Calibration of Stock Betas from Skews of Implied Volatilities

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Capital Asset Pricing Model

The original discrete time CAPM model defined the log price return on individual asset $R_a$ as a linear function of the risk free interest rate $R_f$, the log return of the market $R_M$, and a Gaussian error term:

$$ R_a - R_f = \beta_a (R_M - R_f) + \epsilon_a $$

The beta coefficient $\beta_a$ was originally estimated using historical returns on the asset and market index, by a simple linear regression of asset returns on market returns.

**Fundamental flaw:** it is inherently backward looking, and used in forward looking portfolio construction.
Previous Attempt to Forward Looking Betas

Christoffersen, Jacobs, and Vainberg (2008, McGill University, Canada) have attempted to extract the beta parameter from option prices on the underlying market and asset processes:

\[ \beta_a = \left( \frac{SKEW_a}{SKEW_M} \right)^{\frac{1}{3}} \left( \frac{VAR_a}{VAR_M} \right)^{\frac{1}{2}}, \]

where \( VAR_a \) (resp. \( VAR_M \)), and \( SKEW_a \) (resp. \( SKEW_M \)) are the variance, and the risk-neutral skewness of returns of the asset (resp. of the market).

Then, they use results from Carr and Madan (2001) which relate these moments to options prices (Quad and Cubic) on the asset (resp. on the market).

The advantage of this approach is that option prices are inherently forward looking on the underlying price processes.
Continuous Time CAPM

The market price $M_t$ and an asset price $X_t$ evolve as follows:

\[
\frac{dM_t}{M_t} = \mu dt + \sigma_m dW_t^{(1)},
\]

\[
\frac{dX_t}{X_t} = \beta \frac{dM_t}{M_t} + \sigma dW_t^{(2)},
\]

for constant positive volatilities $\sigma_m$ and $\sigma$. In this model we assume independence between the Brownian motions driving the market and asset price processes: $d\langle W^{(1)}, W^{(2)} \rangle_t = 0$, so that

\[
\text{Cov} \left( \frac{dX_t}{X_t}, \frac{dM_t}{M_t} \right) = \frac{\text{Cov} \left( \beta \frac{dM_t}{M_t} + \sigma dW_t^{(2)}, \frac{dM_t}{M_t} \right)}{\text{Var} \frac{dM_t}{M_t}} = \frac{\text{Cov} \left( \beta \frac{dM_t}{M_t}, \frac{dM_t}{M_t} \right)}{\text{Var} \frac{dM_t}{M_t}} = \beta.
\]
Beta Estimation with Constant Volatility CAPM

Observe that the evolution of \( X_t \) is given by

\[
\frac{dX_t}{X_t} = \beta \mu dt + \beta \sigma_m dW_t^{(1)} + \sigma dW_t^{(2)},
\]

that is a geometric Brownian motion with volatility

\[
\sqrt{\beta^2 \sigma_m^2 + \sigma^2}
\]

Even if this quantity is known, along with the volatility \( \sigma_m \) of the market process, one cannot disentangle \( \beta \) and \( \sigma \). Then, one has to rely on historical returns data.

This drawback, along with the fact that constant volatility does not generate skews, motivates us to introduce stochastic volatility in the model.
Stochastic Volatility in Continuous Time CAPM

We introduce a stochastic volatility component to the market price process, that is we replace $\sigma_m$ by a stochastic process $\sigma_t = f(Y_t)$:

$$
\frac{dM_t}{M_t} = \mu dt + f(Y_t)dW_t^{(1)},
$$

$$
\frac{dX_t}{X_t} = \beta \frac{dM_t}{M_t} + \sigma dW_t^{(2)},
$$

$$
dY_t = \frac{1}{\epsilon} (m - Y_t) dt + \frac{\nu \sqrt{2}}{\sqrt{\epsilon}} dZ_t.
$$

In this model, the volatility process is driven by a mean-reverting OU process $Y_t$ with a large mean-reversion rate $1/\epsilon$ and the invariant (long-run) distribution $\mathcal{N}(m, \nu^2)$. This model also implies stochastic volatility in the asset price through its dependence on the market return. It allows leverage: $d\langle W^{(1)}, Z \rangle_t = \rho \, dt$. However, we continue to assume independence between $W_t^{(2)}$ and the other two Brownian motions $W_t^{(1)}$ and $Z_t$ in order to preserve the interpretation of $\beta$. 

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Pricing Risk-Neutral Measure

The market (or index) and the asset being both tradable, their discounted prices need to be martingales under a pricing risk-neutral measure. Setting

\[ Z_t = \rho dW_t^{(1)} + \sqrt{1 - \rho^2} dW_t^{(3)}, \]

with \((W_t^{(1)}, W_t^{(2)}, W_t^{(3)})\) being three independent BMs, we write:

\[ \frac{dM_t}{M_t} = r dt + f(Y_t) \left( dW_t^{(1)} + \frac{\mu - r}{f(Y_t)} dt \right), \]

\[ \frac{dX_t}{X_t} = r dt + \beta f(Y_t) \left( dW_t^{(1)} + \frac{\mu - r}{f(Y_t)} dt \right) + \sigma \left( dW_t^{(2)} + \frac{(\beta - 1)r}{\sigma} dt \right), \]

\[ dY_t = \frac{1}{\epsilon} (m - Y_t) dt - \frac{\nu \sqrt{2}}{\sqrt{\epsilon}} \left[ \rho \frac{\mu - r}{f(Y_t)} + \sqrt{1 - \rho^2} \gamma(Y_t) \right] dt + \frac{\nu \sqrt{2}}{\sqrt{\epsilon}} \left[ \rho \left( dW_t^{(1)} + \frac{\mu - r}{f(Y_t)} dt \right) + \sqrt{1 - \rho^2} \left( dW_t^{(3)} + \gamma(Y_t) dt \right) \right]. \]
Market price of risk and risk-neutral measure

\( \gamma(Y_t) \) is a **market price of volatility risk**, and we defined the combined market price of risk:

\[
\Lambda(Y_t) = \rho \frac{\mu - r}{f(Y_t)} + \sqrt{1 - \rho^2 \gamma(Y_t)}.
\]

Setting

\[
dW_{t}^{(1)*} = dW_{t}^{(1)} + \frac{\mu - r}{f(Y_t)} dt ,
\]

\[
dW_{t}^{(2)*} = dW_{t}^{(2)} + \frac{(\beta - 1)r}{\sigma} dt ,
\]

\[
dW_{t}^{(3)*} = dW_{t}^{(3)} + \gamma(Y_t) dt ,
\]

by **Girsanov theorem**, there is an equivalent probability \( IP^{*}(\gamma) \) such that \( (W_{t}^{(1)*}, W_{t}^{(2)*}, W_{t}^{(3)*}) \) are independent BMs under \( IP^{*}(\gamma) \), called the **pricing equivalent martingale measure** and determined by the market price of volatility risk \( \gamma \).
Dynamics under the risk-neutral measure

Under $\mathcal{IP}^{\star}(\gamma)$, the model becomes:

$$
\frac{dM_t}{M_t} = rdt + f(Y_t) dW_t^{(1)*},
$$

$$
\frac{dX_t}{X_t} = rdt + \beta f(Y_t) dW_t^{(1)*} + \sigma dW_t^{(2)*},
$$

$$
dY_t = \frac{1}{\varepsilon} (m - Y_t) dt - \frac{\nu \sqrt{2}}{\sqrt{\varepsilon}} \Lambda(Y_t) dt + \frac{\nu \sqrt{2}}{\sqrt{\varepsilon}} dZ_t^*,
$$

$$
Z_t^* = \rho W_t^{(1)*} + \sqrt{1 - \rho^2} W_t^{(3)*}.
$$

We take the point of view that by pricing options on the index $M$ and on the particular asset $X$, the market is “completing itself” and indirectly choosing the market price of volatility risk $\gamma$. 


Market Option Prices

Let $P^{M,\epsilon}$ denote the price of a **European option written on the market index** $M$, with maturity $T$ and payoff $h$, evaluated at time $t < T$ with current value $M_t = \xi$. Then, we have

$$P^{M,\epsilon} = \mathbb{E}^*(\gamma) \left\{ e^{-r(T-t)} h(M_T) \mid \mathcal{F}_t \right\} = P^{M,\epsilon}(t, M_t, Y_t),$$

By the Feynman-Kac formula, the function $P^{M,\epsilon}(t, \xi, y)$ satisfies the **partial differential equation**:

$$\mathcal{L}^\epsilon P^{M,\epsilon} = 0,$$

$$P^{M,\epsilon}(T, \xi, y) = h(\xi),$$

where

$$\mathcal{L}^\epsilon = \frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2.$$
Operator Notation

\[ L_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (m - y) \frac{\partial}{\partial y} \equiv L_{OU} \]

\[ L_1 = \rho \nu \sqrt{2} f(y) \xi \frac{\partial^2}{\partial \xi \partial y} - \nu \sqrt{2} \Lambda(y) \frac{\partial}{\partial y} \]

\[ L_2 = \frac{\partial}{\partial t} + \frac{1}{2} f(y)^2 \xi^2 \frac{\partial^2}{\partial \xi^2} + r(\xi \frac{\partial}{\partial \xi} - \cdot) \equiv L_{BS}(f(y)) \]

Here \( L_{BS}(\sigma) \) denotes the **Black-Scholes operator** with volatility parameter \( \sigma \).

The next step is to expand \( P^{M,\epsilon} \) in powers of \( \sqrt{\epsilon} \)

\[ P^{M,\epsilon} = P_0^M + \sqrt{\epsilon} P_1^M + \epsilon P_2^M + \epsilon^{3/2} P_3^M + \cdots \]
Expansion of the solution

Expanding

\[ \left( \frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2 \right) \left( P_0^M + \sqrt{\epsilon} P_1^M + \epsilon P_2^M + \epsilon^{3/2} P_3^M + \cdots \right) = 0, \]

one cancel the terms in $1/\epsilon$ and $1/\sqrt{\epsilon}$ by choosing $P_0^M$ and $P_1^M$ independent of $y$ (observe that $\mathcal{L}_1$ takes derivatives with respect $y$). The terms of order $\epsilon^0$ lead to

\[ \mathcal{L}_0 P_2^M + \mathcal{L}_2 P_0^M = 0, \]

which is a Poisson equation associated with $\mathcal{L}_0$. The centering condition for this equation is

\[ \langle \mathcal{L}_2 P_0^M \rangle = \langle \mathcal{L}_2 \rangle P_0^M = 0, \]

where $\langle \cdot \rangle$ denotes the averaging with respect to the invariant distribution of $Y_t$ with infinitesimal generator $\mathcal{L}_0$. 
Leading order term

Noting that

\[ \langle \mathcal{L}_2 \rangle = \frac{\partial}{\partial t} + \frac{1}{2} \bar{\sigma}^2 \xi^2 \frac{\partial^2}{\partial \xi^2} + r (\xi \frac{\partial}{\partial \xi} - \cdot) = \mathcal{L}_{BS}(\bar{\sigma}), \]

with \( \bar{\sigma}^2 = \langle f^2 \rangle \), and imposing the terminal condition

\[ P_{0M}(T, \xi) = h(\xi), \]

we deduce that \( P_{0M} \) is the Black-Scholes price of the option computed with the constant effective volatility \( \bar{\sigma} \).

We also have

\[ P_{2M} = -\mathcal{L}^{-1}_0 (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) P_{0M}. \]

so that the terms of order \( \sqrt{\varepsilon} \) lead to

\[ \mathcal{L}_0 P_{3M} + \mathcal{L}_1 P_{2M} + \mathcal{L}_2 P_{1M} = 0, \]

which is again a Poisson equation in \( P_{3M} \) which requires the solvability condition

\[ \langle \mathcal{L}_1 P_{2M} + \mathcal{L}_2 P_{1M} \rangle = 0. \]
Equation for the first correction

\[ \langle \mathcal{L}_2 \rangle P_1^M + \langle \mathcal{L}_1 P_2^M \rangle = \langle \mathcal{L}_2 \rangle P_1^M - \langle \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle P_0^M = 0. \]

Therefore \( P_1^M \) is the solution to the Black-Scholes equation with constant volatility \( \bar{\sigma} \), with a zero terminal condition, and a source term given by \( \langle \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle P_0^M \). In order to compute this source term we introduce a solution \( \phi(y) \) of the Poisson equation

\[ \mathcal{L}_0 \phi(y) = f(y)^2 - \langle f^2 \rangle, \]

so that

\[ \langle \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle P_0^M = \langle \mathcal{L}_1 \mathcal{L}_0^{-1} \left( \frac{1}{2} (f(y)^2 - \langle f^2 \rangle) \xi^2 \frac{\partial^2}{\partial \xi^2} \right) \rangle P_0^M \]

\[ = \langle \mathcal{L}_1 \left( \frac{1}{2} \phi(y) \xi^2 \frac{\partial^2}{\partial \xi^2} \right) \rangle P_0^M = \frac{1}{2} \langle \mathcal{L}_1 \phi \rangle \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} \]

\[ = \frac{\rho \nu}{\sqrt{2}} \langle \phi' f \rangle \xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} \right) - \frac{\nu}{\sqrt{2}} \langle \phi' \Lambda \rangle \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} \]
First correction and market parameters

The first correction term $\sqrt{\varepsilon} P_1^M$ solves the following problem:

$$\langle \mathcal{L}_2 \rangle (\sqrt{\varepsilon} P_1^M) + V_{2}^{M,\varepsilon} \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} + V_{3}^{M,\varepsilon} \xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} \right) = 0,$$

$$\left(\sqrt{\varepsilon} P_1^M\right)(T, \xi) = 0.$$

with

$$V_{2}^{M,\varepsilon} = \frac{\sqrt{\varepsilon \nu}}{\sqrt{2}} \langle \phi' \Lambda \rangle \quad \text{and} \quad V_{3}^{M,\varepsilon} = -\frac{\sqrt{\varepsilon \rho \nu}}{\sqrt{2}} \langle \phi' f \rangle.$$

In fact, the solution is given explicitly by

$$\sqrt{\varepsilon} P_1^M = (T - t) \left( V_{2}^{M,\varepsilon} \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} + V_{3}^{M,\varepsilon} \xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} \right) \right).$$

One can then deduce the price approximation

$$P_{0}^{M,\varepsilon} = P_0^M + (T - t) \left( V_{2}^{M,\varepsilon} \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} + V_{3}^{M,\varepsilon} \xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} \right) \right) + \mathcal{O}(\varepsilon).$$
Parameter Reduction

One of the inherent advantages of this approximation is parameter reduction. While the full stochastic volatility model requires the four parameters \((\epsilon, \nu, \rho, m)\) and the two functions \(f\) and \(\gamma\), our approximated option price requires only the three group parameters:

- The effective historical volatility \(\bar{\sigma}\)
- The volatility level correction \(V_2^{M,\epsilon}\) due to the market price of volatility risk
- The skew parameter \(V_3^{M,\epsilon}\) proportional to \(\rho\)

We can further reduce to only two parameters by noting that \(V_2^{M,\epsilon}\) is associated with a second order derivative with respect to the current market price \(\xi\). As such, it can be considered as a volatility level correction and absorbed into the volatility of the leading order Black-Scholes price.
Adjusted effective volatility

We introduce the adjusted effective volatility $\sigma_{M^*} = \sqrt{\bar{\sigma}^2 + 2V_{2,\varepsilon}}$, and we denote by $P_{M^*}$ the corresponding Black-Scholes option price.

Next, we define the first order correction $\sqrt{\varepsilon}P_{1,M^*}$ solution to

\[ \mathcal{L}_{BS}(\sigma_{M^*}) (\sqrt{\varepsilon} P_{1,M^*}) + V_{3,M^*,\varepsilon} \xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2 P_{0,M^*}}{\partial \xi^2} \right) = 0, \]

\[ (\sqrt{\varepsilon} P_{1,M^*})(T, \xi) = 0. \]

It is indeed given explicitly by

\[ \sqrt{\varepsilon} P_{1,M^*} = (T - t)V_{3,M^*,\varepsilon} \xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2 P_{0,M^*}}{\partial \xi^2} \right), \]

and one can show that the order of accuracy is preserved:

\[ P_{M^*,\varepsilon} = P_{0,M^*} + (T - t)V_{3,M^*,\varepsilon} \xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2 P_{0,M^*}}{\partial \xi^2} \right) + \mathcal{O}(\varepsilon) \]
Proof of order of accuracy

Observe that

$$L_{BS}(\sigma^{M*}) = L_{BS}(\bar{\sigma}) + \frac{1}{2} (2V_{2}^{M,\varepsilon}) \xi^2 \frac{\partial^2}{\partial \xi^2},$$

and therefore

$$L_{BS}(\bar{\sigma})(P_0^M - P_0^{M*}) = V_{2}^{M,\varepsilon} \xi^2 \frac{\partial^2 P_0^{M*}}{\partial \xi^2},$$

$$(P_0^M - P_0^{M*})(T, \xi) = 0.$$ 

Since the source term is $O(\sqrt{\varepsilon})$ because of the $V_{2}^{M,\varepsilon}$ factor, the difference $P_0^M - P_0^{M*}$ is also $O(\sqrt{\varepsilon})$. Next we write

$$|P^{M,\varepsilon} - (P_0^{M*} + \sqrt{\varepsilon} P_1^{M*})| \leq |P^{M,\varepsilon} - (P_0^M + \sqrt{\varepsilon} P_1^M)|$$

$$+ |(P_0^M + \sqrt{\varepsilon} P_1^M) - (P_0^{M*} + \sqrt{\varepsilon} P_1^{M*})|,$$

which, combined with the previous accuracy result, shows that the only quantity left to be controlled is the residual

$$R \equiv (P_0^M + \sqrt{\varepsilon} P_1^M) - (P_0^{M*} + \sqrt{\varepsilon} P_1^{M*}).$$
Proof of order of accuracy (continued)

From the equations satisfied by $P_0^M, \sqrt{\varepsilon} P_1^M, P_0^{M*}, \sqrt{\varepsilon} P_1^{M*}$, it follows that

\[
\mathcal{L}_{BS}(\bar{\sigma})(P_0^M + \sqrt{\varepsilon} P_1^M) + V_2^{M,\varepsilon} \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} + V_3^{M,\varepsilon} \xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} \right) = 0 \]

\[
\mathcal{L}_{BS}(\sigma^{M*})(P_0^{M*} + \sqrt{\varepsilon} P_1^{M*}) + V_3^{M,\varepsilon} \xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2 P_0^{M*}}{\partial \xi^2} \right) = 0. \]

Denoting by

\[
\mathcal{H}^\varepsilon = V_2^{M,\varepsilon} \xi^2 \frac{\partial^2}{\partial \xi^2} + V_3^{M,\varepsilon} \xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2}{\partial \xi^2} \right),
\]

\[
\mathcal{H}^{\varepsilon*} = V_3^{M,\varepsilon} \xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2}{\partial \xi^2} \right),
\]

one deduces that the residual $R$ satisfies the equation:
\[
\mathcal{L}_{BS}(\bar{\sigma})(R) = -\mathcal{H}^\epsilon P_0^M - \left(\mathcal{L}_{BS}(\sigma^{M*}) - V_2^{M,\epsilon} \xi_2^2 \frac{\partial^2}{\partial \xi^2}\right) (P_0^{M*} + \sqrt{\epsilon} P_1^{M*})
\]
\[
= -\mathcal{H}^\epsilon P_0^M + \mathcal{H}^{\epsilon*} P_0^{M*} + V_2^{M,\epsilon} \xi_2^2 \frac{\partial^2}{\partial \xi^2} (P_0^{M*} + \sqrt{\epsilon} P_1^{M*})
\]
\[
= \mathcal{H}^{\epsilon*} (P_0^{M*} - P_0^M) + V_2^{M,\epsilon} \xi_2^2 \frac{\partial^2}{\partial \xi^2} (P_0^{M*} - P_0^M + \sqrt{\epsilon} P_1^{M*})
\]
\[
= \mathcal{O}(\epsilon),
\]
where we have used in the last equality that \(\mathcal{H}^{\epsilon*} = \mathcal{O}(\sqrt{\epsilon})\), \(V_2^{M,\epsilon} = \mathcal{O}(\sqrt{\epsilon})\), \(P_0^{M*} - P_0^M = \mathcal{O}(\sqrt{\epsilon})\), and \(\sqrt{\epsilon} P_1^{M*} = \mathcal{O}(\sqrt{\epsilon})\).

Since \(R\) vanishes at the terminal time \(T\), we deduce \(R = \mathcal{O}(\epsilon)\) which concludes the proof.

The new approximation has now only two parameters to be calibrated \(\sigma^{M*}\) and \(V_3^{M,\epsilon}\), while preserving the accuracy of approximation.

This parameter reduction is essential in the forward-looking calibration procedure presented next.
Asset Option Approximation

Let $P^{a,\varepsilon}$ denote the price of a European option written on the asset $X$, with maturity $T$ and payoff $h$, evaluated at time $t < T$ with current value $X_t = x$. Then, we have

$$P^{a,\varepsilon} = \mathbb{E}^{(\gamma)} \left\{ e^{-r(T-t)} h(X_T) \mid \mathcal{F}_t \right\} = P^{a,\varepsilon}(t, X_t, Y_t).$$

By the Feynman-Kac formula, the function $P^{a,\varepsilon}(t, x, y)$ satisfies the partial differential equation:

$$\mathcal{L}^{a,\varepsilon} P^{a,\varepsilon} = 0,$$

$$P^{a,\varepsilon}(T, x, y) = h(x),$$

where
\[ \mathcal{L}^{a,\epsilon} = \frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}^a_1 + \mathcal{L}^a_2, \]

with

\[ \mathcal{L}_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (m - y) \frac{\partial}{\partial y}, \]

\[ \mathcal{L}^a_1 = \rho \nu \sqrt{2} \beta f(y)x \frac{\partial^2}{\partial x \partial y} - \nu \sqrt{2} \Lambda(y) \frac{\partial}{\partial y}, \]

\[ \mathcal{L}^a_2 = \frac{\partial}{\partial t} + \frac{1}{2} \left( \beta^2 f(y)^2 + \sigma^2 \right) x^2 \frac{\partial^2}{\partial x^2} + r(x \frac{\partial}{\partial x} - \cdot) \equiv \mathcal{L}_{BS} (\sqrt{\beta^2 f(y)^2 + \sigma^2}). \]

Observe that the only differences with options on the market index is the factor \( \beta \) in \( \mathcal{L}^a_1 \), and the modified square volatility \( \beta^2 f(y)^2 + \sigma^2 \) in \( \mathcal{L}^a_2 \). It is easy to see that the only modifications in the approximation are:
1. $\bar{\sigma}^2$ is replaced by $\bar{\sigma}_a^2 = \beta^2 \bar{\sigma}^2 + \sigma^2$

2. $V_2^{M,\varepsilon}$ is replaced by $V_2^{a,\varepsilon} = \beta^2 V_2^{M,\varepsilon} \implies V_2^{a,\varepsilon} = \frac{\beta^2 \sqrt{\varepsilon \nu}}{\sqrt{2}} \langle \phi' \Lambda \rangle$

3. $V_3^{M,\varepsilon}$ is replaced by $V_3^{a,\varepsilon} = \beta^3 V_3^{M,\varepsilon} \implies V_3^{a,\varepsilon} = -\frac{\beta^3 \sqrt{\varepsilon \rho \nu}}{\sqrt{2}} \langle \phi' f \rangle$

4. $\sigma^{M,*}$ is replaced by $\sigma^{a,*} = \sqrt{\beta^2 \bar{\sigma}^2 + \sigma^2 + 2V_2^{a,\varepsilon}}$

5. The option price approximation becomes

$$P^{a,\varepsilon} = P^{a,*}_0 + (T - t)V_3^{a,\varepsilon} x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P^{a,*}_0}{\partial x^2} \right) + O(\varepsilon),$$

where $P^{a,*}_0$ is the Black-Scholes price with volatility $\sigma^{a,*}$

6. **Only the parameters $V_3^{a,\varepsilon}$ and $\sigma^{a,*}$ need to be calibrated**
Beta Estimation

From the expressions for $V_{3}^{M,\varepsilon}$ and $V_{3}^{a,\varepsilon}$, one deduces that

$$V_{3}^{a,\varepsilon} = \beta^{3} V_{3}^{M,\varepsilon}.$$

It is then natural to propose the following estimator for $\beta$:

$$\beta = \left( \frac{V_{3}^{a,\varepsilon}}{V_{3}^{M,\varepsilon}} \right)^{\frac{1}{3}}.$$

Therefore in order to estimate the market beta parameter in a forward looking fashion using the implied skew parameters from option prices we must calibrate our two parameters $V_{3}^{a,\varepsilon}$ and $V_{3}^{M,\varepsilon}$.

Next we show how to do that by using the implied volatility surfaces from options data.
Calibration Method

We know that a first order approximation of an option price (on the market or the individual asset) with time to maturity $\tau = T - t$, and in the presence of fast mean-reverting stochastic volatility, takes the following form:

$$P^\epsilon \sim P_{BS}^* + \tau V_3^\epsilon x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_{BS}^*}{\partial x^2} \right),$$

where $P_{BS}^*$ is the Black-Scholes price with volatility $\sigma^*$. The European call option price $P_{BS}^*$ with current price $x$, time to maturity $\tau$, and strike price $K$ is given by the Black-Scholes formula

$$P_{BS}^* = x N(d_1^*) - Ke^{-r\tau} N(d_2^*),$$

where $N$ is the cumulative standard normal distribution and

$$d_{1,2}^* = \frac{\log(x/K) + (r \pm \frac{1}{2} \sigma^*2) \tau}{\sigma^* \sqrt{\tau}}.$$
Recall the relationship between Vega and Gamma for plain vanilla European options:

\[
\frac{\partial P_{BS}^*}{\partial \sigma} = \tau \sigma^* x^2 \frac{\partial^2 P_{BS}^*}{\partial x^2},
\]

and rewrite our price approximation as

\[
P^\varepsilon \sim P_{BS}^* + \frac{V_3^\varepsilon}{\sigma^*} x \frac{\partial}{\partial x} \left( \frac{\partial P_{BS}^*}{\partial \sigma} \right).
\]

Using the definition of the implied volatility \( P_{BS}(I) = P^\varepsilon \), and expanding the implied volatility as

\[
I = \sigma^* + \sqrt{\varepsilon I_1} + \varepsilon I_2 + \cdots,
\]

we obtain:

\[
P_{BS}(\sigma^*) + \sqrt{\varepsilon I_1} \frac{\partial P_{BS}(\sigma^*)}{\partial \sigma} + \cdots = P_{BS}^* + \frac{V_3^\varepsilon}{\sigma^*} x \frac{\partial}{\partial x} \left( \frac{\partial P_{BS}^*}{\partial \sigma} \right) + \cdots.
\]
By definition $P_{BS}(\sigma^*) = P_{BS}^*$, so that

$$\sqrt{\epsilon} I_1 = \frac{V_3^\epsilon}{\sigma^*} \left( \frac{\partial P_{BS}^*}{\partial \sigma} \right)^{-1} x \frac{\partial}{\partial x} \left( \frac{\partial P_{BS}^*}{\partial \sigma} \right).$$

Using the explicit computation of the Vega

$$\frac{\partial P_{BS}^*}{\partial \sigma} = \frac{x \sqrt{\tau} e^{-d_1^*/2}}{\sqrt{2\pi}},$$

and consequently

$$x \frac{\partial}{\partial x} \left( \frac{\partial P_{BS}^*}{\partial \sigma} \right) = \left( 1 - \frac{d_1^*}{\sigma^* \sqrt{\tau}} \right) \frac{\partial P_{BS}^*}{\partial \sigma},$$

we deduce by using the definition of $d_1^*$:

$$\sqrt{\epsilon} I_1 = \frac{V_3^\epsilon}{\sigma^*} \left( 1 - \frac{d_1^*}{\sigma^* \sqrt{\tau}} \right) = \frac{V_3^\epsilon}{2\sigma^*} \left( 1 - \frac{2r}{\sigma^2} \right) + \frac{V_3^\epsilon}{\sigma^3} \frac{\log(K/x)}{\tau}. $$
Log-Moneyness to Maturity Ratio (LMMR)

Define

\[
LMMR = \frac{\log(K/x)}{\tau},
\]

we obtain the affine LMMR formula

\[
I \sim \sigma^* + \sqrt{\epsilon} I_1 = b^* + a^\epsilon LMMR,
\]

with the intercept \( b^* \) and the slope \( a^\epsilon \) to be fitted to the skew of options data, and related to our model parameters \( \sigma^* \) and \( V_3^\epsilon \) by:

\[
b^* = \sigma^* + \frac{V_3^\epsilon}{2\sigma^*} \left( 1 - \frac{2r}{\sigma^*2} \right),
\]

\[
a^\epsilon = \frac{V_3^\epsilon}{\sigma^*3}.
\]
Calibration Formulas for $V_3^\varepsilon$

We know that $b^*$ and $\sigma^*$ differ from a quantity of order $\sqrt{\varepsilon}$. Therefore by replacing $\sigma^*$ by $b^*$ in the relation $V_3^\varepsilon = a^\varepsilon \sigma^*^3$, the order of accuracy for $V_3^\varepsilon$ is still $\varepsilon$ since $a^\varepsilon$ is also of order $\sqrt{\varepsilon}$. Consequently we deduce

$$V_3^\varepsilon = a^\varepsilon \sigma^*^3 \sim a^\varepsilon b^*^3 \equiv \hat{V}_3^\varepsilon.$$ 

It is indeed also possible to extract $\sigma^*$ as follows.

$$b^* = \sigma^* + \frac{a^\varepsilon \sigma^*^2}{2} \left(1 - \frac{2r}{\sigma^*^2}\right) = \sigma^* - a^\varepsilon \left(r - \frac{\sigma^*^2}{2}\right).$$

Using again the argument that $b^*$ and $\sigma^*$ differ from a quantity of order $\sqrt{\varepsilon}$ and $a^\varepsilon$ is also of order $\sqrt{\varepsilon}$, by replacing $\sigma^*$ by $b^*$ in the last term in the relation above, the order of accuracy is still $\varepsilon$, and we conclude that

$$\sigma^* \sim b^* + a^\varepsilon (r - \frac{b^*^2}{2}) \equiv \hat{\sigma}^*.$$
Beta Calibration

Defining the market fitted parameters as \(a^{M,\epsilon}\) and \(b^{M*}\) and the asset parameters as \(a^{a,\epsilon}\) and \(b^{a*}\), we obtain our main formula:

\[
\hat{\beta} = \left( \frac{\sqrt[3]{V_{3}^{a,\epsilon}}}{\sqrt[3]{V_{3}^{M,\epsilon}}} \right)^{1/3} = \left( \frac{a^{a,\epsilon}}{a^{M,\epsilon}} \right)^{1/3} \left( \frac{b^{a*}}{b^{M*}} \right),
\]

where \(b^{a*} + a^{a,\epsilon} LMMR\) (resp. \(b^{M*} + a^{M,\epsilon} LMMR\)) is the linear fit to the skew of implied volatilities for call options on the individual asset (resp. on the market index).

Observe the similarity with the formula

\[
\beta_{a} = \left( \frac{SKEW_{a}}{SKEW_{M}} \right)^{\frac{1}{3}} \left( \frac{VAR_{a}}{VAR_{M}} \right)^{\frac{1}{2}},
\]

used by Christoffersen, Jacobs, and Vainberg (2008).
**LMMR fit examples**

In the following figure:

Implied volatilities of June 17, 2009 maturity options for the S&P 500 and Amgen, plotted against the option’s *Log-Moneyness to Maturity Ratio (LMMR)*.

These are for February 18, 2009 option prices. The blue line is the affine fit of implied volatilities on *LMMR* by which the $V_3$ parameter is fit. The parameters fit for each series are

**S&P 500 Fit:** $a^{M,\epsilon} = -0.121$ and $b^{M*} = 0.428 \Rightarrow V_3^{M,\epsilon} = -0.0095$

**Amgen Fit:** $a^{a,\epsilon} = -0.128$ and $b^{a*} = 0.434 \Rightarrow V_3^{a,\epsilon} = -0.010$

The beta estimate for Amgen on that day is then 1.03
LMMR fits: S&P500 and Amgen

![Graph showing the relationship between LMMR and Implied Vol for S&P 500 and AMGN.](graph.png)
Forward and Backward Looking Betas

In the following figure:

The **solid blue line** is the **forward looking beta** (y-axis) calibrated on June 17, 2009 expiration call options over the course of 10 market days (x-axis) from February 9, 2009 to February 23, 2009.

The **dashed red line** is the corresponding **historical beta** calibrated on a series of historical prices of the same length as the time to maturity of the options.
THANKS FOR YOUR ATTENTION