Asymptotic Optimal Strategy for Portfolio Optimization in a Slowly Varying Stochastic Environment

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March 9, 2016

Abstract

In this paper, we study the portfolio optimization problem with general utility functions and when the return and volatility of underlying asset are slowly varying. An asymptotic optimal strategy is provided within a specific class of admissible controls under this problem setup. Specifically, we first establish a rigorous first order approximation of the value function associated to a fixed zeroth order suboptimal trading strategy, which is given by the heuristic argument in [J.-P. Fouque, R. Sircar and T. Zariphopoulou, Mathematical Finance, 2016]. Then, we show that this zeroth order suboptimal strategy is asymptotically optimal in a specific family of admissible trading strategies. Finally, we show that our assumptions are satisfied by a particular fully solvable model.

Keywords: Portfolio allocation, stochastic volatility, regular perturbation, asymptotic optimality

1 Introduction

The portfolio optimization problem was first introduced and studied in the continuous-time framework in Merton [1969, 1971], which provided explicit solutions on how to trade stocks and/or how to consume so as to maximize one’s utility, with risky assets following the Black-Scholes-Merton model (that is, geometric Brownian motions with constant returns and constant volatilities), and when the utility function is of specific types (for instance, Constant Relative Risk Aversion (CRRA)). Following these pioneer works, additional constraints were added in this model to mimic real-life investments. This includes transaction cost originally considered by Magill and Constantinides [1976] and a user’s guide by Guasoni and Muhle-Karbe [2013], and investments under drawdown constraint for instance by Grossman and Zhou [1993], Cvitanic and Karatzas [1995] and Elie and Touzi [2008], just to name a few. In the meantime, general models for risky assets have been also considered, among which, Cox and Huang [1989] and Karatzas et al. [1987] first studied the incomplete market case, Zariphopoulou [1999] studied the case where the drift and volatility terms are non-linear functions of the asset price, and Chacko and Viceira [2005] gave a closed-form solution under a particular one-factor stochastic volatility model.

Recently, multiscale factor models for risky assets were considered in the portfolio optimization problem in Fouque et al. [2016], where return and volatility are driven by fast and slow factors. Specifically, the authors heuristically derived the asymptotic approximation to the value function and the optimal strategy for general utility functions. In this paper, we shall focus on the risky asset modeled by only slowly varying stochastic factor, and the reason is twofold: Firstly, slow factor is particularly important in long-term investment, because the effect of fast factor is approximately averaged out in the long time as studied in [Fouque et al., 2016, Section 2]. Secondly, analysis under the model with fast mean-reverting stochastic factor requires singular asymptotic techniques, and more technique details in combining the fast and slow factors, and thus, this will be presented in another paper in preparation (Fouque and Hu [2016]).

We describe the model as below, with dynamics of the underlying asset and slowly varying factor denoted as $S_t$ and $Z_t$ respectively,

$$dS_t = \mu(Z_t) S_t \, dt + \sigma(Z_t) S_t \, dW_t,$$

$$dZ_t = \delta c(Z_t) \, dt + \sqrt{\delta g(Z_t)} \, dW_t^Z,$$  

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where the standard Brownian motions \((W_t, W^Z_t)\) are correlated,
\[
d\langle W, W^Z \rangle_t = \rho \, dt, \quad |\rho| < 1.
\]
Assumptions on the coefficients \(\mu(z), \sigma(z), c(z), g(z)\) of the model will be specified in Section 2.4. In (1.2), \(\delta\) is a small positive parameter that characterizes the slow variation of the process \(Z\). Note that \(Z_t \overset{D}{=} Z_{\delta t}^{(1)}\), where the diffusion process \(Z^{(1)}\) has the following infinitesimal generator, denoted by \(\mathcal{M}\)
\[
\mathcal{M} = \frac{1}{2}g(z)^2 \frac{\partial^2}{\partial z^2} + c(z) \partial_z.
\]
We refer to Fouque et al. [2011] for more details of this model, where asymptotic results in the limit \(\delta \to 0\) are derived for linear problems of option pricing.

Denote by \(X^\pi_t\) the wealth process associated to the Markovian strategy \(\pi\), and in this strategy, the amount of money \(\pi(t, x, z)\) is invested in stock at time \(t\), when the stock price is \(x\), and the level of the slow factor \(Z_t\) is \(z\), with the remaining money held in money market earning a risk-free interest of \(r\). Assuming that the portfolio is self-financing, then \(X^\pi_t\) follows
\[
dX^\pi_t = \pi(t, X^\pi_t, Z_t) \frac{dS_t}{S_t} + r(X^\pi_t - \pi(t, X^\pi_t, Z_t)) \, dt
\]
\[
= (rX^\pi_t + \pi(t, X^\pi_t, Z_t)(\mu(Z_t) - r)) \, dt + \pi(t, X^\pi_t, Z_t)\sigma(Z_t) \, dW_t.
\]
For simplicity and without loss of generality, we assume \(r = 0\) for the rest of the paper, and then
\[
dX^\pi_t = \pi(t, X^\pi_t, Z_t)\mu(Z_t) \, dt + \pi(t, X^\pi_t, Z_t)\sigma(Z_t) \, dW_t.
\]
An investor aims at finding an optimal strategy \(\pi\) which maximizes her terminal expected utility \(\mathbb{E}[U(X^\pi_T)]\), where \(U(x)\) is in a general class of utility functions. Denote by \(V^\delta(t, x, z)\) the value function
\[
V^\delta(t, x, z) = \sup_{\pi \in \mathcal{A}(t, x, z)} \mathbb{E}[U(X^\pi_T)|X^\pi_t = x, Z_t = z],
\]
where the supremum is taken over all admissible strategies \(\mathcal{A}(t, x, z)\),
\[
\mathcal{A}(t, x, z) = \{ \pi : X^\pi_t \text{ in (1.4) stays nonnegative } \forall s \geq t, \text{ given } X_t = x, \text{ and } Z_t = z \}.
\]
The Hamilton-Jacobi-Bellman (HJB) equation for \(V^\delta\) is given by
\[
V^\delta_t + \delta \mathcal{M} V^\delta + \max_{\pi \in \mathcal{A}} \left( \frac{1}{2} \sigma(z)^2 \pi^2 V^\delta_{xx} + \pi \left( \mu(z) V^\delta_x + \sqrt{\delta} \rho g(z) \sigma(z) V^\delta_{zz} \right) \right) = 0.
\]
As in Fouque et al. [2016], we shall assume that \(V^\delta\) is the unique classical solution of (1.7). Maximizing in \(\pi\) and plugging in the optimizer gives the following nonlinear equation,
\[
V^\delta_t + \delta \mathcal{M} V^\delta - \frac{\left( \lambda(z) V^\delta_x + \sqrt{\delta} \rho g(z) V^\delta_{xx} \right)^2}{2V^\delta_{xx}} = 0,
\]
where the optimizer (optimal control) is given in the feedback form by
\[
\pi^* = -\frac{\lambda(z) V^\delta_x}{\sigma(z) V^\delta_{xx}} - \frac{\sqrt{\delta} \rho g(z) V^\delta_{xx}}{\sigma(z) V^\delta_{xx}},
\]
and the Sharpe ratio is \(\lambda(z) = \mu(z) / \sigma(z)\).

The HJB equation (1.7) is fully nonlinear and not explicitly solvable in general. In Fouque et al. [2016], a regular perturbation approach is used to derive an approximation for \(V^\delta\) up to the first order, namely, the value function \(V^\delta\) is formally expanded as follows:
\[
V^\delta = v^{(0)} + \sqrt{\delta} v^{(1)} + \delta v^{(2)} + \cdots,
\]
with \( v^{(0)} \) and \( v^{(1)} \) identified by asymptotic equations. It is also observed in [Fouque et al., 2016, Section 3.2.1] that the zeroth order suboptimal strategy

\[
\pi^{(0)}(t, x, z) = -\frac{\lambda(z) v^{(0)}_x(t, x, z)}{\sigma(z) v^{(0)}_{xx}(t, x, z)},
\]

(1.10)

not only gives the optimal value up to the principal term \( v^{(0)} \), but also up to first order \( \sqrt{\delta} \) correction \( v^{(0)} + \sqrt{\delta} v^{(1)} \).

**Main result.** We first prove that the first order approximation of \( \mathbb{E} \left[ U(X_T^{(0)}) \right] \) is \( v^{(0)} + \sqrt{\delta} v^{(1)} \), where \( X_T^{(0)} \) is the terminal value of the wealth process associated to the strategy \( \pi^{(0)} \) given by (1.10), and \( U(x) \) is in a general class of utility functions with precise assumptions given in Section 2.3. Then we establish the optimality of \( \pi^{(0)} \) in the class of admissible controls of the form \( \pi^{(0)}(t, x, z) + \delta \pi^{(1)}(t, x, z) \), with precise properties given in Section 4. We also solve and analyze a concrete model that satisfies all the assumptions in order to demonstrate that they are reasonable. We remark that the optimality of \( \pi^{(0)} \) in the full class of controls (1.6) remains open.

**Organization of the paper.** In Section 2, we briefly review the classical Merton problem and heuristic results in Fouque et al. [2016]. We also list all the assumptions needed for our theoretical proofs. In Section 3, we apply the regular perturbation technique to the value function associated to the strategy \( \pi^{(0)} \), and we prove its first order accuracy. Then we show \( \pi^{(0)} \) is asymptotically optimal in a smaller class of admissible controls in Section 4. A fully solvable example with a closed-form solution is presented in Section 5, which satisfies all the assumptions listed in Section 2. We make conclusive remarks in Section 6.

## 2 Preliminaries and assumptions

In this section, we review the classical Merton problem, summarize the heuristic results in Fouque et al. [2016], and list the assumptions on the utility function and the state processes needed for later proofs.

### 2.1 Merton problem with constant coefficients

We first discuss the case of \( \mu \) and \( \sigma \) being constant in (1.1), which plays a crucial role in interpreting the leading order value function \( v^{(0)} \) in (1.9) and analysis of the regular perturbation. This problem has been widely studied and completely solved, and we start with some background results.

Let \( X_t \) be the solution to

\[
dX_t = \pi^*(t, X_t) \mu dt + \pi^*(t, X_t) \sigma dW_t,
\]

(2.1)

where \( \pi^*(t, x) \) is the optimal trading strategy, then, \( X_t \) stays nonnegative up to time \( T \), and

\[
\int_0^T |\sigma \pi^*(t, X_t)|^2 dt < \infty, \quad \text{almost surely.}
\]

(2.2)

We refer to [Karatzas and Shreve, 1998, Chapter 3] for details.

Following the notations in Fouque et al. [2016], we denote by \( M(t, x; \lambda) \) the Merton value function, then, one also has the following regularity results.

**Proposition 2.1.** If the investor’s utility \( U(x) \) is \( C^2(0, \infty) \), strictly increasing, strictly concave, and satisfies the Inada and Asymptotic Elasticity conditions:

\[
U'(0^+) = \infty, \quad U'(+\infty) = 0, \quad A\mathbb{E}[U] := \lim_{x \to \infty} \frac{U'(x)}{U(x)} < 1,
\]

then, the Merton value function is strictly increasing, strictly concave in the wealth variable \( x \), and decreasing in the time variable \( t \), and it is the unique \( C^{1,2}([0, T] \times \mathbb{R}^+) \) solution to the HJB equation

\[
M_t + \sup_{\pi} \left\{ \frac{1}{2} \sigma^2 \pi^2 M_{xx} + \mu \pi M_x \right\} = M_t - \frac{1}{2} \lambda^2 \frac{M_x^2}{M_{xx}} = 0, \quad M(T, x; \lambda) = U(x),
\]

(2.3)
where $\lambda = \mu/\sigma$ is the constant Sharpe ratio. It is also continuously differentiable with respect to $\lambda = \mu/\sigma$, and $\pi^* = -\frac{\lambda}{\sigma} \frac{M_t}{M_{xx}}$.

The proof is referred to [Fouque et al., 2016, Section 2.1] where these properties were stated and it was also mentioned that they have been established primarily using Fenchel-Legendre transformation.

The following relation between partial derivatives of $M(t, x; \lambda)$ is provided in [Fouque et al., 2016, Lemma 3.2].

**Lemma 2.2.** The Merton value function $M(t, x; \lambda)$ satisfies the “Vega-Gamma” relation

$$M\lambda = -(T-t)\lambda R^2 M_{xx}, \quad (2.4)$$

where

$$R(t, x; \lambda) = -\frac{M_x(t, x; \lambda)}{M_{xx}(t, x; \lambda)}, \quad (2.5)$$

is the risk-tolerance function.

Note that $R(t, x; \lambda)$ is continuous, strictly positive due to the regularity, concavity and monotonicity of $M(t, x; \lambda)$. As introduced in Fouque et al. [2016], we recall the notation

$$D_k = R(t, x; \lambda)^k \partial_x^k, \quad k = 1, 2, \cdots, \quad (2.6)$$

$$L_{t,x}(\lambda) = \partial_t + \frac{1}{2} \lambda^2 D_2 + \lambda^2 D_1. \quad (2.7)$$

Note that the coefficients of $L_{t,x}(\lambda)$ depend on $R(t, x; \lambda)$, and then on $M(t, x; \lambda)$, and the Merton PDE (2.3) can be re-written as

$$L_{t,x}(\lambda)M(t, x; \lambda) = 0. \quad (2.8)$$

The following proposition regarding the linear operator $L_{t,x}(\lambda)$ will be used repeatedly in Sections 3 and 4.

**Proposition 2.3.** Let $L_{t,x}(\lambda)$ be the operator defined in (2.7), and assume that the utility function $U(x)$ satisfies the conditions in Proposition 2.1 and $U(0+) = 0$ (or finite), then

$$L_{t,x}(\lambda)u(t, x; \lambda) = 0, \quad (2.9)$$

$$u(T, x; \lambda) = U(x).$$

has a unique nonnegative solution.

**Proof.** First, observe that $M(t, x; \lambda)$ is a solution of (2.9). To show uniqueness, we use the following transformation in Fouque et al. [2016],

$$\begin{cases} 
  \xi = -\log M_x(t, x; \lambda) + \frac{1}{2} \lambda^2 (T-t), \\
  t' = t,
\end{cases} \quad (2.10)$$

which is one-to-one since the Jacobian $\left| -\frac{M_x}{M_{xx}} \right|$ stays positive. Define $w(t', \xi; \lambda) = u(t, x; \lambda)$, then $w$ solves:

$$\mathcal{H}w = w_{t'} + \frac{1}{2} \lambda^2 w_{\xi \xi} = 0, \quad w(T, \xi; \lambda) = U(I(e^{-\xi})).$$

Uniqueness of the nonnegative solution then follows from classical results for the heat equation [John, 1982, Chapter 7.1(d)].
2.2 Existing results under slowly varying stochastic factor

In this subsection, we summarize the results in Fouque et al. [2016] that will be used in later sections. Recall the formal expansion of $V^{\delta}$:

$$V^{\delta} = v^{(0)} + \sqrt{\delta}v^{(1)} + \delta v^{(2)} + \cdots,$$

then:

(i) The leading order term $v^{(0)}$ is the unique solution to

$$v^{(0)}_t - \frac{1}{2} \lambda(z)^2 \left( \frac{v^{(0)}_x}{v^{(0)}_{xx}} \right)^2 = 0, \quad v^{(0)}(T, x, z) = U(x),$$

and is the Merton value function associated with the current (or frozen) Sharpe ratio $\lambda(z) = \mu(z)/\sigma(z)$:

$$v^{(0)}(t, x, z) = M(t, x; \lambda(z)).$$

(ii) The first order correction $v^{(1)}$ solves the linear PDE:

$$v^{(1)}_t + \frac{1}{2} \lambda(z)^2 \left( \frac{v^{(0)}_x}{v^{(0)}_{xx}} \right)^2 v^{(1)}_x - \lambda(z)^2 v^{(0)}_x v^{(0)}_{xx} = \rho \lambda(z) g(z) \frac{v^{(0)}_{xx}}{v^{(0)}_{xx}}, \quad v^{(1)}(T, x, z) = 0.$$

Using the notations (2.6)-(2.7), $v^{(1)}$ satisfies

$$\mathcal{L}_{t,x}(\lambda(z))v^{(1)} = -\rho \lambda(z) g(z) D_1 v^{(0)}_z, \quad v^{(1)}(T, x, z) = 0.$$

The linear equation (2.13) with zero terminal condition $v^{(1)}(T, x, z) = 0$ has a unique solution. As a consequence, the following equation:

$$\mathcal{L}_{t,x}(\lambda(z))u(t, x, z) = 0, \quad u(T, x, z) = 0,$$

has a unique the solution $u \equiv 0$.

(iii) By the “Vega-Gamma” relation stated in Lemma 2.2, the $z$-derivative of the leading order term $v^{(0)}$ satisfies:

$$v^{(0)}_z = -(T-t)\lambda(z)\lambda'(z)D_2 v^{(0)},$$

and $v^{(1)}$ is explicitly given in term of $v^{(0)}$ by

$$v^{(1)} = -\frac{1}{2} (T-t) \rho \lambda(z) g(z) \frac{v^{(0)}_{xx} v^{(0)}_{zz}}{v^{(0)}_{xx}}.$$

2.3 Assumptions on the utility $U(x)$

Assumption 2.4. Throughout the paper, we make the following assumptions on the utility $U(x)$:

(i) $U(x)$ is $C^7(0, \infty)$, strictly increasing, strictly concave and satisfying the following conditions (Inada and Asymptotic Elasticity):

$$U'(0+) = \infty, \quad U'(\infty) = 0, \quad AE[U] := \lim_{x \to \infty} x U'(x) / U(x) < 1.$$  

(ii) $U(0+)$ is finite. Without loss of generality, we assume $U(0+) = 0$. 

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(iii) Denote by $R(x)$ the risk tolerance,
\begin{equation}
R(x) := -\frac{U'(x)}{U''(x)}.
\end{equation}
Assume that $R(0) = 0$, $R(x)$ is strictly increasing and $R'(x) < \infty$ on $[0, \infty)$, and there exists $K \in \mathbb{R}^+$, such that for $x \geq 0$, and $2 \leq i \leq 5$,
\begin{equation}
|\partial^i_x R(x)| \leq K.
\end{equation}
(iv) Define the inverse function of the marginal utility $U'(x)$ as $I : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $I(y) = U'^{-1}(y)$, and assume that, for some positive $\alpha$, $I(y)$ satisfies the polynomial growth condition:
\begin{equation}
I(y) \leq \alpha + \kappa y^{-\alpha}.
\end{equation}
Note that the risk tolerance $R(x)$ given by (2.18) is in fact the risk tolerance function $R(t, x; \lambda)$ at terminal time $T$, and that the assumption (2.19) holds for the case $i = 1$ as a consequence stated in the following lemma, although it is made for $2 \leq i \leq 5$.

**Lemma 2.5** (Källblad and Zariphopoulou [2014], Proposition 14). Assume that the risk tolerance $R(x)$ satisfies: $R(0) = 0$, $R(x)$ is strictly increasing and $R'(x) < \infty$ on $[0, \infty)$, and there exists $K \in \mathbb{R}^+$ such that
\begin{equation}
|\partial^2_x R^2(x)| \leq K,
\end{equation}
then
\begin{equation}
R'(x) \leq C \quad \text{and} \quad R(x) \leq C x,
\end{equation}
where $C = \sqrt{K/2}$.

**Lemma 2.6.** The Asymptotic Elasticity condition (2.17) is implied by the following condition:
\begin{equation}
R(x) \leq Cx.
\end{equation}

**Proof.** It follows directly from Proposition B.3 in Schachermayer [2004], which we recall here for completeness.

If $\liminf_{x \rightarrow +\infty} AP[U] = a > 0$, then $AE[U] \leq (1 - a)^+$, where $AP[U]$ is the Arrow-Pratt risk aversion given by:
\begin{equation}
AP[U] = -x \frac{U''(x)}{U'(x)}.
\end{equation}
Now this lemma follows by:
\begin{equation}
a = \liminf_{x \rightarrow +\infty} \left( \frac{x}{R(x)} \right) \geq \liminf_{x \rightarrow +\infty} \frac{x}{C x} = \frac{1}{C} > 0.
\end{equation}

**Remark 2.7.** Assumption 2.4 (ii) is a sufficient assumption, in fact, there are cases where $U(0^+)$ is not finite, but our main Theorem 3.1 still holds. For example, power utility $U(x) = \frac{x^\gamma}{\gamma}$ with $\gamma < 0$, and logarithmic utility $U(x) = \log(x)$. For the first case, the fully non-linear accuracy problem is completely solved in Fouque et al. [2016] by distortion transformation, which linearizes the problem.

By expanding $\partial^2_x R'(x)$ in (2.19) and Lemma 2.5, it is easily shown that Assumption 2.4 (iii) is equivalent to the following conditions on the risk tolerance $R(x)$: a) $R(0) = 0$, and $R(x)$ is strictly increasing on $[0, \infty)$; and b) $|R'(x) (\partial^j_x R^j(x))| \leq K$, $\forall 0 \leq j \leq 4$.

**Proposition 2.8.** The following classes of utility functions satisfy Assumption 2.4:

(i) Average of powers: $U(x) = \int_E x^y \nu(dy)$, where $\nu(dy)$ is a finite positive measure, and the support $E$ is compact, contained in $[0, 1]$ and $\nu(\{0\}) = 0$. Two special cases are:
a) Power utility $U(x) = \frac{1}{\gamma} x^\gamma$, with $\gamma \in (0, 1)$;

b) Mixture of power utilities $U(x) = c_1 \frac{x^{\gamma_1}}{\gamma_1} + c_2 \frac{x^{\gamma_2}}{\gamma_2}$, with $\gamma_1, \gamma_2 \in (0, 1)$ and $c_1, c_2 > 0$.

In both cases, $\nu(dy)$ is a counting measure of point(s) in $[0, 1)$.

(ii) $U(x)$ is given by positive inverse of the marginal utility $I(y) = U^{(-1)}(y) : \mathbb{R}^+ \to \mathbb{R}^+$,

$$I(y) = \int_0^y y^{-s} \nu(ds), \quad (2.23)$$

with $\nu$ being finite and positive on compact support ($N < +\infty$). This is Example 18 in Källblad and Zariphopoulou [2014].

The proof of Proposition 2.8 is left to Appendix A.

**Remark 2.9.** In the first class of utilities, $1 \notin E$ in general, unless further assumptions are prescribed on $\nu(dy)$. For instance if $\nu(dy) = dy$ and $E = [0, 1]$, then, $AE[U] = \lim_{x \to +\infty} \frac{\ln(x) - 1}{\ln(x)} = 1$ which does satisfy (2.17).

**Remark 2.10.** For the power utility $U(x) = \frac{x^\gamma}{\gamma}$, the Arrow-Pratt risk aversion (2.22) is constant and the risk tolerance function (2.5) is linear, given by

$$AP[U] = -x \frac{U''(x)}{U'(x)} = 1 - \frac{1}{\gamma}, \quad R(x) = \frac{x}{1 - \gamma}.$$  

Compared to above, general utilities, such as the mixture of two powers

$$U^{Mix}(x) = c_1 \frac{x^{\gamma_1}}{\gamma_1} + c_2 \frac{x^{\gamma_2}}{\gamma_2}, \quad 0 < \gamma_1 \leq \gamma_2 < 1,$$

produce nonlinear risk aversion functions:

$$AP[U^{Mix}] = \frac{c_1 (1 - \gamma_1) x^{\gamma_1 - \gamma_2} + c_2 (1 - \gamma_2)}{c_1 x^{\gamma_1 - \gamma_2} + c_2}, \quad (2.24)$$

as well as nonlinear risk tolerances,

$$R(x) = \left( \frac{c_1 x^{\gamma_1 - \gamma_2} + c_2}{c_1 (1 - \gamma_1) x^{\gamma_1 - \gamma_2} + c_2 (1 - \gamma_2)} \right) x \sim \begin{cases} \frac{1}{1 - \gamma} \frac{x^{\gamma_2}}{c_1}, & \text{as } x \to 0, \\ \frac{1 - \gamma}{1 - \gamma_1} \frac{x^{\gamma_1}}{c_2}, & \text{as } x \to \infty. \end{cases} \quad (2.25)$$

This is illustrated in Figure 1. Therefore, working with general utility enables us to model nonlinear relation between the relative risk aversion and the wealth (middle plot), and makes our model closer to results from empirical studies on how $AP[U]$ varies with wealth.

![Figure 1: Mixture of power utilities with $\gamma_1 = 0.25$, $\gamma_2 = 0.75$ and $c_1 = c_2 = 1/2$.](image-url)
2.4 Assumptions on the state processes \((X^{\pi(0)}_t, S_t, Z_t)\)

Note that \(z\) is only a parameter in the function \(v^{(0)}(t, x, z)\) given by (2.11), and for fixed \(z\) and \(t\), \(v^{(0)}\) is a concave function that has a linear upper bound. For \(t = 0\), there exists a function \(G(z)\), so that \(v^{(0)}(0, x, z) \leq G(z) + x, \forall (x, z) \in \mathbb{R}^+ \times \mathbb{R}\).

**Assumption 2.11.** We make the following assumptions on the state processes \((X^{\pi(0)}_t, S_t, Z_t)\):

(i) For any starting points \((s, z)\) and fixed \(\delta\), the system of stochastic differential equations (1.1) - (1.2) has a unique strong solution \((S_t, Z_z)\). Moreover, \(\lambda(z)\) is a \(C^1(\mathbb{R})\) function, \(g(z)\) is a \(C^2(\mathbb{R})\) function, and the coefficients \(g(z), c(z), \lambda(z)\) as well as their derivatives \(g'(z), g''(z), \lambda'(z), \lambda''(z)\) and \(\lambda'''(z)\) are at least polynomially growing.

(ii) The process \(Z^{(1)}\) with infinitesimal generator \(M\) defined in (1.3) admits moments of any order uniformly in \(t \leq T\):

\[
\sup_{t \leq T} \left\{ \mathbb{E} \left| Z^{(1)}_t \right|^k \right\} \leq C(T, k). \tag{2.26}
\]

(iii) The process \(G(Z)\) is in \(L^2([0, T] \times \Omega)\) uniformly in \(\delta\), i.e.,

\[
\mathbb{E}_{(0, z)} \left[ \int_0^T G^2(Z_s) \, ds \right] \leq C_1(T, z), \tag{2.27}
\]

where \(C_1(T, z)\) is independent of \(\delta\) and \(Z_s\) follows (1.2) with \(Z_0 = z\).

(iv) The wealth process \(X^{\pi(0)}_t\) is in \(L^2([0, T] \times \Omega)\) uniformly in \(\delta\), i.e.,

\[
\mathbb{E}_{(0, x, z)} \left[ \int_0^T \left( X^{\pi(0)}_s \right)^2 \, ds \right] \leq C_2(T, x, z), \tag{2.28}
\]

where \(C_2(T, x, z)\) is independent of \(\delta\) and \(X^{\pi(0)}_s\) follows

\[
dX^{\pi(0)}_s = \pi^{(0)}(s, X^{\pi(0)}_s, Z_s) \mu(Z_s) \, ds + \pi^{(0)}(s, X^{\pi(0)}_s, Z_s) \sigma(Z_s) \, dW_s, \quad s > 0; \tag{2.29}
\]

\[X^{\pi(0)}_0 = x.\]

**Remark 2.12.** Note that in Assumption 2.11 (i), the word “polynomially growing” is interpreted in different ways depending on the domain of \(g(z), c(z)\) and \(\lambda(z)\). If a function \(h(z) : \mathbb{R} \to \mathbb{R}\), for instance, when \(Z\) is an Ornstein–Uhlenbeck process, then polynomial growth means there exists a integer \(k\) and \(a > 0\), such that

\[|h(z)| \leq a(1 + |z|^k).\]

Otherwise if \(h(z) : \mathbb{R}^+ \to \mathbb{R}\), for example when \(Z\) is a Cox–Ingersoll-Ross process, then it means there exists a integer \(k \in \mathbb{N}\) and \(a > 0\), such that

\[|h(z)| \leq a(1 + z^k + z^{-k}).\]

In Assumption 2.11 (iii), if the diffusion process \(Z\) has exponential moments, then at-most exponential growth of \(G(z)\) ensures (2.27). An explicit example will be given in Section 5.

Now we are ready to present the following estimate, which will be used in the rest of the paper.

**Lemma 2.13.** Under Assumption 2.11 (iii) - (iv), the process \(v^{(0)}(\cdot, X^{\pi(0)}_t, Z_s)\) is in \(L^2([0, T] \times \Omega)\) uniformly in \(\delta\), i.e. \(\forall (t, x, z) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}\):

\[
\mathbb{E}_{(t, x, z)} \left[ \int_t^T \left( v^{(0)}(s, X^{\pi(0)}_s, Z_s) \right)^2 \, ds \right] \leq C_3(T, x, z), \tag{2.30}
\]

where \(v^{(0)}(t, x, z)\) is defined in Section 2.2 and satisfies equation (2.11).
Proof. It follows by the straightforward computation:

\[
\mathbb{E}_{(t,x,z)} \left[ \int_t^T \left( v^{(0)}(s, X_s^{\pi(0)}, Z_s) \right)^2 \, ds \right] \leq \mathbb{E}_{(t,x,z)} \left[ \int_t^T \left( v^{(0)}(0, X_s^{\pi(0)}, Z_s) \right)^2 \, ds \right] \\
\leq \mathbb{E}_{(t,x,z)} \left[ \int_t^T (G(Z_s) + X_s^{\pi(0)})^2 \, ds \right] \\
\leq 2 \left( \mathbb{E}_{(t,x,z)} \left[ \int_t^T G(Z_s) \, ds \right] + \mathbb{E}_{(t,x,z)} \left[ \int_t^T (X_s^{\pi(0)})^2 \, ds \right] \right) \\
\leq 2 \left( \mathbb{E}_{(0,z)} \left[ \int_0^T G(Z_s) \, ds \right] + \mathbb{E}_{(0,x,z)} \left[ \int_0^T (X_s^{\pi(0)})^2 \, ds \right] \right) \\
= 2 \left( C_1(T, z) + C_2(T, x, z) \right) = C_3(T, x, z),
\]

where we have successively used the monotonicity (decreasing property) of \( v^{(0)} \) in \( t \), the concavity of \( v^{(0)} \) in \( x \) and Assumptions 2.11 (iii)-(iv).

\[\square\]

3 First order approximation of the value function \( V^{\pi(0), \delta} \)

In this section, we assume \( \pi^{(0)} = -\frac{\lambda(z)}{\sigma(z)} \frac{v^{(0)}}{v^{(0)}_z} \) is given, and analyze the value function associated to it by perturbation methods. Assume \( \pi^{(0)} \) is admissible, and recall the dynamics of the wealth process associated to the strategy \( \pi^{(0)} \) and the slow factor \( Z_t \):

\[
dX_t^{\pi(0)} = \pi^{(0)}(t, X_t^{\pi(0)}, Z_t) \mu(Z_t) \, dt + \pi^{(0)}(t, X_t^{\pi(0)}, Z_t) \sigma(Z_t) \, dW_t, \\
dZ_t = \delta g(Z_t) \, dt + \sqrt{\delta} c(z) \, dW_t^Z.
\]

Then one defines the value function as the expected utility of terminal wealth:

\[
V^{\pi(0), \delta}(t, x, z) = \mathbb{E} \left\{ U(X_T^{\pi(0)}) | X_t^{\pi(0)} = x, Z_t = z \right\}, \quad \text{(3.1)}
\]

where \( U(\cdot) \) is a general utility function satisfying Assumption 2.4.

Our main result of this section is:

Theorem 3.1. Under assumptions 2.4 and 2.11, the residual function \( E(t, x, z) \) defined by

\[
E(t, x, z) := V^{\pi(0), \delta}(t, x, z) - v^{(0)}(t, x, z) - \sqrt{\delta} v^{(1)}(t, x, z),
\]

is of order \( \delta \). In other words, \( \forall (t, x, z) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R} \), there exists a constant \( C \), such that \( E(t, x, z) \leq C \delta \), where \( C \) may depend on \( (t, x, z) \) but not on \( \delta \).

We recall that a function \( f^\delta(t, x, z) \) is of order \( \delta^k \), denoted by \( f^\delta(t, x, z) \sim \mathcal{O}(\delta^k) \), if \( \forall (t, x, z) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R} \), there exists \( C \) such that \( |f^\delta(t, x, z)| \leq C \delta^k \), where \( C \) may depend on \( (t, x, z) \), but not on \( \delta \). Similarly, we denote \( f^\delta(t, x, z) \sim o(\delta^k) \), if \( \limsup_{\delta \to 0} |f^\delta(t, x, z)|/\delta^k = 0 \).

3.1 Estimate of risk tolerance function \( R(t, x; \lambda(z)) \) and leading order term \( v^{(0)} \)

In this subsection, we derive several properties of the risk tolerance function \( R(t, x; \lambda(z)) \), which will be needed in the proof of Theorem 3.1. Some of the proofs involve lengthy calculations which we put in the appendix.

Proposition 3.2. Let \( I : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be the inverse of marginal utility, and assume it satisfies the growth condition in Assumption 2.4 (iv). Also, define \( H : \mathbb{R} \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^+ \) by

\[
M_x(t, H(x, t, \lambda(z)); \lambda(z)) = \exp \left\{ -x - \frac{1}{2} \lambda^2(z)(T - t) \right\}, \quad \text{(3.2)}
\]

where \( M(t, x; \lambda(z)) \) is the Merton value function. Then:
(i) For each $\lambda(z)$, $H(x,t,\lambda(z))$ is the unique solution to the heat equation,

$$H_t + \frac{1}{2} \lambda^2(z) H_{xx} = 0,$$

with the terminal condition $H(x,T,\lambda(z)) = I(e^{-z})$.

(ii) Moreover, for each $t \in [0,T]$ and $\lambda(z) \in \mathbb{R}$, $H(x,t,\lambda(z))$ is strictly increasing and of full range,

$$\lim_{x \to -\infty} H(x,t,\lambda(z)) = 0 \quad \text{and} \quad \lim_{x \to \infty} H(x,t,\lambda(z)) = \infty. \tag{3.4}$$

(iii) Define the inverse function $H^{-1}(y,t,\lambda(z)) : \mathbb{R}^+ \times [0,T] \times \mathbb{R} \to \mathbb{R}$:

$$H(H^{-1}(y,t,\lambda(z)),t,\lambda(z)) = y,$$

then, for $(t,x,z) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}$, the risk tolerance function $R(t,x;\lambda(z))$ is given by

$$R(t,x;\lambda(z)) = H_x \left( H^{-1}(x,t,\lambda(z)), t, \lambda(z) \right). \tag{3.5}$$

Proof. Similar results under constant $\lambda$ with multiple assets are presented in [Källblad and Zariphopoulou, 2014, Propositions 4 and 6], and here we generalize the statement to our case.

Direct computation shows that $H(x,t,\lambda(z))$ satisfies (3.3). The existence and uniqueness of the solution to (3.3) follows from the nonnegativeness of $H(x,t,\lambda(z))$.

The function $H$ is strictly increasing in $x$ since $H_x(x,t,\lambda(z))$ solves the same heat equation (3.3) with a different terminal condition $H_x(x,T,\lambda(z)) = -e^{-x}I'(e^{-x}) > 0$ which is positive. Standard comparison principle gives the positiveness of $H_x(x,t,\lambda(z))$ for previous time $t < T$. Equation (3.4) follows by using the definition of $H(x,t,\lambda(z))$ in (3.2) and the fact that the value function $M(t,x;\lambda(z))$ satisfies the Inada conditions for all $t < T$ (see Karatzas et al. [1987] for more details).

Therefore the inverse function $H^{(-1)}(y,t,\lambda(z))$ is well defined for any $y \geq 0$, and (3.5) is implied by (3.2) and (2.5). \hfill \Box

Proposition 3.3. Suppose the risk tolerance $R(x) = -\frac{U'(x)}{U''(x)}$ is strictly increasing for all $x$ in $[0,\infty)$ (this is part of Assumption 2.4 (iii)), then, for each $t \in [0,T]$ and $\lambda(z) \in \mathbb{R}$, the risk tolerance function $R(t,x;\lambda(z))$ is strictly increasing in the wealth variable $x$.

Proof. The proof follows the idea in the case of constant $\lambda$ presented in [Källblad and Zariphopoulou, 2014, Proposition 9], which we generalize to the case of variable $\lambda(z)$ here.

Using the relation in (3.5), when $t = T$, we deduce:

$$R'(x) = \frac{H_{xx}(y,T,\lambda(z))}{H_y(y,T,\lambda(z))} \bigg|_{y = H^{(-1)}(x,T,\lambda(z))}. \tag{3.6}$$

Since $R(x)$ and $H(x,T,\lambda(z))$ are strictly increasing, we claim that $H(x,T,\lambda(z))$ is strictly convex, namely, $H_{xx}(x,T,\lambda(z)) > 0$. Standard comparison argument gives the positiveness of $H_{xx}(x,t,\lambda(z))$, for $t < T$. Therefore, $\forall t \in [0,T]$, $R_x(t,x;\lambda(z))$ satisfies

$$R_x(t,x;\lambda(z)) = \frac{H_{xx}(y,t,\lambda(z))}{H_y(y,t,\lambda(z))} \bigg|_{y = H^{(-1)}(x,t,\lambda(z))} > 0,$$

and $R(t,x;\lambda(z))$ is strictly increasing in $x$. \hfill \Box

Proposition 3.4. Under Assumption 2.4, the risk tolerance function $R(t,x,\lambda(z))$ satisfies: $\forall 0 \leq j \leq 4$, $\exists K_j > 0$, such that $\forall (t,x,z) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}$,

$$|R^j(t,x;\lambda(z)) (\partial_x^{j+1} R(t,x;\lambda(z)))| \leq K_j. \tag{3.6}$$
Or equivalently, \( \forall 1 \leq j \leq 5, \) there exists \( \bar{K}_j > 0, \) such that \( \forall (t,x,z) \in [0,T) \times \mathbb{R}^+ \times \mathbb{R}, \)

\[
|\partial^2_x R(t,x;\lambda(z))| \leq \bar{K}_j.
\]

Moreover, for \((t,x,z) \in [0,T) \times \mathbb{R}^+ \times \mathbb{R},\)

\[
R(t,x;\lambda(z)) \leq K_0 x.
\]

**Proof.** The proof of Proposition 3.4 is left to Appendix B. \( \square \)

**Proposition 3.5.** The risk tolerance function \( R(t,x;\lambda(z)) \) satisfies the relation:

\[
R_\lambda = (T-t)\lambda(z)R^2 R_{xx}.
\]

**Proof.** Differentiating (2.15) with respect to \( x \) gives:

\[
\begin{align*}
v_{xx}^{(0)} &= (T-t)\lambda' (R_x v_x^{(0)} + R_{xx} v_{xx}^{(0)}), \\
v_{xx}^{(0)} &= (T-t)\lambda' (R_{xx} v_x^{(0)} + 2R_x v_{xx}^{(0)} + R_{xxx} v_{xxx}^{(0)}).
\end{align*}
\]

The definition (2.5) of \( R(t,x;\lambda(z)) \) and equation (2.12) imply:

\[
R_x = -1 + \frac{v_{xx}^{(0)} v_{xx}^{(0)}}{v_{xx}^{(0)}}.
\]

Differentiating (2.5) with respect to \( z, \) and using the above three equations produces

\[
R_z = \frac{-v_{xx}^{(0)} v_{xx}^{(0)} + v_x^{(0)} v_{xx}^{(0)}}{2 v_{xx}^{(0)}}
\]

\[
= (T-t)\lambda' \frac{-R_x v_x^{(0)} - R_{xx} v_{xx}^{(0)}}{v_{xx}^{(0)}} + (T-t)\lambda' \frac{v_x^{(0)} v_{xx}^{(0)}}{v_{xx}^{(0)}} R_{xx} v_x^{(0)} + 2R_x v_{xx}^{(0)} + R_{xxx} v_{xxx}^{(0)}
\]

\[
= (T-t)\lambda' \left(R_z R - R + R^2 R_{xx} - 2R_x R + R(R_x + 1)\right) = (T-t)\lambda' R^2 R_{xx}.
\]

Then, the chain-rule relation \( R_z = R_\lambda \lambda'(z) \) implies the conclusion. \( \square \)

**Proposition 3.6.** Under Assumption 2.4 (iii) and Assumption 2.11, there exist functions \( d_{i,j}(z) \) and \( \tilde{d}_{i,j}(z) \) at most polynomially growing such that the following inequalities are satisfied:

\[
\begin{align*}
|v_x^{(0)}(t,x,z)| &\leq d_{01}(z)v_x^{(0)}(t,x,z), \quad |v_{xx}^{(0)}(t,x,z)| \leq \tilde{d}_{01}(z)v_x^{(0)}(t,x,z), \\
v_{xxz}^{(0)}(t,x,z) &\leq d_{12}(z)v_{xxz}^{(0)}(t,x,z), \quad |R_z(t,x;\lambda(z))| \leq \tilde{d}_{12}(z)v_{xxz}^{(0)}(t,x,z), \\
v_{xzz}^{(0)}(t,x,z) &\leq d_{13}(z)v_{xzz}^{(0)}(t,x,z), \quad |R_{zz}(t,x;\lambda(z))| \leq \tilde{d}_{13}(z)v_{xzz}^{(0)}(t,x,z).
\end{align*}
\]

**Proof.** The proof of Proposition 3.6 is given in Appendix C where we use Propositions 3.4 and 3.5. \( \square \)
3.2 Proof of Theorem 3.1

The value function $V^{\pi^{(0)},\delta}$ defined in (3.1), satisfies the following PDE:

$$
V_{t}^{\pi^{(0)},\delta} + \delta M V^{\pi^{(0)},\delta} + \frac{1}{2} \sigma^{2}(z) \left( \pi^{(0)} \right)^{2} V_{xx}^{\pi^{(0)},\delta} + \pi^{(0)} \left( \mu(z) V_{x}^{\pi^{(0)},\delta} + \sqrt{\delta} \rho(z) \sigma(z) V_{x}^{\pi^{(0)},\delta} \right) = 0,
$$

(3.9)

$$
V^{\pi^{(0)},\delta}(T, x, z) = U(x).
$$

In the regime $\delta$ small, this is a regular perturbation problem, and the natural expansion takes the form

$$
V^{\pi^{(0)},\delta} = v^{\pi^{(0)},(0)} + \sqrt{\delta} v^{\pi^{(0)},(1)} + \delta v^{\pi^{(0)},(2)} + \cdots,
$$

where $v^{\pi^{(0)},(0)}$ is the leading order term and $v^{\pi^{(0)},(1)}$ is the first order $\sqrt{\delta}$ correction of $V^{\pi^{(0)},\delta}$. Collecting terms of $\mathcal{O}(1)$ in (3.9) gives the linear parabolic equation satisfied by $v^{\pi^{(0)},(0)}$,

$$
v_{t}^{\pi^{(0)},(0)} + \frac{1}{2} \sigma^{2}(z) \left( \pi^{(0)} \right)^{2} V_{xx}^{\pi^{(0)},(0)} + \pi^{(0)} \mu(z) v_{x}^{\pi^{(0)},(0)} = 0,
$$

(3.10)

$$
v^{\pi^{(0)},(0)}(T, x, z) = U(x).
$$

Using the linear operator $\mathcal{L}_{t,x}$ defined in (2.7) and $\pi^{(0)}$ defined in (1.10), (3.10) becomes:

$$
\mathcal{L}_{t,x}(\lambda(z)) v^{\pi^{(0)},(0)} = 0, \quad v^{\pi^{(0)},(0)}(T, x, z) = U(x).
$$

By (2.8) and (12.1), we know that $v^{(0)}$ is a solution to (3.10) and the uniqueness result obtained in Proposition 2.3 implies $v^{\pi^{(0)},(0)} \equiv v^{(0)}$.

Next, collecting terms of $\mathcal{O}(\sqrt{\delta})$ yields

$$
v_{t}^{\pi^{(0)},(1)} + \frac{1}{2} \sigma^{2}(z) \left( \pi^{(0)} \right)^{2} V_{xx}^{\pi^{(0)},(1)} + \pi^{(0)} \mu(z) v_{x}^{\pi^{(0)},(1)} + \rho \pi^{(0)} g(z) \sigma(z) v_{x}^{\pi^{(0)},(0)} = 0,
$$

(3.11)

$$
v^{\pi^{(0)},(1)}(T, x, z) = 0.
$$

Replacing $v^{\pi^{(0)},(0)}$ by $v^{(0)}$ in the above equation gives the same equation (2.13) for $v^{(1)}$. As mentioned in Section 2.2, (2.13) has a unique solution $v^{(1)}$, thus $v^{\pi^{(0)},(1)} \equiv v^{(1)}$. Therefore, the heuristic expansion of $V^{\pi^{(0)},\delta}$ up to the first order is identified as:

$$
V^{\pi^{(0)},\delta} = v^{(0)} + \sqrt{\delta} v^{(1)} + \cdots.
$$

Recall the residual function $E(t, x, z)$ introduced in Theorem 3.1,

$$
E = V^{\pi^{(0)},\delta} - v^{(0)} - \sqrt{\delta} v^{(1)}.
$$

Subtracting (3.10) and (3.11) from (3.9) and using the relations $v^{\pi^{(0)},(0)} \equiv v^{(0)}$, $v^{\pi^{(0)},(1)} \equiv v^{(1)}$, one has

$$
E_{t} + \frac{1}{2} \sigma^{2}(z) \left( \pi^{(0)} \right)^{2} E_{xx} + \pi^{(0)} \mu(z) E_{x} + \delta M E + \sqrt{\delta} \rho \sigma(z) g(z) \pi^{(0)} E_{x}
$$

(3.12)

$$
+ \delta M (v^{(0)} + \sqrt{\delta} v^{(1)}) + \delta \rho \sigma(z) g(z) \pi^{(0)} v_{x}^{(1)} = 0,
$$

$$
E(T, x, z) = 0.
$$

Feynman–Kac formula gives the following probabilistic representation for $E(t, x, z)$

$$
E(t, x, z) = \delta E_{(t,x,z)} \left[ \int_{t}^{T} M v^{(0)}(s, X^{\pi^{(0)}}, Z_{s}) + \sqrt{\delta} M v^{(1)}(s, X^{\pi^{(0)}}, Z_{s}) + \rho \sigma Z_{s} g(Z_{s}) \pi^{(0)} v_{x}^{(1)}(s, X^{\pi^{(0)}}, Z_{s}) \, ds \right]
$$

$$
= \delta (I + \delta^{3/2} \Pi + \delta \rho \Pi).
$$
where $E_{(t,x,z)}[\cdot] = \mathbb{E}[\cdot|X_t = x, Z_t = z]$ and

$$I := E_{(t,x,z)} \left[ \int_t^T c(\bar{Z}(s))v_z^{(0)}(s, X_s^{\pi(0)}, Z_s) + \frac{1}{2} g^2(\bar{Z}(s))v_{zz}^{(0)}(s, X_s^{\pi(0)}, Z_s) \, ds \right], \quad (3.13)$$

$$II := E_{(t,x,z)} \left[ \int_t^T c(\bar{Z}(s))v_z^{(1)}(s, X_s^{\pi(0)}, Z_s) + \frac{1}{2} g^2(\bar{Z}(s))v_{zz}^{(1)}(s, X_s^{\pi(0)}, Z_s) \, ds \right], \quad (3.14)$$

$$III := E_{(t,x,z)} \left[ \int_t^T \lambda(\bar{Z}(s))g(\bar{Z}(s))R(s, X_s^{\pi(0)}; \lambda(\bar{Z}(s)))v_{zz}^{(1)}(s, X_s^{\pi(0)}, Z_s) \, ds \right]. \quad (3.15)$$

In order to show that $E$ is of order $\delta$, it suffices to show that I, II and III are uniformly bounded in $\delta$. The derivation of these bounds are given in the appendix D.

4 Asymptotic Optimality of $\pi^{(0)}$

For a fixed choice of $(\bar{\pi}^0, \bar{\pi}^1, \alpha > 0)$, we introduce the family of admissible trading strategies $\mathcal{A}_0(t, x, z) [\pi^0, \pi^1, \alpha]$ defined by

$$\mathcal{A}_0(t, x, z) [\pi^0, \pi^1, \alpha] = \{ \pi^0 + \delta^\alpha \pi^1 \}_{0 \leq \delta \leq 1}. \quad (4.1)$$

Further conditions will be given in Assumption 4.1.

The goal of this section is to show that the strategy $\pi^{(0)}$ defined in (1.10) asymptotically outperforms every family $\mathcal{A}_0(t, x, z) [\bar{\pi}^0, \bar{\pi}^1, \alpha]$ as precisely stated in our main Theorem 4.5 in Section 4.3.

Denote by $\bar{V}^\delta$ the value function associated to the trading strategy $\pi := \bar{\pi}^0 + \delta^\alpha \bar{\pi}^1 \in \mathcal{A}_0(t, x, z) [\bar{\pi}^0, \bar{\pi}^1, \alpha]$:

$$\bar{V}^\delta = \mathbb{E} [U(X_T^\pi) | X_t^\pi = x, Z_t = z], \quad (4.2)$$

where $X_t^\pi$ is the wealth process following the strategy $\pi$, and $Z_t$ is slowly varying with the same $\delta$:

$$dX_t^\pi = \pi(t, X_t^\pi, Z_t) \mu(Z_t) \, dt + \pi(t, X_t^\pi, Z_t) \sigma(Z_t) \, dW_t, \quad (4.3)$$

$$dZ_t = \delta \sigma(Z_t) \, dt + \sqrt{\delta} g(Z_t) \, dW_t^Z. \quad (4.4)$$

We need to compare $\bar{V}^\delta$ with $V^{\pi^{(0)}}$ defined in (3.1), for which we have established the first order approximation $v^{(0)} + \sqrt{\delta} v^{(1)}$ in Theorem 3.1. This comparison is asymptotic in $\delta$ up to order $\sqrt{\delta}$, and our first step is to obtain the corresponding approximation for $\bar{V}^\delta$. This is done heuristically in Section 4.1 in the two cases $\bar{\pi}^0 = \pi^{(0)}$ and $\bar{\pi}^0 \neq \pi^{(0)}$, and depending on the value of the parameter $\alpha$. The proof of accuracy is given in Section 4.2. Asymptotic optimality of $\pi^{(0)}$ is obtained in Section 4.3.

Assumption 4.1. For a fixed choice of $(\bar{\pi}^0, \bar{\pi}^1, \alpha > 0)$, we require:

(i) The whole family (in $\delta$) of strategies $\{ \pi^0 + \delta^\alpha \pi^1 \}$ is contained in $\mathcal{A}(t, x, z)$;

(ii) Functions $\bar{\pi}^0(t, x, z)$ and $\bar{\pi}^1(t, x, z)$ are continuous on $[0, T] \times \mathbb{R}^+ \times \mathbb{R}$;

(iii) Let $(\bar{X}^t_{s,x})_{t \leq s \leq T}$ be the solution to:

$$d\bar{X}_s = \mu(z)\bar{\pi}^0(s, \bar{X}_s, z) \, ds + \sigma(z)\bar{\pi}^0(s, \bar{X}_s, z) \, dW_s, \quad (4.5)$$

starting at $x$ at time $t$.

By (i), $\bar{X}^t_{s,x}$ is nonnegative and we further assume that it has full support $\mathbb{R}^+$ for any $t < s \leq T$.

Remark 4.2. Notice that $\pi^{(0)}$ defined in (1.10) is continuous on $[0, T] \times \mathbb{R}^+ \times \mathbb{R}$, thus it is natural to require that $\bar{\pi}^0$ and $\bar{\pi}^1$ have the same regularity as $\pi^{(0)}$, that is (ii). Regarding (iii), from Section 2, $\pi^{(0)}$ is the optimal trading strategy for the Merton problem when $\delta = 0$, in which case $Z_t$ is frozen at its initial position $z$. The associated wealth process $\bar{X}^t_{s,x}$ starting at $x$ at time $t$ is the solution to

$$d\bar{X}_s = \mu(z)\pi^{(0)}(s, \bar{X}_s, z) \, ds + \sigma(z)\pi^{(0)}(s, \bar{X}_s, z) \, dW_s, \quad \bar{X}_t = x.$$
Then, from [Källblad and Zariphopoulou, 2014, Proposition 7], one has

\[ \hat{X}^{t,x}_s = H \left( H^{-1}(x, t, \lambda(z)) + \lambda^2(z)(s-t) + \lambda(z)(W_s - W_t), s, \lambda(z) \right), \]

where \( H : \mathbb{R} \times [0, T] \times \mathbb{R} \to \mathbb{R}^+ \) is defined in Proposition 3.2 and is of full range. Consequently, \( \hat{X}^{t,x}_s \) has full support \( \mathbb{R}^+ \), and thus, it is natural to require that \( \hat{X}^{t,x}_s \) has full support \( \mathbb{R}^+ \), that is (iii).

**Remark 4.3.** We have \( A_0(t, x, z) \left[ \pi^0, \pi^1, 0 \right] \equiv A_0(t, x, z) \left[ \pi^0 + \pi^1, 0, \alpha \right], \) so that it is enough to consider \( \alpha > 0 \).

### 4.1 Heuristic Expansion of the Value Function \( \tilde{V}^\delta \)

We look for an expansion of the value function \( \tilde{V}^\delta \) defined in (4.2) of the form

\[ \tilde{V}^\delta = \tilde{v}^{(0)} + \delta \tilde{v}^{(1)} + \delta^2 \tilde{v}^{(2)} + \cdots \]

where \( n \) is the largest integer such that \( n \alpha < 1/2 \). Note that in the case \( \alpha > 1/2, n \) is simply zero. In the derivation, we are interested in identifying the zeroth order term \( \tilde{v}^{(0)} \) and the first non-zero term up to order \( \sqrt{\delta} \). The term following \( \tilde{v}^{(0)} \) will depend on the value of \( \alpha \).

Denote by \( L \) the infinitesimal generator of the state processes \( (X^t_s, Z_t) \) given by (4.3) - (4.4)

\[ L := \delta M + \frac{1}{2} \sigma^2(z) \left( \tilde{\pi}^0 + \delta \tilde{\pi}^1 \right)^2 \partial_{xx} + \left( \tilde{\pi}^0 + \delta \tilde{\pi}^1 \right) \mu(z) \partial_x + \sqrt{\delta} \rho g(z) \sigma(z) \left( \tilde{\pi}^0 + \delta \tilde{\pi}^1 \right) \partial_z, \]

then, the value function \( \tilde{V}^\delta \) defined in (4.2) satisfies

\[ \partial_t \tilde{V}^\delta + L \tilde{V}^\delta = 0, \quad \tilde{V}^\delta(T, x, z) = U(x). \]  

(4.7)

Collecting terms of order one yields the equation satisfied by \( \tilde{v}^{(0)} \)

\[ \tilde{v}_t^{(0)} + \frac{1}{2} \sigma^2(z) \left( \tilde{\pi}^0 \right)^2 \tilde{v}_{xx}^{(0)} + \mu(z) \tilde{v}_x^{(0)} = 0, \]

(4.8)

\[ \tilde{v}^{(0)}(T, x, z) = U(x). \]

The order of approximation will depend on \( \pi^0 \) being identical to \( \pi^{(0)} \) or not.

#### 4.1.1 Case \( \pi^0 \equiv \pi^{(0)} \)

In this case, from the definition (1.10) of \( \pi^{(0)} \), equation (4.8) becomes (2.9) which is also satisfied by \( v^{(0)} \) by (2.12). By Proposition 2.3, we deduce \( \tilde{v}^{(0)} \equiv v^{(0)} \). To identify the term of next order, one needs to discuss case by case:

(i) \( \alpha = 1/2 \). The next order term is \( \tilde{v}^{(1)} \) and it satisfies

\[ \tilde{v}_t^{(1)} + \frac{1}{2} \sigma^2(z) \left( \pi^{(0)} \right)^2 \tilde{v}_{xx}^{(1)} + \pi^{(0)} \mu(z) \tilde{v}_x^{(1)} + \pi^{(0)} \rho g(z) \sigma(z) \tilde{v}_x^{(0)} + \pi^1 \left( \sigma^2(z) \pi^{(0)} \tilde{v}_{xx}^{(0)} + \mu(z) \tilde{v}_x^{(0)} \right) = 0, \]

\[ \tilde{v}^{(1)}(T, x, z) = 0. \]

It reduces to equation (2.13) since we have the relations

\[ \tilde{v}^{(0)} = v^{(0)} \quad \text{and} \quad \sigma^2(z) \pi^{(0)} \tilde{v}_{xx}^{(0)} = -\mu(z) \tilde{v}_x^{(0)}, \]

(4.9)

from the definition (1.10) of \( \pi^{(0)} \). From Section 2.2 item (ii), \( v^{(1)} \) is the unique solution to (2.13) and therefore, we obtain \( \tilde{v}^{(1)} \equiv v^{(1)} \).

(ii) \( \alpha > 1/2 \). The next order is of \( O(\delta^{1/2}) \). By collecting all terms of order \( \delta^{1/2} \), we also obtain that \( \tilde{v}^{(1)} \) satisfies (2.13), and \( \tilde{v}^{(1)} \equiv v^{(1)} \).
(iii) \( \alpha < 1/2 \). The next order correction is \( O(\delta^\alpha) \). Collecting all terms of order \( \delta^\alpha \) in (4.7) yields

\[
\tilde{\nu}_t^\alpha + \frac{1}{2} \sigma^2(z) \left( \pi(0) \right)^2 \tilde{\nu}_{xx}^\alpha + \pi(0) \mu(z)\tilde{\nu}_x^\alpha + \bar{\pi}^1 \left( \sigma^2(z)\pi(0)\tilde{\nu}_{xx}^0 + \mu(z)\tilde{\nu}_x^0 \right) = 0, \quad (4.10)
\]

\( \tilde{\nu}^\alpha(T, x, z) = 0. \)

The last two terms cancel via the relation (4.9), and (4.10) becomes (2.14), which only has the trivial solution, namely \( \tilde{\nu}^\alpha \equiv 0 \). Therefore, we need to identify the next non-vanishing term.

- \( 1/4 < \alpha < 1/2 \). The next order is of \( O(\delta^{1/2}) \), and \( \tilde{\nu}^{(1)} \) satisfies

\[
\tilde{\nu}_t^{(1)} + \frac{1}{2} \sigma^2(z) \left( \pi(0) \right)^2 \tilde{\nu}_{xx}^{(1)} + \pi(0) \mu(z)\tilde{\nu}_x^{(1)} + \frac{1}{2} \sigma^2(z) \left( \bar{\pi}^1 \right)^2 \tilde{\nu}_{xx}^0 + \pi(0) \rho g(z)\sigma(z)\tilde{\nu}_x^0 = 0, \quad (4.11)
\]

\( \tilde{\nu}^{(1)}(T, x, z) = 0, \)

It coincides with (2.13) and we deduce \( \tilde{\nu}^{(1)} = v^{(1)} \).

- \( \alpha = 1/4 \). The next order is of \( O(\delta^{1/4}) \), and the PDE satisfied by \( \tilde{\nu}^{(1)} \) becomes

\[
\tilde{\nu}_t^{(1)} + \frac{1}{2} \sigma^2(z) \left( \pi(0) \right)^2 \tilde{\nu}_{xx}^{(1)} + \pi(0) \mu(z)\tilde{\nu}_x^{(1)} + \frac{1}{2} \sigma^2(z) \left( \bar{\pi}^1 \right)^2 \tilde{\nu}_{xx}^0 + \pi(0) \rho g(z)\sigma(z)\tilde{\nu}_x^0 = 0, \quad (4.12)
\]

\( \tilde{\nu}^{2\alpha}(T, x, z) = 0. \)

Feynman–Kac formula gives:

\[
\tilde{\nu}^{2\alpha}(t, x, z) = \mathbb{E} \left[ \int_t^T \frac{1}{2} \sigma^2(z) \left( \bar{\pi}^1 \right)^2 (s, \tilde{X}_s, z)\tilde{\nu}_x^0(s, \tilde{X}_s, z) \, ds \mid \tilde{X}_t = x \right], \quad (4.13)
\]

with \( \tilde{X}_s \) following (4.5). Notice that, for fixed \( z \), if the source term \( \frac{1}{2} \sigma^2(z) \left( \bar{\pi}^1 \right)^2 \tilde{\nu}_{xx}^0 \) is identically zero after some time \( t_1 \), then, \( \tilde{\nu}^{2\alpha}(t_1, x, z) \) is zero. Therefore, further analysis is needed in order to find the first non-zero term after \( \tilde{\nu}^0 \) at point \( (t, x, z) \). Note that both \( \sigma(z) \) and \( \left( -\tilde{\nu}_{xx}^0 \right) \) are strictly positive (\( \tilde{\nu}^0 = v^0 \) is strictly concave), hence, \( \bar{\pi}^1 \) is the problematic term. Accordingly, we define

\[
t_1(z) = \inf \{ t \in [0, T] : \bar{\pi}^1(u, x, z) = 0, \forall (u, x) \in [t, T] \times \mathbb{R}^+ \},
\]

where we use the convention \( \inf \emptyset = T \). Based on \( t_1(z) \), the following two regions are defined:

\[
\mathcal{K}_1 = \{ (t, x, z) : 0 \leq t < t_1(z), x \in \mathbb{R}^+, z \in \mathbb{R} \}, \quad (4.14)
\]

\[
\mathcal{C}_1 = \{ (t, x, z) : t_1(z) \leq t \leq T, x \in \mathbb{R}^+, z \in \mathbb{R} \}, \quad (4.15)
\]

which form a partition of \([0, T] \times \mathbb{R}^+ \times \mathbb{R} \).

- For any \((t, x, z) \in \mathcal{K}_1\), since \( t < t_1(z) \), there exists a point \((t', x', z) \in [t_1(z)] \times \mathbb{R}^+ \times \{z\} \) such that \( \bar{\pi}^1(t', x', z) \neq 0 \). By continuity of \( \bar{\pi}^1 \), there exist \( \eta > 0 \) and a set \( A := [t', t'+\epsilon] \times [x', x'+\epsilon] \) with \( 0 < \epsilon < t_1(z) - t' \) such that \( |\bar{\pi}^1| \geq \eta \) on \( A \times \{z\} \). By (4.13) and denoting by \( \mu_s \) the
distribution of $\widetilde{X}^{t,x}_s$, we deduce that
\[
\widetilde{v}^{2\alpha}(t, x, z) \leq \frac{1}{2} \sigma^2(z) \int_{x'}^{x'} t' + \epsilon \int_{t'}^{t'} (\pi^{(1)})^2 (s, y, z) d\mu_s(dy) \\
\leq -\frac{1}{2} \sigma^2(z) \eta^2 \int_{x'}^{x'} t' + \epsilon \int_{t'}^{t'} [\bar{v}^{(0)}(s, y, z)] d\mu_s(dy) \leq -\frac{1}{2} \sigma^2(z) \eta^2 \inf_A [-\bar{v}^{(0)}(s, y, z)] \int_{t'}^{t'} \left( \int_{x'}^{x'} \mu_s(dy) \right) ds < 0. \tag{4.16}
\]

The conclusion $\widetilde{v}^{2\alpha}(t, x, z) < 0$ follows from $\bar{v}^{(0)} \equiv v^{(0)}$, strict concavity and continuity of $v^{(0)}$, and the full-support assumption on the distribution $\mu_s$ of $\widetilde{X}^{t,x}_s$.

- For any $(t, x, z) \in C_1$, equation (4.12) becomes (2.14) (since $\bar{v}^{(1)} \equiv 0$ in $C_1$), and consequently, $\bar{v}^{2\alpha}(t, x, z) \equiv 0$. Therefore, we need to analyze the next order term. Recall that $n$ is the largest integer such that $\alpha \eta < 1/2$, and we are in the case $0 < \alpha < 1/4$.

  * If $n = 2$, collecting terms of order $\delta^{1/2}$ and using the facts that $\bar{v}^{2\alpha} \equiv 0$ in $C_1$ and $\bar{v}^{\alpha} \equiv 0$, yields (2.13) for $\bar{v}^{(1)}$, and therefore, $\bar{v}^{(1)} = v^{(1)}$.

  * For $n \geq 3$, namely, the next order is $\delta^{3\alpha}$ and $\alpha < 1/6$, then $\bar{v}^{3\alpha}$ satisfies

\[
\begin{align*}
\bar{v}^{3\alpha} + \frac{1}{2} \sigma^2(z) \left( \bar{v}^{(0)} \right)^2 \bar{v}^{3\alpha} + \bar{v}^{(0)} \mu(z) \bar{v}^{3\alpha} + \sigma^2(z) \pi^{(0)} \bar{v}^{2\alpha} + \mu(z) \bar{v}^{2\alpha} = 0, \tag{4.17}
\end{align*}
\]

Notice that in the above PDE, $z$ is simply a parameter. For fixed $z$, on the region $[t_1(z), T] \times \mathbb{R}^+$, $\bar{v}^{2\alpha}(t, x, z) \equiv 0$ and the above equation reduces to (2.14) again. Therefore, $\bar{v}^{3\alpha}(t, x, z) \equiv 0$ in the region $C_1$. Repeating this argument until $\bar{v}^{\alpha}$, we obtain

\[
\bar{v}^{(i)}(t, x, z) \equiv 0, \quad 2 \leq i \leq n, \quad \forall (t, x, z) \in C_1,
\]

and, as in the case $n = 2$, we conclude $\bar{v}^{(1)} = v^{(1)}$.

We summarize the above discussion in the following table:

<table>
<thead>
<tr>
<th>Value of $\alpha$</th>
<th>Expansion</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 1/2$</td>
<td>$v^{(0)} + \sqrt{\delta} v^{(1)}$</td>
<td></td>
</tr>
<tr>
<td>$\alpha &gt; 1/2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1/4 &lt; \alpha &lt; 1/2$</td>
<td>$v^{(0)} + \sqrt{\delta} v^{(1)}$</td>
<td>$\bar{v}^{(1)}$ satisfies equation (4.11)</td>
</tr>
<tr>
<td>$\alpha = 1/4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0 &lt; \alpha &lt; 1/4$</td>
<td>Region $C_1$: $v^{(0)} + \delta^{2\alpha} \bar{v}^{2\alpha}$</td>
<td>$\bar{v}^{2\alpha}$ satisfies equation (4.12) and (4.16)</td>
</tr>
</tbody>
</table>

### 4.1.2 Case $\bar{v}^{(0)} \neq v^{(0)}$

Recall that the leading order term $\bar{v}^{(0)}$ satisfies (4.8):

\[
\bar{v}^{(0)} + \frac{1}{2} \sigma^2(z) (\bar{v}^{(0)})^2 \bar{v}^{(0)} + \bar{v}^{(0)} \mu(z) \bar{v}^{(0)} = 0, \quad \bar{v}^{(0)}(T, x, z) = U(x).
\]

For $z \in \mathbb{R}$, we introduce

\[
t_0(z) = \inf \left\{ t \geq 0 : \bar{v}^{(0)}(u, x, z) \equiv v^{(0)}(u, x, z), \forall (u, x) \in [t, T] \times \mathbb{R}^+ \right\}, \quad \inf \{ \emptyset \} = T.
\]
Define the regions:

\[ \mathcal{K} = \{(t, x, z) : 0 \leq t < t_0(z), x \in \mathbb{R}^+, z \in \mathbb{R}\}, \]
\[ \mathcal{C} = \{(t, x, z) : t_0(z) \leq t \leq T, x \in \mathbb{R}^+, z \in \mathbb{R}\}. \]  

We claim that in the region \( \mathcal{K} \), \( \bar{\pi}^0 \) and \( \pi^{(0)} \) differ, while in the region \( \mathcal{C} \), \( \bar{\pi}^{(0)} \equiv v^{(0)} \) and we need to identify the next non-vanishing term.

In order to compare \( v^{(0)} \) and \( \bar{v}^{(0)} \), we rewrite the equation (2.11) satisfied by \( v^{(0)} \) as:

\[ v^{(0)}_t + \frac{1}{2} \sigma^2(z) \left( \bar{\pi}^0 \right)^2 v^{(0)}_{xx} + \bar{\pi}^0 \mu(z) v^{(0)}_x - \frac{1}{2} \sigma^2(z) \left( \bar{\pi}^0 - \pi^{(0)} \right)^2 v^{(0)}_{xx} = 0, \]

where we have used the relation \(-\sigma^2(z)\pi^{(0)} v^{(0)}_{xx} = \mu(z) v^{(0)}_x \).

Now, let \( f(t, x, z) \) be the difference of the two leading order terms:

\[ f(t, x, z) = v^{(0)}(t, x, z) - \bar{v}^{(0)}(t, x, z). \]

It satisfies

\[ f_t + \frac{1}{2} \sigma^2(z) \left( \bar{\pi}^0 \right)^2 f_{xx} + \bar{\pi}^0 \mu(z) f_x - \frac{1}{2} \sigma^2(z) \left( \bar{\pi}^0 - \pi^{(0)} \right)^2 f_{xx} = 0, \]
\[ f(T, x, z) = 0. \]

By the Feynman-Kac formula, one has:

\[ f(t, x, z) = -\mathbb{E} \left[ \int_t^T \frac{1}{2} \sigma^2(z) \left( \bar{\pi}^0 - \pi^{(0)} \right)^2 (s, \bar{X}_s, z) v^{(0)}_{xx}(s, \bar{X}_s, z) \, ds \right| \bar{X}_t = x \],

where \( \bar{X}_s \) follows (4.5). Using the argument given in Section 4.1.1 for the case \( 0 < \alpha < 1/4 \), we deduce that the right-hand side in (4.20) is strictly positive. Consequently \( f(t, x, z) > 0 \), and

\[ \bar{v}^{(0)}(t, x, z) < v^{(0)}(t, x, z), \quad \forall (t, x, z) \in \mathcal{K}. \]

Thus, in that case, the next term will not play a role when comparing \( \bar{V}^{\delta} \) and \( V^{\pi^{(0)}}, \delta = v^{(0)} + \sqrt{\delta} \bar{v}^{(1)} + O(\delta) \).

For any \( (t, x, z) \in \mathcal{C} \), since we have \( \bar{\pi}^0 \equiv \pi^{(0)} \) on \( \mathcal{C} \), we can apply here the whole discussion in Section 4.1.1 (on the partition \( \{\mathcal{C} \cap \mathcal{K}_1, \mathcal{C} \cap \mathcal{C}_1\} \) in the case \( 0 < \alpha < 1/4 \)). The expansion results are summarized in the table:

<table>
<thead>
<tr>
<th>Region</th>
<th>Value of ( \alpha )</th>
<th>Expansion</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{K} )</td>
<td>all</td>
<td>( \bar{v}^{(0)} )</td>
<td>( \bar{v}^{(0)} ) satisfies (4.8) and (4.21)</td>
</tr>
<tr>
<td>( \mathcal{C} )</td>
<td>( \alpha = 1/2 )</td>
<td>( v^{(0)} + \sqrt{\delta} \bar{v}^{(1)} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \alpha &gt; 1/2 )</td>
<td>( v^{(0)} + \sqrt{\delta} \bar{v}^{(1)} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( 1/4 &lt; \alpha &lt; 1/2 )</td>
<td>( v^{(0)} + \sqrt{\delta} \bar{v}^{(1)} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \alpha = 1/4 )</td>
<td>( v^{(0)} + \sqrt{\delta} \bar{v}^{(1)} )</td>
<td>( \bar{v}^{(1)} ) satisfies equation (4.11)</td>
</tr>
<tr>
<td>( \mathcal{C} \cap \mathcal{K}_1 )</td>
<td>( 0 &lt; \alpha &lt; 1/4 )</td>
<td>( v^{(0)} + \sqrt{\delta} \bar{v}^{(1)} )</td>
<td>( \bar{v}^{2\alpha} ) satisfies equation (4.12) and (4.16)</td>
</tr>
</tbody>
</table>

### 4.2 Accuracy of Approximations

In order to make rigorous the above expansions, we need additional assumptions listed in Appendix E. They are technical integrability conditions, uniformly in \( \delta \), on the strategies in the class \( \mathcal{A}_0(t, x, z) [\bar{\pi}^0, \bar{\pi}^1, \alpha] \) defined in (4.1) and their associated wealth processes.
Proposition 4.4. Under Assumption 2.4 (i)-(ii), 4.1 and E.1, we obtain the following accuracy results:

Table 3: Accuracy of approximations of \( \hat{V}^\delta \).

<table>
<thead>
<tr>
<th>Case</th>
<th>Region</th>
<th>Value of ( \alpha )</th>
<th>Approximation</th>
<th>Accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi^0 = \pi^{(0)} )</td>
<td>all</td>
<td>( \alpha = 1/2 )</td>
<td>( v(0) + \sqrt{\delta} v(1) )</td>
<td>( O(\delta) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \alpha &gt; 1/2 )</td>
<td>( v(0) + \sqrt{\delta} v(1) )</td>
<td>( O(\delta) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 1/4 &lt; \alpha &lt; 1/2 )</td>
<td>( v(0) + \sqrt{\delta} v(1) )</td>
<td>( O(\delta) )</td>
</tr>
<tr>
<td></td>
<td>( \mathcal{K}_1 )</td>
<td>( \alpha = 1/4 )</td>
<td>( v(0) + \sqrt{\delta} v(1) )</td>
<td>( O(\delta) )</td>
</tr>
<tr>
<td>( \pi^0 \neq \pi^{(0)} )</td>
<td>( \mathcal{C} )</td>
<td>( \alpha = 1/2 )</td>
<td>( v(0) + \sqrt{\delta} v(1) )</td>
<td>( O(\delta) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \alpha &gt; 1/2 )</td>
<td>( v(0) + \sqrt{\delta} v(1) )</td>
<td>( O(\delta) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 1/4 &lt; \alpha &lt; 1/2 )</td>
<td>( v(0) + \sqrt{\delta} v(1) )</td>
<td>( O(\delta) )</td>
</tr>
<tr>
<td></td>
<td>( \mathcal{C} \cap \mathcal{K}_1 )</td>
<td>( \alpha = 1/4 )</td>
<td>( v(0) + \sqrt{\delta} v(1) )</td>
<td>( O(\delta) )</td>
</tr>
<tr>
<td></td>
<td>( \mathcal{C} \cap \mathcal{C}_1 )</td>
<td>( 0 &lt; \alpha &lt; 1/4 )</td>
<td>( v(0) + \sqrt{\delta} v(1) )</td>
<td>( O(\delta) )</td>
</tr>
</tbody>
</table>

where we the meaning of \( \mathcal{O} \) is as in Theorem 3.1.

Proof. Recall that \( \hat{V}^\delta \) satisfies

\[
\hat{V}_t^\delta + \delta \mathcal{M} \hat{V}^\delta + \frac{1}{2} \sigma^2(z) (\pi^0 + \delta \pi^1)^2 \hat{V}_{xx}^\delta + \delta \mu(z) \hat{V}_z^\delta + \sqrt{\delta} \rho g(z) \sigma(z) (\pi^0 + \delta \pi^1) \hat{V}_{xz}^\delta,
\]

\[
\hat{V}^\delta(T, x, z) = U(x).
\]

The proofs of accuracy for the approximations given in Tables 1 and 2 are quite standard, and we sketch them following the order of Table 3. In each case, \( E \) denotes the difference between \( \hat{V}^\delta \) and its approximation.

We start with the case \( \pi^0 = \pi^{(0)} \).

(i) \( \alpha = 1/2 \). Subtracting equation (2.11) and (2.13) from (4.7), we obtain the PDE satisfied by \( E(t, x, z) \):

\[
E_t + \delta \mathcal{M} E + \frac{1}{2} \sigma^2(z) (\pi^0 + \delta \pi^1)^2 E_{xx} + \delta \mu(z) \pi^1 E_z + \sqrt{\delta} \rho g(z) \sigma(z) (\pi^0 + \delta \pi^1) E_{xz} = 0,
\]

\( E(T, x, z) = 0 \).

Then, Feynman–Kac formula produces

\[
E(t, x, z) = \delta \mathbb{E}_{(t, x, z)} \int_t^T \left[ \mathcal{M} v(0)(s, X^\pi_s, Z_s) + \frac{1}{2} \sigma^2(Z_s) (\pi^0 + \delta \pi^1)^2 v_{xx}(s, X^\pi_s, Z_s) \right] ds
\]

\[
+ \delta^3/2 \mathbb{E}_{(t, x, z)} \int_t^T \left[ \mathcal{M} v(0)(s, X^\pi_s, Z_s) + \frac{1}{2} \sigma^2(Z_s) (\pi^0 + \delta \pi^1)^2 v_{xx}(s, X^\pi_s, Z_s) \right] ds
\]

\[
+ \delta \mathbb{E}_{(t, x, z)} \int_t^T \left[ \sigma^2(Z_s) v_{xx}(s, X^\pi_s, Z_s) + \mu(Z_s) \pi^1 v_{x}(s, X^\pi_s, Z_s) \right] ds
\]

\[
+ \delta \rho \mathbb{E}_{(t, x, z)} \int_t^T \left[ g(Z_s) \sigma(Z_s) v_{xx}(s, X^\pi_s, Z_s) + g(Z_s) \sigma(Z_s) \pi^1 v_{x}(s, X^\pi_s, Z_s) \right] ds
\]

Under Assumption E.1 (ia), one has \( E = \mathcal{O}(\delta) \).
(ii) \( \alpha > 1/2 \). Similarly, we have

\[
E_t + LE + \delta M(\nu^{(0)} + \sqrt{\delta}v^{(1)}) + \frac{\delta^{2\alpha}}{2} \sigma(z)^2 \left( \tilde{\pi}^1 \right)^2 \left( v^{(0)}_{xx} + \sqrt{\delta}v^{(1)}_{xx} \right) + \delta^{1/2 + \alpha} \sigma^2(z)\pi^{(0)}\frac{\pi^1 v^{(1)}_{xx}}{\pi^1} \\
+ \delta^{1/2 + \alpha} \mu(z)\pi^1 v^{(1)}_x + \delta \rho g(z)\sigma(z) \left( \pi^{(0)} v^{(1)}_x + \delta^{\alpha-1/2} \pi^1 v^{(0)} + \delta^{\alpha} \pi^1 v^{(1)} \right) = 0,
\]

\[E(T, x, z) = 0.\]

By Feynman–Kac formula and Assumption E.1 (ia), we deduce \( E = O(\delta) \).

(iii) \( 1/4 < \alpha < 1/2 \). We have

\[
E_t + LE + \delta M(\nu^{(0)} + \sqrt{\delta}v^{(1)}) + \frac{\delta^{2\alpha}}{2} \sigma(z)^2 \left( \tilde{\pi}^1 \right)^2 \left( v^{(0)}_{xx} + \sqrt{\delta}v^{(1)}_{xx} \right) + \delta^{1/2 + \alpha} \sigma^2(z)\pi^{(0)}\frac{\pi^1 v^{(1)}_{xx}}{\pi^1} \\
+ \delta^{1/2 + \alpha} \mu(z)\pi^1 v^{(1)}_x + \delta^{1/2 + \alpha} \rho g(z)\sigma(z) \left( \delta^{1/2 - \alpha} \pi^{(0)} v^{(1)}_x + \pi^1 v^{(0)} \right) = 0, \\
E(T, x, z) = 0,
\]

and by Assumption E.1 (ia), we have \( E = O(\delta^{2\alpha}) \).

(iv) \( \alpha = 1/4 \). Subtracting equation (2.11) and (4.11) from (4.7) yield

\[
E_t + LE + \delta M(\nu^{(0)} + \sqrt{\delta}v^{(1)}) + \frac{\delta^{1/4}}{2} \sigma(z)^2 \left( 2\pi^{(0)}\frac{\pi^1}{\pi^1} + \delta^{1/4}(\pi^{(1)})^2 \right) \tilde{\pi}^{(1)}_{xx} + \delta^{1/4} \mu(z)\pi^1 \tilde{\pi}^{(1)}_{xx} \\
+ \delta^{1/4} \rho g(z)\sigma(z) \left( \pi^{(0)} v^{(0)} + \sqrt{\delta} \pi^1 v^{(1)} + \delta^{1/4} \pi^{(0)} \tilde{\pi}^{(1)}_{xx} \right) = 0, \\
E(T, x, z) = 0,
\]

and Assumption E.1 (ic) implies \( E = O(\delta^{3/4}) \).

(v) \( 0 < \alpha < 1/4 \). In the region \( K_1 \), subtracting (2.11) and (4.12) from (4.7) produces

\[
E_t + LE + \delta M(\nu^{(0)} + \sqrt{\delta}v^{(1)}) + \frac{\delta^{3\alpha}}{2} \sigma(z)^2 \left( 2\pi^{(0)}\pi^1 + \delta^{\alpha}(\pi^1)^2 \right) \tilde{\pi}^{2\alpha}_{xx} + \delta^{3\alpha} \mu(z)\pi^1 \tilde{\pi}^{2\alpha}_{xx} \\
+ \sqrt{\delta} \rho g(z)\sigma(z) \left( \pi^{(0)} v^{(0)} + \delta^{\alpha} \pi^1 \right) \left( \nu^{(0)}_{xx} + \delta^{2\alpha} \tilde{\pi}^{2\alpha}_{xx} \right) = 0, \\
E(T, x, z) = 0,
\]

and by Assumption E.1 (ib), one concludes that \( E = O(\delta^{3\alpha \wedge (1/2)}) \).

In the complementary region \( C_1 \), \( E \) satisfies:

\[
E_t + LE + \delta M(\nu^{(0)} + \sqrt{\delta}v^{(1)}) + \frac{\delta^{2\alpha}}{2} \sigma(z)^2 \left( \tilde{\pi}^1 \right)^2 \left( v^{(0)}_{xx} + \sqrt{\delta}v^{(1)}_{xx} \right) + \delta^{1/2 + \alpha} \sigma^2(z)\pi^{(0)}\frac{\pi^1 v^{(1)}_{xx}}{\pi^1} \\
+ \delta^{1/2 + \alpha} \mu(z)\pi^1 v^{(1)}_x + \sqrt{\delta} \rho g(z)\sigma(z) \left( \sqrt{\delta} \pi^{(0)} v^{(1)}_x + \delta^{\alpha} \pi^1 v^{(0)} + \delta^{\alpha+1/2} \pi^1 v^{(1)} \right) = 0, \\
E(T, x, z) = 0.
\]

Note that in the region \( C_1 \), \( \tilde{\pi}^1 = 0 \), and the above equation reduces to:

\[
E_t + LE + \delta M(\nu^{(0)} + \sqrt{\delta}v^{(1)}) + \delta \rho g(z)\sigma(z) \pi^{(0)} v^{(1)}_x = 0, \\
E(T, x, z) = 0,
\]

and then, Assumption E.1 (ib) implies \( E = O(\delta) \).

Now, we turn to the case \( \tilde{\nu}^0 \neq \pi^{(0)} \). In the region \( K \), we know by (4.21) that \( \tilde{\nu}^0 < \nu^{(0)} \). Therefore, \( V^4 - \nu^{(0)} \) is asymptotically of order one and negative. Thus, the next term will not play a role and we
define $E = \bar{V} - \bar{v}$. Subtracting equation (4.8) from (4.7) gives

$$
E_t + \mathcal{L}E + \delta M \bar{v} + \sqrt{\delta} \left( \pi^0 + \delta \pi^1 \right) \rho g(z) \sigma(z) \bar{v}_{xx} + \frac{1}{2} \sigma^2(z) \left( \delta \pi^1 \right)^2 \bar{v}_{xx} + \sigma^2(z) \bar{v}^2 \delta \pi^1 \bar{v}_{xx} + \delta \pi^1 \mu(z) \bar{v} = 0,
$$

$$
E(T, x, z) = 0.
$$

By Assumption E.1 (ii), we conclude that $E = O(\delta^{\alpha(1/2)})$.

Remark that $\pi^0 = \pi^1$ in the region $C$. Therefore, the whole analysis of case $\pi^0 = \pi^1$ can be applied here, except that the case $0 < \alpha < 1/4$, where the accuracy results hold in the partition $\{ C \cap K_1, C \cap C_1 \}$ of $C$. This complete the proof. \(\square\)

### 4.3 Asymptotic Optimality

Our main result in this section is the following:

**Theorem 4.5.** For fixed $(t, x, z)$ and any family of trading strategies $A_0(t, x, z) \left[ \pi^0, \pi^1, \alpha \right]$, then,

$$
\lim_{\delta \to 0} \frac{\bar{V}(t, x, z) - V(\pi^0, \pi^1, \alpha)}{\sqrt{\delta}} \leq 0. \tag{4.22}
$$

That is, the strategy $\pi^0$ which generates $V(\pi^0, \pi^1, \alpha)$, performs asymptotically better up to order $\sqrt{\delta}$ than the family $\{ \pi^0 + \delta \pi^1 \}$ which generates $\bar{V}$.

Additionally, if $\pi^0 \neq \pi^1$, and in the region $K$ defined by (4.18), the strategy $\pi^0$ performs asymptotically better at order one:

$$
\lim_{\delta \to 0} \bar{V}(t, x, z) = \psi(t, x, z) \leq V(\pi^0, \pi^1, \alpha) = \lim_{\delta \to 0} V(\pi^0, \pi^1, \alpha). \tag{4.23}
$$

**Proof.** To compare the asymptotic performance of $\pi^0$ with the family of trading strategies $A_0(t, x, z) \left[ \pi^0, \pi^1, \alpha \right]$, we are essentially comparing the approximations of $\bar{V}$ summarized in Table 3 with the first order approximation $v(0) + \sqrt{\delta} v(1)$ obtained in Theorem 3.1. In each case in Table 3 where the approximation of $\bar{V}$ is $v(0) + \sqrt{\delta} v(1)$, it is easy to check that (4.22) is satisfied and the limit is zero. The remaining five cases are: (a) $\pi^0 = \pi^1$ and $\alpha = 1/4$; (a') $\pi^0 = \pi^1$, in $C_1$, and $\alpha = 1/4$; (b) $\pi^0 = \pi^1$ and $0 < \alpha < 1/4$ in the region $K_1$; (b') $\pi^0 = \pi^1$, in $C_1 \cap K_1$, and $0 < \alpha < 1/4$; and (c) $\pi^0 \neq \pi^1$ in the region $K$.

(a) In the case $\pi^0 = \pi^1$ and $\alpha = 1/4$, the approximation of $\bar{V}$ up to order $\sqrt{\delta}$ is $v(0) + \sqrt{\delta} v(1)$, and it suffices to show that $\bar{V}(1) \leq v(1)$ for all $(t, x, z) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}$. Let $f(t, x, z)$ be the difference

$$
f(t, x, z) = v(1)(t, x, z) - \bar{v}(1)(t, x, z).
$$

Subtracting (4.11) from (2.13) produces

$$
f_t + \frac{1}{2} \sigma^2(z) \left( \pi^0 \right)^2 f_{xx} + \pi^0 \mu(z) f_x - \frac{1}{2} \sigma^2(z) \left( \pi^1 \right)^2 v_{xx} = 0,
$$

$$
f(T, x, z) = 0,
$$

and the representation

$$
f(t, x, z) = -\mathbb{E} \left[ \int_t^T \frac{1}{2} \sigma^2(z) \left( \bar{v}^1 \right)^2 (s, \bar{X}_s, z) v_{xx}(s, \bar{X}_s, z) d\bar{X}_s \bigg| \bar{X}_t = x \right],
$$

where $\bar{X}_t$ follows (4.5). The concavity of $v(0)$ implies $f(t, x, z) \geq 0$ and therefore, (4.22) holds.
(b) In the case $\tilde{\pi}^0 \equiv \pi^{(0)}$ and $0 < \alpha < 1/4$, the approximation of $\tilde{V}^\delta$ is $v^{(0)} + \delta^{2\alpha} \tilde{v}^{2\alpha} + o(\delta^{3\alpha \wedge 1/2})$, where $\tilde{v}^{2\alpha}$ is strictly negative by (4.16). Consequently,

$$\lim_{\delta \to 0} \frac{\tilde{V}^\delta(t,x,z) - V^{\pi(0)}(t,x,z)}{\sqrt{\delta}} = \lim_{\delta \to 0} \frac{\delta^{2\alpha} \tilde{v}^{2\alpha} - \sqrt{\delta} v^{(1)} + o(\delta^{3\alpha \wedge 1/2})}{\sqrt{\delta}} = -\infty,$$

and (4.22) holds.

(c) In the case $\tilde{\pi}^0 \not\equiv \pi^{(0)}$ and $(t,x,z) \in \mathcal{K}$, the approximation of $\tilde{V}^\delta$ is $\tilde{v}^{(0)} + o(1)$, and (4.21) shows that $\tilde{v}^{(0)}$ is strictly less than $v^{(0)}$. Thus, we deduce (4.23).

The proof for the case (a') (resp. (b')) is essentially the same as in (a) (resp. (b)) but in the region $\mathcal{C}$ (resp. $\mathcal{C} \cap \mathcal{K}_1$).

5 A Fully-Solvable Example

In this section, we consider a model studied in Chacko and Viceira [2005] where explicit solutions are derived for the consumption problem over infinite horizon, and in Fouque et al. [2016] where expansions for the terminal wealth problem are derived and accuracy of approximation is proved under power utility with one factor. Our goal is to show that this model satisfies the various assumptions we have made in this paper and, therefore, justify that they are reasonable. The underlying asset $S_t$ and the slowly varying factor $Z_t$ are modeled by:

$$dS_t = \mu S_t dt + \sqrt{\frac{1}{Z_t}} S_t dW_t, \quad (5.1)$$

$$dZ_t = \delta(m - Z_t) dt + \sqrt{\delta} \beta \sqrt{Z_t} dW_t, \quad (5.2)$$

with $\beta > 0$ and $\mu > 0$. The standard Feller condition $\beta^2 \leq 2m$ is assumed to ensure that $Z_t$ stays positive.

In this example, we consider power utilities:

$$U(x) = x^\gamma, \quad 0 < \gamma < 1,$$

for which Assumption 2.4 is satisfied by Proposition 2.8.

This model fits in the class of models (1.1)-(1.2) by identifying the coefficients $\mu(z), \sigma(z), c(z)$ and $g(z)$ as follows:

$$\mu(z) = \mu, \quad \sigma(z) = \sqrt{1/z}, \quad c(z) = m - z, \quad g(z) = \beta \sqrt{z}.$$

For Assumption 2.11 (i)-(ii) of (with state space $(0, \infty)$), we notice that $(Z_t)$ is the unique strong solution to (5.2) and it has finite moments of any order uniformly in $\delta \leq 1$ and $t \leq T$, see for instance [Fouque et al., 2011, Chapter 3].

The process $(S_t)$ is given by:

$$S_t = S_0 \exp \left( \int_0^t \left( \mu - \frac{1}{2Z_s} \right) ds + \int_0^t \sqrt{\frac{1}{Z_s}} dW_s \right).$$

For Assumption 2.11 (iii)-(iv), we first solve (2.11) to obtain $v^{(0)}$ and $\pi^{(0)}$:

$$v_t^{(0)} - \frac{1}{2} \mu^2 z \left( \frac{v_x^{(0)}}{v_{xx}^{(0)}} \right)^2 = 0, \quad v^{(0)}(T,x,z) = \frac{x^\gamma}{\gamma}.$$

The ansatz $v^{(0)}(t,x,z) = \frac{x^\gamma}{\gamma} f(t,z)$ gives the following ODE for $f(t,z)$:

$$f_t(t,z) + \frac{1}{2} \frac{\mu^2 \gamma}{1 - \gamma} f(t,z) = 0, \quad f(T,z) = 1.$$
which admits the explicit solution
\[ f(t, z) = e^{\frac{\mu^2}{2\gamma} z (T - t)}. \]

By Proposition 2.3, we deduce the unique solution
\[ v^{(0)}(t, x, z) = \frac{x\gamma}{\gamma} e^{\frac{\mu^2}{2\gamma} z (T - t)}. \]

Consequently, the zeroth order strategy \( \pi^{(0)} \) and the risk tolerance function \( R(t, x; \lambda(z)) \) are given by
\[
\pi^{(0)}(t, x, z) = \frac{\mu x z}{1 - \gamma}, \quad \text{and} \quad R(t, x; \lambda(z)) = \frac{x}{1 - \gamma}. \tag{5.3}
\]

Note that in this case, the relations on the derivatives of \( v^{(0)} \) in Proposition 3.6 can be verified by direct computation. The verification of Assumption 2.11 (iii)-(iv) will be presented in the next two sections.

5.1 Integrability of the Process \( G(Z) \)

As in Andersen and Piterbarg [2007], one can compute the left-hand sides of (2.27) and (2.28) by solving Riccati equations. For convenience and notations, we recall the classical result:

**Lemma 5.1.** Let \( y(\tau) \) be the solution to the constant coefficient Riccati equation:
\[
y'(\tau) = q_0 + q_1 y(\tau) + q_2 y(\tau)^2, \quad y(0) = 0.
\]

Then \( y(\tau) \) is given by one of the following forms, depending on the sign of \( \Delta = q_1^2 - 4q_0q_2 \):

(i) \( \Delta > 0 \)
\[
y(\tau) = \frac{1}{\alpha} - e^{-\alpha \tau}, \quad \alpha = \sqrt{\Delta}, \quad \alpha_+ = -\frac{q_1 + \alpha}{2q_2}, \quad \alpha_- = -\frac{q_1 - \alpha}{2q_2}. \tag{5.4}
\]

(ii) \( \Delta < 0 \)
\[
y(\tau) = \frac{q_0 \sin(b\tau)}{b \cos(b\tau) - a \sin(b\tau)}, \quad a = q_1/2, \quad b = \sqrt{-\Delta}/2. \tag{5.5}
\]

(iii) \( \Delta = 0 \)
\[
y(\tau) = \frac{a^2 \tau}{q_2 (1 - \alpha \tau)}. \tag{5.6}
\]

As mentioned in Section 2.4, \( v^{(0)}(0, x, z) \) is a concave function, and it has a linear upper bound \( G(z) + x \). To obtain \( G(z) \), we derive: \( \forall x_0 \in \mathbb{R}^+ \),
\[
v^{(0)}(0, x, z) \leq v^{(0)}(0, x_0, z) + \frac{\partial}{\partial x} v^{(0)}(0, x_0, z)(x - x_0) = \frac{x_0^\gamma}{\gamma} e^{\frac{\mu^2 z}{2\gamma} T} - x_0^{-1} e^{\frac{\mu^2 z}{2\gamma} T} x_0 + x_0^{-1} e^{\frac{\mu^2 z}{2\gamma} T} x = \left( \frac{1}{\gamma} - 1 \right) x_0^\gamma e^{\frac{\mu^2 z}{2\gamma} T} + x_0^{-1} e^{\frac{\mu^2 z}{2\gamma} T} x.
\]
Let \( x_0 = e^{\frac{\mu^2 \gamma T}{1 - \gamma}} T \) so that the coefficient in front of \( x \) is 1, and \( G(z) \) can be chosen as:

\[
G(z) = \left( \frac{1}{\gamma} - 1 \right) x_0^\gamma e^{\frac{\mu^2 \gamma T}{1 - \gamma}} T = \left( \frac{1}{\gamma} - 1 \right) e^{\frac{\mu^2 \gamma T}{1 - \gamma}} T.
\]

Then, we shall show that

\[
G(Z_t) = \left( \frac{1}{\gamma} - 1 \right) e^{\frac{\mu^2 \gamma T}{1 - \gamma}} Z_t \in L^2([0, T] \times \Omega),
\]

uniformly in \( \delta \). We have

\[
\mathbb{E}_{(0, z)} \left[ \int_0^T G^2(Z_s) \, ds \right] = \left( \frac{1}{\gamma} - 1 \right)^2 \int_0^T f^\delta(0, z; s) \, ds,
\]

where

\[
f^\delta(t, z; s) = \mathbb{E} \left[ \exp \left( \frac{\mu^2 \gamma T}{1 - \gamma} Z_s \right) \bigg| Z_t = z \right],
\]

solves

\[
f_t^\delta + \frac{\delta}{2} \beta^2 z f_z^\delta z + \delta(m - \beta) f_z^\delta = 0, \quad t \in [0, s),
\]

\[
f^\delta(s, z; s) = e^{wz}, \quad \text{with} \quad w = \frac{\mu^2 \gamma T}{(1 - \gamma)^2}.
\]

This equation admits the solution

\[
f^\delta(t, z; s) = e^{wz + A^\delta(s-t)z + B^\delta(s-t)},
\]

where \( A^\delta(\tau) \) satisfies the Riccati equation:

\[
A^\delta(\tau)' = \frac{\delta}{2} \beta^2 A^\delta(\tau)^2 + (\delta \beta^2 w - \delta) A^\delta(\tau) + \left( \frac{\delta}{2} \beta^2 w^2 - \delta w \right), \quad \tau \in (0, s],
\]

\[
A^\delta(0) = 0,
\]

and \( B^\delta(\tau) \) solves

\[
B^\delta(\tau)' = \delta m(w + A^\delta(\tau)), \quad B^\delta(0) = 0.
\]

In this case, the discriminant \( \Delta = \delta^2 \) is positive, and \( A^\delta(\tau) \) follows case (i) in Lemma 5.1:

\[
A^\delta(\tau) = \frac{-w(1 - e^{-\delta \tau})}{1 - \frac{w - \frac{\beta}{2}}{w - \frac{\beta}{2}}} \exp(-\delta \tau), \quad \tau \in [0, \tau^*(\delta)),
\]

where \([0, \tau^*(\delta)]\) is the domain where \( A^\delta(\tau) \) stays finite. It remains to show that \( A^\delta(\tau) \) and \( B^\delta(\tau) \) are uniformly bounded in \([\delta, \tau) \in [0, \overline{\delta}] \times [0, T] \) for some \( \overline{\delta} \leq 1 \). Note that the boundedness of \( B^\delta(\tau) \) is a consequence of that of \( A^\delta(\tau) \) via equation (5.13). Since \( A^\delta(\tau) \) is continuous on \([0, 1] \times [0, \tau^*(\delta)) \), it suffices to show that i) there exists \( \overline{\delta} \), such that \( \tau^*(\delta) > T \) for \( \delta \leq \overline{\delta} \), and ii) \( \lim_{\delta \to 0} A^\delta(\tau) \) exists. To this end, we examine the following cases:

(a) \( w < \frac{2}{2 \beta} \). The denominator of (5.14) stays above 1, \( \tau^*(\delta) = \infty \), and \( \lim_{\delta \to 0} A^\delta(\tau) = 0 \).

(b) \( w > \frac{2}{\beta} \). Here \( \tau^*(\delta) = -\frac{1}{\beta} \ln \left( \frac{w - \frac{2}{\beta}}{w \beta} \right) \), \( \lim_{\delta \to 0} \tau^*(\delta) = \infty \), and \( \lim_{\delta \to 0} A^\delta(\tau) = 0 \).

(c) \( w = \frac{2}{\beta} \). This case gives the trivial solution \( A^\delta(\tau) \equiv 0 \).

In all cases, \( A^\delta(\tau) \) is uniformly bounded in \([0, \overline{\delta}] \times [0, T] \). Denoting by \( C(T) \) the uniform bound

\[
|A^\delta(\tau)| \leq C(T), \quad \forall (\delta, \tau) \in [0, \overline{\delta}] \times [0, T],
\]

then, following (5.13), we obtain a uniform bound for \( B^\delta(\tau) \):

\[
B^\delta(\tau) \leq \delta m(w + C(T))T \leq m(w + C(T))T.
\]

Therefore, combined with (5.8) and (5.10), we deduce that Assumption 2.11 (iii) is satisfied.
5.2 Moments of the Wealth Process $X_t^{\pi(0)}$

First, using the explicit formula (5.3) for $\pi(0)$, equation (2.29) becomes

$$dX_t^{\pi(0)} = \frac{\mu^2 Z}{1 - \gamma} X_t^{\pi(0)} ds + \frac{\mu \sqrt{Z}}{1 - \gamma} X_t^{\pi(0)} dW_s, \quad s \geq t.$$  \hspace{1cm} (5.15)

In order to control $\mathbb{E}_{(0, x, z)} \left[ \int_0^T X_s^2 ds \right]$, we introduce

$$f^\delta(t, x, z; s) = \mathbb{E} \left[ (X_s^{\pi(0)})^2 \left| X_t = x, Z_t = z \right. \right],$$

which solves

$$f^\delta_t + \frac{\mu^2 z}{1 - \gamma} x f^\delta_x + \frac{\mu^2 z}{2 (1 - \gamma)^2} x^2 f^\delta_{xx} + \delta (m - z) f^\delta_x + \frac{\delta}{2} \beta^2 x f^\delta_z + \frac{\sqrt{\delta \mu \beta}}{1 - \gamma} z x f^\delta_{xz} = 0,$$  \hspace{1cm} (5.16)

$$f^\delta(s, x, z; s) = x^2.$$  \hspace{1cm} (5.17)

The solution is of the form

$$f^\delta(t, x, z; s) = x^2 e^{A^\delta(s-t)z + B^\delta(s-t)},$$

where $A^\delta(\tau)$ satisfies the Riccati equation:

$$A^\delta(\tau)' = \frac{\delta}{2} \beta^2 A^\delta(\tau)^2 + \left( \frac{2 \sqrt{\delta \rho \mu \beta}}{1 - \gamma} - \delta \right) A^\delta(\tau) + \frac{(3 - 2 \gamma) \mu^2}{(1 - \gamma)^2}, \quad \tau \in (0, s], \quad A^\delta(0) = 0,$$  \hspace{1cm} (5.18)

and $B^\delta(\tau)$ solves

$$B^\delta(\tau)' = \delta mA^\delta(\tau), \quad B^\delta(0) = 0.$$  \hspace{1cm} (5.19)

By a similar argument used in Section 5.1, the verification of the uniform bound

$$\mathbb{E}_{(0, x, z)} \left[ \int_0^T X_s^2 ds \right] = \int_0^T f^\delta(0, x, z; s) ds \leq C_2(T, x, z),$$

reduces to i) there exists $\overline{\delta}$, such that $\tau^*(\delta) > T$ for $\delta \leq \overline{\delta}$, (recall that $\tau^*(\delta)$ is defined to be the explosion time) and ii) $\lim_{\delta \to 0} A^\delta(\tau)$ exists. The details are given in Appendix F.

6 Conclusion

In this paper, we have considered the portfolio allocation problem in the context of a slowly varying stochastic environment and when the investor tries to maximize her terminal utility in a general class of utility functions. We proved that the zeroth order strategy identified in Fouque et al. [2016] is in fact asymptotically optimal up to the first order within a specific class of strategies. We have made precise the assumptions needed in order to rigorously establish this asymptotic optimality. These assumptions are on the coefficients of the model, on the utility function, and on the zeroth order value function, that is the solution to the classical Merton problem with constant coefficients. Finally, we analyzed a fully solvable example in order to demonstrate that our assumptions are reasonable.

In an ongoing work, we are establishing the same type of results in the case of a fast varying stochastic environment, and, ultimately, in the case of a model with two factors, one slow and one fast. We also plan to analyze the effect of the first order correction in the strategy on the second order correction of the value function.

Our analysis deals with classical solutions of the partial differential equations involved in the problem, HJB equations for the value functions and linear equations with source for the following terms. A full optimality result would require working with viscosity solutions, and that is also part of our future research.
A Proof of Proposition 2.8

Proof of (i). Without loss of generality, we assume $E = [a, b] \subset [0, 1)$. Notice that $a$ can be zero, but $b$ is strictly less than 1. Define $f(x, y) = x^y$, since $f_z^{(7)}(x, y)$ is continuous in $[x_0 - \delta, x_0 + \delta] \times E$, $\forall x_0 \in (0, \infty)$ and $f_z^{(7)}(x, y)$ is integrable on $E$.

$$U(x) = \int_E f(x, y) \nu(dy)$$

is $C^7(0, \infty)$, Moreover, we have

$$U^{(i)}(x) = \int_E f^{(i)}(x, y) \nu(dy), \text{ for } i \leq 7. \quad (A.1)$$

The monotonicity and concavity follows by the sign of $U'(x)$ and $U''(x)$ in (A.1). $U(0+) = 0$ follows by Dominated Convergence Theorem (DCT). We have:

$$\lim_{x \to 0^+} U'(x) = \lim_{x \to 0^+} \int_{a+\delta}^{b} yx^{y-1} \nu(dy) = \lim_{x \to 0^+} (a + \delta) \left( \frac{1}{x} \right)^{1-b} \nu([a + \delta, b]) = +\infty,$$

for a given $\delta$;

$$\lim_{x \to +\infty} U'(x) = \lim_{x \to +\infty} \int_{a}^{b} y \left( \frac{1}{x} \right)^{1-y} \nu(dy) \leq \lim_{x \to +\infty} b \left( \frac{1}{x} \right)^{1-b} \nu([a, b]) = 0;$$

$$AE[U] = \lim_{x \to +\infty} \frac{x \int_{a}^{b} yx^{y-1} \nu(dy)}{\int_{a}^{b} yx^{y} \nu(dy)} = \lim_{x \to +\infty} \frac{\int_{a}^{b} yx^{y} \nu(dy)}{\int_{a}^{b} yx^{y} \nu(dy)} \leq b < 1,$$

which shows the Inada and Asymptotic Elasticity conditions (2.17). To show Assumption 2.4 (iii) is satisfied, we follow Remark 2.7, and prove the following: a) $R(0) = 0$, $R(x)$ is strictly increasing on $[0, \infty)$; and b) $|R^{(j)}(x)(\partial x^{j+1} R(x))| \leq K$, $\forall 0 \leq j \leq 4$. For convenience, we introduce the short-hand notation

$$\langle f(y) \rangle_x = \int_E f(y) \nu(dy),$$

and in the sequel, we shall omit the subscript $x$ when there is no confusion.

Following from (A.1) and using the short-hand notation, $R(x)$ is given by

$$R(x) = \frac{\int_E yx^{y-1} \nu(dy)}{\int_E y(1-y)x^{y-2} \nu(dy)} = \frac{x \langle y \rangle}{(y(1-y))}. \quad (A.2)$$

Since $1 - y$ is bounded by $1 - b$ and $1 - a$, we deduce

$$\frac{x}{1-a} \leq R(x) \leq \frac{x}{1-b}, \quad (A.3)$$

and obtain $R(0) = 0$ by letting $x \to 0$. Taking derivative in (A.2) gives

$$R'(x) = \frac{\langle y(y+1) \rangle \langle y(1-y) \rangle - \langle y \rangle \langle y^2(1-y) \rangle}{\langle y(1-y) \rangle^2}.$$ 

The positiveness of $R'(x)$ on $[0, \infty)$ follows by

$$R'(x) = \frac{\langle y \rangle \langle y^3 \rangle + \langle y^2 \rangle^2 - \langle y \rangle \langle y^2 \rangle^2}{{\langle y(1-y) \rangle^2}} \geq \frac{\langle y \rangle^2 - \langle y \rangle \langle y^2 \rangle}{{\langle y(1-y) \rangle^2}}$$

$$= \frac{\langle y \rangle}{\langle y(1-y) \rangle} \geq \frac{\langle y \rangle}{(1-a) \langle y \rangle} = \frac{1}{1-a},$$

where we have used $\langle y \rangle \langle y^3 \rangle \geq \langle y^2 \rangle^2$. Thus, $R'(x)$ is bounded below by $\frac{1}{1-a}$ on $[0, \infty)$, and consequently, $R(x)$ is strictly increasing for $x \geq 0$. To show $R'(x) < K$, we derive the upper bound as follows:

$$R'(x) \leq \frac{(b+1)(1-a) \langle y \rangle^2}{(1-b)^2 \langle y \rangle^2} = \frac{(b+1)(1-a)}{(1-b)^2}. \quad (A.4)$$
To show $|R(x)R''(x)| \leq K$, we first compute $R''(x)$:

$$
R''(x) = \frac{1}{x} \left( \langle y^2(y+1) \rangle \frac{2 \langle y^2(1-y) \rangle^2 (y)}{(y(1-y))^3} - \frac{\langle y^2 \rangle \langle y^2(1-y) \rangle + \langle y \rangle \langle y^3(1-y) \rangle + \langle y^2(1-y) \rangle \langle y(y+1) \rangle}{(y(1-y))^2} \right).
$$

Then, the upper bound and lower bound of $R(x)R''(x)$ are computed as follows:

$$
R(x)R''(x) \leq \frac{\langle y \rangle}{(y(1-y))} \left( \frac{\langle b+1 \rangle \langle y^2 \rangle}{(1-b)(y)(y)} + \frac{2(1-b)\langle y^2 \rangle^2 (y)}{(1-b)^3 (y)^3} \right) \leq \frac{1}{1-b} \frac{b+3}{1-b};
$$

$$
R(x)R''(x) \geq - \frac{\langle y \rangle}{(y(1-y))} \left( (1-a) \langle y^2 \rangle^2 + (1-a) \langle y \rangle \langle y^3 \rangle + (1-a)(1+b) \langle y^2 \rangle \langle y \rangle \right) \geq - \frac{(b+3)}{(1-b)^2}.
$$

Combine the above bounds, we have the desired results $|R(x)R''(x)| \leq K$. Similar arguments works for $j = 2, 3, 4$ by straightforward calculations.

The last step is to show the growth condition of $I(y) = U^{(t-1)}(y)$. For $y \geq U'(1) = \int_E y \nu(dy)$, we have $I(y) \leq 1$. The other case $y = U'(x) \leq U'(1)$, where $x \geq 1$,

$$
U'(x) \leq x^{b-1} \int_E y \nu(dy), \quad \forall x \geq 1,
$$

$$
\Rightarrow \quad U'(\left( \left( \frac{t}{\int_E y \nu(dy)} \right)^{1/(b-1)} \right) \leq t, \quad \forall t \leq \int_E y \nu(dy)
$$

$$
\Rightarrow \quad I(y) \leq \kappa y^{-\gamma}, \quad \forall y \leq \int_E y \nu(dy),
$$

where $\kappa = \left( \frac{1}{\int_E y \nu(dy)} \right)^{1/(b-1)}$ is a constant depending solely on $\nu(dy)$, and $\alpha = \frac{1}{1-b} > 1$. Combining the two cases, we have $I(y) \leq \alpha \kappa y^{-\gamma}$.

Proof of (ii). This class of utility functions is defined via the inverse of marginal utility $I(y)$, where $U(x)$ can be recovered by:

$$
U(x) = \int_0^x U'(t) \, dt = \int_0^x I^{(-1)}(t) \, dt.
$$

(A.5)

Then $U(0^+) = 0$ is automatically satisfied. By definition of $I(y)$, $I(y) \in C^\infty(0, \infty)$, so does $U(x)$. The strictly monotonicity and strictly concavity are given by:

$$
U'(x) = I^{(-1)}(x) > 0, \quad U''(x) = \frac{1}{I'(I^{(-1)}(x))} = \left( \int_0^N -s \left( I^{(-1)}(s) \right)^{-s-1} \nu(ds) \right)^{-1} < 0.
$$

By DCT, one has

$$
I(+\infty) = \lim_{y \to +\infty} \int_0^N y^{-s} \nu(ds) = 0,
$$

$$
I(0) = \lim_{y \to 0} \int_0^N y^{-s} \nu(ds) \geq \lim_{y \to 0} \int_y^N y^{-s} \nu(ds) \geq \lim_{y \to 0} y^{-s} \nu[\delta, N] = +\infty,
$$

$$
AE[U] = \lim_{x \to +\infty} x \frac{U'(x)}{U(x)} = \lim_{x \to +\infty} \frac{U'(x) + xU''(x)}{U'(x)} = \lim_{x \to +\infty} 1 - \frac{x}{R(x)} = 1 - \lim_{x \to +\infty} \frac{1}{R'(x)} \leq 1 - \frac{1}{N},
$$

where we have used the fact that $R'(x) \leq N$ derived below.

From Proposition 3.2, $H(x, T, \lambda(z)) = I(e^{-x}) = \int_0^N e^{z \nu} \nu(ds)$, and the risk tolerance $R(x)$ is given by:

$$
R(x) = -\frac{U'(x)}{U''(|x|)} = H_\nu(H^{(-1)}(x, T, \lambda(z))) = \int_0^N \frac{1}{se^{H^{(-1)}(x, T, \lambda(z))(s)}} \nu(ds).
$$

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The fact that $R(0) = 0$ follows by DCT and $H^{(-1)}(0+, T, \lambda(z)) = -\infty$. $R(x)$ has bounded derivative, since:

$$R'(x) = \frac{H_{xx}(H^{(-1)}(x, T, \lambda(z)), T, \lambda(z))}{H_x(H^{(-1)}(x, T, \lambda(z)), T, \lambda(z))} = \frac{\int_0^N e^{H^{(-1)}(x, T, \lambda(z))} s^2 \nu(ds)}{\int_0^N e^{H^{(-1)}(x, T, \lambda(z))} s \nu(ds)} \leq N.$$  

$R(x)$ is strictly increasing, since the numerator stay positive for $x > 0$, i.e. $0 < R'(x) \leq N$. To show $|R(x)R''(x)| \leq K$, one needs

$$R(x)R''(x) + (R'(x))^2 = \frac{1}{2}(R^2(x))'' = \frac{H_{xx}(H^{(-1)}(x, T, \lambda(z)), T, \lambda(z))}{H_x(H^{(-1)}(x, T, \lambda(z)), T, \lambda(z))} = \frac{\int_0^N e^{H^{(-1)}(x, T, \lambda(z))} s^4 \nu(ds)}{\int_0^N e^{H^{(-1)}(x, T, \lambda(z))} s^2 \nu(ds)} \leq N^2.$$  

Since $(R^2(x))''$ and $R'(x)$ are bounded, so does $R(x)R''(x)$. Similar arguments works for $j = 2, 3, 4$ by using the following identities:

$$R^2 R'''' + R^3 + 4RR'''' = \frac{\partial_t^2 H(H^{(-1)}(x, T, \lambda(z)), T, \lambda(z))}{H_x(H^{(-1)}(x, T, \lambda(z)), T, \lambda(z))} = \frac{\int_0^N e^{H^{(-1)}(x, T, \lambda(z))} s^6 \nu(ds)}{\int_0^N e^{H^{(-1)}(x, T, \lambda(z))} s^4 \nu(ds)} \leq N^3$$

$$R^3 R^{(4)} + 7R^2 R'''' + R^4 + 11RR'''' + 4R^2 R'''' = \frac{\partial_t^3 H(H^{(-1)}(x, T, \lambda(z)), T, \lambda(z))}{H_x(H^{(-1)}(x, T, \lambda(z)), T, \lambda(z))} \leq N^4,$$

$$12R^5 + 32R^2 R'''' + 57RR^3 R'''' + R^4 R^{(5)} + 15R R^3 R'''' + 11R^3 R^4 R'''' + 54R^2 R'''' \leq N^5.$$  

Notice that $I(y)$ satisfies the polynomial growth condition due to the following: denote by $\kappa = \nu([0, N])$, if $y \geq 1$, $I(y) \leq \kappa$, otherwise when $y < 1$

$$I(y) \leq \kappa y^{-N}.$$  

Therefore, combining the two cases and defining $\alpha = \max\{N, \kappa\}$ yields the inequality (2.20).

B Proof of Proposition 3.4

To show (3.6), we use the relation (3.5) between the risk tolerance function $R(t, x; \lambda(z))$ and the function $H(x, t, \lambda(z))$ (which is defined in Proposition 3.2), namely

$$R(t, x; \lambda(z)) = H_x(H^{(-1)}(x, t, \lambda(z)), t, \lambda(z)). \quad (B.1)$$

Differentiating (B.1) with respect to $x$, and letting $t = T$ produces:

$$R'(x) = \frac{H_{xx}(y, T, \lambda(z))}{H_x(y, T, \lambda(z))} \bigg|_{y = H^{(-1)}(x, T, \lambda(z))}.$$  

Using $R'(x) \leq C = \sqrt{K/2}$ (proved in Proposition 2.5) gives, for all $x, z \in \mathbb{R},$

$$|H_{xx}(x, T, \lambda(z))| \leq \sqrt{K/2}H_x(x, T, \lambda(z)).$$

Notice that for fixed $\lambda(z)$, both $H_{xx}(x, t, \lambda(z))$ and $H_x(x, t, \lambda(z))$ satisfy the heat equation (3.3). Comparison Principle ensures that the inequality is preserved for $t < T$, i.e. for $(x, t, z) \in \mathbb{R} \times [0, T] \times \mathbb{R},$

$$|H_{xx}(x, t, \lambda(z))| \leq \sqrt{K/2}H_x(x, t, \lambda(z)).$$

Using (B.1) again, we obtain:

$$\partial_x R(t, x; \lambda(z)) = \frac{H_{xx}(y, t, \lambda(z))}{H_x(y, t, \lambda(z))} \bigg|_{y = H^{(-1)}(x, t, \lambda(z))} \leq \sqrt{K/2} := K_0. \quad (B.2)$$
Thus, we have shown (3.6) with $j = 0$.

To complete the proof in the case $j = 1$, we first obtain a relation between derivatives of $R(t, x; \lambda(z))$ and derivatives of $H(x, t, \lambda(z))$:

$$R_x^2(t, x; \lambda(z)) + RR_{xx}(t, x; \lambda(z)) = \frac{1}{2} \partial_x^2 R^2(t, x; \lambda(z)) = \frac{R_{xx}(y, t, \lambda(z))}{H_x(y, t, \lambda(z))} \bigg|_{y = H^{-1}(x, t, \lambda(z))}.$$ (B.3)

Let $t = T$ in the above identity, then, the middle quantity is reduced to $\frac{1}{2} \partial_x^2 R^2(x)$ and is bounded by $K/2$ as assumed in (2.19), so does the ratio of $H_{xx}(x, T, \lambda(z))$ over $H_x(x, T, \lambda(z))$ for all $x \in \mathbb{R}$. Standard Comparison Principle applies and the ratio remains bounded for $t < T$. This results in the boundedness of $\left[R_x^2(t, x; \lambda(z)) + RR_{xx}(t, x; \lambda(z))\right]$. Combining with (B.2), we achieve:

$$|RR_{xx}(t, x; \lambda(z))| \leq K/2 + K_0^2 := K_1.$$

To deal with $j = 2$, we first obtain the identity:

$$R^2 R_{xxx}(t, x; \lambda(z)) + R_x^2(t, x; \lambda(z)) + 4RR_x R_{xx}(t, x; \lambda(z)) = \frac{\partial_x^4 H(y, t, \lambda(z))}{H_x(y, t, \lambda(z))} \bigg|_{y = H^{-1}(x, t, \lambda(z))}.$$ (B.4)

At terminal time $T$, each term on the left-hand side is bounded (cf. Remark 2.7), therefore, the right-hand side is bounded. Then a similar argument based on Comparison Principle gives the following estimate:

$$\left|R^2 R_{xxx}(t, x; \lambda(z))\right| \leq K_2.$$

The remaining cases are completed by replacing (B.4) by the following

$$R^3 R^{(4)}(t, x; \lambda(z)) + 7R^2 R_x R_{xxx}(t, x; \lambda(z)) + R_x^4(t, x; \lambda(z)) + 11RR_x^2 R_{xx}(t, x; \lambda(z)) + 4R^2 R_{xx}^2(t, x; \lambda(z))$$

$$= \frac{\partial_x^6 H(y, t, \lambda(z))}{H_x(y, t, \lambda(z))} \bigg|_{y = H^{-1}(x, t, \lambda(z))},$$

$$12R_x^5(t, x; \lambda(z)) + 32R^2 R_x^2 R_{xxx}(t, x; \lambda(z)) + 57RR_x^2 R_{xx}(t, x; \lambda(z)) + R^4 R^{(5)}(t, x; \lambda(z)) + 15R^3 R_{xx} R_{xxx}(t, x; \lambda(z))$$

$$+ 11R^3 R_x R^{(4)}(t, x; \lambda(z)) + 54R^2 R_x R_{xx}^2(t, x; \lambda(z)) = \frac{\partial_x^8 H(y, t, \lambda(z))}{H_x(y, t, \lambda(z))} \bigg|_{y = H^{-1}(x, t, \lambda(z))},$$

and repeating the same argument.

The bounds $K_j$ are easily obtained by expanding $\partial_x^j R^j(t, x; \lambda(z))$ and using (3.6). To show (3.7), we notice that $R(t, x; \lambda(z))$ is shown to be strictly increasing in Proposition 3.3, and $R^4(t, x; \lambda(z)) \leq K_0$, integrating on both sides with respect to $x$ yields the desired result.

C Proof of Proposition 3.6

In the sequel, to shorten the notation, we shall omit the argument of $v^{(0)}(t, x, z)$ and its derivatives as well as the arguments of the risk tolerance function $R(t, x; \lambda(z))$ and use $R$ when there’s no confusion (note that this is not the risk tolerance $R(x)$ introduced in Assumption 2.4 (iii)). In what follows, we repeatedly use the results in Proposition 3.4 and concavity of $v^{(0)}$.

(1) Recall the “Vega-Gamma” relation in (2.15). A direct calculation gives:

$$\left|v_x^{(0)}\right| = (T - t) |\lambda(z)| \lambda'(z) |Rv_x^{(0)}| \leq (T - t) |\lambda(z)| \lambda'(z) |K_0 xv_x^{(0)}| \leq K_0(T - t) |\lambda(z)| \lambda'(z) |v^{(0)}| = d_{01}(z) v^{(0)},$$

where $d_{01}(z) = K_0(T - t) |\lambda(z)| \lambda'(z)$. 

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(2) Differentiating (2.15) with respect to $z$ brings:
\[
|v_x^{(0)}| = (T-t) \left| \lambda(z)\lambda'(z) \left( R_x v_x^{(0)} + Rv_x^{(0)} \right) \right|
\leq (T-t) |\lambda(z)\lambda'(z)| \left| K_0 v_x^{(0)} + v_x^{(0)} \right| = (K_0 + 1)(T-t) |\lambda(z)\lambda'(z)| v_x^{(0)}
\]
\[= d_{11}(z)v_x^{(0)}, \]
where $d_{11}(z) = (K_0 + 1)(T-t) |\lambda(z)\lambda'(z)|$.

(3) Differentiating $v_x^{(0)}$ with respect to $x$ again produces:
\[
|v_x^{(0)}| = (T-t) \left| \lambda(z)\lambda'(z) \left( R_{xx} v_x^{(0)} + 2R_x v_x^{(0)} + Rv_x^{(0)} \right) \right|
\leq (T-t) |\lambda(z)\lambda'(z)| \left( -R_{xx} Rv_x^{(0)} + 2R_x v_x^{(0)} - (R_x + 1)v_x^{(0)} \right)|
\leq (T-t) |\lambda(z)\lambda'(z)| (K_1 + K_0 + 1) \left| v_x^{(0)} \right|
\]
\[= d_{21}(z) \left| v_x^{(0)} \right|, \]
where $d_{21}(z) = (K_1 + K_0 + 1)(T-t) |\lambda(z)\lambda'(z)|$.

(4) Results in Proposition 3.5 gives:
\[
|R_x| = (T-t) |\lambda(z)\lambda'(z) R_{xx} R^2| \leq (T-t) |\lambda(z)\lambda'(z)| K_1 R = \tilde{d}_{01} R,
\]
where $\tilde{d}_{01}(z) = K_1(T-t) |\lambda(z)\lambda'(z)|$.

(5) Differentiating equation (2.15) with respect to $y$ yields:
\[
|v_y^{(0)}| = (T-t) \left| \left( (\lambda(z)\lambda'(z))_y Rv_x^{(0)} + \lambda(z)\lambda'(z) R_x v_y^{(0)} + \lambda(z)\lambda'(z) Rv_y^{(0)} \right) \right|
\leq (T-t) \left( (|\lambda(z)\lambda'(z)|)^2 + |\lambda(z)\lambda'(z)| (\tilde{d}_{01}(z) + d_{11}(z)) \right) \left| Rv_x^{(0)} \right|
\leq (T-t) \left( (|\lambda(z)\lambda'(z)|)^2 + |\lambda(z)\lambda'(z)| (\tilde{d}_{01}(z) + d_{11}(z)) \right) K_0 v_y^{(0)}
\]
\[= d_{02}(z)v_y^{(0)}, \]
where $d_{02}(z) = K_0(T-t) \left( (|\lambda(z)\lambda'(z)|)^2 + |\lambda(z)\lambda'(z)| (\tilde{d}_{01}(z) + d_{11}(z)) \right)$.

(6) Differentiating equation (3.8) with respect to $x$ gives:
\[
|R_x| = (T-t) |\lambda(z)\lambda'(z) (R^2 R_{xx})_x| = (T-t) |\lambda(z)\lambda'(z)| |R_{xxx} R^2 + 2RR_x R_{xx}|
\leq (T-t) |\lambda(z)\lambda'(z)| ((K_2 + 2K_1 K_0) \tilde{d}_{11}(z)),
\]
where $\tilde{d}_{11}(z) = (K_2 + 2K_1 K_0)(T-t) |\lambda(z)\lambda'(z)|$.

(7) Differentiating $v_x^{(0)}$ given in (2) with respect to $z$, one has:
\[
|v_x^{(0)}| = (T-t) \left| \lambda(z)\lambda'(z) \left( R_x v_x^{(0)} + Rv_x^{(0)} \right) \right|
\leq (T-t) |\lambda(z)\lambda'(z)| \left( K_0 v_x^{(0)} + v_x^{(0)}) \right| \left( K_0 v_x^{(0)} + v_x^{(0)} \right)
\]
\[+ (T-t) |\lambda(z)\lambda'(z)| \left( \tilde{d}_{11}(z)v_x^{(0)} + K_0 d_{11}(z) v_x^{(0)} + \tilde{d}_{01}(z) R |v_x^{(0)}| + Rd_{21}(z) |v_x^{(0)}| \right)
\]
\[= (T-t) \left( (|\lambda(z)\lambda'(z)|) + |\lambda(z)\lambda'(z)| (\tilde{d}_{11}(z) + K_0 d_{11}(z) + \tilde{d}_{01}(z) + d_{21}(z)) \right) v_x^{(0)}
\]
\[= d_{12}(z)v_x^{(0)}, \]
where $d_{12}(z) = (T-t) \left( (|\lambda(z)\lambda'(z)|) + |\lambda(z)\lambda'(z)| (\tilde{d}_{11}(z) + K_0 d_{11}(z) + \tilde{d}_{01}(z) + d_{21}(z)) \right)$.
(8) Differentiating (3.8) with respect to \( z \) gives:

\[
|R_{zz}| = (T - t) \left| (\lambda(z)\lambda'(z))' R^2 R_{xx} + \lambda(z)\lambda'(z)(R^2 R_{xx} + 2RR_x R_{xx}) \right|
\]

\[
\leq (T - t) \left\{ \left| (\lambda(z)\lambda'(z))' K_1 R + |\lambda(z)\lambda'(z)| (\tilde{K}_1(z)R + 2K_1\tilde{d}_0(1)z(R)) \right\} R
\]

\[
= \tilde{d}_0(z)R,
\]

where \( \tilde{d}_0(z) = (T - t) \left\{ \left| (\lambda(z)\lambda'(z))' K_1 + |\lambda(z)\lambda'(z)| (\tilde{K}_1(z) + 2K_1\tilde{d}_0(1)) \right\} \). Here \( \tilde{K}_1(z) \) is the bound of \( RR_{xx} \), i.e. \( \forall (t, x, z) \in [0, T] \times \mathbb{R}^+ \mathbb{R}^+ \),

\[
|R_{zz} R| \leq \tilde{K}_1(z), \quad (C.1)
\]

and is computed by differentiating (3.8) with respect to \( x \) twice:

\[
|R_{zzz} R| = (T - t) |\lambda(z)\lambda'(z)| [R_{zzz} R^3 + 4R^2 R_x R_{zz} + 2RR_x^2 R_{xx} + 2R^2 R_{xx}^2]
\]

\[
\leq (T - t) |\lambda(z)\lambda'(z)| (K_3 + 4K_0K_2 + 2K_0^2K_1 + 2K_1^2) = \tilde{K}_1(z).
\]

(9) Differentiating \( v^{(0)}_{zzz} \) with respect to \( x \) twice, we achieve:

\[
\left| v^{(0)}_{zzzz} \right| = (T - t) \left| (\lambda(z)\lambda'(z))' \left( R_{xx} v^{(0)}_{zz} + R v^{(0)}_{zz} \right)_x + \lambda(z)\lambda'(z) \left( R_{xx} v^{(0)}_{zz} + R v^{(0)}_{xx} + R v^{(0)}_{zz} \right)_x \right|
\]

\[
\leq (T - t) \left[ |(\lambda(z)\lambda'(z))'| R_{xx} v^{(0)}_{zz} + 2R_{xx} v^{(0)}_{zz} + R v^{(0)}_{xx} + R v^{(0)}_{zz} \right]
\]

\[
+ (T - t) |\lambda(z)\lambda'(z)| \left( R_{xx} v^{(0)}_{zz} x + 2R_{xx} v^{(0)}_{zz} + R v^{(0)}_{zz} + 2R v^{(0)}_{zz} + R v^{(0)}_{zz} \right)
\]

\[
\leq (T - t) \left[ |(\lambda(z)\lambda'(z))'| R_{xx} R + 2K_0 + (K_0 + 1) \right] \left| v^{(0)}_{zz} \right| + (T - t) \left| \lambda(z)\lambda'(z) \right|
\]

\[
\cdot \left( R_{xx} R v^{(0)}_{zz} + 2\tilde{d}_{11}(z) + d_{11}(z) \right) \left| v^{(0)}_{zz} \right| + (T - t) \left| \lambda(z)\lambda'(z) \right|
\]

\[
\cdot \left( \tilde{K}_1(z) + 2\tilde{d}_{11}(z) + d_{11}(z)K_1 + 2d_{21}(z)K_0 + \tilde{d}_0(1)z(K_0 + 1) + \tilde{K}_3(z) \right) \left| v^{(0)}_{zz} \right|
\]

\[
= d_{22}(z) \left| v^{(0)}_{zz} \right|,
\]

where

\[
d_{22}(z) = (T - t) \left\{ \left| (\lambda(z)\lambda'(z))' (K_1 + 3K_0 + 1) + |\lambda(z)\lambda'(z)| (\tilde{K}_1(z) + 2\tilde{d}_{11}(z) + d_{11}(z)K_1 + 2d_{21}(z)K_0 + \tilde{d}_0(1)z(K_0 + 1) + \tilde{K}_3(z)) \right\}.
\]

During the derivation we have used the inequalities:

\[
|R_{xx} v^{(0)}_{zz}| = R_{xx} R v^{(0)}_{xx} \leq K_1 \left| v^{(0)}_{zz} \right|, \quad (C.2)
\]

\[
R v^{(0)}_{zzz} = (R + 1) v^{(0)}_{zx} \leq (K_0 + 1) \left| v^{(0)}_{zz} \right|, \quad (C.3)
\]

\[
R v^{(0)}_{zzzz} \leq \tilde{K}_3(z) \left| v^{(0)}_{zz} \right|, \quad (C.4)
\]

To obtain the last one, we first claim:

\[
|R^2 v^{(0)}_{xxzz} | = \left| R_{xx} v^{(0)}_{xx} - (R + 1) v^{(0)}_{zx} + 2(R + 1)^2 v^{(0)}_{zz} \right| \leq (K_1 + 2K_0^2 + 3K_0 + 1) \left| v^{(0)}_{xx} \right|,
\]

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Finally, we differentiate \( v^{(0)}_{zzz} \) with respect to \( z \) again and obtain:

\[
v^{(0)}_{zzz} = (T-t) \left\{ (\lambda(z) \lambda'(z))' (R_{zzz} v^{(0)}_{zz} + R_{rzz} v^{(0)}_{rzz} + 2R_{zr} v^{(0)}_{zr}) + \lambda(z) \lambda'(z) (R_{zzz} v^{(0)}_{zz} + R_{rzz} v^{(0)}_{rzz} + R_{zr} v^{(0)}_{zr}) \right. \\
+ \left. (R_{zzz} v^{(0)}_{zz})' (K_0 + 1) v^{(0)}_{zz} + 2 (\lambda(z) \lambda'(z))' (R_{zzz} v^{(0)}_{zz} + R_{rzz} v^{(0)}_{rzz} + R_{zr} v^{(0)}_{zr}) \right\}
\]

Taking absolute value on both sides,

\[
|v^{(0)}_{zzz}| \leq (T-t) \left\{ |(\lambda(z) \lambda'(z))'| (K_0 + 1) v^{(0)}_{zz} + 2 |(\lambda(z) \lambda'(z))' (R_{zzz} v^{(0)}_{zz} + R_{rzz} v^{(0)}_{rzz} + R_{zr} v^{(0)}_{zr})| \\
+ |(R_{zzz} v^{(0)}_{zz})'| (K_0 + 1) v^{(0)}_{zz} + 2 |(\lambda(z) \lambda'(z))' (R_{zzz} v^{(0)}_{zz} + R_{rzz} v^{(0)}_{rzz} + R_{zr} v^{(0)}_{zr})|ight\}
\]

\( d_{13}(z) \) is easy to identify from the above inequality, in which \( \tilde{K}_{12}(z) \) is a bound uniform in \( t \) and \( x \).

\[
R_{zzz} \leq \tilde{K}_{12}(z).
\] (C.5)

Such \( \tilde{K}_{12}(z) \) exists by the following derivation:

\[
R_{zzz} = (T-t)(\lambda(z) \lambda'(z))' (R^2 R_{zzz} + 2RR_{r} R_{zr}) \\
+ (T-t)\lambda(z) \lambda'(z) (R^2 R_{zzz} + 2RR_{r} R_{zr} + 2RR_{r} R_{zzz} + 2RR_{r} R_{zr} + 2RR_{rr} R_{zzz}).
\]

Every term is bounded (by a function of \( z \)) by previous results, except \( R^2 R_{zzzz} \). For this term, we derive:

\[
|R^2 R_{zzzz}| = (T-t) |\lambda(z) \lambda'(z) (R^4 R^5 + 8R^3 R_{zz} R_{rzzz} + 6R^2 R_{zr} R_{rzzz} + 6R^3 R_{r} R_{rzzzz} + 6R^2 R_{zr} R^2_{zzzz})| \\
\leq (T-t) |\lambda(z) \lambda'(z) (K_4 + 8K_1 K_2 + 6K_0^2 K_2 + 6K_0 K_3 + 6K_0 K_2^2)|.
\]

### D Proof of Boundedness for Theorem 3.1

We first analyze term I in (3.13). The boundedness for the \( z \)-derivatives of \( v^{(0)} \) is given by Proposition 3.6. To bound the \( L^2 \) norm of \( v^{(0)}(\cdot, X_{s}^{0}, Z) \) we rely on Lemma 2.13. In the following we omit the arguments of \( v^{(0)}(s, X_{s}^{0}, Z) \) and its derivatives.

\[
I = E_{t,x,z} \left[ \int_t^T c(Z_s) v^{(0)}(s) + \frac{1}{2} g^2(Z_s) v^{(0)}(s) \, ds \right] = I^{(1)} + \frac{1}{2} I^{(2)}.
\]

\[
|I^{(1)}| \leq E_{t,x,z} \left[ \int_t^T |c(Z_s) v^{(0)}(s)| \, ds \right] \leq E_{t,x,z} \left[ \int_t^T |c(Z_s)| d_{01}(Z_s) |v^{(0)}(s)| \, ds \right] \\
\leq E_{t,z}^{1/2} \left[ \int_t^T c^2(Z_s) d_{01}^2(Z_s) \, ds \right] \frac{E_{t,x,z}^{1/2} \left[ \int_t^T (v^{(0)}(s))^2 \, ds \right]}{E_{t,x,z}^{1/2} \left[ \int_t^T (v^{(0)}(s))^2 \, ds \right]} \\
\leq C(T, z) C_3(T, x, z).
\]
In the calculation above, \( v'(0) \) is replaced by its bound \( d_{01}(z)v'(0) \) derived in Proposition 3.6. By Cauchy-Schwarz inequality, it suffices to bound two expectations. For the first one, we have used the facts that \( c(z) \) and \( d_{01}(z) \) have at most polynomial growth and \( Z_t \) admits moments of any order uniformly in \( \delta \). Proposition 2.13 gives the bound for the second expectation.

The bounds of remaining terms are obtained by the same procedure, and we will sketch the calculation without detailed explanation. Also, in what follows, we omit the arguments of \( R(s, X^{\pi}(s); \lambda(Z_s)) \).

\[
|I^{(2)}| = \mathbb{E}_{(t,x,z)} \left[ \int_t^T g^2(Z_s) |v'_{zzz}(s)| \, ds \right] \leq \mathbb{E}_{(t,x,z)} \left[ \int_t^T g^2(Z_s) d_{02}(Z_s)v'(0) \, ds \right]
\]
\[
\leq \mathbb{E}^{1/2}_{(t,z)} \left[ \int_t^T g^4(Z_s)d_{02}^2(Z_s) \, ds \right] \mathbb{E}^{1/2}_{(t,x,z)} \left[ \int_t^T (v'(0))^2 \, ds \right]
\]
\[
\leq C(T, z)C_3(T, x, z).
\]

Term II in (3.14) and term III in (3.15) contain derivatives in \( z \) of \( v^{(1)} \). To deal with it, we recall the following relation between \( v^{(1)} \) and \( v^{(0)} \) given by equation (2.16):

\[
v^{(1)} = -\frac{1}{2} (T-t) \rho \lambda(z) g(z) v^{(0)}_{xx} v^{(0)}_{zz} = \frac{1}{2} (T-t) \rho \lambda(z) g(z) R v^{(0)}_{xx}.
\]

Differentiating the above equation with respect to \( z \), we are able to rewrite \( v_{zzz}^{(1)} \), \( v_{zz}^{(1)} \) and \( v_{xzz}^{(1)} \) in terms of the risk tolerance function \( R \) and the leading order term \( v^{(0)} \). Then, as before, the derivations are mainly based on Proposition 3.6 and Lemma 2.13, and given as follows:

\[
II = \mathbb{E}_{(t,x,z)} \left[ \int_t^T c(Z_s) v_{zz}(1) + \frac{1}{2} g^2(Z_s)v_{zzz}(1) \, ds \right]
\]
\[
= \frac{1}{2} (T-t) \rho \mathbb{E}_{(t,x,z)} \left[ \int_t^T c(Z_s) \left( \lambda(Z_s) g(Z_s) R v^{(0)}_{xx} \right)_z \, ds \right]
\]
\[
+ \frac{1}{4} (T-t) \rho \mathbb{E}_{(t,x,z)} \left[ \int_t^T g^2(Z_s) \left( \lambda(Z_s) g(Z_s) R v^{(0)}_{xx} \right)_zz \, ds \right]
\]
\[
= \frac{1}{2} (T-t) \rho II^{(1)} + \frac{1}{4} (T-t) \rho II^{(2)}.
\]

\[
II^{(1)} = \mathbb{E}_{(t,x,z)} \left[ \int_t^T c(Z_s) (\lambda(Z_s) g(Z_s))_z R v^{(0)}_{xx} \, ds \right]
\]
\[
+ \mathbb{E}_{(t,x,z)} \left[ \int_t^T c(Z_s) \lambda(Z_s) g(Z_s) \left( R v^{(0)}_{xx} + R v^{(0)}_{xx} (s, X^{\pi}(s), Z_s) \right) \, ds \right]
\]
\[
= II^{(1,1)} + II^{(1,2)} + II^{(1,3)}.
\]

where they are uniformly bounded since

\[
|II^{(1,1)}| \leq \mathbb{E}_{(t,x,z)} \left[ \int_t^T |c(Z_s) (\lambda(Z_s) g(Z_s))_z| K_0 X^{\pi}(0) d_{11}(Z_s)v^{(0)}_{xx} \, ds \right]
\]
\[
\leq K_0 \mathbb{E}^{1/2}_{(t,z)} \left[ \int_t^T c^2(Z_s) d_{11}^2(Z_s) (\lambda(Z_s) g(Z_s))_z^2 \, ds \right] \mathbb{E}^{1/2}_{(t,x,z)} \left[ \int_t^T (v'(0))^2 \, ds \right]
\]
\[
\leq C(T, z)C_3(T, x, z).
\]
\[ \|I(t, x, z)\| \leq E_{(t, x, z)} \left[ \int_t^T |c(Z_s)\lambda(Z_s)g(Z_s)| \, ds \right] \]
\[ \leq K_0 E_{(t, x, z)}^{1/2} \left[ \int_t^T c^2(Z_s)d_0^2(0) + d_1^2(0) \lambda^2(Z_s)g^2(Z_s) \, ds \right] \]
\[ \leq K_0 E_{(t, x, z)}^{1/2} \left[ \int_t^T \left( v(0) \right)^2 \, ds \right] \leq C(T, z)C_3(T, x, z). \]

\[ |I(t, x, z)| \leq E_{(t, x, z)} \left[ \int_t^T |c(Z_s)\lambda(Z_s)g(Z_s)| \, ds \right] \]
\[ \leq K_0 E_{(t, x, z)}^{1/2} \left[ \int_t^T c^2(Z_s)d_0^2(0) + d_2^2(0) \lambda^2(Z_s)g^2(Z_s) \, ds \right] \]
\[ \leq K_0 E_{(t, x, z)}^{1/2} \left[ \int_t^T \left( v(0) \right)^2 \, ds \right] \leq C(T, z)C_3(T, x, z). \]

\[ I(t, x, z) = E_{(t, x, z)} \left[ \int_t^T g^2(Z_s) \left( \lambda(Z_s)g(Z_s) \right)_{z z} \, ds \right] \]
\[ + 2E_{(t, x, z)} \left[ \int_t^T g^2(Z_s) \left( \lambda(Z_s)g(Z_s) \right)_z \left( R_z v_{z z}^{(0)} + R_w v_{z z z}^{(0)}(s, X_s^{(0)}, Z_s) \right) \, ds \right] \]
\[ + E_{(t, x, z)} \left[ \int_t^T g^2(Z_s) \lambda(Z_s)g(Z_s) \left( R_z v_{z z}^{(0)} + 2R_w v_{z z z}^{(0)}(s, X_s^{(0)}, Z_s) \right) \, ds \right] \]
\[ = I(t, x, z) + 2I(t, x, z) + I(t, x, z) + 2I(t, x, z) + I(t, x, z). \]

\[ |I(t, x, z)| \leq E_{(t, x, z)} \left[ \int_t^T |c(Z_s)\lambda(Z_s)g(Z_s)| \, ds \right] \]
\[ \leq K_0 E_{(t, x, z)}^{1/2} \left[ \int_t^T c^2(Z_s)d_0^2(0) + d_1^2(0) \lambda^2(Z_s)g^2(Z_s) \, ds \right] \]
\[ \leq C(T, z)C_3(T, x, z). \]

\[ |I(t, x, z)| \leq E_{(t, x, z)} \left[ \int_t^T |c(Z_s)\lambda(Z_s)g(Z_s)| \, ds \right] \]
\[ \leq K_0 E_{(t, x, z)}^{1/2} \left[ \int_t^T c^2(Z_s)d_0^2(0) + d_1^2(0) \lambda^2(Z_s)g^2(Z_s) \, ds \right] \]
\[ \leq K_0 E_{(t, x, z)}^{1/2} \left[ \int_t^T \left( v(0) \right)^2 \, ds \right] \leq C(T, z)C_3(T, x, z). \]
\[ \mathbf{II}^{(2,4)} \leq \mathbb{E}_{(t,x,z)} \left[ \int_{t}^{T} g^2(Z_s) \left| \lambda(Z_s) g(Z_s) \right| \tilde{d}_{01}(Z_s) R d_{11}(Z_s) v_x^{(0)}(s) \, ds \right] \]
\[ \leq K_0 \mathbb{E}_{(t,z)}^{1/2} \left[ \int_{t}^{T} g^4(Z_s) \tilde{d}_{01}^2(Z_s) d_{11}^2(Z_s) \lambda^2(Z_s) g^2(Z_s) \, ds \right] \mathbb{E}_{(t,x,z)}^{1/2} \left[ \int_{t}^{T} (v^{(0)})^2 \, ds \right] \]
\[ \leq C(T, z) C_3(T, x, z). \]

\[ \mathbf{II}^{(2,5)} \leq \mathbb{E}_{(t,x,z)} \left[ \int_{t}^{T} g^2(Z_s) \left| \lambda(Z_s) g(Z_s) \right| \tilde{d}_{01}(Z_s) R d_{12}(Z_s) v_x^{(0)}(s) \, ds \right] \]
\[ \leq K_0 \mathbb{E}_{(t,z)}^{1/2} \left[ \int_{t}^{T} g^4(Z_s) \tilde{d}_{01}^2(Z_s) d_{12}^2(Z_s) \lambda^2(Z_s) g^2(Z_s) \, ds \right] \mathbb{E}_{(t,x,z)}^{1/2} \left[ \int_{t}^{T} (v^{(0)})^2 \, ds \right] \]
\[ \leq C(T, z) C_3(T, x, z). \]

\[ \mathbf{II}^{(2,6)} \leq \mathbb{E}_{(t,x,z)} \left[ \int_{t}^{T} g^2(Z_s) \left| \lambda(Z_s) g(Z_s) \right| K_0 X_s^{e(0)} d_{13}(Z_s) v_x^{(0)}(s) \, ds \right] \]
\[ \leq K_0 \mathbb{E}_{(t,z)}^{1/2} \left[ \int_{t}^{T} g^4(Z_s) d_{13}^2(Z_s) \lambda^2(Z_s) g^2(Z_s) \, ds \right] \mathbb{E}_{(t,x,z)}^{1/2} \left[ \int_{t}^{T} (v^{(0)})^2 \, ds \right] \]
\[ \leq C(T, z) C_3(T, x, z). \]

Similarly the term III defined in (3.15) becomes:

\[ \text{III} = \mathbb{E}_{(t,x)} \left[ \int_{t}^{T} \lambda(Z_s) g(Z_s) R \left( \frac{1}{2} R - s \rho \lambda(Z_s) g(Z_s) R v_{x}^{(0)} \right) \right] d_x \]
\[ = \frac{1}{2} (T - t) \rho \mathbb{E}_{(t,x)} \left[ \int_{t}^{T} \lambda(Z_s) g(Z_s) R \left( \lambda(Z_s) g(Z_s) \right)^{R v_{x}^{(0)}} + \lambda(Z_s) g(Z_s) \left( R v_{x}^{(0)} + R v_{x}^{(0)} \right) \right] d_x \]
\[ \leq \frac{1}{2} (T - t) \rho \mathbb{E}_{(t,x)} \left[ \int_{t}^{T} \lambda^2(Z_s) g^2(Z_s) \left( R v_{x}^{(0)} + R v_{x}^{(0)} + R v_{x}^{(0)} + R^2 v_{x}^{(0)} \right) \right] d_x \]
\[ \leq \frac{1}{2} (T - t) \rho \left( \text{III}^{(3)} + \text{III}^{(2)} \right) + \frac{1}{2} (T - t) \rho \left( \text{III}^{(3)} + \text{III}^{(4)} + \text{III}^{(5)} + \text{III}^{(6)} \right). \]

Now we analyze them one by one:

\[ \text{III}^{(1)} \leq \mathbb{E}_{(t,x)} \left[ \int_{t}^{T} \left| \lambda(Z_s) g(Z_s) \left( \lambda(Z_s) g(Z_s) \right)^{R v_{x}^{(0)}} \right| \right] d_x \]
\[ \leq \mathbb{E}_{(t,x)} \left[ \int_{t}^{T} \left| \lambda(Z_s) g(Z_s) \left( \lambda(Z_s) g(Z_s) \right)^{R v_{x}^{(0)}} \right| K_0 \mathbb{E}_{(t,z)}^{1/2} \left[ \int_{t}^{T} \lambda^2(Z_s) g^2(Z_s) \right] \mathbb{E}_{(t,x,z)}^{1/2} \left[ \int_{t}^{T} (v^{(0)})^2 \, ds \right] \] (Propositions 3.4 and 3.6)
\[ \leq K_0^2 \mathbb{E}_{(t,z)}^{1/2} \left[ \int_{t}^{T} \left( \lambda(Z_s) g(Z_s) \left( \lambda(Z_s) g(Z_s) \right)^{R v_{x}^{(0)}} \right)^2 \right] d_x \mathbb{E}_{(t,x,z)}^{1/2} \left[ \int_{t}^{T} (v^{(0)})^2 \, ds \right] \] (CS inequality)
\[ \leq C(T, z) \mathbb{E}_{(t,z)}^{1/2} \left[ \int_{t}^{T} \left( v^{(0)}(t, X_s^{e(0)}, Z_s) \right)^2 \right] d_x \] (Bounded moments of $Z_s$ and concavity of $v^{(0)}$)
\[ \leq C(T, z) C_3(T, x, z) \] (Lemma 2.13).
\[ \text{III}^{(2)} \leq E_{(t,x,z)} \left[ \int_t^T \left| \lambda(Z_s)g(Z_s)(\lambda(Z_s)g(Z_s))'RR_{t,x,z}^{(0)} \right| \, ds \right] \\
\leq E_{(t,x,z)} \left[ \int_t^T \left| \lambda(Z_s)g(Z_s)(\lambda(Z_s)g(Z_s))'C X_s^{(0)} Rd(Z_s)v(s) \right| \, ds \right] \\
\leq CE_{(t,x,z)}^{1/2} \left[ \int_t^T \left( \lambda(Z_s)g(Z_s)(\lambda(Z_s)g(Z_s))'d(Z_s) \right)^2 \, ds \right] \mathbb{E}_{(t,x,z)}^{1/2} \left[ \int_t^T \left( X_s^{(0)} v(s) \right)^2 \, ds \right] \\
\leq C(T, z)E_{(t,x,z)}^{1/2} \left[ \int_t^T \left( v(s) \right)^2 \, ds \right] \\
\leq C(T, z)C_3(T, x, z). \\

\[ \text{III}^{(3)} \leq E_{(t,x,z)} \left[ \int_t^T \left| \lambda^2(Z_s)g^2(Z_s)RR_{t,x,z}^{(0)} \right| \, ds \right] \\
\leq E_{(t,x,z)} \left[ \int_t^T \left| \lambda^2(Z_s)g^2(Z_s)C X_s^{(0)} d(Z_s)Rd(Z_s)v(s) \right| \, ds \right] \\
\leq CE_{(t,x,z)}^{1/2} \left[ \int_t^T \lambda^2(Z_s)g^2(Z_s)d(Z_s) \, ds \right] \mathbb{E}_{(t,x,z)}^{1/2} \left[ \int_t^T \left( X_s^{(0)} v(s) \right)^2 \, ds \right] \\
\leq C(T, z)E_{(t,x,z)}^{1/2} \left[ \int_t^T \left( v(s) \right)^2 \, ds \right] \\
\leq C(T, z)C_3(T, x, z). \\

\[ \text{III}^{(4)} \leq E_{(t,x,z)} \left[ \int_t^T \left| \lambda^2(Z_s)g^2(Z_s)RR_{t,x,z}^{(0)} \right| \, ds \right] \\
\leq E_{(t,x,z)} \left[ \int_t^T \left| \lambda^2(Z_s)g^2(Z_s)C X_s^{(0)} Rd(Z_s)v(s) \right| \, ds \right] \\
\leq CE_{(t,x,z)}^{1/2} \left[ \int_t^T \lambda^2(Z_s)g^2(Z_s)d(Z_s) \, ds \right] \mathbb{E}_{(t,x,z)}^{1/2} \left[ \int_t^T \left( X_s^{(0)} v(s) \right)^2 \, ds \right] \\
\leq C(T, z)E_{(t,x,z)}^{1/2} \left[ \int_t^T \left( v(s) \right)^2 \, ds \right] \\
\leq C(T, z)C_3(T, x, z). \\

\[ \text{III}^{(5)} \leq E_{(t,x,z)} \left[ \int_t^T \left| \lambda^2(Z_s)g^2(Z_s)RR_{t,x,z}^{(0)} \right| \, ds \right] \\
\leq E_{(t,x,z)} \left[ \int_t^T \left| \lambda^2(Z_s)g^2(Z_s)C X_s^{(0)} Cd^2(Z_s)v(s) \right| \, ds \right] \\
\leq CE_{(t,x,z)}^{1/2} \left[ \int_t^T \lambda^2(Z_s)g^2(Z_s)d(Z_s) \, ds \right] \mathbb{E}_{(t,x,z)}^{1/2} \left[ \int_t^T \left( X_s^{(0)} v(s) \right)^2 \, ds \right] \\
\leq C(T, z)E_{(t,x,z)}^{1/2} \left[ \int_t^T \left( v(s) \right)^2 \, ds \right] \\
\leq C(T, z)C_3(T, x, z). \]
\[ |\mathrm{III}^{(0)}| \leq \mathbb{E}_{(t,x,z)} \left[ \int_t^T \left| \lambda^2(Z_s)g^2(Z_s)RR_v^{(0)} \right| ds \right] \]
\[ \leq \mathbb{E}_{(t,x,z)} \left[ \int_t^T \lambda_1(Z_s)g^2(Z_s)CX_s^{(0)} R d^4(Z_s)v_x^{(0)} ds \right] \]
\[ \leq C\mathbb{P}_{(t,x,z)}^{1/2} \left[ \int_t^T \lambda_1(Z_s)g^4(Z_s)d^4(Z_s) ds \right] \mathbb{E}_{(t,x,z)}^{1/2} \left[ \int_t^T \left( \chi_s^{(0)} v_x^{(0)} \right)^2 ds \right] \]
\[ \leq C(T,z)\mathbb{P}_{(t,x,z)}^{1/2} \int_t^T \left( v^{(0)} \right)^2 ds \]
\[ \leq C(T,z)\mathbb{P}_{(t,x,z)} \]

E Assumptions in Section 4.2

This set of assumptions is used in deriving Proposition 4.4 where we establish the accuracy of approximation of \( V^\delta \) summarized in Table 3.

Assumption E.1. Let \( \mathcal{A}_0(t, x, z) \left[ \bar{\pi}^0, \bar{\pi}^1, \alpha \right] \) be the family of trading strategies defined in (4.1). Recall that \( X^\pi \) is the wealth generated by the strategy \( \pi = \bar{\pi}^0 + \delta \bar{\pi}^1 \) as defined in (4.3). In order to condense the notation, we systematically omit the argument \((s, X^\sigma, Z_s)\) in what follows. According to the different cases, we further require:

(i) If \( \bar{\pi}^0 = \pi^{(0)} \),

(a) If \( \alpha > 1/4 \), the following quantities are uniformly bounded in \( \delta \):
\[ \mathbb{E}_{(t,x,z)} \int_t^T \mathcal{M}^{(0)} ds, \mathbb{E}_{(t,x,z)} \int_t^T \mathcal{M}^{(1)} ds, \mathbb{E}_{(t,x,z)} \int_t^T \sigma^2(Z_s) \left( \pi^1 \right)^2 v_x^{(0)} ds, \]
\[ \mathbb{E}_{(t,x,z)} \int_t^T \sigma^2(Z_s) \left( \pi^1 \right)^2 v_x^{(1)} ds, \mathbb{E}_{(t,x,z)} \int_t^T \mu(Z_s) \pi^1 v_x^{(1)} ds, \]
\[ \mathbb{E}_{(t,x,z)} \int_t^T g(Z_s)\sigma(Z_s)\pi^{(0)}v_x^{(1)} ds, \mathbb{E}_{(t,x,z)} \int_t^T g(Z_s)\sigma(Z_s)\bar{\pi}^1v_x^{(1)} ds, \]
\[ \mathbb{E}_{(t,x,z)} \int_t^T g(Z_s)\sigma(Z_s)\bar{\pi}^1v_x^{(2)} ds, \mathbb{E}_{(t,x,z)} \int_t^T g(Z_s)\sigma(Z_s)\bar{\pi}^1v_x^{(3)} ds, \]
Here, recall that \( v^{(0)} \) and \( v^{(1)} \) are the leading order term and first order correction of \( V^\delta \) as well as of \( V^{\pi^{(0)},\delta} \), and they satisfy (2.11) and (2.13) respectively.

(b) In the case \( 0 < \alpha < 1/4 \), if \( (t, x, z) \in K_1 \), we need
\[ \mathbb{E}_{(t,x,z)} \int_t^T \mathcal{M}^{(0)} ds, \mathbb{E}_{(t,x,z)} \int_t^T \mathcal{M}^{(2)} ds, \mathbb{E}_{(t,x,z)} \int_t^T \sigma^2(Z_s) \left( \pi^1 \right)^2 \bar{v}_x^{(2)} ds, \]
\[ \mathbb{E}_{(t,x,z)} \int_t^T \sigma^2(Z_s) \left( \pi^1 \right)^2 \bar{v}_x^{(3)} ds, \mathbb{E}_{(t,x,z)} \int_t^T \mu(Z_s) \pi^1 \bar{v}_x^{(2)} ds, \mathbb{E}_{(t,x,z)} \int_t^T \mu(Z_s) \pi^1 \bar{v}_x^{(3)} ds, \]
\[ \mathbb{E}_{(t,x,z)} \int_t^T g(Z_s)\sigma(Z_s)\pi^{(0)}\bar{v}_x^{(0)} ds, \mathbb{E}_{(t,x,z)} \int_t^T g(Z_s)\sigma(Z_s)\bar{\pi}^1\bar{v}_x^{(2)} ds, \]
\[ \mathbb{E}_{(t,x,z)} \int_t^T g(Z_s)\sigma(Z_s)\bar{\pi}^1\bar{v}_x^{(3)} ds, \mathbb{E}_{(t,x,z)} \int_t^T g(Z_s)\sigma(Z_s)\bar{\pi}^1\bar{v}_x^{(4)} ds, \]
to be uniformly bounded in \( \delta \), where \( \bar{v}^{(2)} \) is the coefficient of \( \delta^{2\alpha} \) in the expansion of \( \bar{V}^\delta \) in the case \( 0 < \alpha < 1/4 \), and satisfies the linear PDE (4.12).

Otherwise if \( (t, x, z) \in C_1 \), we only need
\[ \mathbb{E}_{(t,x,z)} \int_t^T \mathcal{M}^{(0)} ds, \mathbb{E}_{(t,x,z)} \int_t^T \mathcal{M}^{(1)} ds, \mathbb{E}_{(t,x,z)} \int_t^T g(Z_s)\sigma(Z_s)\pi^{(0)}v_x^{(1)} ds, \]
to be uniformly bounded.

(c) For the critical case \( \alpha = 1/4 \), we need all the above assumptions in part (ib) with \( \alpha \) replaced by \( 1/4 \), except the sixth one which is not needed. Then, the term \( \bar{v}^{(2)} \) becomes \( \bar{v}^{(3)} \), which is the coefficient of \( \sqrt{\delta} \) in the expansion and satisfies (4.11).

(ii) If \( \bar{\pi}^0 \neq \pi^{(0)} \),

(a) For \( (t, x, z) \) in the region \( K \), the following quantities need to be uniformly bounded in \( \delta \):
\[ \mathbb{E}_{(t,x,z)} \int_t^T \mathcal{M}^{(0)} ds, \mathbb{E}_{(t,x,z)} \int_t^T g(Z_s)\sigma(Z_s) \left( \pi^0 + \delta \pi^1 \right) \left( \bar{\pi}^1 \right)^2 \bar{v}_x^{(0)} ds, \]
\[ \mathbb{E}_{(t,x,z)} \int_t^T \sigma^2(Z_s) \left( \pi^0 \right)^2 \bar{v}_x^{(0)} ds, \mathbb{E}_{(t,x,z)} \int_t^T \mu(Z_s) \pi^1 \bar{v}_x^{(0)} ds, \]
where \( \bar{v}^{(0)} \) is the leading order solution that satisfies (4.8).

(b) For \( (t, x, z) \) in the region \( C \), where \( \bar{\pi}^0 \) and \( \pi^{(0)} \) are identical, requirements are the same as in part (i) with \( (t, x, z) \) restricted to be in \( C \).
F Uniform Bound for $A^\delta(\tau)$ Solution of (5.18)

In order to apply Lemma 5.1 to (5.18), we identify $q_0$, $q_1$, $q_2$ as follows and compute $\Delta$:

$$q_0 = \frac{(3 - 2\gamma)\mu^2}{(1 - \gamma)^2}, \quad q_1 = \frac{2\sqrt{\rho} \mu \beta}{1 - \gamma} - \delta, \quad q_2 = \frac{\delta}{2} \beta^2,$$

$$\Delta = \frac{2\delta \beta^2 \mu^2}{(1 - \gamma)^2} \left(2\rho^2 - 3 + 2\gamma\right) - 4\delta^{3/2} \frac{\rho \mu \beta}{1 - \gamma} + \delta^2.$$ (F.2)

Note that if $\Delta > 0$, by Lemma 5.1 (i), we have $\alpha_+ - \alpha_+ > 0$ and $\alpha_+ + \alpha_- = \text{sgn}(\rho)$ if $\rho \neq 0$ and positive otherwise. It remains to prove i) there exists $\delta$, such that $\tau^*(\delta) > T$ for $\delta \leq \delta_0$ (recall that $\tau^*(\delta)$ is defined to be the explosion time) and ii) $\lim_{\delta \to 0} A^\delta(\tau)$ exists. This is done in the various cases depending on the values of the exponent $\gamma$ of the power utility and the correlation coefficient $\rho$.

Denote by $\delta_0$ a constant such that when $\delta < \delta_0$, the first term of $\Delta$ is dominant, and in case the first term is zero, the second term is dominant. Recall that $0 < \gamma < 1$, we now examine the following cases:

A) $\frac{1}{2} \leq \gamma < 1$.

a) $-1 \leq \rho < -\sqrt{1.5 - \gamma}$. If $\gamma = \frac{1}{2}$, this case is empty.

In this case, $\Delta$ is positive, $A^\delta(\tau)$ lies in type (5.4), and $\tau^*(\delta) = \infty$ for $\delta \in (0, \delta_0]$. The last claim follows by $\alpha_+ > \alpha_- > 0$ and $1 - \frac{a}{\alpha_+} e^{-\alpha_+ \tau} > 0, \forall \tau \in [0, T]$. Therefore we could define $\delta$ to be $\delta_0$, and the limit of $A^\delta(\tau)$ is obtained by straightforward calculation:

$$A^\delta(\tau) = \frac{-q_1 - \alpha}{2q_2} - \frac{\alpha \tau + \mathcal{O}(\delta)}{1 - \frac{-q_1 - \alpha}{q_1 + \alpha}(1 - \alpha \tau + \mathcal{O}(\delta))}$$ (F.3)

$$= \frac{(-q_1 - \alpha)(-q_1 + \alpha)\alpha \tau + \mathcal{O}(\delta)}{2\alpha + \mathcal{O}(\delta)} \to q_0 \tau, \text{ as } \delta \to 0.$$ (F.4)

b) $\rho = -\sqrt{1.5 - \gamma}$.

Under the situation $\rho = -\sqrt{1.5 - \gamma}$, the first term in $\Delta$ vanishes and the second term is dominant, which still gives the positiveness of $\Delta$. So this case is similar to the previous one: positive $\Delta$, $\alpha_+ > \alpha_- > 0$ and $\tau^*(\delta) = \infty$ for $\delta \leq \delta_0 = \delta_0$, except for a slight difference in computing $\lim_{\delta \to 0} A^\delta(\tau) = q_0 \tau$.

c) $-\sqrt{1.5 - \gamma} < \rho < 0$.

In case c) - f), $\Delta$ is negative, thus $A^\delta(\tau)$ lies in type (5.5). In addition, discussion in e)-f) are parallel to case c) and d). For $\rho$ negative, we have $a = q_1/2$ negative, and $\tau^*(\delta) = \frac{\pi}{2} + \frac{1}{b} \arctan\left(\frac{b}{a}\right)$. Now we further restrict the value of $\delta$ to $[0, \delta_0]$, so that $\tau^*(\delta) > T$. In other words, for sufficient small $\delta \leq \delta_0$, $A^\delta(\tau)$ and $B^\delta(\tau)$ still stay finite on $[0, T]$. To this end, it suffices to show $\lim_{\delta \to 0} \tau^*(\delta) = +\infty$:

$$\lim_{\delta \to 0} \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{\sqrt{2(3 - 2\gamma - 2\rho^2)}}{2\rho}\right),$$ (F.5)

$$\lim_{\delta \to 0} \tau^*(\delta) = \lim_{\delta \to 0} \left\{ \frac{\pi}{b} + \frac{1}{b} \arctan\left(\frac{b}{a}\right) \right\} = \lim_{\delta \to 0} \left\{ \frac{\pi}{b} + \frac{\arctan(b/a)}{b} \right\} = +\infty.$$ (F.6)

Therefore $A^\delta(\tau)$ is continuous on $[0, T] \times (0, \delta_0]$, and the problem is reduced to show the finiteness of $\lim_{\delta \to 0} A^\delta(\tau)$. Straightforward calculation yields:

$$\lim_{\delta \to 0} A^\delta(\tau) = \lim_{\delta \to 0} q_0 \frac{b \tau + \mathcal{O}(\delta^2)}{b + \mathcal{O}(\delta)} = q_0 \tau.$$ (F.7)

d) $\rho = 0$.  

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If $\rho = 0$, $a = -\delta/2$ is still negative, and we still have $\tau^*(\delta) = \frac{\pi}{b} + \frac{1}{b} \arctan \left( \frac{\delta}{a} \right)$. The argument here is the same as in case c) except calculations in (F.5) and (F.6).

$$\lim_{\delta \to 0} \arctan \left( \frac{b}{a} \right) = \frac{\pi}{b} \arctan \left( \frac{\delta}{a} \right) = -\pi/2,$$

$$\lim_{\delta \to 0} \tau^*(\delta) = \lim_{\delta \to 0} \left\{ \frac{\pi}{b} + \frac{1}{b} \arctan \left( \frac{b}{a} \right) \right\} = \lim_{\delta \to 0} \left\{ \frac{\pi + \arctan(b/a)}{b} \right\} = +\infty. \quad (F.9)$$

The last step follows by that the top converge to $\pi/2$ and the bottom converges to 0.

e) $0 < \rho < \sqrt{1.5 - \gamma}$.

Now both $a$ and $b$ are positive, and $\tau^*(\delta) = \frac{1}{b} \arctan \left( \frac{\delta}{a} \right)$. Similar calculation as (F.5) and (F.6) yields $\lim_{\delta \to 0} \tau^*(\delta) = +\infty$. So we could still choose $\delta$, such that $\tau^*(\delta) > T$ for $\delta \in (0, \delta]$. Then the problem is still reduced to the finiteness of $\lim_{\delta \to 0} A^4(\tau)$, which is proved by (F.7).

f) $\rho = \sqrt{1.5 - \gamma}$.

In this case the first term of $\Delta$ vanishes and the second term is dominant. Therefore $\Delta$ is negative, and solutions to $A^4(\tau)$ and $B^4(\tau)$ lie in type (5.5). This case is similar to case e), where both $a$ and $b$ are positive and $\tau^*(\delta) = \frac{1}{b} \arctan \left( \frac{\delta}{a} \right)$, except the calculation in (F.5) and (F.6).

$$\frac{b}{a} \geq \sqrt{\Delta} = \frac{2}{q_1} \sqrt{\frac{\rho \mu \beta}{1-\gamma}} + O(\delta^{1/2}) = \sqrt{\frac{1-\gamma}{\rho \mu \beta}} \delta^{1/2} + O(\delta^{1/2}),$$

$$\lim_{\delta \to 0} \tau^*(\delta) = \lim_{\delta \to 0} \left\{ \frac{1}{b} \arctan \left( \frac{b}{a} \right) \right\} = \lim_{\delta \to 0} \left\{ \frac{1}{b} \left( \sqrt{\frac{1-\gamma}{\rho \mu \beta}} \delta^{1/2} + O(\delta^{1/2}) \right) \right\} = +\infty. \quad (F.12)$$

The last step holds since $b$ is of order $\delta^{1/2}$ and $\arctan \left( \frac{\delta}{a} \right)$ is of order $\delta^{1/2}$.

g) $\sqrt{1.5 - \gamma} < \rho \leq 1$. If $\gamma = \frac{1}{2}$, this case is empty.

In this case, $\Delta$ becomes positive again, and $A^4(\tau)$ is of type (5.4). However here $\alpha_- < \alpha_+ < 0$, and $\tau^*(\delta) = -\frac{1}{\alpha} \ln \left( \frac{\alpha_+}{\alpha_-} \right)$. If we could pick $\overline{\tau}$, such that $\tau^*(\delta) > T$, for $\delta \leq \overline{\tau}$, then combining the computation of $\lim_{\delta \to 0} A^4(\tau)$ in a), we still capable to achieve the conclusion that $A^4(\tau)$ is bounded on $[0, T] \times [0, \overline{\tau}]$. Such $\overline{\tau}$ exists since

$$\tau^*(\delta) = -\frac{1}{\alpha} \ln \left( \frac{-q_1 + \alpha}{-q_1 - \alpha} \right) = \frac{\kappa + O(\delta^{1/2})}{\alpha} \to +\infty, \text{ as } \delta \to 0. \quad (F.13)$$

where $\kappa$ is a positive constant free of $\delta$ defined as:

$$\kappa = \lim_{\delta \to 0} \ln \left( \frac{\alpha_+}{\alpha_-} \right). \quad (B)$$

B) $0 \leq \gamma < \frac{1}{2}$, where $\Delta$ is always negative.

a) $-1 \leq \rho < 0$. The discussion here is the same as A)c).

b) $\rho = 0$. The discussion here is the same as A)d).

c) $0 < \rho \leq 1$. The discussion here is the same as A)e).

We have shown that for all cases $A^4(\tau)$ is uniformly bounded in $[0, \overline{\tau}] \times [0, T]$, say by $C(T)$, then again via (5.19), we achieve the uniform bound for $B^4(\tau)$:

$$B^4(\tau) \leq \delta mC(T)T \leq mC(T)T.$$

That concludes the verification of Assumption 2.11 (iv).
References


