Variance Reduction Techniques for Monte Carlo Simulations with Stochastic Volatility Models

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SAMSI
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References:

Variance Reduction for Monte Carlo Simulation in a Stochastic Volatility Environment
(with T. Tullie), Quantitative Finance 2002.

Variance Reduction for Monte Carlo Methods to Evaluate Option Prices under Multi-factor Stochastic Volatility Models
(with S. Han), Quantitative Finance 2004.

A Control Variate Method to Evaluate Option Prices under Multi-Factor Stochastic Volatility Models
(with S. Han), Submitted 2004.


http://www.math.ncsu.edu/~fouque
Monte Carlo Simulations

Essential in

- Computing option prices

\[ \mathbb{E}^* \left\{ e^{-rT} H(S_T) \right\} \approx \frac{1}{N} \sum_{k=1}^{N} e^{-rT} H(S_T^{(k)}) \]

over \( N \) independent realizations.

- Computing measures of risk of portfolios, for instance

\[ \mathbb{P} \left\{ V_T - V_0 < -VaR \right\} \leq \alpha \]
Stochastic Volatility

Essential to account for:

- **Distributions of Returns**
  (under physical measure \( \mathcal{IP} \))

- **Smile/Skew in Implied Volatilities**
  (under risk neutral measure \( \mathcal{IP}^* \))

- **Volatility Time Scales**
  (at least **two factors**: one slow, one fast)
“Parametrization” of the
Implied Volatility Surface $I(t; T, K)$

REQUIRED QUALITIES

• Universal Parsimonous Parameters: Model Independence

• Stability in Time: Predictive Power

• Easy Calibration: Practical Implementation

• Compatibility with Price Dynamics: Applicability to Pricing other Derivatives and Hedging
Model Under Risk Neutral

\[ dS_t = rS_t dt + f(Y_t, Z_t)S_t dW_t^{(0)} \]
\[ dY_t = \left( \alpha (m_f - Y_t) - \nu_f \sqrt{2\alpha} \Lambda_f(Y_t, Z_t) \right) dt \]
\[ + \nu_f \sqrt{2\alpha} \left( \rho_1 dW_t^{(0)} + \sqrt{1 - \rho_1^2} dW_t^{(1)} \right) \]
\[ dZ_t = \left( \delta (m_s - Z_t) - \nu_s \sqrt{2\delta} \Lambda_s(Y_t, Z_t) \right) dt \]
\[ + \nu_s \sqrt{2\delta} \left( \rho_2 dW_t^{(0)} + \rho_{12} dW_t^{(1)} + \sqrt{1 - \rho_2^2 - \rho_{12}^2} dW_t^{(2)} \right) \]

- \( \alpha \) large: \( Y_t \) fast mean reverting on time scale \( 1/\alpha << 1 \)
- \( \delta \) small: \( Z_t \) slow varying on time scale \( 1/\delta >> 1 \)
- \( \Lambda_f \) and \( \Lambda_s \): market prices of volatility risk
- \( |\rho_1| < 1, |\rho_2| < 1, |\rho_{12}| < \sqrt{1 - \rho_2^2} \): correlation coefficients
Options

European Options:

\[ P(t, S_t, Y_t, Z_t) = \mathbb{E}^* \left\{ e^{-r(T-t)} H(S_T) \mid S_t, Y_t, Z_t \right\} \]

Asian Call Options (fixed or floating strike):

\[ P_t = \mathbb{E}^* \left\{ e^{-r(T-t)} (A_T - K_1 S_T - K)^+ \mid \mathcal{F}_t \right\} \]

• Arithmetic Average Asian Option (AAO)

\[ A_T = \frac{1}{T} \int_0^T S_t dt \]

• Geometric Average Asian Option (GAO)

\[ A_T = \exp \left( \frac{1}{T} \int_0^T \ln S_t dt \right) \]

More general sampling functions: \( dt \to d\lambda(t) \)
\[ \text{d}V_t = b(t, V_t) \text{d}t + a(t, V_t) \text{d}\eta_t \]

\[ v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad V_t = \begin{pmatrix} S_t \\ Y_t \\ Z_t \end{pmatrix}, \quad \eta_t = \begin{pmatrix} W_t^{(0)} \\ W_t^{(1)} \\ W_t^{(2)} \end{pmatrix} \]

\[ b(t, v) = \begin{pmatrix} r x \\ \alpha (m_f - y) - \nu_f \sqrt{2\alpha} \Lambda_f(y, z) \\ \delta (m_s - z) - \nu_s \sqrt{2\delta} \Lambda_s(y, z) \end{pmatrix} \]

\[ a(t, v) = \begin{pmatrix} f(y, z)x & 0 & 0 \\ \nu_f \sqrt{2\alpha} \rho_1 & \nu_f \sqrt{2\alpha} \sqrt{1 - \rho_1^2} & 0 \\ \nu_s \sqrt{2\delta} \rho_2 & \nu_s \sqrt{2\delta} \rho_{12} & \nu_s \sqrt{2\delta} \sqrt{1 - \rho_2^2 - \rho_{12}^2} \end{pmatrix} \]
European Monte Carlo

The price $P(t, x, y, z)$ of an European option at time $t$ is given by

$$P(t, v) = \mathbb{E}^* \left\{ e^{-r(T-t)} H(S_T) \mid V_t = v \right\}$$

A basic Monte Carlo approximation is based on calculating the sample mean

$$P(t, x, y, z) \approx \frac{1}{N} \sum_{k=1}^{N} e^{-r(T-t)} H(S_T^{(k)})$$

over $N$ independent realizations of $V$ starting at time $t$ from $v$. 
Importance Sampling

Introduce the martingale

\[ Q_t = \exp \left( \int_0^t h(s, V_s) \cdot d\eta_s + \frac{1}{2} \int_0^t \|h(s, V_s)\|^2 ds \right) \]

and define a new probability \( \tilde{\mathbb{P}} \) equivalent to \( \mathbb{P}^* \) by

\[ \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*} = (Q_T)^{-1} \]

By Girsanov Theorem, under this new measure \( \tilde{\mathbb{P}} \), the process

\[ \tilde{\eta}_t = \eta_t + \int_0^t h(s, V_s) ds \]

is a standard Brownian motion.
Monte Carlo under $\tilde{P}$

\[ P(t, v) = \tilde{\mathbb{E}} \left\{ e^{-r(T-t)} H(S_T) Q_T \mid V_t = v \right\} \]

where

\[ Q_T = \exp \left( \int_0^T h(s, V_s) \cdot d\tilde{\eta}_s - \frac{1}{2} \int_0^T \| h(s, V_s) \|^2 ds \right) \]

and the dynamics of our model becomes

\[ dV_t = \{ b(t, V_t) - a(t, V_t) h(t, V_t) \} dt + a(t, V_t) \, d\tilde{\eta}_t \]

where $\tilde{\eta}_t$ is a standard Brownian motion.

\[ P(t, x, y, z) \approx \frac{1}{N} \sum_{k=1}^{N} e^{-r(T-t)} H(S_T^{(k)}) Q_T^{(k)} \]
Choice of $h$

Applying Ito’s formula to $P(t, V_t)Q_t$ \[ \implies \]

$$H(V_T)Q_T = P(t, v) + \int_t^T Q_s(a' \nabla_v P + P h)(s, V_s)d\tilde{\eta}_s$$

**Variance** of the payoff $H(V_T)Q_T$

$$\text{Var}_P\{H(V_T)Q_T\} = \tilde{\mathbb{E}} \left\{ \int_t^T Q_s^2 ||a' \nabla_v P + P h||^2 ds \right\}.$$  

Indeed, if the quantity $P$ to be computed was known, one could obtain a zero variance by choosing

$$h = -\frac{1}{P} (a' \nabla_v P)$$

Our strategy is to use approximations to the exact value $P$. 
Pricing Equation

\( \delta \ll 1 \) and \( \varepsilon = 1/\alpha \ll 1 \)

\[
P^{\varepsilon,\delta}(t, x, y, z) = \mathbb{E}^* \left\{ e^{-r(T-t)} H(S_T) \mid S_t = x, Y_t = y, Z_t = z \right\}
\]

\[
\left( \frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta} \mathcal{M}_1 + \delta \mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}} \mathcal{M}_3 \right) P^{\varepsilon,\delta} = 0
\]

\[
P^{\varepsilon,\delta}(T, x, y, z) = H(x)
\]

\[
\mathcal{L}_0 = (m_f - y) \frac{\partial}{\partial y} + \nu_f^2 \frac{\partial^2}{\partial y^2}
\]

\[
\mathcal{L}_1 = \nu_f \sqrt{2} \left( \rho_1 f x \frac{\partial^2}{\partial x \partial y} - \Lambda_f \frac{\partial}{\partial y} \right)
\]

\[
\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2} f^2 x^2 \frac{\partial^2}{\partial x^2} + r \left( x \frac{\partial}{\partial x} - \cdot \right)
\]

\[
\mathcal{M}_1 = \nu_s \sqrt{2} \left( -\Lambda_s \frac{\partial}{\partial z} + \rho_2 f x \frac{\partial^2}{\partial x \partial z} \right)
\]

\[
\mathcal{M}_2 = (m_s - z) \frac{\partial}{\partial z} + \nu_s^2 \frac{\partial^2}{\partial z^2}
\]

\[
\mathcal{M}_3 = 2\nu_f \nu_s \left( \rho_1 \rho_2 + \rho_{12} \sqrt{1 - \rho_1^2} \right)
\]
European Options Approximations

Combination of singular and regular perturbations $\Rightarrow$

\[
P^{\varepsilon,\delta}(t, x, y, z) \approx P_{BS}(t, x; T, \bar{\sigma}) + (T - t) \left( V_0^\delta \frac{\partial P_{BS}}{\partial \sigma} + V_1^\delta x \frac{\partial^2 P_{BS}}{\partial x \partial \sigma} \right) \\
+ (T - t) \left( V_2^\varepsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_{BS}}{\partial x^2} \right) \right)
\]

Leading order Black-Scholes price $P_{BS}(t, x; \bar{\sigma}(z))$:

\[
L_{BS}(\bar{\sigma}(z))P_{BS} = 0 \\
P_{BS}(T, x; \bar{\sigma}(z)) = H(x)
\]

at the $z$-dependent effective volatility $\bar{\sigma}(z)$:

\[
\bar{\sigma}^2(z) = \langle f^2(\cdot, z) \rangle
\]

where the brackets denote the average with respect to the invariant distribution $\mathcal{N}(m_f, \nu_f^2)$. 

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The \textbf{small} parameters \((V_0^\delta, V_1^\delta, V_2^\varepsilon, V_3^\varepsilon)\) are given by

\[
V_0^\delta = -\frac{\nu_s \sqrt{\delta}}{\sqrt{2}} \langle \Lambda_s \rangle \bar{\sigma}'
\]

\[
V_1^\delta = \rho_2 \frac{\nu_s \sqrt{\delta}}{\sqrt{2}} \langle f \rangle \bar{\sigma}'
\]

\[
V_2^\varepsilon = \frac{\nu_f \sqrt{\varepsilon}}{\sqrt{2}} \langle \Lambda_f \frac{\partial \phi}{\partial y} \rangle
\]

\[
V_3^\varepsilon = -\rho_1 \frac{\nu_f \sqrt{\varepsilon}}{\sqrt{2}} \langle f \frac{\phi}{\partial y} \rangle
\]

\[
\bar{\sigma}' = d\bar{\sigma}/dz,
\]

and \(\phi(y, z)\) is a solution of the \textbf{Poisson equation}

\[
\mathcal{L}_0 \phi(y, z) = f^2(y, z) - \bar{\sigma}^2(z).
\]

\textbf{Accuracy}:

\begin{itemize}
  \item \textbf{Smooth} payoffs: \quad \text{error} = \mathcal{O}(\varepsilon + \delta)
  
  \item \textbf{Calls} (kinks): \quad \text{error} = \mathcal{O}(\varepsilon \log |\varepsilon| + \delta)
  
  \item \textbf{Digitals} (jumps): \quad \text{error} = \mathcal{O}(\varepsilon^{2/3} \log |\varepsilon| + \delta)
\end{itemize}
Corrections Equations

\[ P_1^\varepsilon(t, x, z) = (T - t) \left( V_2^\varepsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_{BS}}{\partial x^2} \right) \right) \]

solves

\[ \mathcal{L}_{BS}(\bar{\sigma}) P_1^\varepsilon + \left( V_2^\varepsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_{BS}}{\partial x^2} \right) \right) = 0, \quad P_1^\varepsilon(T, x, z) = 0 \]

\[ P_1^\delta(t, x, z) = (T - t) \left( V_0^\delta \frac{\partial P_{BS}}{\partial \sigma} + V_1^\delta x \frac{\partial^2 P_{BS}}{\partial x \partial \sigma} \right) \]

solves

\[ \mathcal{L}_{BS}(\bar{\sigma}) P_1^\delta + 2 \left( V_0^\delta \frac{\partial P_{BS}}{\partial \sigma} + V_1^\delta x \frac{\partial^2 P_{BS}}{\partial x \partial \sigma} \right) = 0, \quad P_1^\delta(T, x) = 0 \]

(for European options: \( \frac{\partial P_{BS}}{\partial \sigma} = (T - t) \sigma x^2 \frac{\partial^2 P_{BS}}{\partial x^2} \))
Computation of $h$ using $P_{BS}$ only

$$h(t, x, y, z) = \frac{-1}{P_{BS}(t, x; \bar{\sigma}(z))} a' \nabla P_{BS}(t, x; \bar{\sigma}(z))$$

$$= \frac{-1}{P_{BS}} \begin{pmatrix} f(y, z)x & \frac{\rho_1 \nu_f \sqrt{2}}{\sqrt{\varepsilon}} & \rho_2 \nu_s \sqrt{2\delta} \\ 0 & \nu_f \frac{\sqrt{2}}{\sqrt{\varepsilon}} \sqrt{1 - \rho_1^2} & \rho_{12} \nu_s \sqrt{2\delta} \\ 0 & 0 & \nu_s \sqrt{2\delta} \sqrt{1 - \rho_2^2 - \rho_{12}^2} \end{pmatrix} \begin{pmatrix} \frac{\partial P_{BS}}{\partial x} \\ \frac{\partial P_{BS}}{\partial y} \\ \frac{\partial P_{BS}}{\partial z} \end{pmatrix}$$

$$= -\frac{\partial P_{BS}}{\partial x} \begin{pmatrix} f(y, z)x \\ 0 \\ 0 \end{pmatrix} - \nu_s \sqrt{2\delta} \frac{\partial P_{BS}}{\partial \sigma} \begin{pmatrix} \rho_2 \\ \rho_{12} \\ \sqrt{1 - \rho_2^2 - \rho_{12}^2} \end{pmatrix}$$

where we have used that $P_{BS}$ does not depend on $y$.

The Vega is given by $\frac{\partial P_{BS}}{\partial \sigma} = x\sqrt{T-t}N'(d_1(x, z))$.\[17\]
Computation of $h$ using $\tilde{P} = P_{BS} + P_1^\varepsilon + P_1^\delta$

$$\tilde{h}(t, x, y, z) = \frac{-1}{\tilde{P}(t, x, z)} a' \nabla \tilde{P}(t, x, z)$$

$$= \frac{-1}{\tilde{P}(t, x, z)} \begin{pmatrix} f(y, z)x & \frac{\rho_1 \nu_f \sqrt{2}}{\sqrt{\varepsilon}} & \rho_2 \nu_s \sqrt{2\delta} \\ 0 & \nu_f \frac{\sqrt{2}}{\sqrt{\varepsilon}} \sqrt{1 - \rho_1^2} & \rho_1 \nu_s \frac{\sqrt{2\delta}}{\nu_s \sqrt{2\delta}} \sqrt{1 - \rho_2^2 - \rho_{12}^2} \\ 0 & 0 & \nu_s \sqrt{2\delta} \sqrt{1 - \rho_2^2 - \rho_{12}^2} \end{pmatrix} \begin{pmatrix} \frac{\partial \tilde{P}}{\partial x} \\ \frac{\partial \tilde{P}}{\partial y} \\ \frac{\partial \tilde{P}}{\partial z} \end{pmatrix}$$

$$\approx \frac{-\partial \tilde{P}}{\partial x} \begin{pmatrix} f(y, z)x \\ 0 \\ 0 \end{pmatrix} - \nu_s \sqrt{2\delta} \frac{\bar{\sigma}'(z) \frac{\partial P_{BS}}{\partial \sigma}}{P_{BS}(t, x; \bar{\sigma}(z))} \begin{pmatrix} \rho_2 \\ \rho_1 \nu_s \frac{\sqrt{2\delta}}{\nu_s \sqrt{2\delta}} \sqrt{1 - \rho_2^2 - \rho_{12}^2} \end{pmatrix}$$

where we have used again that $\tilde{P}$ does not depend on $y$, and neglected terms of higher order.
## Numerical Results (European Calls)

<table>
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<tr>
<th>$r$</th>
<th>$m_f$</th>
<th>$m_s$</th>
<th>$\nu_f$</th>
<th>$\nu_s$</th>
<th>$\rho_1$</th>
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<th>$\Lambda_f$</th>
<th>$\Lambda_s$</th>
<th>$f(y, z)$</th>
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<th>$S_0$</th>
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<th>$K$</th>
<th>$T$ years</th>
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<th>$P_{BS}$</th>
<th>$\tilde{P}$</th>
<th>$P^{IS}(P_{BS})$</th>
<th>$P^{IS}(\tilde{P})$</th>
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<td>10.779</td>
<td>11.069</td>
<td>.00400 (11.13)</td>
<td>.00099 (11.03)</td>
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<td>.00070 (11.03)</td>
<td>.00070 (10.99)</td>
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<td>.00128 (11.09)</td>
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<tr>
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<td>10.779</td>
<td>12.20</td>
<td>.00384 (12.03)</td>
<td>.00243 (11.60)</td>
</tr>
</tbody>
</table>

5000 realizations (Euler scheme with time step $\Delta t = 0.005$).
Two-Step Variance Reduction for Asian Options

\[
P(t, S_t, Y_t, Z_t, I_t) = \mathbb{E}^* \left\{ e^{-r(T-t)} \left( \frac{I_T}{T} - K_1 S_T - K \right)^+ \mid S_t, Y_t, Z_t, I_t \right\}
\]

where \( I_t = \int_0^t S_u du \) (or \( dI_t = S_t dt \)).

Basic Monte Carlo for AAO’s

\[
P \approx P^{MC} = \frac{e^{-r(T-t)}}{N} \sum_{k=1}^{N} \left( \frac{I_{T}^{(k)}}{T} - K_1 S_{T}^{(k)} - K \right)^+
\]

- Unlike the case of European options, the approximated prices of AAOs do not have closed-form solutions (F-Han 2003).

- Importance sampling using approximated AAO prices requires numerical PDE solutions \( \Rightarrow \) not efficient.

- \( \Rightarrow \) two-step variance reduction strategy by combining control variates and importance sampling.
Control Variates for Arithmetic Average Asian Options

- **Constant volatility**: Boyle-Broadie-Glasserman proposed a variance reduction method for AAOs based on using GAOs as control variates:

  \[ P_{A}^{CV} = P_{A}^{MC} + \lambda(P_{G}^{MC} - P_{G}) \]

- \( P_{G}^{MC} \) is the unbiased Monte Carlo estimator of the GAO price computed using the same run as for \( P_{A}^{MC} \).

- The company price \( P_{G} \), i.e. the counterpart geometric average Asian option, has an **analytic solution**.

- The parameter \( \lambda \) is chosen to minimize the sample variance. For Asian options, \( \lambda \) is often chosen equal to -1.
Two-Step Strategy in the Stochastic Volatility Case

• Implement GAOs control variates.

• No closed-form solutions for GAOs.

• Evaluate GAOs by Monte Carlo using the importance sampling variance reduction technique presented and tested in the European options case.

\[ P_G(t, V_t, L_t) = \mathbb{IE}^* \left\{ e^{-r(T-t)} \left( \exp \left( \frac{L_T}{T} \right) - K_1 S_T - K \right)^+ \mid V_t, L_t \right\} \]

where \( V_t = (S_t, Y_t, Z_t) \), and the running sum process \( (L_t) \) is given by

\[ dL_t = \ln S_t \, dt \]
Approximate GAOs Prices (fixed strike case $K_1 = 0$)

\[ P_G(t, x, y, z, L) \approx \tilde{P}_G(t, x, z, L) \]

where

\[
\tilde{P}_G = P_0^{fix} - V_0^\delta (T - t) \sqrt{2} \frac{\partial P_0^{fix}}{\partial \sigma} + V_1^\delta (T - t) x \frac{\partial^2 P_0^{fix}}{\partial x \partial \sigma} \\
- \frac{V_2^\varepsilon (T - t)^2}{2} \frac{\partial P_0^{fix}}{\partial x} + \frac{(V_2^\varepsilon - V_3^\varepsilon) (T - t)^3}{3} \frac{\partial^2 P_0^{fix}}{\partial x^2} + \frac{V_3^\varepsilon (T - t)^4}{4} \frac{\partial^3 P_0^{fix}}{\partial x^3}
\]

The leading order term is the homogenized “Black-Scholes” price:

\[
P_0^{fix}(t, x, L; \bar{\sigma}) = \exp \left( \frac{L - t \ln x}{T} + \ln x + R \right) N(d_1) - K e^{-r(T-t)} N(d_2)
\]

\[
\frac{\partial P_0^{fix}}{\partial \sigma} = \frac{T - t}{3} \bar{\sigma} x^2 \frac{\partial^2 P_0^{fix}}{\partial x^2} - \frac{T - t}{6} \bar{\sigma} x \frac{\partial P_0^{fix}}{\partial x}
\]

with formulas for $R(t, T, z)$, $d_1(x, z, L)$ and $d_2(x, z, L)$. 
Computation of $h_G$

Approximate the “perfect” $h_G$

$$h_G(t, x, y, z, L) = -\frac{1}{P_G} a' \nabla_{(v, L)} P_G$$

by approximating $P_G(t, x, y, z, L)$ by $\tilde{P}_G(t, x, z, L) \implies$

$$\tilde{h}_G(t, x, z, L) = \left(-\frac{\partial \tilde{P}_G}{\partial x}\right) \begin{pmatrix} f(y, z)x \\ 0 \\ 0 \end{pmatrix} - \nu_s \sqrt{2\delta} \frac{\bar{\sigma}(z) \partial P^f_{0x}}{P^f_{0x}(t, x, z, L)} \begin{pmatrix} \rho_2 \\ \rho_{12} \\ \sqrt{1 - \rho^2_2 - \rho^2_{12}} \end{pmatrix}$$
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<td>1</td>
<td>0.058433 (7.31)</td>
<td>0.014814 (6.96)</td>
</tr>
</tbody>
</table>
AAOs Monte Carlo by C.V. GAOs and I.S.

\[ P_A^{CV} = P_A^{MC} - 1 \left( P_G^{MC} - P_G^{IS}(\overline{P_G}) \right) \]

with \( \alpha = 75 \) and \( \delta = 0.1 \)

Variance reduced from \((1.5411)10^{-4}\) to \((1.6201)10^{-6}\) with respective sample means 8.4604 and 8.4965 \((\Delta t = 0.005, N = 5000)\).
Control Variates

Conventional control variate setup with $m$ multiple controls:

$$P_{CV} \triangleq P_{MC} + \sum_{i=1}^{m} \lambda_i (\hat{P}_C^i - P_C^i)$$

- $P_{MC}$: unbiased estimator of $P$ obtained by the sample mean of outputs from $N$ i.i.d. realizations.

- Each $\hat{P}_C^i$ represents a sample mean obtained from the same realizations than those used in $P_{MC}$.

- $\hat{P}_C^i$ is an unbiased estimator of $P_C^i$ which admits a closed-form expression.

The control variate $P_{CV}$ is an unbiased estimator of $P$.

The control parameters $\lambda_i$’s need to be chosen to minimize the variance of $P_{CV}$ (see Glasserman).
Building New Control Variates

Using Ito’s formula, the discounted option price satisfies

\[ d \left( e^{-rs}P(s, S_s, Y_s, Z_s) \right) = e^{-rs} \left( a' \nabla P \right)(s, S_s, Y_s, Z_s) \cdot d\eta_s \]

Integrating between the initial time 0 and the maturity time \( T \), and using the terminal condition \( P(T, S_T, Y_T, Z_T) = H(S_T) \), we have

\[ P(0, S_0, Y_0, Z_0) = e^{-rT}H(S_T) \]

\[ - \int_0^T e^{-rs} (a' \nabla P)(s, S_s, Y_s, Z_s) \cdot d\eta_s \]

This suggests that the martingale term is the “perfect” control variate. However, it requires the unknown pricing function \( P(s, x, y, z) \) through its gradient in the stochastic integrals.
Using Approximations

The approximation

\[ P(t, x, y, z) \approx \tilde{P}(t, x, z) \]

\[ \tilde{P} = P_{BS} + P_1^\varepsilon + P_1^\delta \]

suggests multiple control variates given by the centered martingales:

\[ M_1(\tilde{P}) = \int_0^T e^{-rs} \frac{\partial \tilde{P}}{\partial x}(s, S_s, Z_s) f(Y_s, Z_s) S_s dW_s^{(0)*} \]

\[ M_2(\tilde{P}) = \nu_2 \sqrt{2\delta} \int_0^T e^{-rs} \frac{\partial \tilde{P}}{\partial z}(s, S_s, Z_s) dW_s^* \]

obtained by computing \( a' \nabla P \approx a' \nabla \tilde{P} \), using that \( \tilde{P} \) does not depend on the variable \( y \) and the notation

\[ W_s^* = \rho_2 W_s^{(0)*} + \rho_{12} W_s^{(1)*} + \sqrt{1 - \rho_2^2 - \rho_{12}^2} W_s^{(2)*} \]
Implementation of Control Variates

\[ P^{MC} = \frac{1}{N} \sum_{k=1}^{N} e^{-rT} H(S^{(k)}_T) \]

\[ \hat{P}_i^C = \frac{1}{N} \sum_{k=1}^{N} M_i^{(k)}(\tilde{P}) \quad \lambda_i = -1 \quad i = 1, 2 \]

where the superscript corresponds to the \( k \)th realization.
Since \( P^1_C = P^2_C = 0 \), we deduce

\[ P^{CV} = P^{MC} - \hat{P}_1^C - \hat{P}_2^C \]

The variance of this estimator is given by the sum of the quadratic variations of the martingales \( M_1(P - \tilde{P}) \) and \( M_2(P - \tilde{P}) \). It involves only the gradient of \( P - \tilde{P} \) in contrast with the importance sampling method where division by \( P \) is required.
## Numerical Results: One-Factor Model

<table>
<thead>
<tr>
<th>$r$</th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$\nu_1$</th>
<th>$\nu_2$</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
<th>$\rho_{12}$</th>
<th>$\Lambda_1$</th>
<th>$\lambda_2$</th>
<th>$f(y, z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>-2.6</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-0.3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>exp(y)</td>
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<table>
<thead>
<tr>
<th>$S_0$</th>
<th>$Y_0$</th>
<th>$Z_0$</th>
<th>$K$</th>
<th>$T$ years</th>
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<tr>
<td>110</td>
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<td>1</td>
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<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$V_{MC}/V_{IS}(\hat{P})$</th>
<th>$V_{MC}/V_{CV}(P_{BS})$</th>
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<tr>
<td>0.5</td>
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<tr>
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<td>29.9266</td>
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<td>5</td>
<td>19.3333</td>
<td>31.8946</td>
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<tr>
<td>10</td>
<td>29.6250</td>
<td>51.4791</td>
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<tr>
<td>25</td>
<td>36.7143</td>
<td>100.1556</td>
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<tr>
<td>50</td>
<td>48.0000</td>
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</tr>
<tr>
<td>100</td>
<td>106.3333</td>
<td>229.9224</td>
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**Numerical Results: Two-Factor Model**

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<th>$\nu_1$</th>
<th>$\nu_2$</th>
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<th>$\rho_2$</th>
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<th>$f(y, z)$</th>
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<td>-0.8</td>
<td>0.5</td>
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<td>-0.2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\exp(y + z)$</td>
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</table>

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>$Y_0$</th>
<th>$Z_0$</th>
<th>$K$</th>
<th>$T$ years</th>
</tr>
</thead>
<tbody>
<tr>
<td>55</td>
<td>-1</td>
<td>-1</td>
<td>50</td>
<td>1</td>
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<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\delta$</th>
<th>$V^{MC}/V^{IS}(\tilde{P})$</th>
<th>$V^{MC}/V^{CV}(P_{BS})$</th>
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<tr>
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<tr>
<td>100</td>
<td>0.01</td>
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Conclusions and Ongoing Research

- **Monte Carlo simulations** play an important role in computational finance.

- **Variance reduction** techniques are essential.

- **Approximations** obtained by **perturbation methods** can be used.

- **Control variates** are more efficient than **importance sampling**.

- **Path dependent options** remain challenging problems.