SPECTRAL DECOMPOSITION OF OPTION PRICES IN FAST MEAN-REVERTING STOCHASTIC VOLATILITY MODELS

JEAN-PIERRE FOUQUE∗, SEBASTIAN JAIMUNGAL†, AND MATTHEW J. LORIG‡

Abstract. Using spectral decomposition techniques and singular perturbation theory, we develop a systematic method to approximate the prices of a variety of options in a fast mean-reverting stochastic volatility setting. Four examples are provided in order to demonstrate the versatility of our method. These include: European options, up-and-out options, double-barrier knock-out options, and options which pay a rebate upon hitting a boundary. For European options, our method is shown to produce option price approximations which are equivalent to those developed in [5].

Key words. Stochastic volatility, fast mean-reversion, asymptotics, spectral theory, barrier options, rebate options.

AMS subject classifications. 60H30, 65N25, 91B25, 91G20

1. Introduction. Since it was originally analyzed in the context of Sturm-Liouville operators, spectral theory has enjoyed wide popularity in both science and engineering. In physics, for example, the stationary-state wave functions are simply the eigenfunctions of the time-independent Schrödinger equation. Electrical engineers are well-versed in the theory of Fourier series and Fourier transforms. It is not surprising then, that techniques from spectral theory have found their place in finance as well.

For instance, in [13], eigenfunction methods are used to price European-style options in a Black-Scholes setting. The authors of [9] use eigenfunction techniques in the context of bond pricing. Spectral decomposition techniques have been particularly successful at aiding in the development of analytic pricing formulas for a variety of exotic options. For example, in [17], Fourier series methods are used to obtain closed-form expressions for prices of double-barrier options in a Black-Scholes setting. And in [3, 15, 16] spectral decomposition techniques are used to obtain analytic option prices—both European and path-dependent—where the underlying and short rate are controlled by a one-dimensional diffusion. Additionally, the authors of [1] use spectral theory to evaluate both bonds and options in a unified credit-equity framework.

Like spectral theory, stochastic volatility models have become an indispensable tool in mathematical finance. By and large, this is due to the fact that two of the earliest and most well-known stochastic volatility models—the Heston model [10] and Hull-White model [12]—capture the most salient features of the implied volatility surface while at the same time maintaining analytic tractability. Stochastic volatility models have become so popular, in fact, that entire books have been written on the subject [5, 8, 14].

It seems natural, then, to try to employ elements from spectral theory in a stochastic volatility setting. Yet, other than the spectral decomposition of various volatility processes, which is expertly done in [14], there is a surprising lack of literature in this area. In particular, the authors of this paper are unaware of any literature that uses...
spectral methods to price double-barrier options in a stochastic volatility setting in which the Brownian motions driving the stock and volatility are correlated. The difficulty with using spectral analysis when the stock price and volatility are correlated arises because spectral decomposition techniques work best when there is some sort of symmetry inherent in the problem being studied. This symmetry is broken when the stock price and volatility are correlated via two Brownian motions. Yet we know that correlation between the stock price and the volatility is important because it is needed in order to capture the skew of the implied volatility at the money [5, 8].

In this paper, we apply techniques from spectral theory to a class of fast mean-reverting stochastic volatility models in which the stock price and volatility are correlated via two Brownian motions. Such models, first studied in [5] are important because volatility has been empirically shown to operate on short time-scales [6, 11]. In particular, we provide a general method for deriving the approximate price of a variety of path-dependent and European options. We do this by reducing the option-pricing problem to the problem of solving a few, simple, eigenproblems.

The remainder of this paper is organized as follows. In section 2 we introduce a class of fast mean-reverting models. In section 2.1 we examine the option-pricing problem, derive a pricing PDE, review the separation of variables technique, and show how this leads to an eigenvalue equation. In section 3 an asymptotic expansion is performed with respect to the small parameter, $\epsilon$, which represents the characteristic time-scale of the volatility process. This leads to $O(\epsilon^0)$ and $O(\sqrt{\epsilon})$ eigenvalue equations. A useful transformation is then performed in section 3.1 in order to ensure that the $O(\epsilon^0)$ eigenfunctions are orthonormal. The important matter of boundary conditions is discussed in section 3.2. In section 3.3, we show how to use the terminal boundary condition, given by the payoff of the option at maturity, in order to find the undetermined coefficients of the separation of variables technique. Section 4.1 outlines a step-by-step method to obtain the approximate price of a variety of path-dependent and European options. Examples of how to use this method are provided in sections 4.2, 4.3, 4.4 and 4.5. Finally, an appendix is provided in order to clarify some derivations (appendix A) and discuss some numerical issues (appendix B).

2. A Class of Fast Mean-Reverting Models. We consider a class of fast mean-reverting stochastic volatility models, whose dynamics under the physical measure, $\mathbb{P}$, is given by the following system of SDE’s

\[
\begin{align*}
    dS_t &= \mu S_t \, dt + f(Y^\epsilon_t) S_t \, dW_t, \\
    X_t &= \log S_t, \\
    dX_t &= \left( \mu - \frac{1}{2} f^2(Y^\epsilon_t) \right) \, dt + f(Y^\epsilon_t) \, dW_t, \\
    dY^\epsilon_t &= \frac{1}{\epsilon} (m - Y^\epsilon_t) \, dt + \frac{\nu \sqrt{2}}{\sqrt{\epsilon}} \, dB_t, \\
    d\langle W, B \rangle_t &= \rho \, dt.
\end{align*}
\]

Here, $W_t$ and $B_t$ are Brownian motions with instantaneous correlation $\rho$, where $\rho^2 \leq 1$. The process $S_t$ represents the price of a non-dividend paying asset (stock, index, etc.), which evolves as a diffusion with constant drift $\mu$ and stochastic volatility $f(Y^\epsilon_t)$. For convenience, we introduce $X_t = \log S_t$. The dynamics of $X_t$ are obtained using Ito’s Lemma. The process $Y^\epsilon_t$ appears as an Ornstein-Uhlenbeck (OU) process running
on time-scale $\epsilon$. Throughout this paper, we will indicate explicit dependence on the parameter $\epsilon$ with a superscript. The parameter $\epsilon$ is intended to be small (i.e. $0 < \epsilon \ll 1$) so that the rate of mean-reversion of the OU process is large. It is in this sense that the volatility is fast mean-reverting. We note that $Y_t^{\epsilon}$ has an invariant distribution, $Y_\infty^{\epsilon} \sim \mathcal{N}(m, \nu^2)$, which is independent of $\epsilon$. It is not necessary to specify the precise form of $f(y)$, as it will not play a role in the asymptotic analysis that follows. However, we assume that there exist constants $c_1$ and $c_2$ such that $0 < c_1 \leq f(y) \leq c_2 < \infty$ for all $y \in \mathbb{R}$. Likewise, the particular choice of $Y_t^{\epsilon}$ as an OU process is not crucial for our analysis. The essential aspect of $Y_t^{\epsilon}$ is that it be an ergodic process with a unique invariant distribution, which is independent of $\epsilon$.

We introduce a money market account $M_t = M_0 e^{rt}$, which grows at a constant risk-free rate, $r > 0$. In order for the model to be arbitrage-free, we assume the existence of a class of equivalent martingale measures (EMM’s) $\tilde{\mathbb{P}}^A$, which are indexed by the market price of volatility risk $A$. Furthermore, we suppose the market price of volatility risk is a function of $Y_t^{\epsilon}$ only (i.e. $A = A(Y_t^{\epsilon})$). Under these assumptions, the class of stochastic volatility models takes the following form

$$dS_t = rS_t dt + f(Y_t^{\epsilon})S_t d\tilde{W}_t,$$

$$dX_t = \left( r - \frac{1}{2} f^2(Y_t^{\epsilon}) \right) dt + f(Y_t^{\epsilon}) d\tilde{W}_t,$$

$$dY_t^{\epsilon} = \left[ \frac{1}{\epsilon} (m - Y_t^{\epsilon}) - \frac{\nu \sqrt{2}}{\sqrt{\epsilon}} \Lambda(Y_t^{\epsilon}) \right] dt + \frac{\nu \sqrt{2}}{\sqrt{\epsilon}} d\tilde{B}_t,$$

$$d\langle \tilde{W}, \tilde{B} \rangle_t = \rho dt.$$

Here, $\tilde{W}_t$ and $\tilde{B}_t$ are Brownian motions under $\tilde{\mathbb{P}}^A$. We note that, as it should be, $(S_t/M_t)$ is a martingale with respect to the canonical filtration of the Brownian motions. It is not necessary to specify the precise form of $\Lambda(y)$. However, we assume the existence of a constant $c_3$ such that $|\Lambda(y)| \leq c_3 < \infty$ for all $y \in \mathbb{R}$. In all, the relatively weak conditions on the functions, $f(y)$ and $\Lambda(y)$, as well as the the condition that $Y_t^{\epsilon}$ be ergodic, admit a large class of fast mean-reverting stochastic volatility models.

For the remainder of this paper we will assume that the market chooses a unique EMM $\tilde{\mathbb{P}}^A = \tilde{\mathbb{P}}$, under which all options can be priced consistently.

### 2.1. Pricing PDE.

We consider an option maturing at finite time $T > t$, whose payoff can be expressed as $h(X_\tau)$ where

$$\tau = \inf \{ s > 0 : X_s \notin (x_l, x_u) \} \wedge T, \quad (2.1)$$

$$h : [x_l, x_u] \to \mathbb{R}.$$ 

Here, $-\infty \leq x_l < x_u \leq \infty$. Options which fall into this category include European options, single- and double-barrier knock-out options and options which pay a rebate upon hitting a barrier or have a random payoff at maturity. According to risk-neutral pricing, the value of the option, $P_t^\epsilon$, at time $t < \tau$ is given by

$$P_t^\epsilon = M_t \tilde{\mathbb{E}} \left[ \frac{h(X_\tau)}{M_\tau} \left| X_t, Y_t^{\epsilon} \right. \right] =: P^\epsilon(t, X_t, Y_t^{\epsilon}),$$

where $\tilde{\mathbb{E}}$ is the expectation operator under the risk-neutral measure, $\tilde{\mathbb{P}}$. Note, in defining the function $P^\epsilon(t, x, y)$ we have used the Markov property of the joint process
\((X_t, Y_t^*)\). It is well-known that \(P^\epsilon(t, x, y)\) satisfies the following partial differential equation (PDE) and boundary conditions (BC’s)

\[
\left(\partial_t - r + \mathcal{L}_{X,Y}^\epsilon\right)P^\epsilon = 0, \quad (2.2)
\]

\[
P^\epsilon(T, x, y) = h(x), \quad (2.3)
\]

\[
\lim_{x \to x_l} P^\epsilon(t, x, y) = h(x_l) =: R_l, \quad \text{(optional)} \quad (2.4)
\]

\[
\lim_{x \to x_u} P^\epsilon(t, x, y) = h(x_u) =: R_u. \quad \text{(optional)} \quad (2.5)
\]

Equation (2.3), the terminal BC, is derived from the option payoff at maturity. The lower BC, equation (2.4), is derived from the value of the option \(R_l\) the first time the log of the underlying hits a barrier at \(x_l < X_t\). For example, if the option “knocks out” when the log of the underlying reaches a level \(x_l\), then \(\lim_{x \to x_l} P^\epsilon(t, x, y) = 0\).

If the option pays a rebate \(R_l\) when the log of the underlying reaches a level \(x_l\) then \(\lim_{x \to x_l} P^\epsilon(t, x, y) = R_l\). The upper BC, equation (2.5), is derived analogously. Note that we have required a terminal BC. However, the upper and lower BC’s are optional.

In the case of a European option, for example, we would omit equations (2.4) and (2.5) and we would formally take \(x_l = -\infty\) and \(x_u = \infty\) and set \(R_l = R_u = 0\). Likewise, for an up-and-out option we would omit BC (2.4) and formally set \(x_l = -\infty\) and \(R_l = 0\).

We shall seek solution to the boundary value problem (BVP), defined by equations (2.2-2.5), via the method of separation of variables. To begin, we look for solutions of the PDE

\[
\left(\partial_t - r + \mathcal{L}_{X,Y}^\epsilon\right)u^\epsilon = 0, \quad (2.6)
\]

of the form

\[
u^\epsilon(t, x, y) = g^\epsilon(t)\Psi^\epsilon(x,y). \quad (2.7)
\]

We assume \(u^\epsilon(t, x, y) \neq 0\) for all \(t, x\) and \(y\). Now, inserting (2.7) into (2.6) and dividing through by \(g^\epsilon(t)\Psi^\epsilon(x,y)\), we find

\[
\frac{-(\partial_t - r)g^\epsilon}{g^\epsilon} = \mathcal{L}_{X,Y}^\epsilon \Psi^\epsilon. \quad (2.8)
\]

Since the left hand side of (2.8) depends only on \(t\), and the right hand side depends only on \(x\) and \(y\), we conclude that both sides must be equal to a constant, which we denote as \(\lambda_q^\epsilon\). We then have

\[
-(\partial_t - r)g_q^\epsilon = \lambda_q^\epsilon g_q^\epsilon, \quad (2.9)
\]

\[
\mathcal{L}_{X,Y}^\epsilon \Psi_q^\epsilon = \lambda_q^\epsilon \Psi_q^\epsilon. \quad (2.10)
\]

The index \(q\) indicates that there are a number of different values that \(\lambda_q^\epsilon\) could take. We include the subscript \(q\) on \(g_q^\epsilon(t)\) and \(\Psi_q^\epsilon(x,y)\) in order to indicate their dependence on the constant \(\lambda_q^\epsilon\). Equation (2.10), along with appropriate BC’s at \(x_l\) and \(x_u\), define an eigenvalue problem. In general, if either of \(x_l = -\infty\) or \(x_u = \infty\), then the spectrum of \(\lambda_q^\epsilon\) will be continuous. To indicate this, we will use the Greek letters \(\omega, \nu, \mu\) in place of \(q\) (i.e. \(q \rightarrow \{\omega, \nu, \mu\}\) for continuous spectrum). On the other hand, if \(-\infty < x_l < x_u < \infty\), then the spectrum of \(\lambda_q^\epsilon\) will be discrete. In this case, we will use the the Latin letters \(m, n, k\) in place of \(q\) (i.e. \(q \rightarrow \{m, n, k\}\) for discrete...
When we do not wish to distinguish between the discrete and continuous spectrum cases, we will leave \( q \) as it is.

Although it is not necessary, it is convenient to impose the boundary condition
\[
g_q(\epsilon) = 1. \quad \text{Upon doing this, the solution to equation (2.9), which is valid for any } \lambda_q^r, \text{ is straightforward}
\]
\[
g_q(\epsilon) = \epsilon^{(\lambda_q^r-r)(T-t)}. \quad (2.11)
\]

Of course, we do not expect that option prices can be written as a simple product
\[
P(\epsilon)(t,x,y) = g_q(\epsilon)(t)\Psi_q(x,y). \quad \text{Such an assumption would not allow sufficient flexibility to match boundary conditions (2.3-2.5). Rather, we suppose that}
\]
\[
P(\epsilon)(t,x,y) \text{ can be written as a linear combination of product solutions. Specifically, our proposed solutions for the continuous and discrete spectrum cases are}
\]
\[
P(\epsilon)(t,x,y) = \Psi_q^r(x,y) + \int A_q^r g_q(\epsilon)(t)\Psi_q(x,y) d\omega, \quad \text{(continuous spectrum)} \quad (2.12)
\]
\[
P(\epsilon)(t,x,y) = \Psi_q^r(x,y) + \sum_m A_q^m g_q(\epsilon)(t)\Psi_q(x,y). \quad \text{(discrete spectrum)} \quad (2.13)
\]

Here, the pair \( \{\Psi_q^r(x,y), \lambda_q^r\} \) is a solution to the eigenvalue equation (2.10), which satisfies BCs
\[
\Psi_q^r(x_l,y) = 0 \text{ if } x_l > -\infty, \quad (2.14)
\]
\[
\Psi_q^r(x_u,y) = 0 \text{ if } x_u < \infty. \quad (2.15)
\]

\( \Psi_q^r(x,y) \) is a solution to equation (2.10) with \( \lambda_q^r = r =: \lambda_q^r \), which satisfies BC's
\[
\lim_{x \to x_l} \Psi_q^r(x,y) = R_l, \quad (2.16)
\]
\[
\lim_{x \to x_u} \Psi_q^r(x,y) = R_u. \quad (2.17)
\]

We call \( \Psi_q^r(x,y) \) the “steady-state” solution of equation (2.6) since there is no dependence on \( t \).

In equations (2.12, 2.13), the function \( g_q(\epsilon)(t) \) is given by (2.11) and the \( A_q^r \) are coefficients, which are independent of \( t \) and \( x,y \). By the linearity of the operator \( (\partial_t - r + \mathcal{L}_{X,Y}) \) and by BC's (2.14-2.17), the proposed solutions (2.12, 2.13) will automatically satisfy PDE (2.2) and BCs (2.4, 2.5). What remains is to choose the \( A_q^r \) such that the terminal boundary condition (2.3) is satisfied as well.

3. Asymptotic Analysis. For general \( f(y) \) and \( \Lambda(y) \) there is no analytic set of solutions to the eigenvalue problem \( \mathcal{L}_{X,Y}x^r \Psi_q^r = \lambda_q^r \Psi_q^r \). However, we note that \( \mathcal{L}_{X,Y} \) can be conveniently decomposed in powers of \( \sqrt{\epsilon} \) as
\[
\mathcal{L}_{X,Y} = \frac{1}{\epsilon} \mathcal{L}^{(-2)} + \frac{1}{\sqrt{\epsilon}} \mathcal{L}^{(-1)} + \mathcal{L}^{(0)},
\]
\[
\mathcal{L}^{(-2)} = (m-y) \partial_u + \nu^2 \partial^2_{yy},
\]
\[
\mathcal{L}^{(-1)} = \rho \sqrt{2f(y)} \partial_y^2 - \nu \sqrt{2\Lambda(y)} \partial_y,
\]
\[
\mathcal{L}^{(0)} = \left( r - \frac{1}{2} f^2(y) \right) \partial_x + \frac{1}{2} f^2(y) \partial_{xx}^2. \quad (3.1)
\]
This decomposition suggests a singular perturbative approach. To this end, we expand \( \Psi^\epsilon_q \) and \( \lambda^\epsilon_q \) in powers of \( \sqrt{\epsilon} \). We have

\[
\Psi^\epsilon_q = \Psi^{(0)}_q + \sqrt{\epsilon} \Psi^{(1)}_q + \epsilon \Psi^{(2)}_q + \ldots,
\]

\[
\lambda^\epsilon_q = \lambda^{(0)}_q + \sqrt{\epsilon} \lambda^{(1)}_q + \epsilon \lambda^{(2)}_q + \ldots.
\]

Since \( g^\epsilon_q(t) \) and \( A^\epsilon_q \) depend on \( \epsilon \), to be consistent, we must expand them as well. We have

\[
A^\epsilon_q = A^{(0)}_q + \sqrt{\epsilon} A^{(1)}_q + \ldots,
\]

\[
g^\epsilon_q(t) = g^{(0)}_q + \sqrt{\epsilon} g^{(1)}_q + \ldots,
\]

where (3.2) and (3.3) are obtained by expanding (2.11) in powers of \( \sqrt{\epsilon} \). We insert the above expansions into (2.12, 2.13) and obtain

\[
P^\epsilon = P^{(0)} + \sqrt{\epsilon} P^{(1)} + \ldots,
\]

where \( P^{(0)} \) is given by

\[
P^{(0)} = \Psi^{(0)}_r + \int A^{(0)}_m g^{(0)}_m \Psi^{(0)} d\omega, \quad \text{(continuous spectrum)}
\]

\[
P^{(0)} = \Psi^{(0)}_r + \sum_m A^{(0)}_m g^{(0)}_m \Psi^{(0)}_m, \quad \text{(discrete spectrum)}
\]

and \( P^{(1)} \) is given by

\[
P^{(1)} = \Psi^{(1)}_r + \int \left( A^{(1)}_m g^{(0)}_m \Psi^{(0)} + A^{(0)}_m g^{(1)}_m \Psi^{(0)} + A^{(0)}_m g^{(0)}_m \Psi^{(1)} \right) d\omega, \quad \text{(continuous spectrum)}
\]

\[
P^{(1)} = \Psi^{(1)}_r + \sum_m \left( A^{(1)}_m g^{(0)}_m \Psi^{(0)} + A^{(0)}_m g^{(1)}_m \Psi^{(0)} + A^{(0)}_m g^{(0)}_m \Psi^{(1)} \right), \quad \text{(discrete spectrum)}
\]

Returning to the eigenvalue problem, we insert the expansions of \( \Psi^\epsilon_q \) and \( \lambda^\epsilon_q \) into \( \mathcal{L}_{X,Y} \Psi^\epsilon_q = \lambda^\epsilon_q \Psi^\epsilon_q \) and collect terms of like-powers of \( \sqrt{\epsilon} \). At \( \mathcal{O}(1/\epsilon) \) we have

\[
\mathcal{L}^{(-2)} \Psi^{(0)}_q = 0.
\]

Noting that both terms in \( \mathcal{L}^{(-2)} \) contain derivatives with respect to \( y \), we see that equation (3.8) has a solution of the form

\[
\Psi^{(0)}_q = \Psi^{(0)}_q(x).
\]

Similarly, collecting terms of \( \mathcal{O}(1/\sqrt{\epsilon}) \) we find

\[
\mathcal{L}^{(-2)} \Psi^{(1)}_q + \mathcal{L}^{(-1)} \Psi^{(0)}_q = 0 \quad \Rightarrow \quad \mathcal{L}^{(-2)} \Psi^{(1)}_q = 0.
\]
Note that $\mathcal{L}^{-1}\Psi_q^{(0)} = 0$ since both terms in $\mathcal{L}^{-1}$ contain derivatives with respect to $y$. As such, we see that equation (3.9) has a solution of the form

$$\Psi_q^{(1)} = \Psi_q^{(1)}(x).$$

Continuing the asymptotic analysis, at $O(\varepsilon^0)$ we have

$$L^{(-2)}\Psi_q^{(2)} + L^{(-1)}\Psi_q^{(1)} + L^{(0)}\Psi_q^{(0)} = \lambda_q^{(0)}\Psi_q^{(0)},$$

$$\Rightarrow L^{(-2)}\Psi_q^{(2)} + L^{(0)}\Psi_q^{(0)} = \lambda_q^{(0)}\Psi_q^{(0)},$$

$$\Rightarrow L^{(-2)}\Psi_q^{(2)} = (\lambda_q^{(0)} - L^{(0)})\Psi_q^{(0)},$$

(3.10)

where we have used $L^{-1}\Psi_q^{(1)} = 0$ for the same reason that $L^{(-1)}\Psi_q^{(0)} = 0$. Equation (3.10) is a Poisson equation for $\Psi_q^{(2)}$ in the variable $y$ with respect to the operator $L^{(-2)} = \varepsilon L_{\gamma}$. The operator $\varepsilon L_{\gamma}$ is the infinitesimal generator of $Y_\gamma^1$ under the physical measure $\mathbb{P}$. In order for an equation of the form $L_{\gamma}^* u(y) = v(y)$ to have a solution with reasonable growth at infinity, the following centering condition must hold

$$\langle v \rangle := \int v(y)dF_{\gamma}(y) = 0.$$

Here, $F_{\gamma}$ is the distribution of $Y_\gamma^\infty$ under $\mathbb{P}$. Throughout this paper, the notation $\langle \cdot \rangle$ will always indicate averaging with respect to the invariant distribution $F_{\gamma}$. In equation (3.10), the role of $v(y)$ is played by $(\lambda_q^{(0)} - L^{(0)})\Psi_q^{(0)}(y)$. Hence, in order for $\Psi_q^{(2)}$ to have a solution with reasonable growth at infinity, we must have

$$\langle (\lambda_q^{(0)} - L^{(0)})\Psi_q^{(0)} \rangle = 0,$$

$$\langle L^{(0)} \rangle \Psi_q^{(0)} = \lambda_q^{(0)}\Psi_q^{(0)}.$$  

(3.11)

Equation (3.11) is one of two key results of this section. It says that $\lambda_q^{(0)}$ is an eigenvalue of $\langle L^{(0)} \rangle$ and $\Psi_q^{(0)}(x)$ is the corresponding eigenfunction. Returning to the asymptotic analysis, at $O(\sqrt{\tau})$ we see

$$L^{(-2)}\Psi_q^{(3)} + L^{(-1)}\Psi_q^{(2)} + L^{(0)}\Psi_q^{(1)} = \lambda_q^{(0)}\Psi_q^{(1)} + \lambda_q^{(1)}\Psi_q^{(0)},$$

$$\Rightarrow L^{(-2)}\Psi_q^{(3)} = -L^{(-1)}\Psi_q^{(2)} + (\lambda_q^{(0)} - L^{(0)})\Psi_q^{(1)} + \lambda_q^{(1)}\Psi_q^{(0)}.$$  

(3.12)

Equation (3.12) is a Poisson equation for $\Psi_q^{(3)}$ in the variable $y$ with respect to the operator $L^{(-2)}$. Therefore, in order to ensure reasonable growth of $\Psi_q^{(3)}$ at infinity, we must enforce the following centering condition

$$\langle -L^{(-1)}\Psi_q^{(2)} + (\lambda_q^{(0)} - L^{(0)})\Psi_q^{(1)} + \lambda_q^{(1)}\Psi_q^{(0)} \rangle = 0,$$

$$\Rightarrow -\langle L^{(-1)}\Psi_q^{(2)} \rangle + (\lambda_q^{(0)} - L^{(0)})\Psi_q^{(1)} + \lambda_q^{(1)}\Psi_q^{(0)} = 0.$$  

(3.13)

Equation (3.13) is the second key result of this section. As we will see, this equation will enable us to find the $O(\sqrt{\tau})$ eigenvalues and eigenfunctions. This is as far as we will take the asymptotic analysis.
3.1. A Self-Adjoint Transformation. Recall that $\Psi^{(0)}(x)$ satisfies equation (3.11), which we repeat here for clarity

$$\langle L^{(0)} \rangle \Psi^{(0)} = \lambda^{(0)} \Psi^{(0)},$$

where $\langle L^{(0)} \rangle$ is given by

$$\langle L^{(0)} \rangle = \left( r - \frac{1}{2} \sigma^2 \right) \partial_x + \frac{1}{2} \sigma^2 \partial_{xx},$$

$$\sigma^2 = \langle f^2 \rangle.$$

Equation (3.14), along with boundary conditions at $x_l$ and $x_u$, define an eigenvalue problem. As it stands, equation (3.14) is fairly innocuous; it is simply a linear second order ODE with constant coefficients. However, because of the $\partial_x$ term, $\langle L^{(0)} \rangle$ is not self-adjoint in $L^2((x_l, x_u), (\cdot, \cdot))$, where we have defined the inner product

$$(u, v) := \int_{x_l}^{x_u} \overline{u(x)} v(x) dx.$$

Hence, the eigenfunctions $\Psi^{(0)}_q(x)$ will not be orthonormal \(^1\). It will greatly simplify the analysis that follows if we have a set of orthonormal eigenfunctions with which to work. Hence, we will make the following useful transformation. We suppose $\Psi^{(0)}_q(x)$ can be written

$$\Psi^{(0)}_q(x) = e^{cx} \psi^{(0)}_q(x),$$

$$c = -\frac{(r - \frac{1}{2} \sigma^2)}{\sigma^2}.$$  

One can easily show

$$\langle L^{(0)} \rangle \Psi^{(0)} = e^{cx} \tilde{L}^{(0)} \psi^{(0)}_q,$$

where

$$\tilde{L}^{(0)} = -\frac{1}{2} \Gamma^2 + \frac{1}{2} \sigma^2 \partial_{xx},$$

$$\Gamma^2 = \frac{(r - \frac{1}{2} \sigma^2)^2}{\sigma^2} = \sigma^2 c^2.$$  

An important consequence of this transformation is that $\langle L^{(0)} \rangle \Psi^{(0)}_q = \lambda^{(0)} \psi^{(0)}_q$ if and only if

$$\tilde{L}^{(0)} \psi^{(0)}_q = \lambda^{(0)} \psi^{(0)}_q,$$

or, if

$$\partial^2_{xx} \psi^{(0)}_q = \lambda^{(0)} \psi^{(0)}_q,$$

\(^1\) Alternatively, would could work with the weighted inner product $(u, v) := \int_{x_l}^{x_u} \overline{u(x)} v(x) e^{-2cx} dx$, where $c$ is given by equation (3.16). On this inner product space, the operator $\langle L^{(0)} \rangle$ is orthonormal.
where
\[
\lambda^{(0)}_q = \frac{2\lambda^{(0)}_q + \Gamma^2}{\sigma^2}, \quad \lambda^{(0)}_q = -\frac{1}{2}\Gamma^2 + \frac{1}{2}\pi^2\lambda^{(0)}_q.
\] (3.20)

Equation (3.19) is simple to solve for a variety of different boundary conditions. Furthermore, because \(\partial_{xx}^2\) and \(\tilde{L}^{(0)}\) are self-adjoint, the transformed eigenfunctions \(\psi^{(0)}_q\), properly normalized, form a complete orthonormal basis in \(L^2((x_l,x_u), (\cdot, \cdot))\). i.e.

\[
\begin{align*}
\left(\psi^{(0)}_\omega; \psi^{(0)}_\nu\right) &= \delta(\omega - \nu), \quad \text{(continuous spectrum)} \\
\left(\psi^{(0)}_m; \psi^{(0)}_n\right) &= \delta_{m,n}, \quad \text{(discrete spectrum)}
\end{align*}
\]

Now, recall that \(\Psi^{(2)}_q(x,y)\) satisfies equation (3.10), which we repeat here for clarity

\[
\mathcal{L}^{(-2)}\Psi^{(2)}_q = \left(\lambda^{(0)}_q - \mathcal{L}^{(0)}\right)\Psi^{(0)}_q.
\]

Therefore, using equation (3.11) we have

\[
\mathcal{L}^{(-2)}\Psi^{(2)}_q = \left(\left(\mathcal{L}^{(0)}\right) - \mathcal{L}^{(0)}\right)\Psi^{(0)}_q
= \frac{1}{2} (\sigma^2 - f^2) \left(\partial_{xx}^2 - \partial_x\right)\Psi^{(0)}_q.
\]

At this point, it is convenient to introduce a function \(\phi(y)\), which satisfies

\[
\mathcal{L}^{(-2)}\phi = f^2 - \sigma^2.
\]

Upon doing this, we may express \(\Psi^{(2)}_q(x,y)\) as

\[
\Psi^{(2)}_q(x,y) = -\frac{1}{2} \phi(y) \left(\partial_{xx}^2 - \partial_x\right)\Psi^{(0)}_q(x).
\] (3.21)

Now, recall that \(\Psi^{(1)}_q\) satisfies equation (3.13), which we repeat here for clarity

\[
\left(\mathcal{L}^{(-1)}\Psi^{(2)}_q\right) = \left(\lambda^{(0)}_q - \mathcal{L}^{(0)}\right)\Psi^{(0)}_q + \lambda^{(1)}_q\Psi^{(1)}_q.
\] (3.22)

Let us focus on the term \(\left(\mathcal{L}^{(-1)}\Psi^{(2)}_q\right)\) term for a moment. Using equations (3.1, 3.15, 3.21), we have

\[
\left(\mathcal{L}^{(-1)}\Psi^{(2)}_q\right) = \left(\left(\rho\sqrt{2}f(y)\partial_{xy}^2 - \rho\sqrt{2}\Lambda(y)\partial_y\right)\left(-\frac{1}{2} \phi(y) \left(\partial_{xx}^2 - \partial_x\right)\Psi^{(0)}_q(x)\right)\right)
= (V_3 \left(\partial_{xxx}^2 - \partial_{xx}^2\right) + V_2 (\partial_{xx}^2 - \partial_x)\right) e^{\frac{\sqrt{2}y}{\sigma}}\Psi^{(0)}_q(x),
\]

where we have defined group parameters

\[
V_2 = \frac{\nu}{\sqrt{2}} \langle \Lambda\phi' \rangle.
\] (3.23)

\[
V_3 = \frac{-\rho\nu}{\sqrt{2}} \langle f\phi' \rangle.
\] (3.24)
With a little more work, we derive
\[
\langle \mathcal{L}^{(1)} \psi_q^{(2)} \rangle = e^{cx} (V_2 \chi + V_3 \eta_q) \partial_x \psi_q^{(0)} + e^{cx} (V_2 \xi_q + V_3 \gamma_q) \psi_q^{(0)},
\]
where
\[
\chi = 2c - 1,
\eta_q = \lambda_q^{(0)} + 3c^2 - 2c,
\xi_q = \lambda_q^{(0)} + c^2 - c,
\gamma_q = (3c - 1) \lambda_q^{(0)} + c^3 - c^2.
\]

Next, we look at the term, \( \lambda_q^{(0)} - \langle \mathcal{L}^{(0)} \rangle \) \( \Psi_q^{(1)} \), from equation (3.22). As we did with \( \Psi_q^{(0)}(x) \) we will assume \( \Psi_q^{(1)}(x) \) can be written
\[
\Psi_q^{(1)}(x) = e^{cx} \psi_q^{(1)}(x),
\]
so that
\[
\left( \lambda_q^{(0)} - \langle \mathcal{L}^{(0)} \rangle \right) \Psi_q^{(1)} = e^{cx} \left( \lambda_q^{(0)} - \tilde{\mathcal{L}}^{(0)} \right) \psi_q^{(1)}. \quad (3.30)
\]

Now, inserting equations (3.15, 3.25, 3.30) into (3.22) and canceling a common factor of \( e^{cx} \) we find
\[
(V_2 \chi + V_3 \eta_q) \partial_x \psi_q^{(0)} + (V_2 \xi_q + V_3 \gamma_q) \psi_q^{(0)} = \left( \lambda_q^{(0)} - \tilde{\mathcal{L}}^{(0)} \right) \psi_q^{(1)} + \lambda_q^{(1)} \psi_q^{(0)}. \quad (3.31)
\]

Once one finds \( \psi_q^{(0)}(x) \) and \( \lambda_q^{(0)} \), one can use equation (3.31) to solve for \( \psi_q^{(1)}(x) \) and \( \lambda_q^{(1)} \). It is sometimes useful to write equation (3.31) as
\[
(V_2 \chi + V_3 \eta_q) \partial_x \psi_q^{(0)} + (V_2 \xi_q + V_3 \gamma_q) \psi_q^{(0)} = \frac{\sigma^2}{2} \left( \lambda_q^{(0)} - \partial_x^2 \right) \psi_q^{(1)} + \lambda_q^{(1)} \psi_q^{(0)}. \quad (3.32)
\]

Upon rearranging some terms, we see that (3.31) has a similar structure to that of (3.18)
\[
\tilde{\mathcal{L}}^{(0)} \psi_q^{(1)} = \lambda_q^{(0)} \psi_q^{(1)} - \left( (V_2 \chi + V_3 \eta_q) \partial_x \psi_q^{(0)} + (V_2 \xi_q + V_3 \gamma_q) \psi_q^{(0)} + \lambda_q^{(1)} \psi_q^{(0)} \right).
\]

That is, when \( \tilde{\mathcal{L}}^{(0)} \) acts upon \( \psi_q^{(1)} \) it yields the \( \mathcal{O}(\epsilon^0) \) eigenvalue \( \lambda_q^{(0)} \) times \( \psi_q^{(1)} \) minus a source term—given in parenthesis—which does not appear in (3.18).

We note that, by repeating the analysis of sections 3 and 3.1 for the special case \( \lambda_q = r \), one obtains the following results
\[
\psi_r = \psi_r^{(0)} + \sqrt{\epsilon} \psi_r^{(1)} + \ldots = e^{cx} \left( \psi_r^{(0)} + \sqrt{\epsilon} \psi_r^{(1)} + \ldots \right),
\lambda_r = \lambda_r^{(0)} + \sqrt{\epsilon} \lambda_r^{(1)} + \ldots = r + \sqrt{\epsilon} \cdot 0 + \ldots,
\lambda_r^{(1)} = \frac{2r + \Gamma^2}{\sigma^2},
\]
where $\psi_r^{(0)}(x)$ satisfies

$$
\partial_{xx}^2 \psi_r^{(0)} = \lambda_r^{(0)} \psi_r^{(0)},
$$

(3.33)

and $\psi_r^{(1)}(x)$ satisfies

$$
(V_2\chi + V_3\eta_r) \partial_x \psi_r^{(0)} + (V_2\xi_r + V_3\gamma_r) \psi_r^{(0)} = \frac{\sigma^2}{2} \left( \lambda_r^{(0)} - \partial_{xx}^2 \right) \psi_r^{(1)}.
$$

(3.34)

The four boxed equations (3.18, 3.31, 3.33, 3.34) serve as the main results of section 3. The “difficult work” in solving the option-pricing problem considered in section 2.1 is in finding solutions to the eigenproblems (3.18, 3.31) and the steady-state equations (3.33, 3.34). Once this is done, finding expressions for $P^{(0)}$ and $P^{(1)}$ is formulaic.

### 3.2. Boundary Conditions.

We must now specify appropriate BC’s for $\psi_q^{(0)}(x)$, $\psi_q^{(1)}(x)$, $\psi_r^{(0)}(x)$ and $\psi_r^{(1)}(x)$. From equation (2.14, 2.15) we see that the $O(\epsilon^0)$ BC’s for $\psi_q^{(0)}(x)$ are

$$
\psi_q^{(0)}(x_l) = 0 \text{ if } x_l > -\infty, \\
\psi_q^{(0)}(x_u) = 0 \text{ if } x_u < \infty,
$$

(3.35, 3.36)

and, the $O(\sqrt{\epsilon})$ BC’s for $\psi_q^{(1)}(x)$ are given by

$$
\psi_q^{(1)}(x_l) = 0 \text{ if } x_l > -\infty, \\
\psi_q^{(1)}(x_u) = 0 \text{ if } x_u < \infty.
$$

(3.37, 3.38)

We can obtain the $O(\epsilon^0)$ BC’s for $\psi_r^{(0)}(x)$ from equation (2.16, 2.17). Recalling that $\Psi_r^{(0)}(x) = e^{cx} \psi_r^{(0)}(x)$, we have

$$
\lim_{x \to x_l} e^{cx} \psi_r^{(0)}(x) = R_l, \\
\lim_{x \to x_u} e^{cx} \psi_r^{(0)}(x) = R_u.
$$

(3.39, 3.40)

Likewise, the $O(\sqrt{\epsilon})$ BC’s for $\psi_r^{(1)}(x)$ are given by

$$
\lim_{x \to x_l} e^{cx} \psi_r^{(1)}(x) = 0, \\
\lim_{x \to x_u} e^{cx} \psi_r^{(1)}(x) = 0.
$$

(3.41, 3.42)

We now examine the terminal boundary condition $P(t, x, y) = h(x)$. It will be useful to define

$$
Q^e(t, x, y) = P^e(t, x, y) - \Psi_r^e(x, y) = \begin{cases} 
\int A_m g_m^e(t) \psi_m^e(x, y) d\omega, & \text{(continuous spectrum)} \\
\sum_m A_m g_m^e(t) \psi_m^e(x, y), & \text{(discrete spectrum)}
\end{cases}
$$

Expanding $Q^e(t, x, y)$ as $Q^{(0)}(t, x) + \sqrt{\epsilon} Q^{(1)}(t, x) + \ldots$ we identify

$$
Q^{(0)}(t, x) = P^{(0)}(t, x) - \Psi_r^{(0)}(x) = \begin{cases} 
e^{cx} \int A_m g_m^{(0)} \psi_m^{(0)} d\omega, & \text{(continuous spectrum)} \\
e^{cx} \sum_m A_m g_m^{(0)} \psi_m^{(0)}, & \text{(discrete spectrum)}
\end{cases}
$$

(3.43)
Note that we have used \( \Psi^0_q = e^{rx} \psi^0_q \) and \( \Psi^1_q = e^{rx} \psi^1_q \). Now, expanding the terminal BC as \(^2\)

\[
P(T, x, y) = h(x) + \sqrt{\epsilon} \cdot 0 + \ldots,
\]

we have the \( O(\epsilon^0) \) and \( O(\sqrt{\epsilon}) \) terminal BC’s

\[
Q^0(T, x) = P^0(T, x) - \Psi^0_r(x) = h(x) - \Psi^0_r(x), \quad \quad (3.46)
\]

\[
Q^1(T, x) = P^1(T, x) - \Psi^1_r(x) = 0 - \Psi^1_r(x). \quad \quad (3.47)
\]

Finally, recalling that \( g^0_q(T) = 1 \) and \( g^1_q(T) = 0 \), and using equations (3.43, 3.45, 3.46, 3.47), we have

\[
\begin{align*}
&h(x) - \Psi^0_r(x) = \begin{cases} 
e^{-rx} \int A^0_0 \psi^0_q(x) d\omega \quad \text{continuous spectrum} \\
&\ne^{-rx} \sum_m A^0_m \psi^0_m(x) \quad \text{discrete spectrum} \end{cases} \quad (3.48) \\
&-\Psi^1_r(x) = \begin{cases} 
e^{-rx} \int (A^1_0 \psi^0_q(x) + A^0_0 \psi^1_q(x)) d\omega \quad \text{continuous spectrum} \\
&\ne^{-rx} \sum_m (A^1_m \psi^0_m(x) + A^0_m \psi^1_m(x)) \quad \text{discrete spectrum} \end{cases} \quad (3.49)
\end{align*}
\]

Equations (3.48, 3.49) will be extremely useful in determining \( A^0_q \) and \( A^1_q \).

3.3. Expressions for \( A^0_q \) and \( A^1_q \). In this section we will derive expressions for \( A^0_q \) and \( A^1_q \). The results we derive are independent of whether we work in the continuous or discrete spectrum setting. However, for simplicity, we will work in the continuous spectrum setting. As a reminder, we will replace the eigenvalue index \( q \) with the Greek letters \( \omega, \nu, \mu \).

Our first order of business is to find an expression for \( A^0_q \). To do this, we recall equation (3.48)

\[
h(x) - \Psi^0_r(x) = \ne^{-rx} \int A^0_0 \psi^0_q(x) d\omega.
\]

Note that \( \Psi^0_r(x) = e^{rx} \psi^0_q(x) \). Now, using the orthonormality of the \( O(\epsilon^0) \) eigenfunctions, \( \psi^0_q, \psi^0_q \), we have

\[
\begin{align*}
\left( \psi^0_q(x), e^{-rx} h(x) - \psi^0_q(x), \psi^0_q(x) \right) &= \int A^0_0 (\psi^0_q(x), \psi^0_q(x)) d\omega \\
&= \int A^0_0 \delta(\nu - \omega) d\omega \\
&= A^0_0.
\end{align*}
\]

\(^2\)Keeping the entire terminal BC in the \( O(\epsilon^0) \) term is consistent with the choice made in [5]. We will need this in order to establish the equivalence of our method to that of [5].
A similar calculation in the discrete spectrum setting yields an analogous result. See appendix A for details. We summarize continuous and discrete spectrum expressions for \( A_q^{(0)} \) here

\[
A_q^{(0)} = \left( \psi^{(0)}_\omega(x), e^{-\epsilon x}h(x) \right) - \left( \psi^{(0)}_\nu(x), \psi^{(1)}_r(x) \right) \quad \text{(continuous spectrum),}
\]
\[
A_m^{(0)} = \left( \psi^{(0)}_m(x), e^{-\epsilon x}h(x) \right) - \left( \psi^{(0)}_r(x), \psi^{(1)}_m(x) \right) \quad \text{(discrete spectrum).}
\]

We may use similar techniques to derive a result for \( A_q^{(1)} \). We recall equation (3.49)

\[-\psi^{(1)}_r(x) = e^{-\epsilon x} \int \left( A^{(1)}_\omega \psi^{(0)}_\omega(x) + A^{(0)}_\omega \psi^{(1)}_\omega(x) \right) d\omega,
\]

where \( \psi^{(1)}_r(x) = e^{\epsilon x} \psi^{(1)}_r(x) \). Hence

\[-\left( \psi^{(0)}_\nu(x), \psi^{(1)}_r(x) \right) = \int \left( A^{(1)}_\omega \left( \psi^{(0)}_\nu(x), \psi^{(0)}_\omega(x) \right) + A^{(0)}_\omega \left( \psi^{(0)}_\nu(x), \psi^{(1)}_\omega(x) \right) \right) d\omega
\]

\[= \int \left( A^{(1)}_\omega \delta(\nu - \omega) + A^{(0)}_\omega \left( \psi^{(0)}_\nu(x), \psi^{(1)}_\omega(x) \right) \right) d\omega
\]

\[= A^{(1)}_\nu + \int A^{(0)}_\omega \left( \psi^{(0)}_\nu(x), \psi^{(1)}_\omega(x) \right) d\omega.
\]

Once again, the discrete spectrum result is analogous. We summarize the expressions for \( A_q^{(1)} \) below

\[
A^{(1)}_\omega = -\left( \psi^{(0)}_\omega(x), \psi^{(1)}_r(x) \right) - \int A^{(0)}_\nu \left( \psi^{(0)}_\nu(x), \psi^{(1)}_r(x) \right) d\nu \quad \text{(continuous spectrum)}
\]
\[
A^{(1)}_m = -\left( \psi^{(0)}_m(x), \psi^{(1)}_r(x) \right) - \sum_n A^{(0)}_n \left( \psi^{(0)}_m(x), \psi^{(1)}_n(x) \right) \quad \text{(discrete spectrum)}
\]

4. Practical Implementation. In this section we present a systematic method to derive the approximate price of any option that has payoff \( h(X_\tau) \) where \( \tau \) is given by (2.1). We then show how to implement this method with four examples. We conclude by discussing how to calibrate the class of fast mean-reverting models considered in this paper using market data.

4.1. A General Method for Pricing Options. The systematic method to price options with a variety of BC’s is composed of five straightforward steps.

**Step 1: Find Expressions for \( \psi^{(0)}_r(x) \) and \( \psi^{(1)}_r(x) \).** Here, \( \psi^{(0)}_r(x) \) must solve equation (3.33) with BC’s (3.39, 3.40), and \( \psi^{(1)}_r(x) \) must solve equation (3.34) with BC’s (3.39, 3.40).

**Step 2: Solve the \( O(\epsilon^0) \) Eigenvalue Equation to Obtain \( \psi^{(0)}_q(x) \) and \( \lambda^{(0)}_q \).** This is simply a matter of solving the eigenvalue equation (3.18) or, equivalently, equation (3.19), with BC’s (3.35, 3.36).

**Step 3: Solve the \( O(\sqrt{\epsilon}) \) Eigenvalue Equation to Obtain \( \psi^{(1)}_q(x) \) and \( \lambda^{(1)}_q \).** Here, \( \psi^{(1)}_q(x) \) and \( \lambda^{(1)}_q \) must satisfy equation (3.31) or, equivalently, equation (3.32), as well as BC’s (3.41, 3.42). As we will show with some examples, this may require expressing \( \psi^{(1)}_q(x) \) as a linear combination of the orthonormal basis functions \( \psi^{(0)}_q(x) \).
Step 4: Find Expressions for \( A_q^{(0)} \) and \( A_q^{(1)} \). Having found expressions for \( \psi_r^{(0)}(x) \), \( \psi_r^{(1)}(x) \), \( \psi_q^{(0)}(x) \) and \( \psi_q^{(1)}(x) \), one can obtain \( A_q^{(0)} \) and \( A_q^{(1)} \) from equations (3.50, 3.51) and (3.52, 3.53) respectively.

Step 5: Write Expressions for \( P^{(0)}(t, x) \) and \( P^{(1)}(t, x) \). Recalling that \( g_q^{(0)}(t) \) and \( g_q^{(1)}(t) \) are given by equations (3.2) and (3.3), one may now express \( P^{(0)}(t, x) \) and \( P^{(1)}(t, x) \) using equations (3.4, 3.5, 3.6, 3.7).

This concludes the systematic method to price options. We now provide four option-pricing examples in order to show the versatility of this method. These examples will make the above procedure clear.

4.2. Example: European-Style Options. In this section we consider a European-style option expiring at fixed time \( T \) with payoff \( h(X_T) \). The appropriate BC for \( P^{(0)}(t, x, y) \) is

\[
P^{(0)}(T, x, y) = h(x).
\]

Because there is no upper or lower BC, we formally set \( x_l = -\infty \), \( x_u = +\infty \) and \( R_l = R_u = 0 \). Since neither \( x_l \) nor \( x_u \) is finite, we will work in the continuous spectrum framework. As a reminder, we will replace \( q \) with its Greek counterparts, \( \{\omega, \nu, \mu\} \).

Step 1: Find Expressions for \( \psi_{r}^{(0)}(x) \) and \( \psi_{r}^{(1)}(x) \). From equation (3.33, 3.39, 3.40), we see \( \psi_{r}^{(0)}(x) \) must satisfy

\[
\frac{\partial^2}{\partial x^2} \psi_{r}^{(0)} = \lambda_{r}^{(0)} \psi_{r}^{(0)},
\]

\[
\lim_{x \to \pm \infty} e^{cx} \psi_{r}^{(0)}(x) = 0.
\]

Likewise, from equations (3.34, 3.41, 3.42), we see that \( \psi_{r}^{(1)}(x) \) must satisfy

\[
(V_2 \chi + V_3 \eta_{r}) \frac{\partial}{\partial x} \psi_{r}^{(0)} + (V_2 \xi_{r} + V_3 \gamma_{r}) \psi_{r}^{(0)} = \frac{\sigma^2}{2} \left( \lambda_{r}^{(0)} - \frac{\partial^2}{\partial x^2} \right) \psi_{r}^{(1)},
\]

\[
\lim_{x \to \pm \infty} e^{cx} \psi_{r}^{(1)}(x) = 0.
\]

The obvious solutions are

\[
\psi_{r}^{(0)}(x) = 0,
\]

\[
\psi_{r}^{(1)}(x) = 0.
\]

Step 2: Solve the \( O(\epsilon^0) \) Eigenvalue Equation to Obtain \( \psi_{\omega}^{(0)}(x) \) and \( \lambda_{\omega}^{(0)} \). We must find a set of solutions to equation (3.19), which we restate below for clarity

\[
\frac{\partial^2}{\partial x^2} \psi_{\omega}^{(0)} = \lambda_{\omega}^{(0)} \psi_{\omega}^{(0)}.
\]

We must also ensure that \( \lim_{x \to \pm \infty} \left| \psi_{\omega}^{(0)}(x) \right|^2 < \infty \). One can easily verify the following set of solutions to equation (4.2)

\[
\psi_{\omega}^{(0)}(x) = \frac{1}{\sqrt{2\pi}} e^{i\omega x},
\]

\[
\lambda_{\omega}^{(0)} = -\omega^2 \quad \Rightarrow \quad \lambda_{\omega}^{(0)} = -\frac{1}{2} \left( \Gamma^2 + \sigma^2 \omega^2 \right).
\]
Note, the second part of equation (4.4) follows from equation (3.20). Also, the normalization constant $\frac{1}{\sqrt{2\pi}}$ was chosen to give the following orthogonality relation on $L^2((\mathbb{R}, \cdot, \cdot))$

$$
\left( \psi^{(0)}_\omega, \psi^{(0)}_\nu \right) = \delta(\omega - \nu). \quad (4.5)
$$

**Step 3: Solve the $O(\sqrt{t})$ Eigenvalue Equation to Obtain $\psi^{(1)}(x)$ and $\lambda^{(1)}$.**

The pair \( \{ \psi^{(1)}(x), \lambda^{(1)} \} \) must satisfy equation (3.31), which we repeat for clarity

$$
(V_2 \chi + V_3 \eta_\omega) \partial_x \psi^{(0)}_\omega + (V_2 \xi_\omega + V_3 \gamma_\omega) \psi^{(0)}_\omega = \left( \lambda^{(0)}_\omega - \tilde{\mathcal{L}}^{(0)} \right) \psi^{(1)}_\omega + \lambda^{(1)} \psi^{(0)}_\omega. \quad (4.6)
$$

Having already obtained expressions for $\psi^{(0)}_\omega(x)$ and $\lambda^{(0)}_\omega$, we are now in a position to find expressions for $\psi^{(1)}(x)$ and $\lambda^{(1)}$. Inserting equations (4.3, 4.4) into (4.6) we find the following set of solutions for $\psi^{(1)}_\omega(x)$ and $\lambda^{(1)}_\omega$

$$
\psi^{(1)}_\omega(x) = \psi^{(0)}_\omega(x), \quad (4.7)
\lambda^{(1)}_\omega = V_3 \beta_\omega + V_2 \zeta_\omega, \quad (4.8)
\beta_\omega = (c + i\omega)^2 - (c + i\omega)^2, \\
\zeta_\omega = (c + i\omega)^2 - (c + i\omega).
$$

It is worth noting that the result $\psi^{(1)}_\omega(x) = \psi^{(0)}_\omega(x)$ is a special case, which holds only for European-style options. As we will see, this result leads to a remarkably simple expression for $P^{(1)}(t,x)$.

**Step 4: Find Expressions for $A^{(0)}_\omega$ and $A^{(1)}_\omega$.** Noting that $\psi^{(1)}_\omega(x) = 0$, equation (3.50) becomes

$$
A^{(0)}_\omega = \left( \psi^{(0)}_\omega(x), e^{-c x} h(x) \right). \quad (4.9)
$$

Likewise, using $\psi^{(1)}_\omega(x) = 0$, equation (3.52) becomes

$$
A^{(1)}_\omega = - \int A^{(0)}_\nu \left( \psi^{(0)}_\omega(x), \psi^{(1)}_\nu(x) \right) d\nu. \quad (4.10)
$$

Now, recalling (4.7), and using (4.5), we have

$$
A^{(1)}_\omega = - \int A^{(0)}_\nu \left( \psi^{(0)}_\omega(x), \psi^{(0)}_\nu(x) \right) d\nu \\
= - \int A^{(0)}_\nu \delta(\omega - \nu) d\nu, \\
= - A^{(0)}_\omega. \quad (4.10)
$$

**Step 5: Write Expressions for $P^{(0)}(t,x)$ and $P^{(1)}(t,x)$.** Having obtained expressions for $A^{(0)}_\omega$ and $A^{(1)}_\omega$, we are now in position to write convenient expressions for $P^{(0)}(t,x)$ and $P^{(1)}(t,x)$. From (3.4), we have

$$
P^{(0)}(t,x) = e^{c t} \int A^{(0)}_\omega g^{(0)}_\omega(t) \psi^{(0)}_\omega(x) d\omega, \quad (4.11)
$$
where \( A^{(0)}_\omega \) is given by (4.9), \( g^{(0)}_\omega(t) \) is given by (3.2), \( \psi^{(0)}(x) \) is given by (4.3) and \( \lambda^{(0)}_\omega \) is given by (4.4).

As previously noted, because \( \psi^{(1)}_\omega(x) = \psi^{(0)}_\omega(x) \), the expression for \( P^{(1)}(t, x) \) is remarkably simple. In fact, it is quite similar to that of \( P^{(0)}(t, x) \). From (3.6), we have

\[
P^{(1)}(t, x) = e^{-rT} \int \left( A^{(1)}_\omega g^{(0)}_\omega(t)\psi^{(0)}_\omega(x) + A^{(0)}_\omega g^{(1)}_\omega(t)\psi^{(0)}_\omega(x) + A^{(0)}_\omega g^{(0)}_\omega(t)\psi^{(1)}_\omega(x) \right) d\omega
\]

\[
e^{-rT} \int A^{(0)}_\omega g^{(1)}_\omega(t)\psi^{(0)}_\omega(x)d\omega,
\]

where \( g^{(1)}_\omega(t) \) is given by (3.3) and \( \lambda^{(1)}_\omega \) is given by (4.8). Note that we have used \( A^{(1)}_\omega = -A^{(0)}_\omega \) and \( \psi^{(0)}_\omega(x) = \psi^{(1)}_\omega(x) \).

**Equivalence to Black-Scholes and Fouque-Papanicolaou-Sircar [5].** In this section, we will show that \( P^{(0)}(t, x) \) corresponds to the Black-Scholes price of a European option with \( \sigma^2 = \sigma^2 \). We will also show that \( P^{(1)}(t, x) \), the correction to the Black-Scholes price due to fast mean-reversion of the volatility, is the same as that obtained in [5].

In the Black-Scholes model, the underlying is assumed to follow geometric Brownian motion with risk-neutral drift \( r \) and volatility \( \sigma \). The Black-Scholes price \( P_{BS}(t, x) \) of a European option can then be expressed as an integral of the discounted option payoff \( e^{-r(T-t)}h(y) \) with respect to the risk-neutral density of the log of the underlying

\[
P_{BS}(t, x) = e^{-r(T-t)} \int h(y)\Phi(y-x)dy,
\]

\[
\Phi(y-x) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp \left( -\frac{(y-x-(T-t)(r-\sigma^2/2))^2}{2\sigma^2(T-t)} \right).
\]

Although it is less common to do so, \( P_{BS}(t, x) \) may also be expressed as a Fourier transform

\[
P_{BS}(t, x) = \frac{1}{\sqrt{2\pi}} \int u(\omega, t, x)e^{i\omega x}d\omega
\]

where \( u(\omega, t, x) \) is a function which depends on the option payoff, the current time \( t \), and the current value of the log of the underlying \( x \). It turns out that for \( \sigma^2 = \sigma^2 \), the function \( u(\omega, t, x) \) is exactly given by

\[
u(\omega, t, x) = e^{\sigma^2 T}g^{(0)}_\omega(t).
\]

Please see [14] for details. Thus, using \( \psi^{(0)}_\omega(x) = \frac{1}{\sqrt{2\pi}} e^{i\omega x} \), we see that

\[
P_{BS}(t, x) = e^{-rT} \int A^{(0)}_\omega g^{(0)}_\omega(t)\psi^{(0)}_\omega(x)d\omega
\]

\[
= P^{(0)}(t, x),
\]

which establishes the equivalence of the \( O(\epsilon^0) \) price of a European option \( P^{(0)}(t, x) \) to the Black-Scholes price \( P_{BS}(t, x) \).

We now focus on the \( O(\sqrt{\epsilon}) \) correction to the Black-Scholes price. In [5], Fouque et. al. state that \( P^{(1)} \), the \( O(\sqrt{\epsilon}) \) correction to the Black-Scholes price of a European
option due to fast mean-reversion of the volatility, can be obtained by solving the following PDE

\[ L^{BS} P^{(1)} = -\mathcal{H} P^{BS}, \quad (4.14) \]

\[ L^{BS} = -\partial_\tau - r + \left\langle L^{(0)} \right\rangle, \quad (4.15) \]

\[ \mathcal{H} = V_2 (\partial_{xx}^2 - \partial_x) + V_3 (\partial_{xxx}^2 - \partial_{xx}^2), \quad (4.16) \]

with terminal BC given by \( P^{(1)}(T,x) = 0 \). Note that we have introduced \( \tau = T - t \).

Using \( P^{(0)}(t,x) = P^{BS}(t,x) \), expressions (4.11, 4.12) for \( P^{(0)}(t,x) \) and \( P^{(1)}(t,x) \), as well as

\[ \partial_\tau P^{(1)} = e^{cz} \int A^{(0)}(0) g^{(0)}(0) \left( 1 + \tau \left( \lambda^{(0)}_\omega - r \right) \right) \psi^{(0)}_\omega d\omega, \]

\[ \left\langle L^{(0)} \right\rangle P^{(1)} = e^{cz} \int A^{(0)}(0) g^{(0)}(0) \lambda^{(0)}_\omega \psi^{(0)}_\omega d\omega, \]

\[ \mathcal{H} P_0 = e^{cz} \int A^{(0)}(0) g^{(0)}(0) \psi^{(0)}_\omega d\omega, \]

one can easily show that our expression for \( P^{(1)}(t,x) \) satisfies PDE (4.14). As \( P^{(1)}(t,x) \) also satisfies the terminal BC \( P^{(1)}(T,x) = 0 \), for European-style options, this establishes the equivalence of \( P^{(1)}(t,x) \) derived using our spectral approach, to the function \( P^{(1)}(t,x) \) as derived in [5].

Conveniently, this also establishes the accuracy of our approximation. According to [5], we have the following result for European options with smooth and bounded payoffs: for all \( t < T \) and \( x,y \in \mathbb{R} \)

\[ \left| P^r(t,x,y) - \left( P^{(0)}(t,x) + \sqrt{\mathcal{H}} P^{(1)}(t,x) \right) \right| = O(\epsilon). \]

From [7], we also have the following result for European call options: for all \( t < T \) and \( x,y \in \mathbb{R} \)

\[ \left| P^c(t,x,y) - \left( P^{(0)}(t,x) + \sqrt{\mathcal{H}} P^{(1)}(t,x) \right) \right| = O(\epsilon |\log \epsilon|) \]

**European Call Option.** The payoff for a European Call, expiring at time \( T \), with strike price \( K \), is given by \((S_T - K)^+\). In log notation we have

\[ h(x) = (e^x - e^k)^+, \]

\[ k = \log K. \]

Using equation (4.9) we have

\[ A^{(0)}_\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix} (e^x - e^k)^+ e^{-cx} dx \quad (4.17) \]

\[ = \frac{1}{\sqrt{2\pi}} \frac{e^{-k(c+\omega-1)}}{(c+i\omega-1)(c+i\omega)}. \quad (4.18) \]

We note that the integral (4.17) will not converge unless

\[ \text{Im}[\omega] < c - 1. \]
Thus, in deriving result (4.18), we have implicitly assumed \( \omega = \omega_T + i \omega_1 \) and fixed \( \omega_1 < c - 1 \). This procedure is justified because \( \lambda^{(0)}_{\omega_T + i \omega_1} \) and \( \psi^{(0)}_{\omega_T + i \omega_1} (x) \) are still acceptable solutions to the zeroth order eigensystem. Because of the condition we have placed on \( \omega_1 \), when evaluating integrals (4.11, 4.12) one must make sure to set \( \omega = \omega_T + i \omega_1 \) and \( d\omega = d\omega_T \).

As we conclude this section, we note that \( P^{(1)} (t, x) \) is a linear function of the two group parameters, \( V_2 \) and \( V_3 \). It is useful to absorb a factor of \( \sqrt{\tau} \) into the group parameters so that \( \sqrt{\tau} P^{(1)} (t, x) \) is a linear function of \( V_2' \) and \( V_3' \) where

\[
V_2' = \sqrt{\epsilon} \frac{\nu'}{\sqrt{2}} \langle \lambda \phi' \rangle ,
\]

\[
V_3' = -\sqrt{\epsilon} \frac{\rho \nu'}{\sqrt{2}} \langle f \phi' \rangle .
\]

This cosmetic change makes it clear that, although adding a fast mean-reverting factor of volatility introduced two functions and five parameters to the Black-Scholes model—\( f(y), \Lambda(y), \epsilon, m, \nu, \rho, y \)—the net effect of these additions is entirely captured by the two group parameters defined above.

The parameters \( V_2' \) and \( V_3' \) are the same as those defined in [5], where it is shown that, for European Call options, \( V_2' \) corresponds to an adjustment of the overall level of the implied volatility, and \( V_3' \) controls the at the money skew of the implied volatility. This structure leads to a remarkably simple calibration procedure, which we outline in section 4.6.

Figure 4.1 demonstrates the effect of \( V_2' \) and \( V_3' \) on the price of a European call option. Figure 4.2 shows the effect of \( V_2' \) and \( V_3' \) on implied volatility.

**Fig. 4.1. Prices of European call options are plotted as a function of \( S_t \), the current price of the underlying. In these subfigures, \( T - t = 1/2 \), \( r = 0.05 \), \( \sqrt{\tau} = 0.34 \) and \( K = 2 \). In subfigure (a), we set \( V_2' = 0 \), and vary \( V_3' \) from \(-0.03\) (red, dot-dashed) to \(0.03\) (blue, dashed). In subfigure (b), we set \( V_2' = 0 \), and vary \( V_3' \) from \(-0.01\) (red, dot-dashed) to \(0.01\) (blue, dashed). In both subfigures the solid line corresponds to the Black-Scholes price of the option (i.e. \( V_2' = V_3' = 0 \)).**

**4.3. Example: Up-and-Out Option.** In this section we consider an up-and-out style option. Such an option “knocks out” (i.e. becomes worthless) if the log stock price \( X_t \) reaches a level \( x_u \) prior to the time of maturity \( T \). If the log stock price remains below the level \( x_u \) for all \( t < T \), then the option has payoff \( h(X_T) \). For an up-and-out style option, the appropriate BC’s for \( P^e (t, x, y) \) are as follows

\[
P^e (T, x, y) = h(x) .
\]

\[
\lim_{x \to x_u} P^e (t, x, y) = 0 .
\]
Fig. 4.2. Implied volatilities of European call options are plotted as a function of strike price $K$. In these subfigures, $T-t = 1/2$, $r = 0.05$, $\sqrt{\sigma^2} = 0.34$ and $S_t = 2$. In subfigure (a), we set $V_2^* = 0$, and vary $V_2^*$ from $-0.01$ (red, dot-dashed) to $0.01$ (blue, dashed). In subfigure (b), we set $V_2^* = 0$, and vary $V_2^*$ from $-0.01$ (red, dot-dashed) to $0.01$ (blue, dashed). In both subfigures the solid line corresponds to $I = \pi$ (i.e. $V_2^* = V_3^* = 0$).

We note that there is no lower BC. Hence, we formally set $x_l = -\infty$ and $R_l = 0$. As $x_l$ is not finite, we will be working in the continuous spectrum framework. Hence, we will replace $q$ with its Greek counterparts, $\{\omega, \nu, \mu\}$.

**Step 1: Find Expressions for $\psi_r^{(0)}(x)$ and $\psi_r^{(1)}(x)$**. From equation (3.33, 3.39, 3.40), we see $\psi_r^{(0)}(x)$ must satisfy

$$\frac{\partial^2}{\partial x^2} \psi_r^{(0)} = \lambda_r^{(0)} \psi_r^{(0)},$$

$$\lim_{x \to -\infty} \psi_r^{(0)}(x) = 0,$$

$$\lim_{x \to x_u} \psi_r^{(0)}(x) = 0.$$

Likewise, from equations (3.34, 3.41, 3.42), we see that $\psi_r^{(1)}(x)$ must satisfy

$$(V_2 \chi + V_3 \eta_r) \frac{\partial}{\partial x} \psi_r^{(0)} + (V_2 \xi_r + V_3 \gamma_r) \psi_r^{(0)} = \frac{\sigma^2}{2} \left(\lambda_r^{(0)} - \frac{\partial^2}{\partial x^2}\right) \psi_r^{(1)},$$

$$\lim_{x \to -\infty} \psi_r^{(1)}(x) = 0,$$

$$\lim_{x \to x_u} \psi_r^{(1)}(x) = 0.$$

We have the following solutions

$$\psi_r^{(0)}(x) = 0,$$

$$\psi_r^{(1)}(x) = 0.$$

**Step 2: Solve the $O(\epsilon^0)$ Eigenvalue Equation to Obtain $\psi_\omega^{(0)}(x)$ and $\lambda_\omega^{(0)}$.** We must find a set of solutions to equation (3.18), or equivalently, equation (3.19), which satisfy BC (3.36). We have

$$\frac{\partial^2}{\partial x^2} \psi_\omega^{(0)} = \lambda_\omega^{(0)} \psi_\omega^{(0)},$$

$$\psi_\omega^{(0)}(x_u) = 0.$$
We must also have \( \lim_{x \to -\infty} \left| \psi^{(0)}(x) \right| < \infty \). One can easily verify the following set of solutions to equations (4.23, 4.24)

\[
\psi^{(0)}(x) = \sqrt{\frac{2}{\pi}} \sin(\omega (x - x_l)) \quad \omega > 0, 
\]

\[
\lambda^{(0)}_{\omega} = -\omega^2 \quad \Rightarrow \quad \lambda^{(0)}_{\omega} = -\frac{1}{2} (\Gamma^2 + \sigma^2 \omega^2). 
\]

Note that we have enforced \( \omega > 0 \) since \( \psi^{(0)}(0) \omega(x) = -\psi^{(0)}(x) \). The normalization constant \( \sqrt{2/\pi} \) was chosen to give the following inner product relations in \( L^2((-\infty, x_u),(\cdot, \cdot)) \)

\[
\left( \psi^{(0)}_\omega, \psi^{(0)}_{\nu} \right) = \delta(\omega - \nu),
\]

\[
\left( \psi^{(0)}_\omega, \partial_x \psi^{(0)}_{\nu} \right) = \frac{-2\omega\nu}{\pi(\omega^2 - \nu^2)}.
\]

**Step 3: Solve the \( O(\sqrt{\epsilon}) \) Eigenvalue Equation to Obtain \( \psi^{(1)}_\omega(x) \) and \( \lambda^{(1)}_\omega \).**

The pair \( \{ \psi^{(1)}_\omega(x), \lambda^{(1)}_\omega \} \) must satisfy equation (3.31) as well as BC (3.38), which we repeat here for clarity

\[
(V_2 \chi + V_3 \eta_\omega) \partial_x \psi^{(0)}_\omega + (V_2 \xi_\omega + V_3 \gamma_\omega) \psi^{(0)}_\omega = \left( \lambda^{(0)}_{\omega} - \tilde{L}^{(0)} \right) \psi^{(1)}_\omega + \lambda^{(1)}_{\omega} \psi^{(0)}_\omega, 
\]

\[
\psi^{(1)}_\omega(x_u) = 0. 
\]

For European-style options, we found that \( \psi^{(1)}_\omega(x) = \psi^{(0)}_\omega(x) \). However, for up-and-out style options, this is no longer the case. Still, finding solutions for \( \psi^{(1)}_\omega(x) \) and \( \lambda^{(1)}_\omega \) is relatively straight-forward. To begin, we recall that \( \{ \psi^{(0)}(x) \} \) form an orthonormal basis in \( L^2((-\infty, x_u),(\cdot, \cdot)) \). As such, we may express \( \psi^{(1)}_\omega(x) \) as

\[
\psi^{(1)}_\omega(x) = \int a^{(1)}_{\omega,\nu} \psi^{(0)}_{\nu}(x) d\nu, 
\]

where \( a^{(1)}_{\omega,\nu} \) are, as of yet, undetermined coefficients. Note, by construction, \( \psi^{(1)}_\omega(x) \) satisfies BC (4.28). Also note that

\[
\left( \lambda^{(0)}_{\omega} - \tilde{L}^{(0)} \right) \psi^{(1)}_\omega = \int a^{(1)}_{\omega,\nu} \left( \lambda^{(0)}_{\omega} - \tilde{L}^{(0)} \right) \psi^{(0)}_{\nu} d\nu 
\]

\[
= \int a^{(1)}_{\omega,\nu} \left( \lambda^{(0)}_{\omega} - \lambda^{(0)}_{\nu} \right) \psi^{(0)}_{\nu} d\nu. 
\]

Inserting (4.30) into (4.27) yields

\[
(V_2 \chi + V_3 \eta_\omega) \partial_x \psi^{(0)}_\omega + (V_2 \xi_\omega + V_3 \gamma_\omega) \psi^{(0)}_\omega = \int a^{(1)}_{\omega,\nu} \left( \lambda^{(0)}_{\omega} - \lambda^{(0)}_{\nu} \right) \psi^{(0)}_{\nu}(x) d\nu + \lambda^{(1)}_{\omega} \psi^{(0)}_\omega. 
\]
Hence

\[
(V_2 \chi + V_3 \eta_\omega) \left( \psi_\mu(0), \partial_x \psi_\omega(0) \right) + (V_2 \xi_\omega + V_3 \gamma_\omega) \left( \psi_\mu(0), \psi_\omega(0) \right) \\
= \int a_{\omega, \mu}^{(1)} \left( \lambda_\omega(0) - \lambda_\mu(0) \right) \left( \psi_\mu(0), \psi_\omega(0) \right) d\nu + \lambda_\omega^{(1)} \left( \psi_\mu(0), \psi_\omega(0) \right),
\]

\Rightarrow

\[
(V_2 \chi + V_3 \eta_\omega) \left( \psi_\mu(0), \partial_x \psi_\omega(0) \right) + (V_2 \xi_\omega + V_3 \gamma_\omega) \delta(\mu - \omega) \\
= \int a_{\omega, \mu}^{(1)} \left( \lambda_\omega(0) - \lambda_\mu(0) \right) \delta(\mu - \nu) d\nu + \lambda_\omega^{(1)} \delta(\mu - \omega) \\
= a_{\omega, \mu}^{(1)} \left( \lambda_\omega(0) - \lambda_\mu(0) \right) + \lambda_\omega^{(1)} \delta(\mu - \omega). \tag{4.31}
\]

For \( \mu \neq \omega \), we have \( \delta(\mu - \omega) = 0 \). Therefore, we must have

\[
(V_2 \chi + V_3 \eta_\omega) \left( \psi_\mu(0), \partial_x \psi_\omega(0) \right) = a_{\omega, \mu}^{(1)} \left( \lambda_\omega(0) - \lambda_\mu(0) \right),
\]

\Rightarrow

\[
a_{\omega, \mu}^{(1)} = \frac{(V_2 \chi + V_3 \eta_\omega) \left( \psi_\mu(0), \partial_x \psi_\omega(0) \right)}{(\lambda_\omega(0) - \lambda_\mu(0))} \tag{4.32}
\]

where \( \chi \) and \( \eta_\omega \) are given by equations (3.26, 3.27). Note that we are free to set \( a_{\omega, \omega}^{(1)} = 0 \) since any other choice for \( a_{\omega, \omega}^{(1)} \) would simply amount to a renormalization of \( \psi_\omega(0) \) (i.e. \( \psi_\omega(0) \rightarrow (1 + \sqrt{\tau} a_{\omega, \omega}^{(1)}) \psi_\omega(0) \)). Now, inserting equation (4.32) into (4.31) and integrating with respect to \( \omega \) we find

\[
\int (V_2 \chi + V_3 \eta_\omega) \left( \psi_\mu(0), \partial_x \psi_\omega(0) \right) d\omega + \int (V_2 \xi_\omega + V_3 \gamma_\omega) \delta(\mu - \omega) d\omega \\
= \int \frac{(V_2 \chi + V_3 \eta_\omega) \left( \psi_\mu(0), \partial_x \psi_\omega(0) \right)}{(\lambda_\omega(0) - \lambda_\mu(0))} (\lambda_\omega(0) - \lambda_\mu(0)) d\omega + \int \lambda_\omega^{(1)} \delta(\mu - \omega) d\omega,
\]

\Rightarrow

\[
\lambda_\omega^{(1)} = V_2 \xi_\mu + V_3 \gamma_\mu. \tag{4.33}
\]

where \( \xi_\mu \) and \( \gamma_\mu \) are given by equations (3.28, 3.29).

**Step 4: Find Expressions for \( A_\omega^{(0)} \) and \( A_\omega^{(1)} \).** Because \( \psi_\tau(0)(x) = 0 \), equation (3.50) becomes

\[
A_\omega^{(0)} = \left( \psi_\omega(0)(x), e^{-\omega x} h(x) \right). \tag{4.34}
\]

Likewise, noting \( \psi_\tau^{(1)}(x) = 0 \), equation (3.49) becomes

\[
A_\omega^{(1)} = -\int A_\nu^{(0)} \left( \psi_\omega^{(1)}(x), \psi_\nu^{(1)}(x) \right) d\nu.
\]

Now, using equation (4.29), we have

\[
A_\omega^{(1)} = -\int A_\nu^{(0)} \left( \psi_\omega^{(0)}(x), \psi_\nu^{(0)}(x) \right) d\mu d\nu \\
= -\int A_\nu^{(0)} \int a_{\nu, \mu}^{(1)} \left( \psi_\omega^{(0)}(x), \psi_\nu^{(0)}(x) \right) d\mu d\nu \\
= -\int A_\nu^{(0)} a_{\nu, \omega}^{(1)} d\nu \tag{4.35}
\]

where \( a_{\nu, \omega}^{(1)} \) is given by (4.32) and \( A_\omega^{(0)} \) is given by (4.34).
Step 5: Write Expressions for $P^{(0)}(t, x)$ and $P^{(1)}(t, x)$. Using equation (3.4), we have

$$P^{(0)}(t, x) = e^{cx} \int A^{(0)}_{\omega} g^{(0)}_{\omega}(t) \psi^{(0)}_{\omega}(x) d\omega,$$  \hspace{1cm} (4.36)

where $A^{(0)}_{\omega}$ is given by (4.34), $g^{(0)}_{\omega}(t)$ is given by (3.2), $\psi^{(0)}_{\omega}(x)$ is given by (4.25) and $\lambda^{(0)}_{\omega}$ is given by (4.26). Likewise, using equation (3.6), as well as equations (4.29, 4.35), we obtain the following expression for $P^{(1)}(t, x)$

$$P^{(1)}(t, x) = e^{cx} \int A^{(1)}_{\omega} g^{(1)}_{\omega}(t) \psi^{(1)}_{\omega}(x) d\omega$$  

where $g^{(1)}_{\omega}(t)$ is given by (3.3), $\lambda^{(1)}_{\omega}$ is given by (4.33), and $a^{(1)}_{\omega}$ is given by (4.32). Note, care must be taken when numerically evaluating the double integral in equation (4.37) as a singularity appears along the line $\omega = \nu$. For details on how to remove this singularity, see appendix B.

Equivalence to Black-Scholes and Fouque-Papanicolaou-Sircar [5]. In order to show that $P^{(1)}(t, x)$ corresponds to the Black-Scholes price of a double-barrier knock-out option with $\sigma^2 = \sigma_0^2$ and to show that $P^{(1)}(t, x)$, the correction to the Black-Scholes price due to fast mean-reversion of the volatility, is the same as that obtained by the methods developed in [5], please refer to section 4.4.

Up-and-Out Call. Assuming the option has not knocked out prior to the expiration date $T$, the payoff for an up-and-out call is given by $(S_T - K)^+$. Using log notation we have

$$h(x) = (e^x - e^K)^+, \quad k = \log K.$$  

For obvious reasons, we assume $k < x_a$. Using equation (4.34), which is an inner product in $L^2((\infty, x_a), \langle \cdot, \cdot \rangle)$, we have

$$A^{(0)}_{\omega} = \int_{-\infty}^{x_a} \frac{1}{\pi} e^{\omega(x-x_a)} (e^x - e^K)^+ e^{-cx} dx$$

$$= e^{-cx_a} \int_{-\infty}^{x_a} \frac{1}{\pi} \left( \frac{e^k}{e^x + \omega^2} - \frac{e^{x_a}}{1 - e^x + \omega^2} \right) + e^{(1-c)k} \left( \frac{\chi \partial_x \psi_{\omega}(k) + \xi \psi_{\omega}(k)}{(e^x + \omega^2)((1 - e^x + \omega^2))} \right).$$

Together with equations (4.32) and (4.33), $P^{(0)}(t, x)$ and $P^{(1)}(t, x)$ can now be obtained using equations (4.36, 4.37).

As was the case with European options, we note that $\sqrt{t}P^{(1)}(t, x)$ is a linear function of $V^2_2$ and $V^3_3$, the group parameters defined in equations (4.19, 4.20). Figure 4.3 demonstrates the effect of parameters $V^2_2$ and $V^3_3$ on the price of an up-and-out call option.
Fig. 4.3. Prices of up-and-out call options are plotted as a function of $S_t$, the current price of the underlying. In these subfigures, $T - t = 1/12$, $r = 0.05$, $\sqrt{\sigma^2} = 0.34$, $K = 2$ and $S_u = \exp(x_u) = 2.5$. In subfigure (a), we set $V_2 = 0$, and vary $V_2^+$ from $-0.01$ (red, dot-dashed) to $0.01$ (blue, dashed). In subfigure (b), we set $V_2^+ = 0$, and vary $V_3^-$ from $-0.001$ (red, dot-dashed) to $0.001$ (blue, dashed). In both subfigures the solid line corresponds to the Black-Scholes price of the option (i.e. $V_2^+ = V_3^- = 0$).

4.4. Example: Double-Barrier Knock-Out Option. In this section we consider a double-barrier knock-out option. Such an option “knocks out” (i.e. becomes worthless) if the log of the underlying hits a level $x_l < X_t$ or $x_u > X_t$ prior to the expiration date $T$. If the option does not knock out prior to expiration it has a payoff $h(X_T)$. The appropriate boundary conditions for $P^r(t,x,y)$ are as follows

$$P^r(T,x,y) = h(x),$$
$$\lim_{x \to x_l} P^r(t,x,y) = 0,$$
$$\lim_{x \to x_u} P^r(t,x,y) = 0.$$

Because $-\infty < x_l < x_u < +\infty$, we work in the discrete spectrum setting. Thus, throughout this section we will replace $q$ with the Latin letters $\{k,n,m\}^3$.

**Step 1: Find Expressions for $\psi_r^{(0)}(x)$ and $\psi_r^{(1)}(x)$**. From equation (3.33, 3.39, 3.40), we see $\psi_r^{(0)}(x)$ must satisfy

$$\partial_{xx}^2 \psi_r^{(0)} = \lambda_r^{(0)} \psi_r^{(0)},$$
$$\lim_{x \to x_l} \psi_r^{(0)}(x) = 0,$$
$$\lim_{x \to x_u} \psi_r^{(0)}(x) = 0.$$

Likewise, from equations (3.34, 3.41, 3.42), we see that $\psi_r^{(1)}(x)$ must satisfy

$$(V_2 \chi + V_3 \eta_r) \partial_x \psi_r^{(0)} + (V_2\xi_r + V_3\gamma_r) \psi_r^{(0)} = \frac{\sigma^2}{2} \left( \lambda_r^{(0)} - \partial_{xx}^2 \right) \psi_r^{(1)},$$
$$\lim_{x \to x_l} \psi_r^{(1)}(x) = 0,$$
$$\lim_{x \to x_u} \psi_r^{(1)}(x) = 0.$$

It should be clear from the context when we use $k$ as a replacement index for $q$ and when we use $k = \log K$. 

23
We have the following solutions

\[ \psi_{\ell}(0)(x) = 0, \]
\[ \psi_{\ell}(1)(x) = 0. \]

**Step 2: Solve the O(\(\epsilon^0\)) Eigenvalue Equation to Obtain \(\psi_m(0)(x)\) and \(\lambda_m(0)\).**

The pair \(\{\psi_m(0)(x), \lambda_m(0)\}\) must satisfy equation (3.19) and BC’s (3.35, 3.36), which we restate here for clarity

\[ \partial_x^2 \psi_m(0) = \lambda_m(0) \psi_m(0), \tag{4.41} \]
\[ \psi_m(0)(x_l) = 0, \tag{4.42} \]
\[ \psi_m(0)(x_u) = 0. \tag{4.43} \]

One can easily verify the following set of solutions to equations (4.41, 4.42, 4.43)

\[ \psi_m(0)(x) = \sqrt{\frac{2}{x_u - x_l}} \sin (\alpha_m(x - x_l)), \tag{4.44} \]
\[ \lambda_m(0) = -\alpha_m^2 \Rightarrow \lambda_m(0) = -\frac{1}{2} \left( \Gamma^2 + \sigma^2 \alpha_m^2 \right), \tag{4.45} \]
\[ \alpha_m = \frac{m \pi}{x_u - x_l}, \quad m \in \mathbb{Z}^+. \tag{4.46} \]

Note, the RHS of equation (4.45) follows from equation (3.20). Also, the normalization constant \(\sqrt{\frac{2}{x_u - x_l}}\) was chosen to give the following inner product relations on \(L^2((x_l, x_u), (\cdot, \cdot))\)

\[ (\psi_n(0), \psi_m(0)) = \delta_{n,m}, \]
\[ (\psi_n(0), \partial_x \psi_m(0)) = \begin{cases} \frac{2nm(-1)^{n+m}}{(m^2 - n^2)(x_u - x_l)} & n \neq m, \\ 0 & n = m. \end{cases} \]

**Step 3: Solve the O(\(\sqrt{\epsilon}\)) Eigenvalue Equation to Obtain \(\psi_m(1)(x)\) and \(\lambda_m(1)\).**

We must find a set of solutions to equation (3.31) which satisfy BC’s (3.37, 3.38). We have

\[ (V_2 \chi + V_3 \eta_m) \partial_x \psi_m(0) + (V_2 \xi_m + V_3 \gamma_m) \psi_m(0) = \left( \lambda_m(0) - \tilde{\lambda}(0) \right) \psi_m(1) + \lambda_m(1) \psi_m(0), \tag{4.47} \]
\[ \psi_m(1)(x_l) = 0, \tag{4.48} \]
\[ \psi_m(1)(x_u) = 0. \tag{4.49} \]

To begin, we recall that the \(\psi_m(0)(x)\) form a complete set of orthonormal basis functions in \(L^2((x_l, x_u), (\cdot, \cdot))\). Hence, we will write \(\psi_m(1)(x)\) as a linear combination of these basis functions. We have

\[ \psi_m(1)(x) = \sum_n a_{m,n} \psi_n(0)(x). \tag{4.50} \]
Note that, by construction, $\psi_m^{(1)}(x)$ satisfies BC’s (4.48, 4.49). It is also useful to note
\[
\left(\lambda_m^{(0)} - \tilde{\lambda}_m^{(0)}\right) \psi_m^{(1)} = \sum_n a_{m,n}^{(1)} \left(\lambda_m^{(0)} - \tilde{\lambda}_m^{(0)}\right) \psi_n^{(0)} = \sum_n a_{m,n}^{(1)} \left(\lambda_m^{(0)} - \lambda_n^{(0)}\right) \psi_n^{(0)}. \tag{4.51}
\]
Inserting (4.51) into (4.47) yields
\[
(V_2 \gamma + V_3 \eta_m) \partial_x \psi_m^{(0)} + (V_2 \xi_m + V_3 \gamma_m) \psi_m^{(0)} = \sum_n a_{m,n}^{(1)} \left(\lambda_m^{(0)} - \lambda_n^{(0)}\right) \psi_n^{(0)}(x) + \lambda_m^{(1)} \psi_m^{(0)}.
\]
Hence
\[
(V_2 \gamma + V_3 \eta_m) \left(\psi_k^{(0)}, \partial_x \psi_m^{(0)}\right) + (V_2 \xi_m + V_3 \gamma_m) \left(\psi_k^{(0)}, \psi_m^{(0)}\right)
= \sum_n a_{m,n}^{(1)} \left(\lambda_m^{(0)} - \lambda_n^{(0)}\right) \left(\psi_k^{(0)}, \psi_n^{(0)}\right) + \lambda_m^{(1)} \left(\psi_k^{(0)}, \psi_m^{(0)}\right),
\]
\[
\Rightarrow (V_2 \gamma + V_3 \eta_m) \left(\psi_k^{(0)}, \partial_x \psi_m^{(0)}\right) + (V_2 \xi_m + V_3 \gamma_m) \delta_{k,m}
= \sum_n a_{m,n}^{(1)} \left(\lambda_m^{(0)} - \lambda_n^{(0)}\right) \delta_{k,n} + \lambda_m^{(1)} \delta_{k,m}
= a_{m,k}^{(1)} \left(\lambda_m^{(0)} - \lambda_k^{(0)}\right) + \lambda_m^{(1)} \delta_{k,m}. \tag{4.52}
\]
For $m \neq k$, we have
\[
(V_2 \gamma + V_3 \eta_m) \left(\psi_k^{(0)}, \partial_x \psi_m^{(0)}\right) = a_{m,k}^{(1)} \left(\lambda_m^{(0)} - \lambda_k^{(0)}\right),
\]
\[
\Rightarrow a_{m,k}^{(1)} = \frac{(V_2 \gamma + V_3 \eta_m) \left(\psi_k^{(0)}, \partial_x \psi_m^{(0)}\right)}{\lambda_m^{(0)} - \lambda_k^{(0)}}, \tag{4.53}
\]
where $\gamma$ and $\eta_m$ are given by equations (3.26, 3.27). This is completely analogous to the result we derived for $a_{k,\nu}^{(1)}$ in the up-and-out options case (see equation (4.32) for comparison). We choose $a_{k,k}^{(1)} = 0$, as any other choice would simply amount to a renormalization of $\psi_k^{(0)}(x)$ (i.e. $\psi_k^{(0)}(x) \to 1 + \sqrt{a_{k,k}^{(1)}} \psi_k^{(0)}(x)$). Now, inserting equation (4.53) into (4.52) and summing over $m$ we find
\[
\sum_m (V_2 \gamma + V_3 \eta_m) \left(\psi_k^{(0)}, \partial_x \psi_m^{(0)}\right) + \sum_m (V_2 \xi_m + V_3 \gamma_m) \delta_{k,m}
= \sum_m \frac{(V_2 \gamma + V_3 \eta_m) \left(\psi_k^{(0)}, \partial_x \psi_m^{(0)}\right)}{\lambda_m^{(0)} - \lambda_k^{(0)}} \left(\lambda_m^{(0)} - \lambda_k^{(0)}\right) + \sum_m \lambda_m^{(1)} \delta_{k,m},
\]
\[
\Rightarrow \lambda_k^{(1)} = V_2 \xi_k + V_3 \gamma_k, \tag{4.54}
\]
where $\xi_k$ and $\gamma_k$ are given by equations (3.28, 3.29). Note the similarity between equations (4.54) and (4.33).
Step 4: Find Expressions for $A_m^{(0)}$ and $A_m^{(1)}$. Noting $\psi_r^{(0)}(x) = 0$, equation (3.51) becomes

$$A_m^{(0)} = \left(\psi_m^{(0)}(x), e^{-c_x h(x)}\right).$$  \hfill (4.55)

We can obtain an expression for $A_m^{(1)}$ from equation (3.53). Since $\psi_r^{(1)}(x) = 0$, we have

$$A_m^{(1)} = -\sum_n A_n^{(0)} \left(\psi_m^{(0)}(x), \psi_n^{(1)}(x)\right).$$

Now, using equation (4.50), we have

$$A_m^{(1)} = -\sum_n A_n^{(0)} \sum_k a_{n,k}^{(1)} \left(\psi_m^{(0)}(x), \psi_k^{(1)}(x)\right)$$
$$= -\sum_n A_n^{(0)} \sum_k a_{n,k}^{(1)} \delta_{m,k}$$
$$= -\sum_n A_n^{(0)} a_{n,m}^{(1)}$$ \hfill (4.56)

where $a_{n,m}^{(1)}$ is given by (4.53) and $A_n^{(0)}$ is given by (4.55).

Step 5: Write Expressions for $P^{(0)}(t,x)$ and $P^{(1)}(t,x)$. Using equation (3.5), it is now straightforward to write an expression for $P^{(0)}(t,x)$. Recalling $\Psi_r^{(0)}(x) = 0$, we have

$$P^{(0)}(t,x) = e^{c_x} \sum_m A_m^{(0)} g_m^{(0)}(t) \psi_m^{(0)}(x),$$ \hfill (4.57)

where, $A_m^{(0)}$ is given by (4.55), $g_m^{(0)}(t)$ is given by (3.2), $\psi_m^{(0)}(x)$ is given by (4.44) and $\lambda_m^{(0)}$ is given by (4.45). Now, using equation (3.7) for $P^{(1)}(t,x)$, as well as equations (4.50, 4.56), we derive

$$P^{(1)}(t,x) = e^{c_x} \sum_m \left( A_m^{(1)} g_m^{(0)}(t) \psi_m^{(0)}(x) + A_m^{(0)} g_m^{(1)}(t) \psi_m^{(0)}(x) + A_m^{(0)} g_0^{(0)}(t) \psi_m^{(0)}(x) \right)$$
$$= e^{c_x} \sum_m \left( -\sum_n A_n^{(0)} a_{n,m}^{(1)} g_m^{(0)}(t) \psi_m^{(0)}(x) + A_m^{(0)} g_m^{(1)}(t) \psi_m^{(0)}(x) + A_m^{(0)} g_0^{(0)}(t) \sum_n a_{n,m}^{(1)} \psi_m^{(0)}(x) \right)$$
$$= e^{c_x} \sum_m A_m^{(0)} g_m^{(1)}(t) \psi_m^{(0)}(x) + e^{c_x} \sum_m \sum_n g_m^{(0)}(t) \left( A_m^{(0)} a_{n,m}^{(1)} \psi_m^{(0)}(x) - A_m^{(0)} a_{n,m}^{(1)} \psi_m^{(0)}(x) \right),$$ \hfill (4.58)

where $g_m^{(1)}(t)$ is given by (3.3), $\lambda_m^{(1)}$ is given by (4.54), and $a_{n,m}^{(1)}$ is given by (4.53).

Note, numerically evaluating (4.57) and (4.58) requires truncating the series at some finite $m$. However, this does not pose a major problem because both $g_m^{(0)}(t)$ and $g_m^{(1)}(t)$ contain a factor of the form $\sim \exp \left(-C(T-t)m^2\right)$, where $C = \sigma^2 \pi^2/(x_u-x_l)^2$. Hence, as long as we are in a regime where $T-t >> \epsilon$, the error due to truncating the series will be small.
Equivalence to Black-Scholes and Fouque-Papanicolaou-Sircar [5]. In this section, we will show that $P^{(0)}(t, x)$ corresponds to the Black-Scholes price of a double-barrier knock-out option with $\sigma^2 = \Psi^2$. We will also show that $P^{(1)}(t, x)$, the correction to the Black-Scholes price due to fast mean-reversion of the volatility, is the same as the correction that would be obtained by the methods developed in [5], had the authors done the calculation.

We remind the reader that, in the Black-Scholes model, the underlying is assumed to follow geometric Brownian motion with risk-neutral drift $r$ and volatility $\sigma$. The Black-Scholes price $P^{BS}(t, x)$ of a double-barrier knock-out option can then be expressed as an integral of the discounted option payoff $e^{-r(T-t)}h(y)$ with respect to the risk-neutral density $\Phi(s, y; t, x)$ of the log of the underlying, with the absorbing BC's: $\Phi(s, x; t, x) = 0$ and $\Phi(s, x_u; t, x) = 0$ (see, for example, [2,17]). We have

$$P^{BS}(t, x) = e^{-r(T-t)} \int_{x_1}^{x_u} h(y) \Phi(T, y; t, x) dy,$$

(4.59)

$$\Phi(T, y; t, x) = \sum_m e^{\lambda_m^{(0)}(T-t) - r(y-x)} \psi_m^{(0)}(y) \psi_m^{(0)}(x).$$

It is a simple rearrangement of terms to show that $P^{BS}(t, x)$, given by (4.59), is equivalent to $P^{(0)}(t, x)$, given by (4.57).

Although the authors of [5] do not address double-barrier knock-out options, it is clear from their methodology that, had they done the calculation, $P^{(1)}(t, x)$ would satisfy the following PDE and BC's

$$\mathcal{L}^{BS} P^{(1)} = -\mathcal{H} P^{BS},$$

$$P^{(1)}(T, x) = 0,$$

$$P^{(1)}(t, x_1) = 0,$$

$$P^{(1)}(t, x_u) = 0,$$

where $\mathcal{L}^{BS}$ is given by (4.15) and $\mathcal{H}$ is given by (4.16). By construction, $P^{(1)}(T, x)$, given by (4.58), satisfies the above BC's. Additionally, using

$$P^{(1)} = \sum_m \left( A_m^{(1)} g_m^{(0)} \psi_m^{(0)} + A_m^{(0)} \tau \lambda_m^{(1)} g_m^{(0)} \psi_m^{(0)} + A_m^{(0)} g_m^{(0)} \psi_m^{(1)} \right),$$

$$\partial_t P^{(1)} = \sum_m \left( A_m^{(1)} \left( \lambda_m^{(0)} - r \right) g_m^{(0)} \psi_m^{(0)} + A_m^{(0)} \lambda_m^{(1)} \left( 1 + \tau \left( \lambda_m^{(0)} - r \right) \right) g_m^{(0)} \psi_m^{(1)} \right),$$

$$\left\langle \mathcal{L}^{(0)} \right\rangle P^{(1)} = \sum_m \left( A_m^{(1)} g_m^{(0)} \lambda_m^{(0)} \psi_m^{(0)} + A_m^{(0)} \tau \lambda_m^{(1)} g_m^{(0)} \lambda_m^{(0)} \psi_m^{(0)} \right.$$  

$$+ A_m^{(0)} g_m^{(0)} \left( \lambda_m^{(0)} \psi_m^{(1)} + \lambda_m^{(1)} \psi_m^{(0)} - \mathcal{H} \psi_m^{(0)} \right),$$

one can easily show that $P^{(1)}(T, x)$ satisfies $\mathcal{L}^{BS} P^{(1)} = -\mathcal{H} P^{BS}$. For double-barrier knock-out options, this establishes the equivalence of the spectral method developed in this paper to that of Fouque et. al in [5].

Double-Barrier Call. Assuming the option has not knocked-out prior to the expiration date $T$, the payoff for a double-barrier call with strike price $K$ is given by $(S_T - K)^+$. In log notation, we have

$$h(x) = (e^x - e^k)^+,$$

$$k = \log K.$$
For the sake of simplicity, we assume \( x_l < k < x_u \). Using equation (4.55), which is an inner product in \( L^2((x_l, x_u), (\cdot, \cdot)) \), we have

\[
A_n^{(0)} = \sqrt{\frac{2}{x_u - x_l}} \int_{x_l}^{x_u} \sin \left( \frac{n\pi(x - x_l)}{x_u - x_l} \right) e^{-c^2} (e^x - e^k)^+ dx
\]

\[
= \sqrt{\frac{2}{x_u - x_l}} \left( (-1)^{n+1} e^{(1-c)x_l} \alpha_n + \frac{(-1)^n e^{k-cx_u} \alpha_n}{\alpha_n^2 + c^2} \right)
\]

\[
+ \epsilon^{(1-c)k} \left( \frac{(c-1)\psi_n(k) + \partial_x \psi_n(k)}{\alpha_n^2 + (c-1)^2} + \frac{c\psi_n(k) - \partial_x \psi_n(k)}{\alpha_n^2 + c^2} \right)
\]

It is now straightforward to calculate \( P^{(0)}(t, x) \) and \( P^{(1)}(t, x) \) using equations (4.57, 4.58).

Once again, we point out the linear dependence of \( P^{(1)}(t, x) \) on group parameters \( V^2_p \) and \( V^2_3 \), which are defined in (4.19, 4.20). Figure 4.4 demonstrates the effect of the parameters \( V_2^p \) and \( V_3^p \) on the price of a double-barrier call option.

**Fig. 4.4.** Prices of double-barrier knock-out call options are plotted as a function of \( S_t \), the current price of the underlying. In these subfigures, \( T - t = 1/12, r = 0.05, \sqrt{T} = 0.34, K = 2, S_l = \exp(x_l) = 1.5 \) and \( S_u = \exp(x_u) = 2.5 \). In subfigure (a), we set \( V_2^p = 0 \), and vary \( V_3^p \) from \(-0.01 \) (red, dot-dashed) to \( 0.01 \) (blue, dashed). In subfigure (b), we set \( V_2^p = 0 \), and vary \( V_3^p \) from \(-0.001 \) (red, dot-dashed) to \( 0.001 \) (blue, dashed). In both subfigures the solid line corresponds to the Black-Scholes price of the option (i.e. \( V_2^p = V_3^p = 0 \)).

### 4.5. Example: Double-Barrier Option with a Rebate.

In this section we consider an option which pays a rebate \( R_l \) if the log of the underlying reaches a level \( x_u > X_t \) or a rebate, \( R_t \), if the log of the underlying reaches a level \( x_l < X_t \). If the underlying does not reach either level prior to the expiration date, \( T \), then the options has a terminal payoff of \( h(x) \). The appropriate boundary conditions for \( P^p(t, x, y) \) are as follows

\[
P^p(t, x, y) = h(x), \quad \text{(4.60)}
\]

\[
\lim_{x \to x_l} P^p(t, x, y) = R_l, \quad \text{(4.61)}
\]

\[
\lim_{x \to x_u} P^p(t, x, y) = R_u. \quad \text{(4.62)}
\]

Because \(-\infty < x_l < x_u < \infty \) we work in the discrete spectrum setting. Thus, we will use representations (3.5, 3.7) for \( P^{(0)}(t, x) \) and \( P^{(1)}(t, x) \) respectively. Also, we will
use the notation $q \to \{l, m, n\}$ to indicate that we are working with a discrete set of eigenvalues.

**Step 1: Find Expressions for $\psi_r^{(0)}(x)$ and $\psi_r^{(1)}(x)$**. From equations (3.33, 3.39, 3.40), we see $\psi_r^{(0)}(x)$ must satisfy

$$
\partial_{xx}^2 \psi_r^{(0)} = \lambda''_r \psi_r^{(0)}, \\
\lim_{x \to x_l} e^{\gamma(x)} \psi_r^{(0)}(x) = R_l, \\
\lim_{x \to x_u} e^{\gamma(x)} \psi_r^{(0)}(x) = R_u.
$$

And, from equations (3.34, 3.41, 3.42), we see that $\psi_r^{(1)}(x)$ must satisfy

$$(V_2 \chi + V_3 \eta_r) \partial_x \psi_r^{(0)} + (V_2 \xi_r + V_3 \gamma_r) \psi_r^{(0)} = \frac{\sigma^2}{2} \left( \lambda''_r - \partial_{xx}^2 \right) \psi_r^{(1)}, \quad (4.63)
$$

The expression for $\psi_r^{(0)}(x)$ is

$$
\psi_r^{(0)}(x) = \frac{e^{-cx} R_u \sinh \left( \sqrt{\lambda''_r} (x - x_l) \right) + e^{-cx} R_l \sinh \left( \sqrt{\lambda''_r} (x_u - x) \right)}{\sinh \left( \sqrt{\lambda''_r} (x_u - x_l) \right)}, \quad (4.64)
$$

and the expression for $\psi_r^{(1)}(x)$ is

$$
\psi_r^{(1)}(x) = \frac{(V_2 \chi + V_3 \eta_r)}{2\sigma^2} R_l' (x - x_l) \sinh \left( \sqrt{\lambda''_r} (x - x_u) \right) + R_u' (x_u - x) \sinh \left( \sqrt{\lambda''_r} (x - x_l) \right)
$$

\[+ \frac{(V_2 \xi_r + V_3 \gamma_r)}{2\sigma^2} \sqrt{\lambda''_r} \left( e^{2\sqrt{\lambda''_r} x_l} - e^{2\sqrt{\lambda''_r} x_u} \right) \left( R_l' (x - x_l) \left( e^{\sqrt{\lambda''_r} (x - x_l) - \sqrt{\lambda''_r} (x - x_u)} \right) + R_u' (x + x_l - 2x_u) \left( e^{\sqrt{\lambda''_r} (x + x_u) - \sqrt{\lambda''_r} (x - x_u)} \right) \right) \]

\[+ R_u' (x - x_u) \left( e^{\sqrt{\lambda''_r} (x - x_u) + 2x_l} - e^{\sqrt{\lambda''_r} (x - x_u) + 2x_l} \right) \]

\[+ R_u' (x - 2x_l + x_u) \left( -e^{\sqrt{\lambda''_r} (x + x_l) + 2x_l} + e^{\sqrt{\lambda''_r} (x + x_l + 2x_l)} \right) \right), \quad (4.65)
$$

where

$$
R_l' = \frac{e^{-cx} R_l \sinh \left( \sqrt{\lambda''_r} (x_u - x) \right)}{\sinh \left( \sqrt{\lambda''_r} (x_u - x_l) \right)}, \quad R_u' = \frac{e^{-cx} R_u \sinh \left( \sqrt{\lambda''_r} (x - x_l) \right)}{\sinh \left( \sqrt{\lambda''_r} (x_u - x_l) \right)}.
$$
Steps 2 and 3: Solve the $O(\epsilon^0)$ and $O(\sqrt{\epsilon})$ Eigenvalue Equations to Obtain $\psi_m^{(0)}(x)$, $\lambda_m^{(0)}$, $\psi_m^{(1)}(x)$ and $\lambda_m^{(1)}$. From equations (3.35, 3.36, 3.37, 3.38), we see that the BC’s for $\psi_m^{(0)}(x)$, and $\psi_m^{(1)}(x)$ are

$$
\psi_m^{(0)}(x_l) = 0, \quad \psi_m^{(0)}(x_u) = 0, \quad \psi_m^{(1)}(x_l) = 0, \quad \psi_m^{(1)}(x_u) = 0.
$$

These are exactly the same BC’s as we had in the double-barrier option case (see equations (4.42, 4.43, 4.48, 4.49) for comparison). Hence, $\psi_m^{(0)}(x)$, $\lambda_m^{(0)}$, $\psi_m^{(1)}(x)$ and $\lambda_m^{(1)}$ are given by equations (4.44, 4.45, 4.50, 4.54) respectively.

**Step 4: Find Expressions for $A_m^{(0)}$ and $A_m^{(1)}$.** From equation (3.51), we have

$$
A_m^{(0)} = \left( \psi_m^{(0)}(x), e^{-cx} h(x) \right) - \left( \psi_m^{(0)}(x), \psi_r^{(0)}(x) \right),
$$

(4.66)

where $\psi_m^{(0)}(x)$ is given by equation (4.44) and $\psi_r^{(0)}(x)$ is given by equation (4.64). Now, to find an expression for $A_m^{(1)}$ we recall equation (3.53)

$$
A_m^{(1)} = - \left( \psi_m^{(0)}(x), \psi_r^{(1)}(x) \right) - \sum_n A_n^{(0)} \left( \psi_m^{(0)}(x), \psi_n^{(1)}(x) \right).
$$

Hence, using equation (4.50), we have

$$
A_m^{(1)} = - \left( \psi_m^{(0)}(x), \psi_r^{(1)}(x) \right) - \sum_n A_n^{(0)} \sum_k a_{n,k}^{(1)} \left( \psi_m^{(0)}(x), \psi_k^{(0)}(x) \right)
$$

$$
= - \left( \psi_m^{(0)}(x), \psi_r^{(1)}(x) \right) - \sum_n A_n^{(0)} \sum_k a_{n,k}^{(1)} \delta_{m,k}
$$

$$
= - \left( \psi_m^{(0)}(x), \psi_r^{(1)}(x) \right) - \sum_n A_n^{(1)} a_{n,m},
$$

(4.67)

where $\psi_r^{(1)}(x)$ solves equation (4.63), and $a_{n,m}^{(1)}$ is given by equation (4.53).

**Step 5: Write Expressions for $P^{(0)}(t,x)$ and $P^{(1)}(t,x)$.** It is now straightforward to write an expression for $P^{(0)}(t,x)$. From equation (3.5) we have

$$
P^{(0)}(t,x) = e^{cx} \psi_r^{(0)}(x) + e^{cx} \sum_m A_m^{(0)} g_m^{(0)}(t) \psi_m^{(0)}(x),
$$

(4.68)

where, $\psi_r^{(0)}(x)$ is given by (4.64), $A_m^{(0)}$ is given by (4.66), $g_m^{(0)}(t)$ is given by (3.2), $\psi_m^{(0)}(x)$ is given by (4.44) and $A_m^{(0)}$ is given by (4.45). From equation (3.7), we obtain the following expression for $P^{(1)}(t,x)$

$$
P^{(1)}(t,x) = e^{cx} \psi_r^{(1)}(x) + e^{cx} \sum_m A_m^{(1)} g_m^{(1)}(t) \psi_m^{(1)}(x)
$$

$$
+ e^{cx} \sum_m \sum_n g_m^{(0)}(t) \left( A_m^{(0)} a_{m,n}^{(1)} \psi_n^{(0)}(x) - A_m^{(0)} a_{n,m}^{(1)} \psi_m^{(0)}(x) \right)
$$

$$
- e^{cx} \sum_m g_m^{(0)}(t) \left( \psi_m^{(0)}(x), \psi_r^{(1)}(x) \right) \psi_m^{(0)}(x),
$$

(4.69)
where \( \psi^{(1)}(x) \) is given by (4.65), \( g^{(1)}_n(t) \) is given by (3.3), \( \lambda^{(1)}_m \) is given by (4.54), and \( a^{(1)}_{n,m} \) is given by (4.53).

As with all the options examined in this paper, \( \sqrt{t} \psi^{(1)}(t, x) \) is a linear function of group parameters \( V_2 \) and \( V_3 \). Figure 4.5 demonstrates the effect of the parameters \( V_2 \) and \( V_3 \) on the price of a double-barrier option with a rebate.

![Diagram](image)

**Fig. 4.5.** Prices of rebate options are plotted as a function of \( S_t \), the current price of the underlying. In these subfigures, \( T - t = 1/4, \) \( r = 0.05, \) \( \sqrt{\sigma^2} = 0.34, \) \( S_t = \text{exp}(x_l) = 1.5, \) \( S_u = \text{exp}(x_u) = 2.5, \) \( R_l = 2 \) and \( R_u = 4. \) In subfigure (a), we set \( V_3 = 0, \) and vary \( V_2 \) from \( -0.01 \) (red, dot-dashed) to \( 0.01 \) (blue, dashed). In subfigure (b), we set \( V_2 = 0, \) and vary \( V_3 \) from \( -0.001 \) (red, dot-dashed) to \( 0.001 \) (blue, dashed). In both subfigures the solid line corresponds to the Black-Scholes price of the option (i.e. \( V_2 = V_3 = 0 \)).

4.6. Calibration. In this section we will briefly discuss how one can calibrate the class of fast mean-reverting models considered in this paper using market data.

Although European calls and puts are commonly sold on major indices and stocks, the more exotic options considered in this paper (e.g. single- and double-barrier options, rebate options, etc.) are traded with considerable less frequency. Calibrating these thinly traded exotic options using market data is problematic for a number of reasons. First, one may not have confidence that the market price of an exotic option with an open interest of 10 is in some sense “correct” when compared to the market price of a European call option with an open interest of 10,000. Second, even if one has confidence in the market prices of exotic options, there may not be enough options available on the market with which one can calibrate. For example, a trader may wish to give the price of a double-barrier knock-out option with a lower barrier of 15 and upper barrier of 20 and a strike of 17, yet he may only have access to the prices of double-barrier knock-out options with lower barriers of 12 and upper barriers of 22.

One of the great advantages of the option-pricing framework developed in this paper is that, although the fast mean-reverting volatility process adds five parameters \( (m, \epsilon, \nu, \rho, y) \) and two unspecified functions \( (f \) and \( \Lambda) \) to the Black-Scholes framework, specific knowledge of these parameters and functions is not needed to give the approximate price of a given option. Instead, the parameters and functions listed above are replaced two group parameters \( V_2^2 \) and \( V_3^2 \), given by (4.19, 4.20). What is more, \( V_2^2 \) and \( V_3^2 \) are defined consistently throughout this paper irrespective of the type of options being considered. That is, the definitions of \( V_2^2 \) and \( V_3^2 \) that are used to give the approximate price of a European call option are the same definitions of \( V_2^2 \) and \( V_3^2 \) that are used to give the approximate price of e.g. a double-barrier knock-out option. Thus, we can circumnavigate the problem of pricing thinly traded exotic options in the following way.
1. Using (liquid) European call options, fit observed implied volatilities $I_{ij}$ as an affine function of LMMR$_{ij}$

$$I_{ij} = b + a \text{LMMR}_{ij}, \quad \text{LMMR}_{ij} = \log(K_{ij}/S_t)/(T_i - t),$$

where $I_{ij}$ is defined implicitly through

$$P_{BS}(T_i, K_{ij}, I_{ij}) = P_{Market}(T_i, K_{ij}).$$

2. The group parameters $V^2$ and $V^3$ are then given by solving

$$b = \sigma^* + \frac{V^3}{2\sigma^*} \left(1 - \frac{2x}{(\sigma^*)^2}\right), \quad a = \frac{V^3}{(\sigma^*)^3}, \quad \sigma^* = \sqrt{\sigma^2 + 2V^2}. $$

We note that $\sigma$, the average level of volatility of the underlying, which can be obtained from historical returns data, is needed to determine $V^2$. The justification for steps 1 and 2 are derived in [5].

3. Use the obtained values for $\sigma^*$, $V^2$ and $V^3$ to give approximate prices for (illiquid) exotic options.

By performing the steps outlined above, one may obtain the approximate price of a variety of options in a manner which is consistent within the option-pricing framework of this paper, and which is consistent with option prices on the market.

We note that relying on historical returns to estimate $\sigma$ can be problematic. Thus, it is desirable to avoid this task. In fact, estimating $\sigma$ is not strictly necessary in order to calibrate the class of models discussed in this paper to market data. Recall that $P^{(0)}(t, x) = P^{BS}(t, x)$ with $\sigma = \sigma$ and $P^{(1)}(t, x)$ satisfies $\mathcal{L}^{BS} P^{(1)} = -H P^{(0)}$ where $\mathcal{L}^{BS}$ and $H$ are given by (4.15) and (4.16) respectively. By defining $P_*^{(0)}(t, x) = P^{BS}(t, x)$ with $\sigma = \sigma^*$ and by defining $P_*^{(1)}(t, x)$ as the solution to

$$\mathcal{L}^{BS}_* P_*^{(1)} = -\mathcal{H}_* P_*^{(0)},$$

$$\mathcal{L}^{BS}_* = -\partial_t - r + \langle \mathcal{L}^{(0)} \rangle_*, \quad \mathcal{L}^{(0)}_* = \left(r - \frac{1}{2}(\sigma^*)^2\right) \partial_x + \frac{1}{2}(\sigma^*)^2 \partial^2_{xx},$$

$$\mathcal{H}_* = V_3 \left(\partial^3_{xxx} - \partial^2_{xx}\right),$$

with appropriate BC’s, one finds that the approximate price $P_*^{(0)} + P_*^{(1)}$ has the same accuracy as $P^{(0)} + \sqrt{\varepsilon} P^{(1)}$. The obvious advantage of reformulating the problem in terms of $P_*^{(0)}$ and $P_*^{(1)}$ is that the dependence on $V^2$ and $\sigma$ is entirely captured by $\sigma^*$, thus eliminating the need to estimate $\sigma$.

5. Conclusion. Using elements from spectral analysis and singular perturbation theory, we have presented a systematic way to obtain the approximate price of a variety of European and path-dependent options in a fast mean-reverting stochastic volatility setting. In essence, we have reduced the option-pricing problem to a few simple eigenvalue problems. One key feature of our technique is that we were able to maintain correlation between the stock-price and volatility processes via two Brownian motions and still produce pricing formulas for double-barrier options. To our knowledge, this has yet to be done in literature. Extending our techniques to more sophisticated models is an on-going process. A logical next step, for example, would
be to add a fast mean-reverting factor of volatility to a well-established model such as CEV or Heston as done in [4].

Appendix A. Discrete-Spectrum Derivation of $A_m^{(0)}$ and $A_m^{(1)}$. In this section, we derive the discrete spectrum expressions for $A_m^{(0)}$ and $A_m^{(1)}$, which we stated in section 3.3 without a derivation. Our first order of business is to find an expression for $A_m^{(0)}$. To do this, we recall equation (3.48)

$$h(x) - \Psi_r^{(0)}(x) = e^{cx} \sum_m A_m^{(0)} \psi_m^{(0)}(x).$$

Note that $\Psi_r^{(0)}(x) = e^{cx} \psi_r^{(0)}(x)$. Now, using the orthonormality of the $O(\epsilon^0)$ eigenfunctions, $(\psi_n^{(0)}, \psi_m^{(0)}) = \delta_{n,m}$, we have

$$(\psi_n^{(0)}(x), e^{-cx} h(x)) - (\psi_n^{(0)}(x), \psi_r^{(0)}(x)) = \sum_m A_m^{(0)} \left( \psi_n^{(0)}(x), \psi_m^{(0)}(x) \right)$$

$$= \sum_m A_m^{(0)} \delta_{n,m}$$

$$= A_n^{(0)},$$

which is the result quoted in equation (3.51). Now, we recall equation (3.49)

$$-\Psi_r^{(1)}(x) = e^{cx} \sum_m \left( A_m^{(1)} \psi_m^{(0)}(x) + A_m^{(0)} \psi_m^{(1)}(x) \right),$$

where $\Psi_r^{(1)}(x) = e^{cx} \psi_r^{(1)}(x)$. Hence

$$- (\psi_n^{(0)}(x), \psi_r^{(1)}(x)) = \sum_m \left( A_m^{(1)} \left( \psi_n^{(0)}(x), \psi_m^{(0)}(x) \right) + A_m^{(0)} \left( \psi_n^{(0)}(x), \psi_m^{(1)}(x) \right) \right)$$

$$= \sum_m \left( A_m^{(1)} \delta(n - m) + A_m^{(0)} \left( \psi_n^{(0)}(x), \psi_m^{(1)}(x) \right) \right)$$

$$= A_n^{(1)} + \sum_m A_m^{(0)} \left( \psi_n^{(0)}(x), \psi_m^{(1)}(x) \right).$$

Therefore

$$A_m^{(1)} = - (\psi_n^{(0)}(x), \psi_r^{(1)}(x)) - \sum_n A_n^{(0)} \left( \psi_n^{(0)}(x), \psi_n^{(1)}(x) \right),$$

which is the result quoted in equation (3.53).

Appendix B. Addressing Numerical Integration Difficulties. In this section, we demonstrate how to accurately evaluate the double integral in equation (4.37).

$$I = \int_0^\infty \int_0^\infty g_\nu^{(0)} \left( A_\omega^{(0)} a_\nu^{(1)} \psi_\nu^{(0)} - A_\nu^{(0)} a_\omega^{(1)} \psi_\omega^{(0)} \right) d\nu d\omega. \quad (B.1)$$

The difficulty in numerically evaluating (B.1) is that, for most $A_\omega^{(0)}$, the integrand blows up as $\nu \to \omega$. This is due to the factor of $1/(\omega^2 - \nu^2)$ which appears in $a_\nu^{(1)}$ (refer to equation 4.32 for details). Thus, as it is written in equation (B.1), numerically evaluating $I$ would require adding and subtracting some very large numbers, which
most numerical integrators are not very well-equipped to do. Thankfully, there are a few numerical tricks we can perform in order to facilitate numerical evaluation of (B.1). To begin, we will establish some notation. Let

\[
h(\omega, \nu) = A^{(0)}_{\omega} a^{(1)}_{\omega, \nu} \psi^{(0)}_{\nu} - A^{(0)}_{\nu} a^{(1)}_{\nu, \omega} \psi^{(0)}_{\omega},
\]

(B.2)

and, make the following change of variables

\[
\omega(u, v) = \frac{1}{\sqrt{2}} (u - v), \\
\nu(u, v) = \frac{1}{\sqrt{2}} (u + v).
\]

Now, we define

\[
G(u, v) := g(\omega(u, v)), \\
H(u, v) := h(\omega(u, v), \nu(u, v)),
\]

so that

\[
I = \int_{0}^{\infty} \int_{-u}^{u} G(u, v) H(u, v) dv du.
\]

(B.3)

So far, everything we have done is cosmetic; the integrand of equation (B.3) still blows up near \(v = 0\) (which corresponds to \(\omega = \nu\)). Note, however, that \(H(u, v) = -H(u, -v)\). As such, we may write equation (B.3) as

\[
I = \int_{0}^{\infty} \int_{0}^{u} H(u, v) (G(u, v) - G(u, -v)) dv du.
\]

(B.4)

The integrand in equation (B.4) is well-behaved throughout its domain. Figure B.1 illustrates how this simple trick smoothens out the singularity.

REFERENCES


Fig. B.1. In subfigure (a) we plot the integrand of equation (B.3), $G(u, su)H(u, su)$, for $s \in (-1, 1)$. In subfigure (b) we plot the integrand of equation (B.4), $H(u, su)\{G(u, su) - G(u, -su)\}$, for $s \in (0, 1)$. In both plots, the solid black line corresponds to $u = 4$, the dot-dashed red line corresponds to $u = 8$, and the dashed blue line corresponds to $u = 16$. Note that the integrand of equation (B.4) is well-behaved, whereas the integrand of equation (B.3) blows up at $s = 0$. For both plots, we chose the following parameters: $T - t = 1/2$, $x = 0$, $r = 0.05$, $\sqrt{\sigma^2} = 0.34$, $R_u = 4$, $V_2 = 1$ and $V_3 = 1$.