to 0.

Recall the result regarding the regularity of the solution of the BSB equation in [18] [6].

**Theorem 6.2.** If \(\varphi\) is locally Lipschitz continuous and \(\varphi\), and \(\varphi'\) have polynomial growth, then the solution \(V^\varepsilon\) of (4.1) belongs to \(C^{1,2}_p([0,T] \times \mathbb{R})\). Moreover, \(\partial_x^2 V^\varepsilon(t,x)\) is Hölder continuous in \(x\) with an exponent \(\kappa \in (0,1]\) for any \(t \in [0,T]\).

**Remark 6.1.** All the constants in the polynomial controls and Hölder continuity only depend on the bounds of the volatility interval \([\sigma, \sigma + \varepsilon]\). Since we are only interested in the cases where \(\varepsilon\)'s are small, we can choose these constants including Hölder exponent to be universal, i.e. independent of \(\varepsilon\); see [18] [6] or [11].

We recapitulate the theorem as follows
\[
\begin{cases}
|V^\varepsilon(t,x)| \leq B_0(1 + |x|^{b_0}), \\
|\partial_x V^\varepsilon(t,x)| \leq B_1(1 + |x|^{b_1}), \\
|\partial_x^2 V^\varepsilon(t,x)| \leq B_2(1 + |x|^{b_2}), \\
|\partial_x^2 V^\varepsilon(t,x) - \partial_x^2 V^\varepsilon(t,y)| \leq B_3|x-y|^\kappa,
\end{cases}
\]
for any \(t \in [0,T]\) where all constants \(B_0, B_1, B_2, B_3\) and \(b_0, b_1, b_2, \kappa\) are universal, i.e. independent of \(\varepsilon\).

**Lemma 6.3.** Given \(\varphi(x)\) which satisfies the condition (6.2), \(\partial_x^2 V^\varepsilon(t,\cdot)\), as a family of functions of \(x\) indexed by \(\varepsilon\), uniformly converges to \(\partial_x^2 V_0(t,\cdot)\) on the compact set \(K_\rho\) as \(\varepsilon\) tends to 0 for any fixed \(t \in [0,T]\).

**Proof.** The proof is similar to that of Theorem 5.2.5 in [10]. Indeed, because \(V^\varepsilon\) has up to second order partial derivative in \(x\), by following the arguments in the proof of the Theorem 5.2.5 in [10] we conclude that \(\partial_x V^\varepsilon\) converges to \(\partial_x V_0\) as \(\varepsilon\) tends to 0 , uniformly for all \(x \in K_\rho\).

However, \(V^\varepsilon(t,\cdot)\) is only twice differentiable with respect to \(x\). In order to obtain the uniform convergence of the sequence of second partial derivatives \(\partial_x^2 V^\varepsilon\), we first observe from (6.17) that \(\partial_x^2 V^\varepsilon\) is a family of uniformly bounded and Hölder continuous functions of \(x\) on \(K_\rho\). It implies that \(\partial_x^2 V^\varepsilon\) is equi-continuous. Then, there exists a sub-sequence \(\partial_x^2 V^\varepsilon'\) which converges uniformly on \(K_\rho\). Together with the convergence of \(\partial_x V^\varepsilon\) to \(\partial_x V_0\), we conclude that the \(\partial_x^2 V^\varepsilon'\) uniformly converges to \(\partial_x^2 V_0\) for all \(x \in K_\rho\).
Since the limit $\partial^2_x V_0$ is independent of the choice of sub-sequence $\{\partial^2_x V^{\varepsilon}\}$, we claim that $\{\partial^2_x V^{\varepsilon}\}$ converges uniformly to $\partial^2_x V_0$. Indeed, if there is a sub-sequence $\{\partial^2_x V^{\varepsilon''}\}$ does not converges to $\partial^2_x V_0$, then according to the Arzela-Ascoli Theorem $\{\partial^2_x V^{\varepsilon''}\}$ has a uniformly convergent sub-sequence $\{\partial^2_x V^{\varepsilon'''}\}$. Again, together with the convergence of $\{\partial_x V^{\varepsilon}\}$ to $\partial_x V_0$, $\{\partial^2_x V^{\varepsilon'''}\}$ has to converge to $\partial^2_x V_0$, which is a contradiction with the assumption. Therefore, the lemma follows.

Let
\[
S^{\varepsilon}_t = \left\{ x \big| \partial^2_x V^{\varepsilon}(t, x) \partial^2_x V_0(t, x) \leq 0, \quad \exists\varepsilon' \leq \varepsilon \right\} \cap K_{\rho}.
\]

Note that $\{S^{\varepsilon}_t\}_{\varepsilon}$ as a family of sets indexed by $\varepsilon$ is non-increasing as $\varepsilon$ decreases to 0. Define
\[
S^0_t := \lim_{\varepsilon \downarrow 0} S^{\varepsilon}_t.
\]

**Lemma 6.4.** Given $\varphi(x)$ which satisfies the condition (6.2), for any fixed $t \in [0, T)$
\[
S^0_t = \left\{ x \in K_{\rho} \big| \partial^2_x V_0(t, x) = 0 \right\}.
\]

**Proof.** Notice that if $x \in K_{\rho}$ such that $\partial^2_x V_0(t, x) = 0$, then $x \in S^{\varepsilon}_t$ for all $\varepsilon > 0$. It implies that
\[
S^0_t \supseteq \left\{ x \in K_{\rho} \big| \partial^2_x V_0(t, x) = 0 \right\}.
\]

On the other hand, if $\partial^2_x V_0(t, x) > 0$ for any $x \in K_{\rho}$, then due to the uniform convergence of $\{\partial^2_x V^{\varepsilon}(t, \cdot)\}$ there exists $\varepsilon_0 > 0$ such that $\partial^2_x V^{\varepsilon}(t, x) > 0$ for all $\varepsilon < \varepsilon_0$. Hence, $\partial^2_x V^{\varepsilon}(t, x) \partial^2_x V_0(t, x) > 0$ for all $\varepsilon < \varepsilon_0$, i.e., $x \notin S^{\varepsilon}_t \forall \varepsilon < \varepsilon_0$. It is followed by $x \notin S^0_t$.

Similarly, we can prove that any $x \in K_{\rho}$ such that $\partial^2_x V_0(t, x) < 0$ does not lie in $S^0_t$ either. Therefore, we can claim that
\[
S^0_t \subseteq \left\{ x \in K_{\rho} \big| \partial^2_x V_0(t, x) = 0 \right\}.
\]

For any fixed $t \in [0, T)$, we take $\tilde{S}^{\varepsilon}_t := \text{the closure of } S^{\varepsilon}_t \text{ for each } \varepsilon$. For this sequence of closed, bounded and non-increasing sets, we define its limit
\[
\tilde{S}^0_t := \lim_{\varepsilon \downarrow 0} \tilde{S}^{\varepsilon}_t.
\]

Due to the continuity of $\partial^2_x V_0(t, x)$ in $x$, it is true that $\tilde{S}^0_t = S^0_t$ for any fixed $t \in [0, T)$. The following lemma tells us that the same relationship holds between $\tilde{S}^{\varepsilon}_t$ and $S^{\varepsilon}_t$.

**Lemma 6.5.** Given $\varphi(x)$ which satisfies the condition (6.2) and any fixed $t \in [0, T)$, it is true that
\[
\tilde{S}^{\varepsilon}_t = S^{\varepsilon}_t.
\]

**Proof.** To prove the lemma, it suffices to show that $\forall x_0 \in \tilde{S}^{\varepsilon}_t$, either $x_0 \in S^{\varepsilon}_t$ or $x_0 \in S^0_t$. Indeed, if it is true, then together with the fact that $S^{\varepsilon}_t \supseteq S^0_t$ (by the monotonicity of $\{S^{\varepsilon}_t\}$) we can see that $\tilde{S}^{\varepsilon}_t = S^{\varepsilon}_t$, i.e. $S^{\varepsilon}_t$ is a closed set.

For any $x_0 \in \tilde{S}^{\varepsilon}_t$, according to the definition of closure, there exists a sequence $\{x_n\} \subset S^{\varepsilon}_t$ such that $\lim_{n \to \infty} x_n = x_0$. For each $x_n$, there is a $\varepsilon_n \leq \varepsilon$ such that
\[
\partial^2_x V^{\varepsilon_n}(t, x_n) \partial^2_x V_0(t, x_n) \leq 0.
\]

Since $0 < \varepsilon_n \leq \varepsilon$ for all $n > 0$, there exist a $\varepsilon_0 \in [0, \varepsilon]$ and a sub-sequence $\{\varepsilon_n\}$ such that $\lim_{n \to \infty} \varepsilon_n = \varepsilon_0$. 

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We take the corresponding sub-sequence \( \{x_{n'}\} \) which converges to \( x_0 \) by assumption. Then,
\[
\lim_{n' \to \infty} \partial_x^2 V^{x_{n'}}(t, x_{n'})\partial_x^2 V_0(t, x_{n'}) = \partial_x^2 V^{x_0}(t, x_0)\partial_x^2 V_0(t, x_0) \leq 0. \tag{6.18}
\]
Indeed,
\[
\begin{align*}
|\partial_x^2 V^{x_{n'}}(t, x_{n'})\partial_x^2 V_0(t, x_{n'}) - & \partial_x^2 V^{x_0}(t, x_0)\partial_x^2 V_0(t, x_0)| \\
\leq |\partial_x^2 V^{x_{n'}}(t, x_{n'})\partial_x^2 V_0(t, x_{n'}) - & \partial_x^2 V^{x_0}(t, x_0)\partial_x^2 V_0(t, x_{n'})| \\
+ & |\partial_x^2 V^{x_{n'}}(t, x_0)\partial_x^2 V_0(t, x_{n'}) - \partial_x^2 V^{x_0}(t, x_0)\partial_x^2 V_0(t, x_{n'})| \\
\leq & \partial_x^2 V_0(t, x_{n'})B_3|x_{n'} - x_0|\epsilon + \partial_x^2 V^{x_0}(t, x_0)B_3|x_{n'} - x_0|\epsilon
\end{align*}
\tag{6.19}
\]
where the last inequality is due to H"older continuity of \( \partial_x^2 V^\epsilon \) for all \( \epsilon \). Then, due to the continuity of \( \partial_x^2 V_0(t, \cdot), \partial_x^2 V_0(t, x_{n'}) \) converges to \( \partial_x^2 V_0(t, x_0) \) as \( n' \) tends to \( \infty \). Based on the fact of the uniform boundedness of \( \{\partial_x^2 V^{x_{n'}}(t, \cdot)\} \) on \( K_\rho \) and convergence of \( \{\partial_x^2 V^{x_{n'}}(t, x_0)\} \) to \( \partial_x^2 V^{x_0}(t, x_0) \), it is clear that all three terms in (6.19) converges to 0. Therefore, (6.18) follows. We discuss two possible cases for the value of \( \epsilon_0 \).

1. Case \( \epsilon_0 > 0 \): We also know that \( \epsilon_0 \leq \epsilon \), since \( \epsilon_{n'} \leq \epsilon \). According to the definition of \( S^\epsilon_1, x_0 \in S^\epsilon_1 \).

2. Case \( \epsilon_0 = 0 \): From the fact (6.18), \( (\partial_x^2 V_0(t, x_0))^2 \leq 0 \). Therefore, \( \partial_x^2 V_0(t, x_0) = 0 \), i.e. \( x \in S^\epsilon_0 \) due to the lemma 6.5.

Therefore, the lemma is proved. \( \square \)

Note that any \( x \in S^\epsilon_0 \) is a zero point of \( \partial_x^2 V_0 \) at time \( t \). Let \( U(t, x) := \partial_x^2 V_0(t, x) \). We shall consider zero set of \( U(t, x) \) for any fixed \( t \in [0, T] \). With the assumption (6.2) on \( \varphi \), \( V_0 \) has up to fourth derivative with respect to \( x \). Therefore, we can derive the equation for \( U(t, x) \) from (4.2) as follows
\[
\begin{align*}
\partial_t U + \sigma^2 U + 2\sigma x \partial_x U + \frac{1}{2} \sigma^2 x^2 \partial_x^2 U &= 0, \\
U(T) &= \varphi''.
\end{align*}
\]
Let \( x := \log y, \hat{U}(t, y) := U(t, x) \). Then, \( \hat{U}(t, x) \) solves the following PDE
\[
\begin{align*}
\partial_t \hat{U} + \frac{3}{2} \sigma^2 \partial_y \hat{U} + \frac{1}{2} \sigma^2 \partial_y^2 \hat{U} + \sigma^2 \partial_y^4 \hat{U} &= 0.
\end{align*}
\]
Notice that all coefficients in the above equation are constant. Therefore, Theorem B in [1] and the remark below it are applicable to \( \hat{U} \). They directly implies that the size of the zero set of \( \hat{U} \)
\[
Z_t = \left\{ y \mid \hat{U}(t, y) = 0 \right\},
\]
is non-increasing as variable \( t \) goes from \( T \) to 0. Noting that the change of variable \( x = \log y \) is a one-to-one mapping, we can conclude the following proposition.

**Proposition 6.3.** If \( \varphi \) satisfies (6.2), then \( \partial_x^2 V_0 \) has at most the same number of zero points as \( \varphi'' \) does for any fixed \( t \).

At this point, we can conclude that if \( \varphi'' \) has a finite number of zero points then the \( g_{d,e} \) will be vanishing as \( e \) decreases to 0. However, to achieve the goal that expectation of compact part goes to 0, we still need to show that the law of variable \( X_{s,e}^r \) does not give a positive probability to any single point for any fixed \( s \in [t, T] \).
Recall the main Theorem 1.1 in [14] that for every $t \in (0, T)$, the marginal law of

$$M_t = \int_0^t \alpha_s dW_s$$

does not weight points, where $(\alpha_t)$ is any progressively measurable process such that

$$0 < \sigma \leq \alpha_t \leq \sigma + \varepsilon.$$

The following lemma is a simple extension to the above result.

**Lemma 6.6.** Let $(X_t^{*\varepsilon})$ solves the SDE (5.1). For any $t \in (0, T]$, $X_t^{*\varepsilon}$ does not weight points.

**Proof.** Due to the transformation applied in the section 5.1, we only need to prove the claim for the process $(Y_t^{*\varepsilon})$ which solves

$$dY_t^{*\varepsilon} = -\frac{1}{2} (\alpha_t^{*\varepsilon})^2 dt + \alpha_t^{*\varepsilon} d\tilde{W}_t,$$

on the probability space $(\Omega, \mathcal{F}, \tilde{Q})$.

Let

$$\xi_t = \exp \left( -\int_0^t (\alpha_s^{*\varepsilon})^2 \frac{ds}{8} + \int_0^t \alpha_s^{*\varepsilon} dW_s \right), \quad \text{for } t \leq T.$$

Define a measure $\tilde{Q}$ on $\mathcal{F}_T$ by

$$d\tilde{Q} = \xi_T dQ.$$

According to Girsanov’s theorem, under the measure $\tilde{Q}$

$$\tilde{W}_t = -\int_0^t \alpha_s^{*\varepsilon} \frac{ds}{2} + W_t$$

is a Brownian motion and $(Y_t^{*\varepsilon})$ has the following dynamic

$$dY_t^{*\varepsilon} = \alpha_t^{*\varepsilon} d\tilde{W}_t.$$

Note that the worst case scenario volatility process for $\chi$, $(\alpha_t^{*\varepsilon})$ is a adapted and bounded process. According to the Novikov condition, $(\xi_t)$ is a martingale, and therefore $Q$ and $\tilde{Q}$ are two equivalent measures.

From the Theorem 1.1 in [14], we learn that the law of $Y_t^{*\varepsilon}$ does not weight points under the measure $\tilde{Q}$, for any $t \in [0, T]$. Due to equivalence between $Q$ and $\tilde{Q}$, we can claim that $Y_t^{*\varepsilon}$ does not weight points under the measure $Q$. Therefore, the lemma follows. \(\square\)

For given $s \in [t, T]$, we can not directly use the continuity of a probability measure to claim that $\lim_{\varepsilon \downarrow 0} Q_{tx} \left[ X_s^{*\varepsilon} \in \bar{S}_s^0 \right] = Q_{tx} \left[ X_s^{*0} \in \bar{S}_s^0 \right]$, because both the process $(X_t^{*\varepsilon})$ and the set $\bar{S}_t^\varepsilon$ depend on $\varepsilon$. Therefore, we define a capacity from the laws of a family of random variables $\{X_s^{*\varepsilon}\}_\varepsilon$ as follows

$$c(A) := \sup_{\varepsilon \in [0, 1]} Q_{tx} \left( X_s^{*\varepsilon} \in A \right),$$

for any $A \in \mathcal{B}(\mathbb{R})$.

**Proposition 6.4.** $X_s^{*\varepsilon}$ converges weakly to $X_s$ for any $s > 0$.

**Proof.** It is a direct implication of the lemma 3.1. \(\square\)

If $\phi''$ has a finite number of zero points, then due to the lemma 6.6 $Q_{tx} \left( X_s^{*\varepsilon} \in \bar{S}_s^0 \right) = 0$ for any $\varepsilon > 0$. This fact directly leads to $c(S_s^0) = 0$. Due to the weak convergence of $\{X_s^{*\varepsilon}\}$, the family of laws of $\{X_s^{*\varepsilon}\}$ is weakly compact.
Lemma 6.7. If \( \varphi \) satisfies the condition (6.2) and \( \varphi'' \) has a finite number of zero points, then
\[
\lim_{\varepsilon \to 0} Q_{tx} \left[ X^s_{*\varepsilon} \in \bar{S}^\varepsilon_s \right] = 0
\]
for any \( s \in (t, T) \).

Proof. First, we observe that
\[
0 \leq Q_{tx} \left( X^s_{*\varepsilon} \in \bar{S}^\varepsilon_s \right) \leq c(\bar{S}^\varepsilon_s).
\]

Notice that \( \{ \bar{S}^\varepsilon_s \} \) is a sequence of decreasing closed sets and converges to \( \bar{S}^0_s \) as \( \varepsilon \) goes to 0. Because of the weak compactness of the laws of \( \{ X^s_{*\varepsilon} \} \) and Lemma 8 in [7], it can be seen that
\[
c(\bar{S}^\varepsilon_s) \downarrow c(\bar{S}^0_s) = 0.
\]
Then, the lemma is true. \( \square \)

Theorem 6.3. If \( \varphi \) satisfies (6.2) and \( \varphi'' \) has a finite number of zero points, then there exists \( \varepsilon_0 > 0 \) such that \( E_{tx} [|ii|] < \delta \) for any fixed \( \rho > 0 \), any fixed point \( (t, x) \in (0, T] \times \mathbb{R} \) and \( \forall \delta > 0 \) as long as \( \varepsilon < \varepsilon_0 \).

Proof. Recall from (6.15) that
\[
E_{tx}[i] = E_{tx} \left[ \int_t^T g^{d,\varepsilon}(t, X^s_{*\varepsilon}) \mathbb{I}_{K_{\varepsilon}(X^s_{*\varepsilon})} \sigma \left( X^s_{*\varepsilon} \right)^2 \partial^2_s V_0(s, X^s_{*\varepsilon}) ds \right].
\]

Note that \( g^{d,\varepsilon}(t, x) \) can only take three possible values: \( \{-1, 0, 1\} \). Indeed,
\[
|g^{d,\varepsilon}(t, x)| = \begin{cases} 1 & \text{if } \partial^2_s V_0(t, x) \partial^2_s V_0(t, x) < 0, \\ 0 & \text{if } \partial^2_s V_0(t, x) \partial^2_s V_0(t, x) \geq 0. \\ \end{cases}
\]

Therefore, due to (6.16) it follows that
\[
E_{tx}[|ii|] = E_{tx} \left[ \int_t^T |g^{d,\varepsilon}(s, X^s_{*\varepsilon})| \mathbb{I}_{K_{\varepsilon}(s)} \left( X^s_{*\varepsilon} \right)^2 \left| \partial^2_s V_0(s, X^s_{*\varepsilon}) \right| ds \right] 
\leq \sigma M_1 \left[ E_{tx} \int_t^T (X^s_{*\varepsilon})^4 (1 + |X^s_{*\varepsilon}|^m)^2 ds \right]^{1/2} \left[ \int_t^T Q_{tx}(X^s_{*\varepsilon} \in \bar{S}^\varepsilon_s) ds \right]^{1/2}
\leq \sigma M_1 \sqrt{M_8(1 + |x|^{2m+4})} \left[ \int_t^T Q_{tx}(X^s_{*\varepsilon} \in \bar{S}^\varepsilon_s) ds \right]^{1/2}.
\]

Due to the lemma 6.7, it follows that
\[
\int_t^T Q_{tx}(X^s_{*\varepsilon} \in \bar{S}^\varepsilon_s) ds \to 0,
\]
as \( \varepsilon \downarrow 0 \). Therefore, the lemma is concluded. \( \square \)

Now, we are ready to claim the main result in this section.

Theorem 6.4. If \( \varphi \) satisfies (6.2) and \( \varphi'' \) has a finite number of zero points, then \( \lim_{\varepsilon \to 0} I_1 = 0 \) for any fixed \( (t, x) \in [0, T] \times \mathbb{R} \).

Proof. Note that
\[
|I_1| \leq E_{tx}[|ii|] + E_{tx}[|iii|].
\]

For any \( \delta > 0 \), due to the proposition 6.2 there exists \( \rho_0(t, x, \delta) > 0 \) such that \( E_{tx}[|ii|] < \delta \) for all \( \rho > \rho_0(t, x, \delta) \).

By the theorem 6.3, for the given \( \rho_0(t, x, \delta) \) and \( \delta \) there exist \( \varepsilon_0(t, x, \rho_0(t, x, \delta)) \) such that \( E_{tx}[|ii|] < \delta \) for any \( \varepsilon < \varepsilon_0(t, x, \rho_0(t, x, \delta)) \). Therefore, the theorem follows. \( \square \)
From the proofs, we essentially derived that \( \bar{\alpha} = \sigma + \bar{\varepsilon} \bar{g} \) is a good approximation of the worst case scenario volatility \( \alpha^{\star, \varepsilon} \); see their definitions in the section 5. Together with the properties of the law of the asset price process in the worst case scenario, we proved the main theorems that \( Z^\varepsilon(t, x) \) goes to 0 as \( \varepsilon \downarrow 0 \), for any \( (t, x) \in [0, T) \times \mathbb{R} \). Therefore, we can construct an approximation of \( V^\varepsilon \) of order \( o(\varepsilon) \): \( V_0 + \varepsilon V_1 \). The performance of this approximation procedure is studied numerically in the next section.

7 Numerical results

In this section, we will work on a non-trivial example: a symmetric European butterfly with the following payoff function

\[
\phi(x) = (x - 90)^+ - 2(100 - x)^+ + (x - 110)^+ \tag{7.1}
\]

which is neither convex nor concave; see the Figure 1.

![Figure 1: The payoff function of a symmetric European butterfly](image)

Even though the payoff function (7.1) does not satisfy the conditions of the main theorem 4.1, we can consider a regularization \( \bar{\phi} \) of \( \phi \), which satisfies the conditions of the Theorem 4.1. Moreover, \( \bar{\phi} \) can be chosen sufficiently close to \( \phi \) such that

\[
\|V_0(\phi) + \varepsilon V_1(\phi) - [V_0(\bar{\phi}) + \varepsilon V_1(\bar{\phi})]\| \ll \varepsilon.
\]

That is, the approximation of \( \bar{\phi} \): \( V_0(\bar{\phi}) + \varepsilon V_1(\bar{\phi}) \) is a good proxy for that of \( \phi \).

Indeed, we numerically compute the worst case scenario price \( V^\varepsilon(\phi) \), by the scheme provided in [15]. It is proved by Barles [4] that the numerical solution from that scheme is locally uniformly convergent to \( V^\varepsilon \), the unique viscosity solution of (4.1), as the scheme becomes finer. We also compare the numerically computed worst case scenario price with its approximation: \( V_0(\phi) + \varepsilon V_1(\phi) \), where \( V_0(\phi) \) is given by the Black-Scholes formula and \( V_1(\phi) \) is numerically computed by a simple difference scheme according to the equation (4.3). Because the scheme for computing \( V^\varepsilon(\phi) \) uses Newton iteration in each time step to deal with the nonlinearity, our approximation is computed a lot more efficiently. For visual comparison of the worst case scenario prices with corresponding approximations, we show complete numerical results for a very small \( \varepsilon = 0.006 \), a small \( \varepsilon = 0.01 \), and a \( \varepsilon = 0.05 \).
which is not so small. Throughout all the experiments, we set $\sigma = 0.15$ and $T = 0.25$. When $\varepsilon = 0.05$, the upper bound of the volatility interval is $\sigma + \varepsilon = 0.20$, which is $1/3$ larger than the base volatility level $\sigma = 0.15$. In other words, even if $\varepsilon = 0.05$ is small, 5% volatility is significant. From Figure 2, we note that the worst case scenario prices are higher than the Black-Scholes prices. That is, we need extra cash to super-replicate the option when facing the model ambiguity. It also can be noted that the first order corrected prices capture the main feature of the worst case scenario prices $V^\varepsilon$ for different values of $\varepsilon$.

Figure 2: The black curve represents the worst case scenario prices and the red curve represents the Black-Scholes prices; the blue curve represents the difference between the worst case scenario prices and its Black-Scholes prices and the green curve is $\varepsilon \star V_1$; all curves are plotted against asset prices.

To see the trend of the error of our approximation as $\varepsilon$ increases, we choose 8 equally spaced values from 0 to 0.05 for $\varepsilon$. For each $\varepsilon$, we compute the error of the approximation, which is defined by

$$\text{error}(\varepsilon) = \sup_x |V^\varepsilon(0, x) - V_0(0, x) - \varepsilon V_1(0, x)|.$$  \hfill (7.2)


Figure 3: Error for different values of \( \varepsilon \).

As shown in the Figure 3 the error increases as \( \varepsilon \) becomes larger, so the second order approximation will be needed to improve accuracy for large values of \( \varepsilon \). However, there is not an abrupt change in the range of \( \varepsilon \) we choose.

8 Conclusion

In this paper, we have studied the asymptotic behavior of the worst case scenario option prices as the degree of model ambiguity vanishes. This study not only helps us understand how a linear expectation turns into a sublinear expectation, but also gives us an approximation procedure of worst case scenario option prices when \( \varepsilon \) is small. From the numerical results, we see that the approximation procedure works well even when the upper bound volatility is 1/3 larger than the lower bound.

Note that the worst case scenario price is often computed to evaluate the risk in a portfolio. Our approximation procedure improves the efficiency of this evaluation, because it avoids the Newton iteration which is employed in the scheme for \( V^{\varepsilon} \). Moreover, the worst case scenario price \( V^{\varepsilon} \) has to be re-computed for a new value of \( \varepsilon \). However, the equations (4.2) and (4.3) for \( V_0 \) and \( V_1 \) are independent of \( \varepsilon \), so the approximation only requires us to compute \( V_0 \) and \( V_1 \) once for all small values of \( \varepsilon \).

References


