Option Pricing under Hybrid Stochastic and Local Volatility

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Abstract This paper proposes a new option pricing model which can be thought of as a hybrid stochastic volatility and local volatility model. This model is built on the local volatility of the constant elasticity of variance (CEV) model multiplied by stochastic volatility which is driven by a fast mean-reverting Ornstein-Uhlenbeck process. The formal asymptotic approximations for option prices are studied to extend the existing CEV pricing formula. From the results, we show that our model improves the downside of the CEV model in terms of dynamics of implied volatilities with respect to underlying asset price by resolving possible hedging instability problem. The implied volatility structure and calibration of our option pricing model are also discussed.

Keywords: stochastic volatility, constant elasticity of variance, asymptotic analysis, option pricing, implied volatility

1. Introduction

The geometric Brownian motion assumption for underlying asset price in the standard Black-Scholes model (1973) is well-known not to capture the accumulated empirical evidence in financial industry. A main draw back in the assumption of the Black-Scholes model is in flat implied volatilities, which is contradictory to empirical results showing that implied volatilities of the equity options exhibit the smile or skew curve. For example, Rubinstein (1985) before the 1987 crash and by Jackwerth and Rubinstein (1996) after the crash belong to those representative results. Before the 1987 crash, the geometry of implied volatilities against the strike price was often observed to be U-shaped with minimum at or near the the money but, after the crash, the shape of smile curve becomes more typical.

Among those several ways of overcoming the draw back and extending the geometric Brownian motion to incorporate the smile effect, there is a way that makes volatility depend on underlying asset price as well as time. One renowned model in this category is the constant elasticity of variance diffusion model. This model was initially studied by Cox (1975), and Cox and Ross (1976) and was designed and developed to incorporate the negative correlation between the underlying asset price change and volatility change. The CEV diffusion has been applied to exotic options as well as standard options by many authors. For example, see Boyle and Tian (1999), Boyle et al. (1999), Davydov and Linetsky (2001, 2003), and Linetsky (2004) for studies of barrier and/or lookback options. See Beckers (1980), Emanuel and MacBeth (1982), Schroder (1989), Delbaen and Shirakawa (2002), Heath and Platen (2002), and Carr and Linetsky (2006) for general option pricing studies. The book by Jeanblanc, Yor, and Chesney (2006) may serve as a general reference.

One disadvantage in local volatility models is, however, that volatility and underlying risky asset price changes are perfectly correlated either positively or negatively depending on the chosen elasticity parameter, whereas empirical studies show that prices may decrease when volatility increases or vice versa but it seems that there is no definite correlation all the time. For example, Harvey

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(2001) and Ghysels et al. (1996) show that it is time varying. Based on this observation together with the renowned contribution of stochastic volatility formulation to option pricing, we introduce an external process, apart from the risky asset process itself, driving volatility. Then new volatility is given by the multiplication of a function of the new process and the local CEV volatility so that the new option pricing formulation can be thought of as a hybrid stochastic volatility and local volatility model.

Many financial models seem to like to have a feature that is given by a mean-reverting process. The process is characterized by its typical time to get back to the mean level of its long-run distribution (the invariant distribution of the process). We take a fast mean-reverting stochastic process as the external process mentioned above so that if mean reversion is very fast, then the underlying asset price process should be close to the CEV diffusion. This means that if mean reversion is extremely fast, then the CEV model is a good approximation. However, for fast but finite mean reversion, the CEV model needs to be corrected to account for stochastic volatility. This is the main concern of this paper.

The methodology to be used here is asymptotic analysis based upon a mean-reverting Ornstein-Uhlenbeck (OU) diffusion which decorrelates rapidly while fluctuating on a fast time-scale. The choice of fast mean-reverting OU process provides us analytic advantage. This is basically related to averaging principle and ergodic theorem or, more directly, asymptotic diffusion limit theory of stochastic differential equations with a small parameter which has been initiated by Khasminskii (1966) and developed by Papanicolaou (1978), Asch et al. (1991), Kim (2004) and Cerrai (2009). This type of theory has been effectively applied to some ‘pure’ stochastic volatility models in financial mathematics. Fouque et al. (2000) considered a class of underlying asset price models in which volatility is a function of fast mean-reverting OU diffusion and give a certain model independent correction extending the standard Black-Scholes formula. This correction is universal in this class of models and it is given by two group parameters which can be easily calibrated from the observed implied volatility surface. The analysis has been effectively applied to many topics in finance including option pricing, interest rate derivatives and credit derivatives.

This paper is organized as follows. In Section 2, we review the CEV option price formula and formulate an option pricing problem as a stochastic volatility extension of the CEV model in terms of a singularly perturbed PDE. In Section 3, we use singular perturbation technique on the option pricing PDE to derive the corrected price of the CEV option price and also give an error estimate to verify mathematical rigor for our approximation. In Section 4, we study the implied volatility structure and the calibration of our option pricing model. Concluding remarks are given in Section 5.

2. The CEV Formula - Review

In this section we review briefly the well-known CEV option price formula.

The dynamics of underlying asset price of the renowned CEV model is given by the SDE

$$dX_t = \mu X_t dt + \sigma X_t^\theta dW_t,$$

where $\mu$, $\sigma$ and $\theta$ are constants. In this model, volatility is a function of underlying asset price which is given by $\sigma X_t^{\theta - 1}$. Depending upon the elasticity parameter $\frac{\theta}{2}$, this model can reduce to the well-known popular option pricing models. For example, If $\theta = 2$, it reduces to the Black-Scholes
model with constant volatility $\sigma$ (1973). If $\theta = 1$, it becomes the square root model or Cox-Ross model (1976). If $\theta = 0$, then it is the absolute model. Note that if $\theta < 2$, the volatility and the stock price move inversely. If $\theta > 2$, the volatility and the stock price move in the same direction.

There is an analytic form of option price formula for the CEV diffusion solving (1). It was first given by Cox (1975) using the transition probability density function for the CEV diffusion. He did not show the derivation of the transition probability density function but it is not difficult to obtain it. Basically, one transforms $X_t$ into $Y_t$ via $Y_t = X_t^{2-\theta}, \theta \neq 2$ and then apply the Ito lemma and the Feynman-Kac formula. Then using the well-known result on the parabolic PDE by Feller (1951) one can obtain the transition probability density function as follows:

$$f(\tilde{x}, T; x, t) = (2 - \theta) k \frac{1}{2\pi \sigma^2 (2 - \theta)} e^{-u-v} I_{\frac{1}{2\theta}} (2(uv)^{\frac{1}{2}}), \quad \theta \neq 2$$

where

$$k = \frac{2r}{\sigma^2 (2 - \theta)(e^{r(2-\theta)(T-t)} - 1)},$$

$$u = kx^{2-\theta} e^{r(2-\theta)(T-t)},$$

$$v = k\tilde{x}^{2-\theta}$$

and $I_q(x)$ is the modified Bessel function of the first kind of order $q$ defined by

$$I_q(x) = \sum_{n=0}^{\infty} \frac{(\frac{x}{2})^{2n+q}}{n! \Gamma(n+1+q)}$$

Once the transition probability density function is obtained, the call option pricing formula for the CEV model will follow as sum of two integrals:

$$C_{CEV}(t, x) = \int_{-\infty}^{\infty} e^{-r(T-t)}(\tilde{x} - K)^+ f(\tilde{x}, T; x, t) d\tilde{x}$$

$$= e^{-r(T-t)} \int_{K}^{\infty} \tilde{x} f(\tilde{x}, T; x, t) d\tilde{x} - e^{-r(T-t)} K \int_{K}^{\infty} f(\tilde{x}, T; x, t) d\tilde{x}.$$ 

The computation of these integrals was done by Schroder (1989) and the result is given by the following theorem. Of course, the put price can be given by the put-call parity.

**Theorem 2.1:** The call option price $C_{CEV}$ for $X_t = x$ is given by

$$C_{CEV}(t, x) = \sum_{n=0}^{\infty} g(n+1, u) G(n+1 + \frac{1}{2-\theta}, kK^{2-\theta})$$

$$-K e^{-r(T-t)} \sum_{n=0}^{\infty} g(n+1 + \frac{1}{2-\theta}, u) G(n+1, kK^{2-\theta}),$$

where $G(m, v)$ and $g(m, u)$ are the complementary Gamma distribution and its density given by

$$G(m, v) = \int_{v}^{\infty} g(m, u) du, \quad g(m, u) = \frac{e^{-u}u^{m-1}}{\Gamma(m)},$$

respectively.

**Proof:** Refer to Schroder (1989) or Chen and Lee (1993).
Alternatively, the call option price can be obtained by solving the pricing PDE

\[
\partial_t C + \frac{1}{2} \sigma^2 x^\theta \partial^2_{xx} C + r(x \partial_x C - C) = 0,
\]

\[
C(T, x) = (x - K)^+.
\]

This PDE can be solved, for example, by the Green's function method after being reduced to a simple form. See Lipton (2001) for details.

Sometimes it is more convenient to rewrite the above option formula in the form which directly generalizes the standard Black-Scholes formula (see, for example, Lipton (2001)):

\[
C_{CEV}(t, x) = e^{-r(T-t)x} \int_K^{\infty} \left( \frac{x}{y} \right)^{\frac{1}{2(2-\theta)}} e^{-(x+y)} I_{\frac{1}{2-\theta}}(2\sqrt{xy}) \, dy
\]

\[
+ e^{-r(T-t)K} \int_K^{\infty} \left( \frac{y}{x} \right)^{\frac{1}{2(2-\theta)}} e^{-(x+y)} I_{\frac{1}{2-\theta}}(2\sqrt{xy}) \, dy,
\]

where

\[
\tilde{x} = \frac{2xe^{r(2-\theta)(T-t)}}{(2-\theta)^2 \chi},
\]
\[
\chi = \frac{\sigma^2}{(2-\theta)r} (e^{r(2-\theta)T} - e^{r(2-\theta)t}),
\]
\[
\tilde{K} = \frac{2K^{2-\theta}}{(2-\theta)^2 \chi}.
\]

Note that by using the asymptotic formula

\[
I_{\frac{1}{2-\theta}} \left( \frac{x}{\beta^2} \right) \sim \frac{\beta e^{x/\beta^2 - 1/8x}}{\sqrt{2\pi x}}, \quad \beta \to 0
\]

for the Bessel function, one can show that the CEV call price \( C_{CEV} \) goes to the usual Black-Scholes price \( C_{BS} \) as \( \theta \) goes to 2.

3. Problem Formulation

In this section, we establish a new option pricing model that extends the local volatility of the CEV model to incorporate stochastic volatility.

First, the local volatility term \( X_t^\theta \) of the CEV model is changed into such a form that it is a product of a function of external process \( Y_t \) and \( X_t^\theta \) so that the volatility of the asset price depends on both internal and external noise sources. In this form, it would be natural to choose \( Y_t \) as a mean-reverting diffusion process since the newly formed volatility gets back to the mean level that corresponds to the CEV case. The mean-reverting Ornstein-Uhlenbeck process is an example of such process.

As an underlying asset price model in this paper, therefore, we take the SDEs

\[
dX_t = \mu X_t dt + f(Y_t)X_t^\theta dW_t
\]
\[
dY_t = \alpha(m - Y_t)dt + \beta d\tilde{Z}_t,
\]
where \( f(y) \) is a sufficiently smooth function and the Brownian motion \( \hat{Z}_t \) is correlated with \( W_t \) such that \( d\langle W, \hat{Z} \rangle_t = \rho dt \). In terms of the instantaneous correlation coefficient \( \rho \), we write \( \hat{Z}_t = \rho W_t + \sqrt{1 - \rho^2} Z_t \) for convenience. Here, \( Z_t \) is a standard Brownian motion independent of \( W_t \).

It is well-known from the Ito formula that the solution of (8) is a Gaussian process given by

\[
Y_t = m + (Y_0 - m)e^{-\alpha t} + \beta \int_0^t e^{-\alpha(t-s)} d\hat{Z}_s
\]

and thus \( Y_t \sim N(m + (Y_0 - m)e^{-\alpha t}, \beta^22^{-\alpha}) \) leading to invariant distribution given by \( N(m, \beta^22^{-\alpha}) \). Later, we will use notation \( \langle \cdot \rangle \) for the average with respect to the invariant distribution, i.e.,

\[
\langle g(Y_t) \rangle = \frac{1}{\sqrt{2\pi \nu^2}} \int_{-\infty}^{+\infty} g(y)e^{-\frac{(y-m)^2}{2\nu^2}} dy, \quad \nu^2 \equiv \frac{\beta^2}{2\alpha}
\]

for arbitrary function \( g \). Notice that \( \nu^2 \) denotes the variance of the invariant distribution of \( Y_t \).

Once an equivalent martingale measure \( Q \), under which the discounted asset price \( e^{-rT}X_t \) is a martingale, is provided, the option price is given by the formula

\[
P(t, x, y) = E^Q[e^{-r(T-t)}h(X_T)|X_t = x, Y_t = y]
\]

under \( Q \). The existence of this martingale measure is guaranteed by the well-known Girsanov theorem, which further provides that processes defined by

\[
W_t^* = W_t + \int_0^t \frac{\mu - r}{f(Y_s)X_s^{\theta/2}} ds,
\]
\[
Z_t^* = Z_t + \int_0^t \gamma_s ds
\]

are independent standard Brownian motion under \( Q \), where \( \gamma_t \) is arbitrary adapted process to be determined. We will use notation \( Q^\gamma \) in stead of \( Q \) to emphasize the dependence on \( \gamma \). It may raise a question whether Novikov’s condition is satisfied on \( W^* \). This issue could be resolved by giving dependence of \( \mu \) on the process \((X_t, Y_t)\). We, however, omit the details in this paper.

We first change the subjective SDEs (7)-(8) into the risk-neutral version as follows. Under an equivalent martingale measure \( Q^\gamma \), we have

\[
dX_t = rX_t dt + f(Y_t)X_t^\theta dW_t^*,
\]
\[
dY_t = (\alpha(m - Y_t) - \beta \Lambda(t, y)) dt + \beta d\hat{Z}_t^*,
\]

where \( \hat{Z}_t^* = \rho W_t^* + \sqrt{1 - \rho^2} Z_t^* \) and \( \Lambda \) (the market price of volatility risk) is assumed to be independent of \( x \).

Then the Feynman-Kac formula leads to the following pricing PDE for the option price \( P \).

\[
P_t + \frac{1}{2} f^2(y) x^\theta P_{xx} + \rho \beta f(y) x^\theta P_{xy} + \frac{1}{2} \beta^2 P_{yy}
\]
\[
+ rxP_x + (\alpha(m - y) - \beta \Lambda(t, y))P_y - rP = 0.
\]
4. **Asymptotic Option Pricing**

Now, to develop an asymptotic pricing theory on fast mean-reversion, we introduce a small parameter $\epsilon$ to denote the inverse of the rate of mean reversion $\alpha$:

$$
\epsilon = \frac{1}{\alpha}
$$

which is also the typical correlation time of the OU process $Y_t$. We assume that $\nu$ (the standard deviation of the invariant distribution of $Y_t$), which was introduced in Section 3 as $\nu = \frac{\beta}{\sqrt{2 \alpha}}$, remains fixed in scale as $\epsilon$ becomes zero. Thus we have $\alpha \sim O(\epsilon^{-1})$, $\beta \sim O(\epsilon^{-1/2})$, and $\nu \sim O(1)$.

After rewritten in terms of $\epsilon$, the PDE (12) becomes

$$
\left( \frac{1}{\epsilon} L_0 + \frac{1}{\sqrt{\epsilon}} L_1 + L_2 \right) P_\epsilon = 0
$$

where

$$
L_0 = \nu \partial_{yy}^2 + (m - y) \partial_y,
$$

$$
L_1 = \sqrt{2} \rho \nu f(y) x^\theta \partial_{xy}^2 - \sqrt{2} \nu \Lambda(t, y) \partial_y,
$$

$$
L_2 = \partial_t + \frac{1}{2} f(y) x^\theta \partial_{xx}^2 + r(x \partial_x - \cdot).
$$

Here, the operator $\alpha L_0$ is the same as the infinitesimal generator of the OU process $Y_t$ acts only on the $y$ variable. The operator $L_1$ is the sum of two terms; the first one is the mixed partial differential operator due to the correlation of the two Brownian motions $W^*$ and $\hat{Z}^*$ and the second one is the first-order differential operator with respect to $y$ due to the market price of elasticity risk. Finally, the third operator $L_2$ is a generalized version of the classical Black-Scholes operator at the volatility level $\sigma(x, y) = f(y) x^{\theta - 1}$ in stead of constant volatility.

Before we solve the problem (14)-(17), we write a useful lemma about the solvability (centering) condition on the Poisson equation related to $L_0$ as follows.

**Lemma 4.1:** If solution to the Poisson equation

$$
L_0 \chi(y) + \psi(y) = 0
$$

exists, then the condition $\langle \psi \rangle = 0$ must be satisfied, where $\langle \cdot \rangle$ is the expectation with respect to the invariant distribution of $Y_t$. If then, solutions of (18) are given by the form

$$
\chi(y) = \int_0^t E^y[\psi(Y_t)] \, dt + \text{constant}.
$$

**Proof:** Refer to Fouque et al. (2000). $\Box$

Although the problem (14)-(17) is a singular perturbation problem, we are able to obtain a limit of $P_\epsilon$ as $\epsilon$ goes to zero and also characterize the first correction for small but nonzero $\epsilon$. In order to do it, we first expand $P_\epsilon$ in powers of $\sqrt{\epsilon}$:

$$
P_\epsilon = P_0 + \sqrt{\epsilon} P_1 + \epsilon P_2 + \cdots
$$

Here, the choice of the power unit $\sqrt{\epsilon}$ in the power series expansion was determined by the method of matching coefficient.
Substituting (20) into (14) yields
\[ \frac{1}{\epsilon} \mathcal{L}_0 P_0 + \frac{1}{\sqrt{\epsilon}} (\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0) \]
\[ + (\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0) + \sqrt{\epsilon} (\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1) + \cdots = 0, \]
which holds for arbitrary \( \epsilon > 0 \).

Under some reasonable growth condition, one can show that the first corrected price \( P_0 + \sqrt{\epsilon} P_1 \) is independent of the unobserved variable \( y \) as follows.

**Theorem 4.1** Assume that \( P_0 \) and \( P_1 \) do not grow so much as
\[ \frac{\partial P_i}{\partial y} \sim e^{\frac{x^2}{2}}, \quad y \to \infty, \quad i = 0, 1. \]
Then \( P_0 \) and \( P_1 \) do not depend on the variable \( y \).

**Proof:** From the asymptotic expansion (21), we first have
\[ \mathcal{L}_0 P_0 = 0. \]
Solving this equation yields
\[ P_0(t, x, y) = c_1(t, x) \int_0^y e^{\frac{(y-z)^2}{2\sigma^2}} dz + c_2(t, x) \]
for some functions \( c_1 \) and \( c_2 \) independent of \( y \). From the imposed growth condition on \( P_0 \), \( c_1 = 0 \) can be chosen. This leads that \( P_0(t, x, y) \) becomes a function of only \( t \) and \( x \). We denote it by
\[ P_0 = P_0(t, x). \]
Since each term of the operator \( \mathcal{L}_1 \) contains \( y \)-derivative, the \( y \)-independence of \( P_0 \) yields \( \mathcal{L}_1 P_0 = 0 \). On the other hand, from the expansion (21), \( \mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0 \) holds. These facts are put together and yield \( \mathcal{L}_0 P_1 = 0 \) so that \( P_1 \) also is a function of only \( t \) and \( x \);
\[ P_1 = P_1(t, x). \]
\( \square \)

We next derive a PDE that the leading order price \( P_0 \) satisfies based upon the observation that \( P_0 \) and \( P_1 \) do not depend on the current level \( y \) of the process \( Y_t \).

**Theorem 4.2:** Under the growth condition on \( P_i \) \( (i = 0, 1) \) expressed in Theorem 4.1, the leading term \( P_0(t, x) \) is given by the solution of the PDE
\[ \partial_t P_0 + \frac{1}{2} \bar{\sigma}^2 x^2 \partial^2_{xx} P_0 + r(x \partial_x P_0 - P_0) = 0 \]
with the terminal condition \( P_0(T, x) = h(x) \), where \( \bar{\sigma} = \sqrt{(\bar{f}^2)} \). So, \( P_0 \) (Call) is the same as the price in the CEV formula in Theorem 2.1 while only \( \sigma \) is replaced with the effective coefficient \( \bar{\sigma} \).

**Proof:** From the expansion (21), the PDE
\[ \mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0 = 0 \]
holds. Since each term of the operator $L_1$ contains $y$-derivative, $L_1P_1 = 0$ holds so that (26) leads to

\[(27) \quad L_0P_2 + L_2P_0 = 0\]

which is a Poisson equation. Then the centering condition must be satisfied. From Lemma 3.1 with $\psi = L_2P_0$, the leading order $P_0(t, x)$ has to satisfy the PDE

\[(28) \quad \langle L_2 \rangle P_0 = 0\]

with the terminal condition $P_0(T, x) = h(x)$, where

\[\langle L_2 \rangle = \partial_t + \frac{1}{2}(f^2)x^\theta \partial^2_x + r(x\partial_x - \cdot).\]

Thus $P_0$ solves the PDE (25). □

Next, we derive the first correction $P_1(t, x)$ and estimate the error of the approximation $P_0 + \sqrt{\epsilon}P_1$.

For convenience, we need notation

(29) \quad $V_3 = \frac{\rho\nu}{\sqrt{2}} \langle f\psi' \rangle$,

(30) \quad $V_2 = -\nu \sqrt{2} \langle \Lambda\psi' \rangle$,

where $\psi(y)$ is solution of the Poisson equation

(31) \quad $L_0\psi = \nu^2\psi''' + (m - y)\psi' = f^2 - \langle f^2 \rangle$.

Note that $V_3$ and $V_2$ are constants.

**Theorem 4.3:** The first correction $P_1(t, x)$ satisfies the PDE

\[(32) \quad \partial_t P_1 + \frac{1}{2}\sigma^2 x^\theta \partial^2_x P_1 + r(x\partial_x P_1 - P_1) = V_3x^\frac{\theta}{2} \partial_x (x^\theta \partial^2_x P_0) + V_2x^\theta \partial^2_x P_0 + r(x\partial_x - \cdot)\]

with the final condition $P_1(T, x) = 0$, where $V_3$ and $V_2$ are given by (29) and (30), respectively. Further, the solution $P_1(t, x)$ ($\theta \neq 2$) of (32) is given by

\[(33) \quad P_1(t, x) = -\int_0^\tau \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma^2(\tau - z)} \exp\{\frac{-s^2}{2\sigma^2(\tau - z)}\exp\{\frac{-(s - y)^2}{2\sigma^2(\tau - z)}\}
\cdot\exp\{(-\frac{1}{2}(\frac{a}{y^2} - \frac{a}{s^2}) + \frac{1}{2\sigma^2}(\frac{a}{y} + by)^2 + \frac{1}{2\sigma^2}(\frac{a}{s} + bs)^2)z + \frac{1}{\sigma^2}(a\ln\frac{y}{s} + \frac{b}{2}(y^2 - s^2))}\}
\cdot h(y^{-\frac{1}{2}\theta + 1}, T - z) \, dy \, dz,$

where $\tau = T - t$, $s = x^{-\frac{1}{2}\theta + 1}$ and

\[\tilde{\sigma} = \frac{1}{\sqrt{2}}(\frac{1}{2}\theta + 1), \quad a = \frac{1}{2}\sigma^2(-\frac{1}{2}\theta + 1)(-\frac{1}{2}\theta), \quad b = r(-\frac{1}{2}\theta + 1), \quad h(x, t) = V_3x^\frac{\theta}{2} \partial_x (x^\theta \partial^2_x P_0) + V_2x^\theta \partial^2_x P_0 + r(x\partial_x - \cdot).\]
Proof: From (17) and (28)
\[
L_2 P_0 = L_2 P_0 - \langle L_2 P_0 \rangle
= \frac{1}{2} x^\theta (f^2 - \langle f^2 \rangle) \partial_x^2 P_0
\]
holds. Then (27) and (31) lead to
\[
P_2 = -L_0^{-1}(L_2 P_0)
= -\frac{1}{2} x^\theta L_0^{-1} (f^2 - \langle f^2 \rangle) \partial_x^2 P_0
= -\frac{1}{2} (\psi(y) + c(t, x)) x^\theta \partial_x^2 P_0
\]
for arbitrary function \(c(t, x)\) independent of \(y\).

On the other hand, from the expansion (21)
\[
L_0 P_3 + L_1 P_2 + L_2 P_1 = 0
\]
holds. This is a Poisson PDE for \(P_3\). Thus the centering condition \(\langle L_1 P_2 + L_2 P_1 \rangle = 0\) has to be satisfied. Then using the result (34) we obtain
\[
\langle L_2 \rangle P_1 = \langle L_2 P_1 \rangle = -\langle L_1 P_2 \rangle = \frac{1}{2} \langle L_1 \psi \rangle x^\theta \partial_x^2 P_0.
\]
Applying (16), (17), (29) and (30) to this result, we obtain the PDE (32).

Now, we solve the PDE (32) by following step. First, we change (32) with the terminal condition \(P_1(T, x) = 0\) into the initial value problem
\[
\begin{align*}
- \frac{\partial P_1}{\partial \tau} + \frac{1}{2} \sigma^2 \frac{\partial^2 P_1}{\partial s^2} + \left( \frac{a}{s} + bs \right) \frac{\partial P_1}{\partial s} - rP_1 &= \tilde{h}(s, \tau), \\
P_1(s, 0) &= 0,
\end{align*}
\]
(35)
where \(\tilde{h}(s, \tau) = h(s^{-\frac{1}{2\alpha+1}}, \tau)\). Then by the change of dependent variables
\[
P_1(\tau, s) = v(s)g(\tau)u(\tau, s),
\]
where
\[
v(s) = \exp \left\{ -\frac{1}{\sigma^2} \left( a \ln s + \frac{b}{2} s^2 \right) \right\},
\]
\[
g(\tau) = \exp \left[ \left\{ \frac{1}{2} \left( \frac{a}{s^2} - b \right) - \frac{1}{2\sigma^2} (\frac{a}{s} + bs)^2 - r \right\} \tau \right],
\]
the PDE (35) becomes
\[
\frac{\partial u}{\partial \tau} = \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial s^2} + F(s, \tau),
\]
(36)
\[
u(s, 0) = 0,
\]
where
\[
F(s, \tau) = -\exp\left\{ -\frac{1}{2} \left( \frac{a}{s^2} - b \right) + \frac{1}{2\sigma^2} (\frac{a}{s} + bs)^2 + r \right\} \tau + \frac{1}{\sigma^2} (a \ln s + \frac{b}{2} s^2) \tilde{h}(s, \tau).
\]
Equation (36) is a well-known form of diffusion equation whose solution is given by

$$u(\tau, s) = \int_0^{\tau} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\tilde{\sigma}^2(\tau - z)}} \exp\left\{ -\frac{(s - y)^2}{2\tilde{\sigma}^2(\tau - z)} \right\} F(y, z) dy dz.$$

Going back to the original variables, therefore, the solution of the PDE (32) is given by (33). □

5. Accuracy - Error Estimate

Combining the leading order $P_0$ and the first correction term $\tilde{P}_1 = \sqrt{\epsilon} P_1$, we have the approximation $P_0 + \tilde{P}_1$ to the original price $P$. Naturally, the next concern is about accuracy of the approximation $P_0 + \tilde{P}_1$ to the price $P$.

First, we write a well-known property of solutions to the Poisson equation (18) about their boundedness or growth.

**Lemma 5.1** If there are constant $C$ and integer $n \neq 0$ such that

$$|\psi(y)| \leq C(1 + |y|^n), \quad (37)$$

then solutions of the Poisson equation (18), i.e. $L_0\chi(y) + \psi(y) = 0$, satisfy

$$|\chi(y)| \leq C_1(1 + |y|^n) \quad (38)$$

for some constant $C_1$. If $n = 0$, then $|\chi(y)| \leq C_1(1 + \log(1 + |y|))$.

**Proof:** Refer to Fouque et al. (2000). □

Now, we are ready to estimate the error of the approximation $P_0 + \tilde{P}_1$ to the price $P$.

**Theorem 5.1** If each of $f$ and $\Lambda$ is bounded by a constant independent of $\epsilon$, then we have

$$P^\epsilon - (P_0 + \tilde{P}_1) = O(\epsilon). \quad (39)$$

**Proof:** We first define $Z^\epsilon(t, x, y)$ by

$$Z^\epsilon = \epsilon P_2 + \epsilon \sqrt{\epsilon} P_3 - [P^\epsilon - (P_0 + \tilde{P}_1)].$$

Once we show that $Z^\epsilon = O(\epsilon)$, the estimate (39) will follow immediately.

In general, $h$ is not smooth as we assume in this proof. However, the generalization of the proof to the case of European vanilla call or put option is possible as shown in Fouque et al. (2003). The detailed proof is omitted here.

Using the operators in (15)-(17), we define

$$\mathcal{L}^\epsilon = \frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2$$
and apply the operator $\mathcal{L}^\epsilon$ to $Z^\epsilon$ to obtain

$$\mathcal{L}^\epsilon Z^\epsilon = \mathcal{L}^\epsilon (P_0 + \sqrt{\epsilon}P_1 + \epsilon P_2 + \epsilon\sqrt{\epsilon}P_3) - P^\epsilon$$

$$= \mathcal{L}^\epsilon (P_0 + \sqrt{\epsilon}P_1 + \epsilon P_2 + \epsilon\sqrt{\epsilon}P_3)$$

$$= \frac{1}{\epsilon} \mathcal{L}_0 P_0 + \frac{1}{\sqrt{\epsilon}} (\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0) + (\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0)$$

$$+ \sqrt{\epsilon} (\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1) + \epsilon (\mathcal{L}_1 P_3 + \mathcal{L}_2 P_2) + \epsilon\sqrt{\epsilon} \mathcal{L}_2 P_3$$

$$= \epsilon (\mathcal{L}_1 P_3 + \mathcal{L}_2 P_2) + \epsilon\sqrt{\epsilon} \mathcal{L}_2 P_3. \tag{40}$$

Here, we have used $\mathcal{L}^\epsilon P^\epsilon = 0$ and the fact that the expansion (21) holds for arbitrary $\epsilon$. The result (40) yields $\mathcal{L}^\epsilon Z^\epsilon = O(\epsilon)$.

Now, let $F^\epsilon$ and $G^\epsilon$ denote

$$F^\epsilon(t, x, y) = \mathcal{L}_1 P_3 + \mathcal{L}_2 P_2 + \sqrt{\epsilon} \mathcal{L}_2 P_3, \tag{41}$$

$$G^\epsilon(x, y) = P_2(T, x, y) + \sqrt{\epsilon} P_3(T, x, y), \tag{42}$$

respectively. Then from (40)-(42) we have a parabolic PDE for $Z^\epsilon$ of the form

$$\mathcal{L}^\epsilon Z^\epsilon - \epsilon F^\epsilon = 0$$

with the final condition $Z^\epsilon(T, x, y) = \epsilon G^\epsilon(x, y)$. Applying the Feynman-Kac formula to this PDE yields

$$Z^\epsilon = \epsilon E^\epsilon \left[ e^{-r(T-t)} G^\epsilon - \int_t^T e^{-r(s-t)} F^\epsilon \, ds \mid X_t^\epsilon = x, Y_t^\epsilon = y \right]$$

in a risk-neutral world. On the other hand, since $P_2$ and $P_3$ composing $F^\epsilon$ and $G^\epsilon$ are solutions of the Poisson equations $\mathcal{L}_0 P_2 + \mathcal{L}_2 P_0 = 0$ and $\mathcal{L}_0 P_3 + (\mathcal{L}_1 P_2 + \mathcal{L}_2 P_1) = 0$, respectively, $F^\epsilon$ and $G^\epsilon$ are bounded uniformly in $x$ and at most linearly growing in $|y|$ from Lemma 5.1. Consequently, the desired error estimate $Z^\epsilon = O(\epsilon)$ follows. □

6. IMPLIED VOLATILITIES AND CALIBRATION

To obtain the first correction term $\tilde{P}_1$, whose PDE is given by Theorem 4.3, it is required to have the parameters $V_3$ and $V_2$. Practically, the value of the parameter $\theta$ is close to 2. So, in this section, we employ another perturbation for $\theta$ and fit the parameters to observed prices of Call options for different strikes and maturities.

For convenience, we use the implied volatility $I$ defined by

$$C_{BS}(t, x; K, T; I) = P(\text{Approximated Observed Price}).$$

After we choose $I_0(K, T)$ such that

$$C_{BS}(t, x; K, T; I_0) = P_0(t, x) = C_{CEV}(\tilde{\sigma}),$$

if we use expansion $I = I_0 + \sqrt{\epsilon} I_1 + \cdots$ on the left-hand side and our approximation $P = P_0 + \tilde{P}_1 + \cdots$ on the right-hand side, then

$$C_{BS}(t, x; K, T; I_0) + \sqrt{\epsilon} I_1 \frac{\partial C_{BS}}{\partial \sigma}(t, x; K, T; I_0) + \cdots = P_0(t, x) + \tilde{P}_1(t, x) + \cdots,$$
Table 1. The Elasticity for SPX

<table>
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<th>Number</th>
<th>Maturity(day)</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>70</td>
<td>2.0138</td>
</tr>
<tr>
<td>2</td>
<td>130</td>
<td>2.0063</td>
</tr>
<tr>
<td>3</td>
<td>220</td>
<td>2.0111</td>
</tr>
<tr>
<td>4</td>
<td>490</td>
<td>2.0075</td>
</tr>
<tr>
<td>5</td>
<td>860</td>
<td>2.0121</td>
</tr>
</tbody>
</table>

which yields that the implied volatility is given by

\begin{equation}
I = I_0 + \tilde{P}_1(t, x) \left( \frac{\partial C_{BS}}{\partial \sigma}(t, x; K, T; I_0) \right)^{-1} + O(1/\alpha).
\end{equation}

where $\frac{\partial C_{BS}}{\partial \sigma}$ (Vega) is well-known to be given by

\[
\frac{\partial C_{BS}}{\partial \sigma}(t, x; K, T; I_0) = \frac{xe^{-d_1^2/2}\sqrt{T-t}}{\sqrt{2\pi}},
\]

\[
d_1 = \frac{\log(x/K) + (r + \frac{1}{2}I_0^2)(T-t)}{I_0\sqrt{T-t}}.
\]

On the other hand, we note that when $\theta = 2$, $P_0$ is the Black-Scholes Call option price with volatility $\tilde{\sigma}$, denoted by $C_{BS}(\tilde{\sigma})$, and $\tilde{P}_1$ is explicitly given by

\[
\tilde{P}_1 = -(T-t)H_0,
\]

where

\begin{equation}
H_0 = A(V^*_3, V^*_2)C_{BS}(\tilde{\sigma})
\end{equation}

with

\[
V^*_3 = \frac{\nu}{\sqrt{2\alpha}}(f\psi'),
\]

\[
V^*_2 = -\frac{\nu}{\sqrt{2\alpha}}(A\psi').
\]

Here, we used a differential operator $A(\alpha, \beta)$ defined by

\[
A(\alpha, \beta) = \alpha x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2}{\partial x^2} \right) + \beta x^2 \frac{\partial^2}{\partial x^2},
\]

and $\psi(y)$ as a solution of the Poisson equation (31). Note that to compute $A(V^*_3, V^*_2)$ it is required to have Gamma and Speed of the Black-Scholes price. So, fitting (43) to the implied volatilities one can obtain $V^*_3$ and $V^*_2$.

Practically, the elasticity $\theta$ is close to 2. See Table 1. So, from now on we approximate $P_0$ and $\tilde{P}_1$ near $\theta = 2$. We use notation

\[
\phi = 2 - \theta
\]

here. We expand $P_0$ as

\[
P_0 = C_{CEV}(\bar{\sigma}) = C_{BS}(\bar{\sigma}) + \phi P_{0,1} + \phi^2 P_{0,2} + \cdots
\]
and $H$, defined by $H(t, x) = A(V_3^*, V_2^*)P_0$, and $\tilde{P}_1$, respectively, as
\[
H = H_0 + \phi H_1 + \phi^2 H_2 + \cdots \\
\tilde{P}_1 = -(T - t)H_0 + \phi \tilde{P}_{1,1} + \phi^2 \tilde{P}_{1,2} + \cdots.
\]
Then from Theorem 4.2 $P_{0,1}$ satisfies the Black-Scholes equation with a non-zero source but with a zero terminal condition which is given by
\[
\mathcal{L}_{BS}(\tilde{\sigma})P_{0,1} = \frac{1}{2} \sigma^2 x^2 \ln x \frac{\partial^2 C_{BS}}{\partial x^2}, \\
P_{0,1}(T, x) = 0,
\]
and $\mathcal{L}_{BS}(\tilde{\sigma})H_0 = 0$ as it should be, and from Theorem 4.3 $\tilde{P}_{1,1}$ is given by
\[
\mathcal{L}_{BS}(\tilde{\sigma})\tilde{P}_{1,1} = H_1 - \frac{1}{2} \sigma^2 (T - t)x^2 \ln x \frac{\partial^2 H_0}{\partial x^2}, \\
\tilde{P}_{1,1}(T, x) = 0,
\]
where $H_0$ is (44) and $H_1$ is given by
\[
H_1 = A(V_3^*, V_2^*)P_{0,1}.
\]
Up to order $\phi^2$, therefore, $P_0$ is given by
\[
P_0 = C_{BS}(\tilde{\sigma}) + \phi P_{0,1},
\]
where $P_{0,1}$ is the solution of (45), and the correction $\tilde{P}_1$ is given by
\[
\tilde{P}_1 = -(T - t)A(V_3^*, V_2^*)C_{BS}(\tilde{\sigma}) + \phi \tilde{P}_{1,1},
\]
where $\tilde{P}_{1,1}$ is the solution of (46), respectively.

Synthesizing the discussion above, pricing methodology can be implemented as follows in the case near $\theta = 2$. After estimating $\tilde{\sigma}$ (the effective historical volatility) from the risky asset price returns and confirming from the historical asset price returns that volatility is fast mean-reverting, fit (43) to the implied volatility surface $I$ to obtain $V_{i}^*$, $i = 2, 3$. Then the Call price of a European option corrected for stochastic volatility is approximated by
\[
P_0 + \tilde{P}_1 = C_{BS}(\tilde{\sigma}) - (T - t)A(V_3^*, V_2^*)C_{BS}(\tilde{\sigma}) + (2 - \theta)(P_{0,1} + \tilde{P}_{1,1}),
\]
where $P_{0,1}$ and $\tilde{P}_{1,1}$ are given (in integral form) by
\[
P_{0,1} = \frac{1}{2} \tilde{\sigma}^2 E^* \left[ - \int_t^T X_s^2 \ln X_s \frac{\partial^2 C_{BS}(\tilde{\sigma})}{\partial x^2} \, ds \right], \\
\tilde{P}_{1,1} = E^* \left[ - \frac{1}{\sqrt{\alpha}} \int_t^T A(V_3^*, V_2^*)P_{0,1} \, ds \\
+ \frac{1}{2} \tilde{\sigma}^2 \int_t^T (T - s)X_s^2 \ln X_s \frac{\partial^2}{\partial x^2} \left( A(V_3^*, V_2^*)C_{BS}(\tilde{\sigma}) \right) \, ds \right],
\]
respectively.

Our corrected price $P_0 + \tilde{P}_1$ has been constructed based upon the new volatility given by the multiplication of a function of the OU process and the local CEV volatility. So, let us call our model as SVCEV model. Figure 1 and 2 show the proof of our new option pricing formulation in terms of implied volatilities. Here, the red line is market data. The blue line is SVCEV fit. Also, Figure 3 demonstrates that the implied volatilities move in the desirable direction with respect to
the underlying asset price, which would be contrary to the CEV model. In figure 3, we plot the implied volatility curve using calibration results $\sigma = 0.11$, $V_2 = -0.0611$, $V_3 = -0.1472$, $\theta = 2.011$ for SPX (Days to Maturity 220).

7. Conclusion

Our new option pricing model based upon a hybrid structure of stochastic volatility and constant elasticity of variance has demonstrated the improvement of the downside of the CEV model in that the implied volatilities move in accordance with the real market phenomena with respect to the underlying asset price. This is an important result in the sense that otherwise it may create a possible hedging instability problem. Also, the geometric structure of the implied volatilities shows a smile fitting the market data well. Consequently, the new underlying asset price model may serve
as a sound alternative one replacing the CEV model for other financial problems such as credit risk and portfolio optimization apart from option pricing discussed in this paper.

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