Applications to
Fixed Income and Credit Markets

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Fixed Income

Perturbations around Vasicek (for instance)

to account for:

- **Volatility Time Scales**
- **Fit to Yield Curves**

Reference:

Stochastic Volatility Corrections for Interest Rate Derivatives

Mathematical Finance 14(2), April 2004

Constant Volatility Vasicek Model

Under the physical probability $I\!P$:

$$d\bar{r}_t = a(\bar{r}_\infty - \bar{r}_t)dt + \bar{\sigma}d\bar{W}_t$$

Under the risk-neutral pricing probability $I\!P^\star$:

$$d\bar{r}_t = a(r^\star - \bar{r}_t)dt + \bar{\sigma}d\bar{W}_t^\star$$

with a constant **market price of interest rate risk** λ:

$$r^\star = \bar{r}_\infty - \frac{\lambda\bar{\sigma}}{a}$$

Bonds Prices

$$\Lambda(t,T) = \mathbb{E}^\star\left\{e^{-\int_t^T \bar{r}_s ds}|\mathcal{F}_t\right\} = \mathbb{E}^\star\left\{e^{-\int_t^T \bar{r}_s ds}|\bar{r}_t\right\} = \bar{P}(t,\bar{r}_t;T)$$

Vasicek PDE:

$$\frac{\partial \bar{P}}{\partial t} + \frac{1}{2}\bar{\sigma}^2\frac{\partial^2 \bar{P}}{\partial x^2} + a(r^\star - x)\frac{\partial \bar{P}}{\partial x} - x\bar{P} = 0$$

with the **terminal condition** $\bar{P}(t,x;T) = 1$.

Introduce the **time-to-maturity** $\tau = T - t$ and seek a solution of the form:

$$\bar{P}(T-\tau,x;T) = A(\tau)e^{-B(\tau)x}$$

by solving linear ODE's with $A(0) = 1$ and $B(0) = 0$.

Affine Yields

$$
\begin{aligned}
B(\tau) &= \frac{1-e^{-a\tau}}{a} \\
A(\tau) &= \exp\left\{-\left[R_\infty \tau - R_\infty \frac{1-e^{-a\tau}}{a} + \frac{\bar{\sigma}^2}{4a^3}\left(1-e^{-a\tau}\right)^2\right]\right\}
\end{aligned}
$$

with

$$
R_\infty = r^\star - \frac{\bar{\sigma}^2}{2a^2} = \bar{r}_\infty - \frac{\lambda\bar{\sigma}}{a} - \frac{\bar{\sigma}^2}{2a^2}
$$

Yield Curve:

$$
\begin{aligned}
R(t,\tau) &= -\frac{1}{\tau}\log\left(\Lambda(t,t+\tau)\right) \\
&= -B(\tau)\bar{r}_t + \log A(\tau) \\
&= R_\infty - (R_\infty - \bar{r}_t)\frac{1-e^{-a\tau}}{a\tau} + \frac{\bar{\sigma}^2}{4a^3\tau}\left(1-e^{-a\tau}\right)^2
\end{aligned}
$$

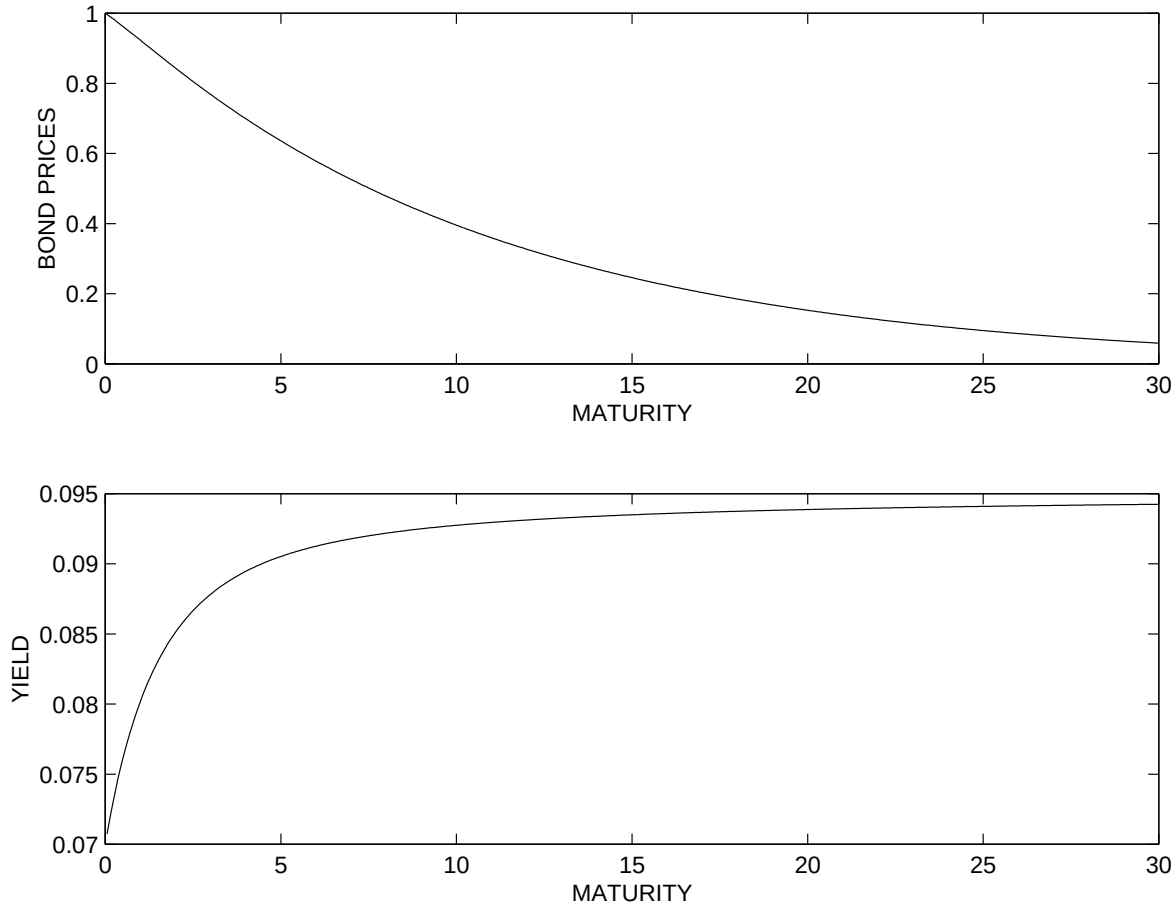


Figure 1: *Bond prices (top) and cblue Yield curve (bottom) in the Vasicek model with* $a = 1$, $r^\star = 0.1$ *and* $\bar{\sigma} = 0.1$. *Maturity* τ *runs from* 0 *to* 30 *years.* $R_\infty = 0.095$ *and the initial rate is* $x = 0.07$.

Bond Options Prices

Example: a **Call Option** with **strike** K and **maturity** T_0 written on a zero-coupon bond with maturity $T > T_0$.

The **payoff**

$$h(\Lambda(T_0, T)) = (\Lambda(T_0, T) - K)^+$$

is a function of $\bar{r}_{T_0}$ since $\Lambda(T_0, T) = \bar{P}(T_0, \bar{r}_{T_0}; T)$

Call Option Price:

$$\bar{C}(t, x; T, T_0) = \mathbb{E}^\star \left\{ e^{-\int_t^{T_0} \bar{r}_s ds} h\left(\Lambda(T_0, T)\right) | \bar{r}_t = x \right\}$$

solution of Vasicek PDE with terminal condition at $t = T_0$:

$$\bar{C}(T_0, x; T, T_0) = (\bar{P}(T_0, x; T) - K)^+$$

$$\bar{C}(t, x; T, T_0) = \bar{P}(t, x; T) N(h_1) - K \bar{P}(t, x; T_0) N(h_2)$$

Stochastic Volatility Vasicek Models

Under the physical measure:

$$dr_t = a(r_\infty - r_t)dt + f(Y_t)dW_t$$

where f is a positive function of a **mean-reverting volatility driving process** Y_t.

Example: Y_t is an OU process:

$$dY_t = \alpha(m - Y_t)dt + \nu\sqrt{2\alpha}d\hat{Z}_t$$

where $\hat{Z}_t$ is a Brownian motion possibly **correlated** to the Brownian motion W_t driving the short rate:

$$\hat{Z}_t = \rho W_t + \sqrt{1-\rho^2}\, Z_t$$

(W_t, Z_t) independent Brownian motions.

Stochastic Volatility Vasicek Pricing Models

Under the risk-neutral pricing probability $I\!P^{\star(\lambda,\gamma)}$:

$$
\begin{aligned}
dr_t &= \left(a(r_\infty - r_t) - \lambda(Y_t) f(Y_t)\right) dt + f(Y_t) dW_t^\star \\
dY_t &= \left(\alpha(m - Y_t) - \nu\sqrt{2\alpha}\left[\rho\lambda(Y_t) + \gamma(Y_t)\sqrt{1-\rho^2}\right]\right) dt \\
&\quad + \nu\sqrt{2\alpha}\left(\rho dW_t^\star + \sqrt{1-\rho^2}\, dZ_t^\star\right)
\end{aligned}
$$

for bounded market prices of risk $\lambda(y)$ and $\gamma(y)$.

Under **fast mean-reversion**:

α is large

Bond Pricing

$$P(t,x,y;T) = \mathbb{E}^{\star(\lambda,\gamma)}\left\{ e^{-\int_t^T r_s ds} | r_t = x, Y_t = y \right\}$$

$$\begin{aligned}
\frac{\partial P}{\partial t} &+ \frac{1}{2} f(y)^2 \frac{\partial^2 P}{\partial x^2} + \left(a(r_\infty - x) - \lambda(y) f(y)\right) \frac{\partial P}{\partial x} - xP \\
&+ \alpha \left(\nu^2 \frac{\partial^2 P}{\partial y^2} + (m-y) \frac{\partial P}{\partial y} \right) \\
&+ \nu\sqrt{2\alpha} \left(\rho f(y) \frac{\partial^2 P}{\partial x \partial y} - \left[\rho\lambda(y) + \gamma(y)\sqrt{1-\rho^2} \right] \frac{\partial P}{\partial y} \right) = 0
\end{aligned}$$

with the terminal condition $P(T,x,y;T) = 1$ for every x and y.

Expand :

$$P^\varepsilon = P_0 + \sqrt{\varepsilon} P_1 + \varepsilon P_2 + \varepsilon\sqrt{\varepsilon} P_3 + \cdots \qquad \varepsilon = 1/\alpha$$

Leading Order Term

$$\frac{\partial P_0}{\partial t} + \frac{1}{2}\bar{\sigma}^2 x^2 \frac{\partial^2 P_0}{\partial x^2} + a\left(r^\star - x\right)\frac{\partial P_0}{\partial x} - xP_0 = 0$$

Effective volatility $\bar{\sigma}^2 = \left\langle f^2 \right\rangle$ and $r^\star = r_\infty - \langle \lambda f \rangle / a$

The zero order term $P_0(t,x)$ **is the Vasicek bond price**

$$P_0(T-\tau, x; T) = \bar{P}(T-\tau, x; T) = A(\tau)e^{-B(\tau)x}$$

computed with the constant parameters $(a, r^\star, \bar{\sigma})$**.**

The Correction $\tilde{P}_1 = \sqrt{\varepsilon} P_1$

The correction $\tilde{P}_1$ solves the source problem:

$$\mathcal{L}_{Vasicek}(a, r^\star, \bar{\sigma})\tilde{P}_1 = \left(V_1 \frac{\partial}{\partial x} + V_2 \frac{\partial^2}{\partial x^2} + V_3 \frac{\partial^3}{\partial x^3} \right) P_0$$

with the zero terminal condition $\tilde{P}_1(T, x) = 0$.

It involves the **constant** quantities, **small** of order $1/\sqrt{\alpha}$

$$\begin{aligned}
V_3 &= \frac{\nu}{\sqrt{2\alpha}} \rho \langle f \phi' \rangle \\
V_2 &= -\frac{\nu}{\sqrt{2\alpha}} \left(\rho \langle \lambda \phi' \rangle + \sqrt{1-\rho^2} \langle \gamma \phi' \rangle \right) - \nu \rho \sqrt{\frac{2}{\alpha}} \langle f \psi' \rangle \\
V_1 &= \nu \sqrt{\frac{2}{\alpha}} \left(\rho \langle \lambda \psi' \rangle + \sqrt{1-\rho^2} \langle \gamma \psi' \rangle \right)
\end{aligned}$$

The Correction $\tilde{P}_1$: explicit computation

Using the variable $\tau = T - t$ and the explicit form $P_0 = Ae^{-Bx}$:

$$\begin{aligned}\frac{\partial \tilde{P}_1}{\partial \tau} &= \frac{1}{2}\bar{\sigma}^2 \frac{\partial^2 \tilde{P}_1}{\partial x^2} + \hat{a}(r^\star - x)\frac{\partial \tilde{P}_1}{\partial x} - x\tilde{P}_1 \\ &\quad + A(\tau)e^{-B(\tau)x}\left(V_3 B(\tau)^3 - V_2 B(\tau)^2 + V_1 B(\tau)\right)\end{aligned}$$

We seek a solution of the form $\tilde{P}_1(T-\tau, x; T) = D(\tau)A(\tau)e^{-B(\tau)x}$ with the condition $D(0) = 0$ so that $\tilde{P}_1(T, x; T) = 0$

We get:

$$D' = V_3 B^3 - V_2 B^2 + V_1 B$$

and

$$\begin{aligned}D(\tau) &= \frac{V_3}{\hat{a}^3}\left(\tau - B(\tau) - \frac{1}{2}\hat{a}B(\tau)^2 - \frac{1}{3}\hat{a}^2 B(\tau)^3\right) \\ &- \frac{V_2}{\hat{a}^2}\left(\tau - B(\tau) - \frac{1}{2}\hat{a}B(\tau)^2\right) + \frac{V_1}{\hat{a}}\left(\tau - B(\tau)\right)\end{aligned}$$

Summary

The corrected bond price is given by

$$\begin{aligned} P(T-\tau, x, y; T) &\approx P_0(T-\tau, x; T) + \tilde{P}_1(T-\tau, x; T) \\ &= A(\tau)\,(1 + D(\tau))\, e^{-B(\tau)x} \end{aligned}$$

where D is a small factor of order $1/\sqrt{\alpha}$.

The **error**

$$|P^{\varepsilon}(t, x, y; T) - \Big(P_0(t, x : T) + \tilde{P}_1(t, x; T)\Big)|$$

is of order $1/\alpha$.

Corrections for bond options prices are also obtained.

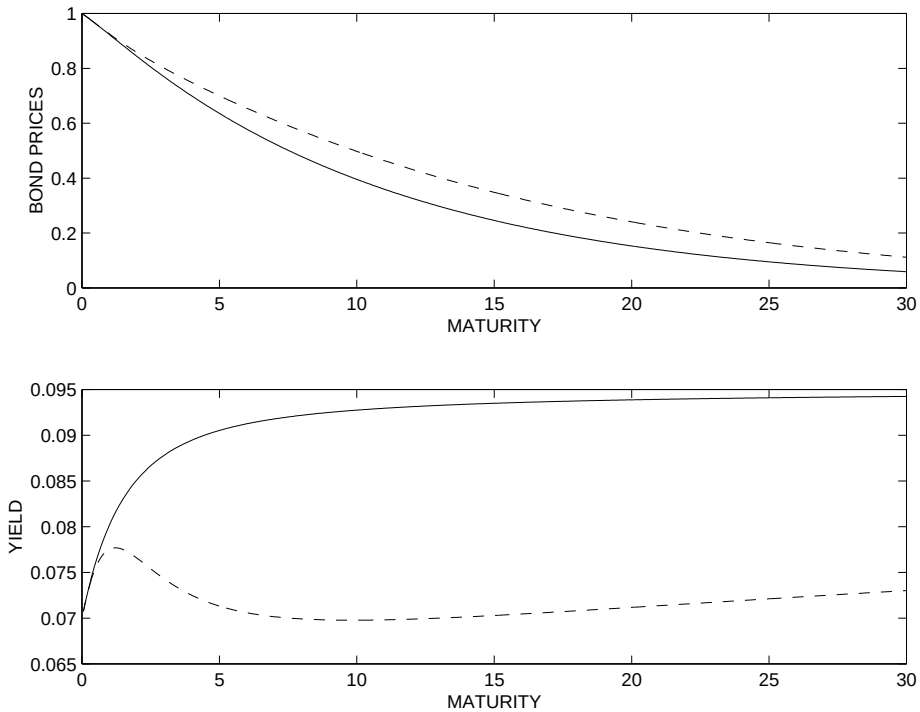


Figure 2: *Top: bond prices and corrected bond prices (dotted curve). Bottom: yield curve and corrected yield curve (dotted curve) in the simulated Vasicek model (constant and stochastic volatility) with:* $a = 1$, $r^\star = 0.1$ *and* $\bar{\sigma} = 0.1$ *as in Figure* 3. *Correction:* $V_3 = 1/\sqrt{\alpha}$ $(\rho \neq 0)$, $\alpha = 10^3$ *and* $\lambda = \gamma = 0$ *implying* $V_1 = 0$ *and* $V_2 = 0$. *Maturity* τ *runs from* 0 *to* 30 *years and the initial rate is* $x = 0.07$.

Model Parameters	**Correction Parameters**
Rate of mean-reversion of short-rate: a	a
Long-run mean under $\mathbb{P}$: $\bar{r}_\infty$	$r^\star$
Specific volatility distribution: $f(\cdot)$	Mean volatility $\bar{\sigma}$
Rate of mean-reversion of volatility : α	Group parameter V_1
Mean-level of (Y_t): m	Group parameter V_2
"V-vol": β	Group parameter V_3
Correlation: ρ	
Interest-rate risk premium: $\lambda(\cdot)$	
Volatility risk premium: $\gamma(\cdot)$	

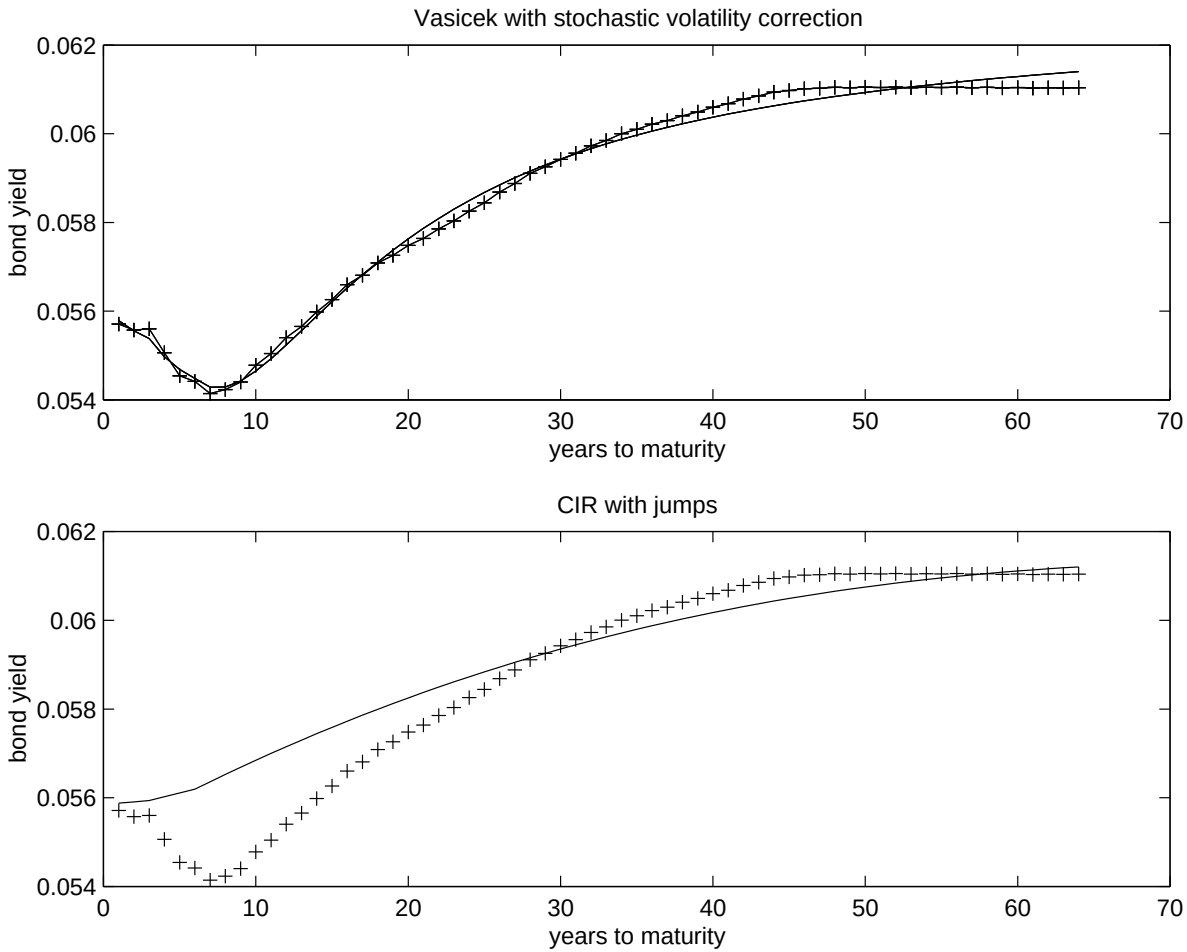


Figure 3: *Snapshot of the yield curve fit with the stochastic volatility corrected Vasicek model (top) and with the single factor CIR model and down jumps (bottom) for September 6, 1998.*

Credit

Perturbations around Merton/Black-Cox

(in the context of the structural approach for instance)

to account for:

- **Volatility Time Scales in Default Times**
- **Fit to Yield Spreads**

References:

Stochastic Volatility Effects on Defaultable Bonds

Applied Mathematical Finance 2006

with R. Sircar and K. Solna

Modeling Correlated Defaults:

First Passage Model under Stochastic Volatility

Journal of Computational Finance 2008

with B. Wignall and X. Zhou

Defaultable Bonds

In the first passage structural approach, the payoff of a defaultable zero-coupon bond written on a risky asset X is

$$h(X) \quad = \quad \mathbf{1}_{\{\inf_{0\leq s\leq T} X_s > B\}}.$$

By no-arbitrage, the value of the bond is

$$\begin{aligned} P^B(t,T) &= \mathbb{E}^\star\left\{e^{-r(T-t)}\mathbf{1}_{\{\inf_{0\leq s\leq T} X_s > B\}} \mid \mathcal{F}_t\right\} \\ &= \mathbf{1}_{\{\inf_{0\leq s\leq t} X_s > B\}} e^{-r(T-t)}\mathbb{E}^\star\left\{\mathbf{1}_{\{\inf_{t\leq s\leq T} X_s > B\}} \mid \mathcal{F}_t\right\}, \end{aligned}$$

Using the **predictable stopping time** $\tau_t = \inf\{s \geq t, X_s \leq B\}$:

$$\mathbb{E}^\star\left\{\mathbf{1}_{\{\inf_{t\leq s\leq T} X_s > B\}} \mid \mathcal{F}_t\right\} \quad = \quad \mathbb{P}^\star\{\tau_t > T \mid \mathcal{F}_t\}.$$

This defaultable zero-coupon bond is in fact a binary down-an-out barrier option where the barrier level and the strike price coincide.

Constant Volatility: Merton's Approach

$$
\begin{aligned}
dX_t &= rX_t dt + \sigma X_t dW_t^\star \\
X_t &= X_0 \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma W_t^\star \right).
\end{aligned}
$$

In the Merton's approach, default occurs if $X_T < B$:

Defaultable bond = European digital option

$$
\begin{aligned}
u^d(t,x) &= \mathbb{E}^\star \left\{ e^{-r\tau} \mathbf{1}_{\{X_T > B\}} \mid X_t = x \right\} = e^{-r\tau} \mathbb{P}^\star \left\{ X_T > B \mid X_t = x \right\} \\
&= e^{-r\tau} N(d_2(\tau))
\end{aligned}
$$

with the usual notation $\tau = T - t$ and the *distance to default*:

$$
d_2(\tau) = \frac{\log\left(\frac{x}{B}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}
$$

Constant Volatility: Black-Cox Approach

$$\mathbb{E}^\star\left\{\mathbf{1}_{\{\inf_{t\le s\le T} X_s > B\}} \mid \mathcal{F}_t\right\}$$
$$= \mathbb{P}^\star\left\{\inf_{t\le s\le T}\left((r-\frac{\sigma^2}{2})(s-t)+\sigma(W_s^\star - W_t^\star)\right) > \log\left(\frac{B}{x}\right) \mid X_t = x\right\}$$

computed using distribution of minimum, or using PDE's:

$$\mathbb{E}^\star\left\{e^{-r(T-t)}\mathbf{1}_{\{\inf_{t\le s\le T} X_s > B\}} \mid \mathcal{F}_t\right\} \quad = \quad u(t, X_t)$$

where $u(t,x)$ is the solution of the following problem

$$\mathcal{L}_{BS}(\sigma)u = 0 \ \text{ on } \ x > B,\, t < T$$
$$u(t,B) = 0 \quad \text{for any } \ t \le T$$
$$u(T,x) = 1 \ \text{ for } \ x > B,$$

which is to be solved for $x > B$.

Constant Volatility: Barrier Options

Using the European digital pricing function $u^d(t,x)$

$$\mathcal{L}_{BS}(\sigma)u^d = 0 \ \text{ on } \ x > 0,\, t < T$$
$$u^d(T,x) = 1 \ \text{ for } \ x > B, \ \text{ and } 0 \text{ otherwise}$$

By the method of images one has:

$$\begin{aligned} u(t,x) &= u^d(t,x) - \left(\frac{x}{B}\right)^{1-\frac{2r}{\sigma^2}} u^d\left(t, \frac{B^2}{x}\right) \\ &= e^{-r(T-t)}\left(N(d_2^+(T-t)) - \left(\frac{x}{B}\right)^{1-\frac{2r}{\sigma^2}} N(d_2^-(T-t))\right) \end{aligned}$$

where we denote

$$d_2^{\pm}(\tau) = \frac{\pm\log\left(\frac{x}{B}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

Yield Spreads Curve

The *yield spread* $Y(0,T)$ at time zero is defined by

$$e^{-Y(0,T)T} = \frac{P^B(0,T)}{P(0,T)},$$

where $P(0,T)$ is the default free zero-coupon bond price given here, in the case of constant interest rate r, by $P(0,T) = e^{-rT}$, and $P^B(0,T) = u(0,x)$, leading to the formula

$$Y(0,T) = -\frac{1}{T}\log\left(N\left(d_2(T)\right) - \left(\frac{x}{B}\right)^{1-\frac{2r}{\sigma^2}} N\left(d_2^-(T)\right)\right)$$

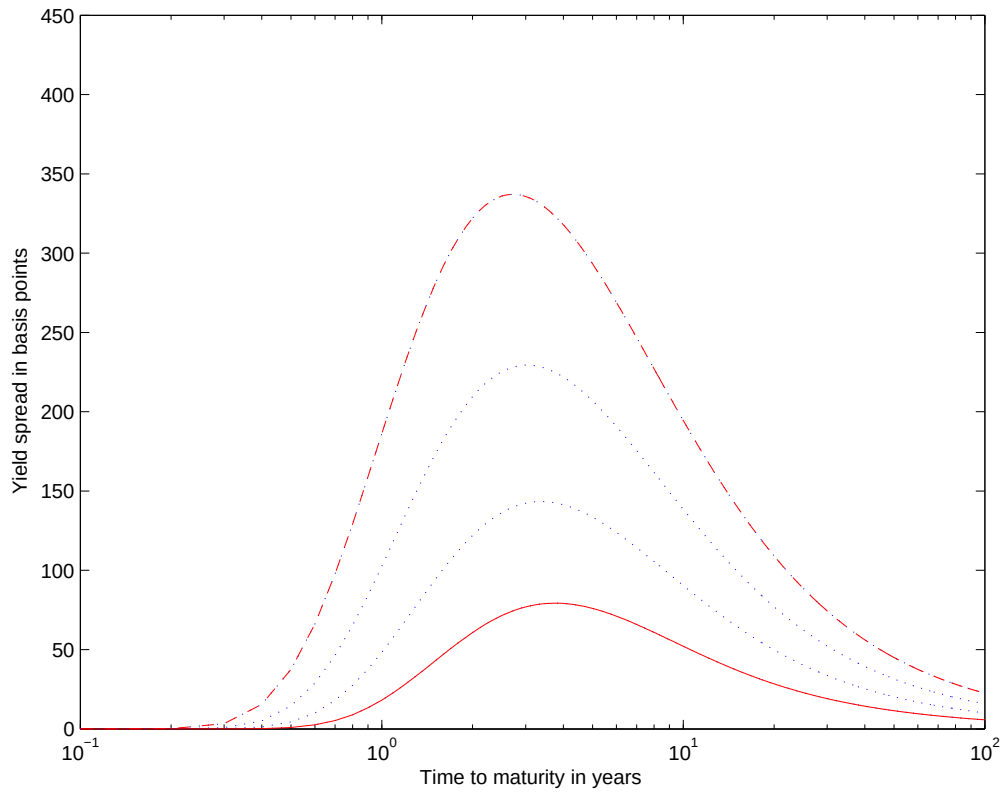


Figure 4: *The figure shows the sensitivity of the yield spread curve to the volatility level. The ratio of the initial value to the default level* x/B *is set to* 1.3*, the interest rate* r *is 6% and the curves increase with the values of* σ*: 10%, 11%, 12% and 13% (time to maturity in unit of years, plotted on the log scale; the yield spread is quoted in basis points)*

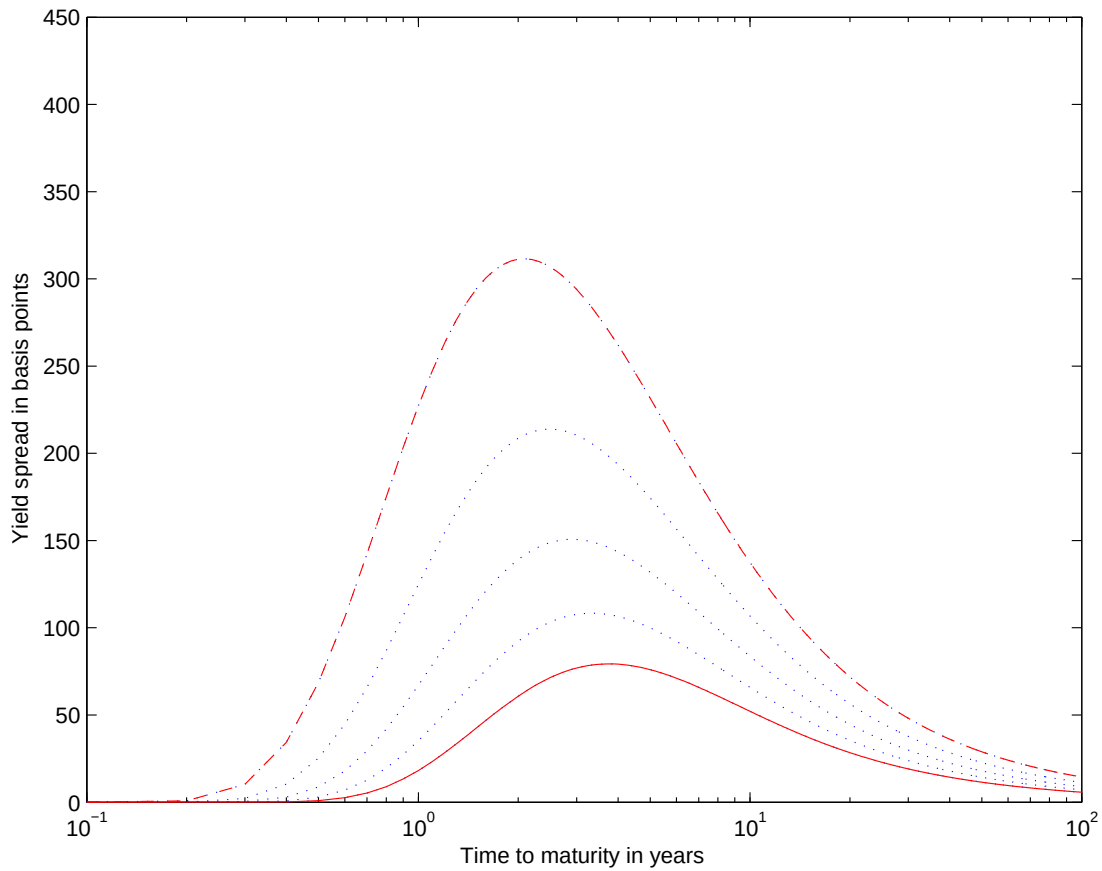


Figure 5: *This figure shows the sensitivity of the yield spread to the leverage level. The volatility level is set to 10%, the interest rate is 6%. The curves increases with the decreasing ratios* x/B: $(1.3, 1.275, 1.25, 1.225, 1.2)$.

Challenge: Yields at Short Maturities

As stated by Eom et.al. (empirical analysis 2001), **the challenge for theoretical pricing models is to raise the average predicted spread relative to crude models such as the constant volatility model, without overstating the risks associated with volatility or leverage.**

Several approaches (**within structural models**) have been proposed that aims at the modeling in this regard. These include

- Introduction of jumps (Zhou,...)
- Stochastic interest rate (Longstaff-Schwartz,...)
- Imperfect information (on X_t) (Duffie-Lando,...)
- Imperfect information (on B) (Giesecke)

Stochastic Volatility Models

$$
\begin{aligned}
dX_t &= \mu X_t dt + f(Y_t) X_t \, dW_t^{(0)} \\
dY_t &= \alpha(m - Y_t) dt + \nu \sqrt{2\alpha} \, dW_t^{(1)}
\end{aligned}
$$

where we assume that

- f non-decreasing, $0 < c_1 \leq f \leq c_2$
- Invariant distribution of Y: $\mathcal{N}(m, \nu^2)$ independent of α
- $\alpha > 0$ is the *rate of mean reversion* of Y
- The standard Brownian motions $W^{(0)}$ and $W^{(1)}$ are correlated

$$
d \left\langle W^{(0)}, W^{(1)} \right\rangle_t = \rho_1 \, dt
$$

Stochastic Volatility Models under $\mathbb{P}^\star$

In order to price defaultable bonds under this model for the underlying we rewrite it under a risk neutral measure $\mathbb{P}^\star$, chosen by the market through the market price of volatility risk Λ_1, as follows

$$\begin{aligned} dX_t &= rX_t dt + f(Y_t)X_t\, dW_t^{(0)\star}, \\ dY_t &= \Big(\alpha(m - Y_t) - \nu\sqrt{2\alpha}\Lambda_1(Y_t)\Big)\, dt + \nu\sqrt{2\alpha}\, dW_t^{(1)\star}. \end{aligned}$$

Here $W^{(0)\star}$ and $W^{(1)\star}$ are standard Brownian motions under $\mathbb{P}^\star$ correlated as $W^{(0)}$ and $W^{(1)}$. We assume that the market price of volatility risk Λ_1 is bounded and a function of y only.

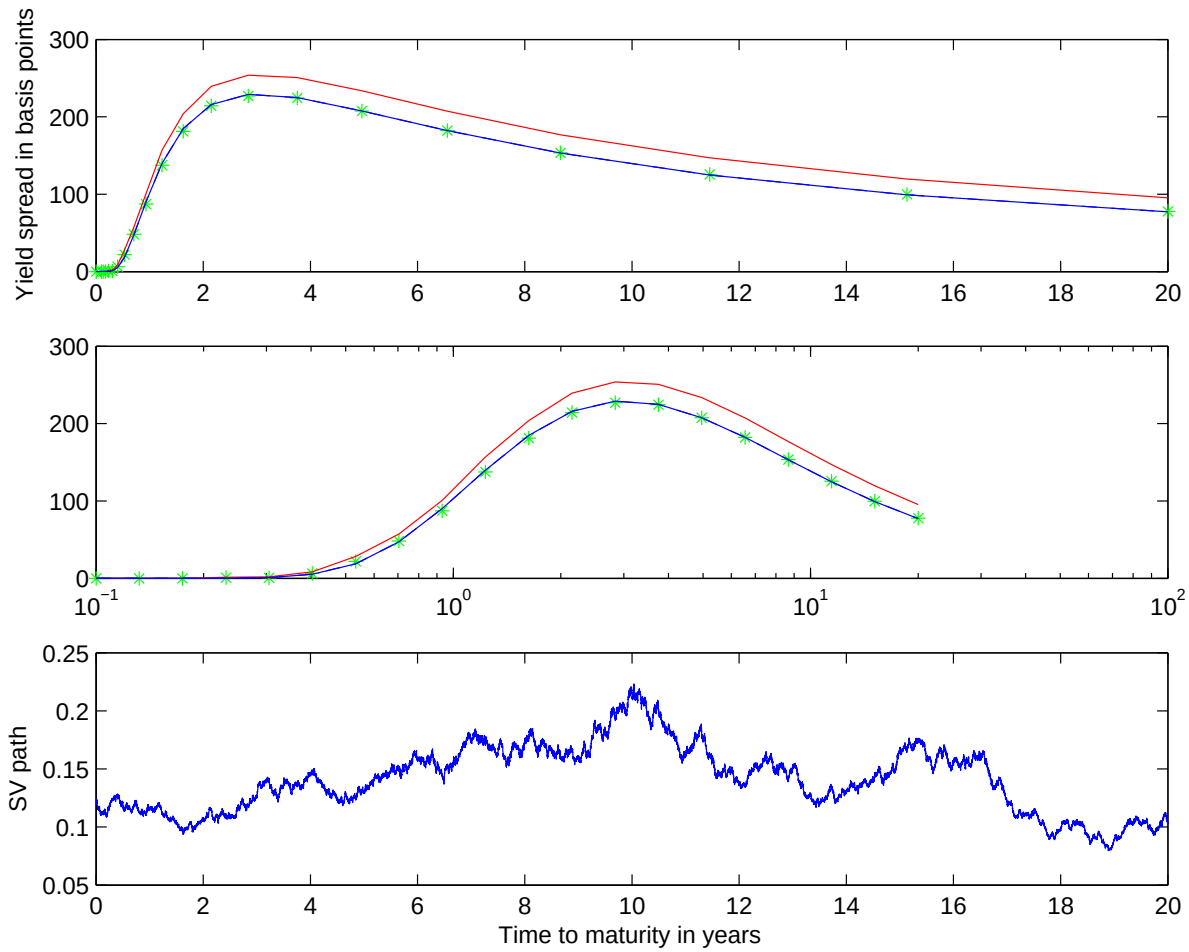


Figure 6: **Uncorrelated slowly mean-reverting stochastic volatility**: $\alpha = 0.05$ *and* $\rho_1 = 0$.

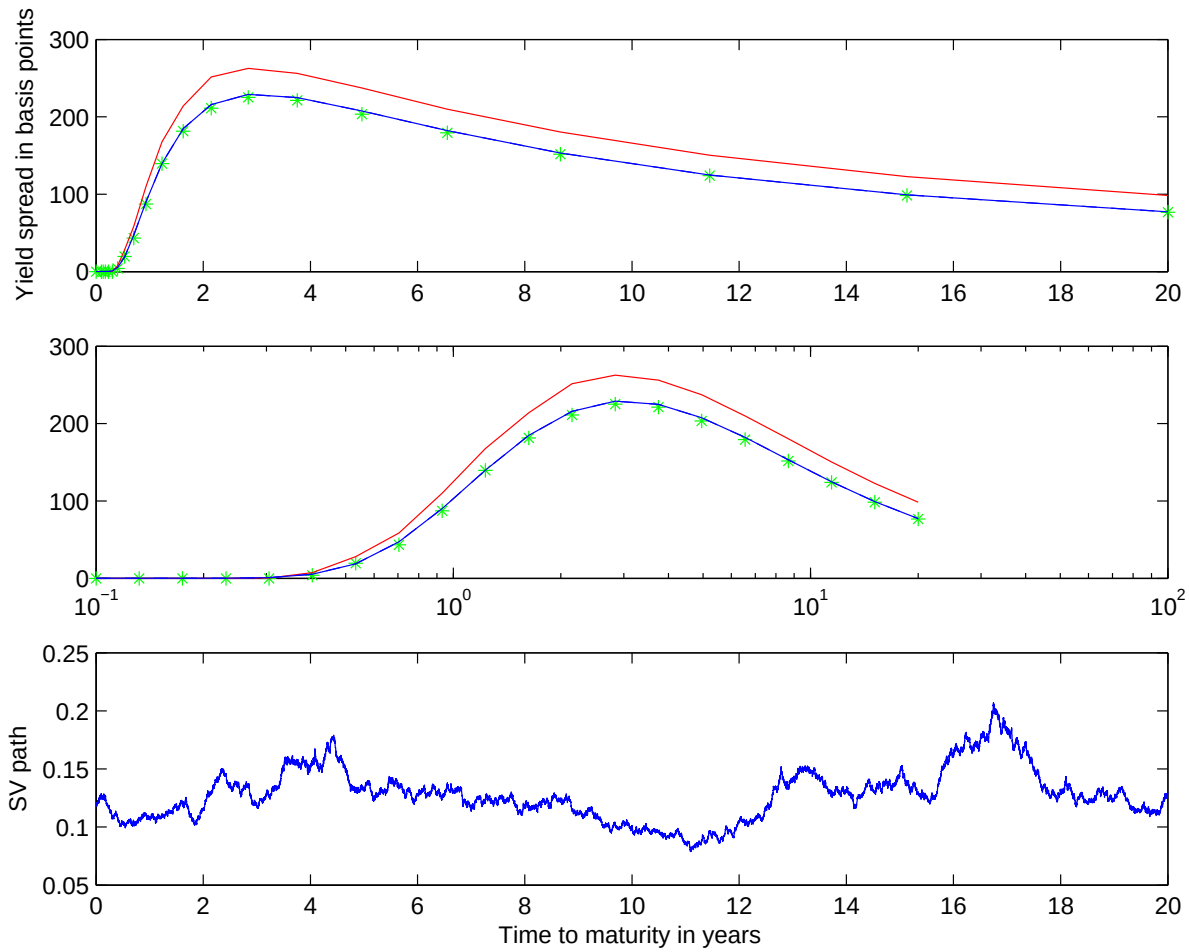


Figure 7: **Correlated slowly mean-reverting stochastic volatility**: $\alpha = 0.05$ *and* $\rho_1 = -0.05$.

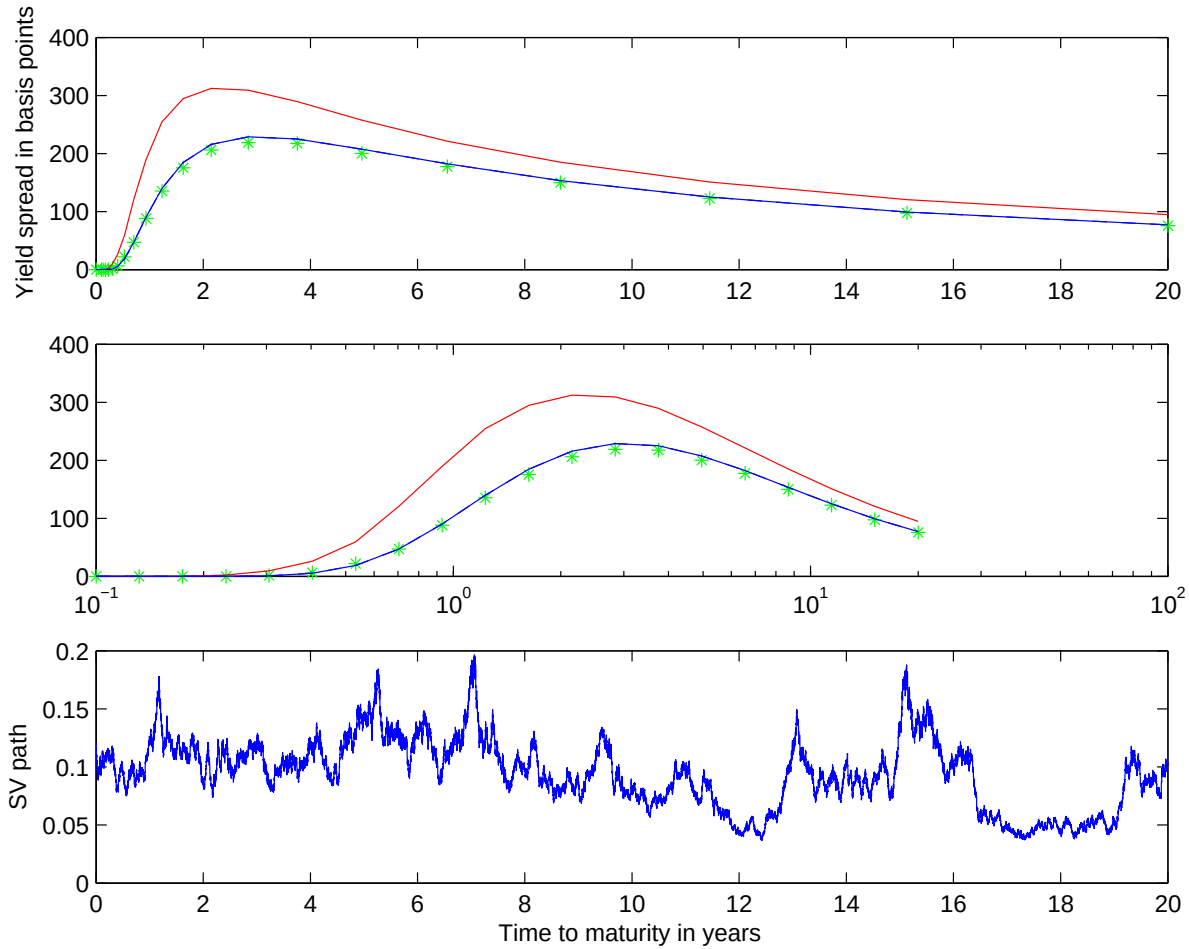


Figure 8: **Uncorrelated stochastic volatility:** $\alpha = 0.5$ **and** $\rho_1 = 0$**.**

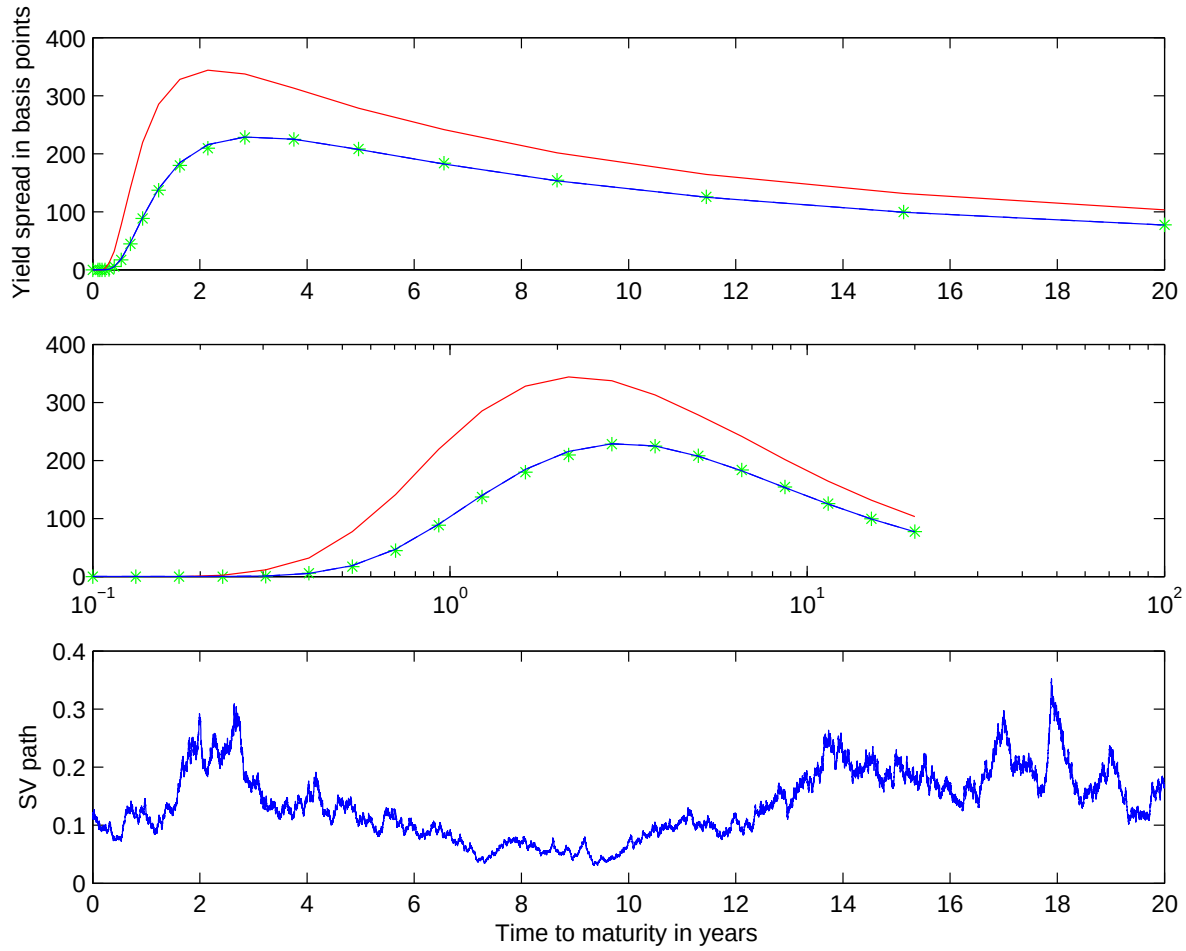


Figure 9: **Correlated stochastic volatility:** $\alpha = 0.5$ **and** $\rho_1 = -0.05$**.**

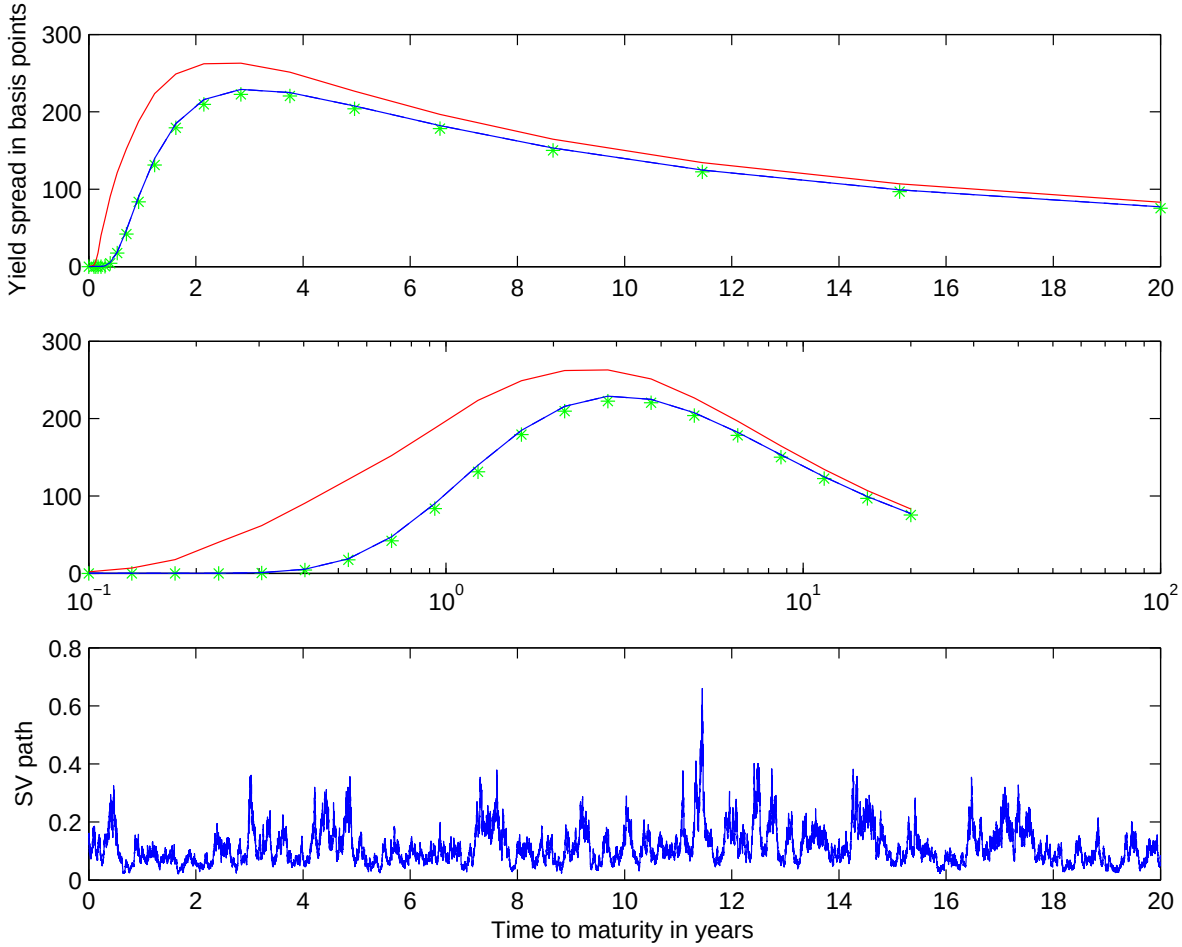


Figure 10: **Uncorrelated fast mean-reverting stochastic volatility:** $\alpha = 10$ **and** $\rho_1 = 0$**.**

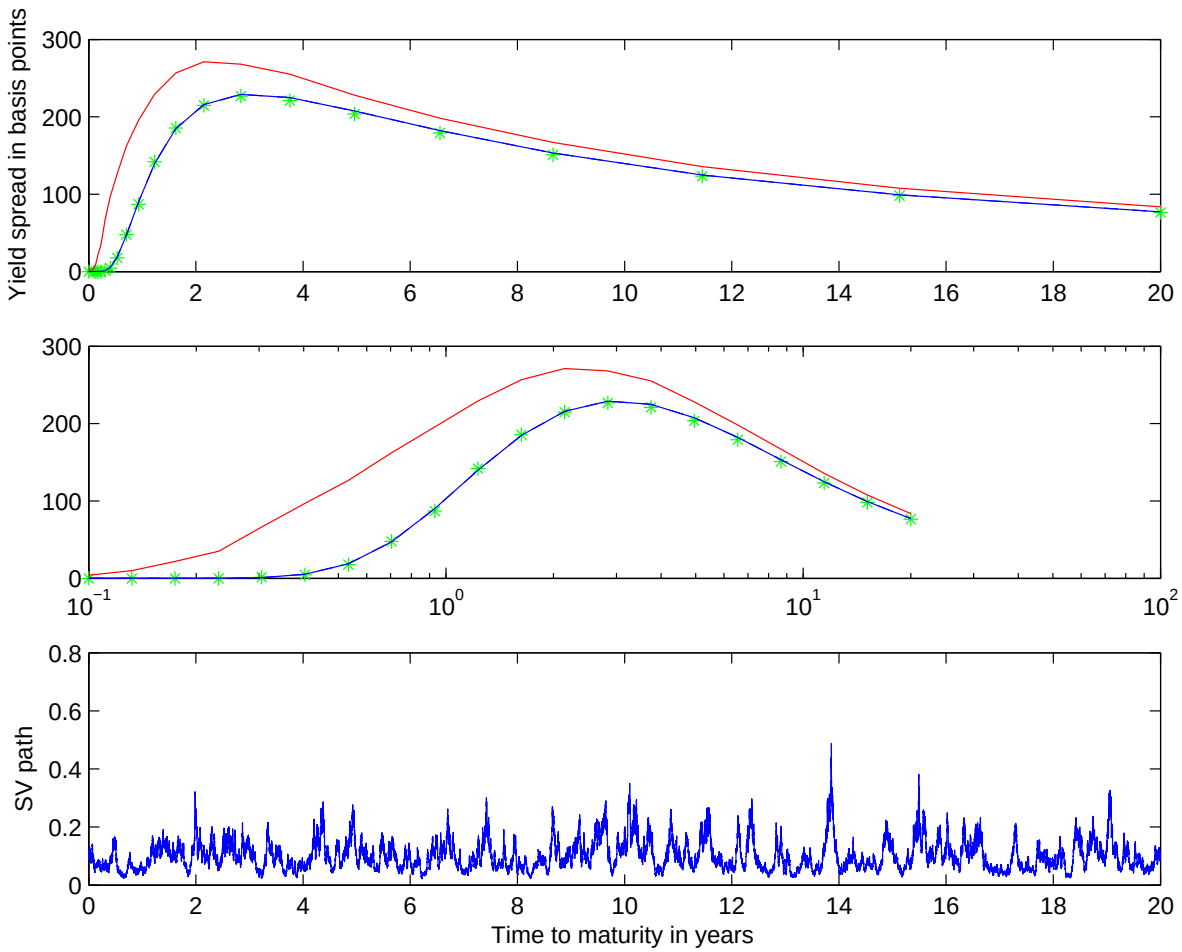


Figure 11: **Correlated fast mean-reverting stochastic volatility:** $\alpha = 10$ **and** $\rho_1 = -0.05$**.**

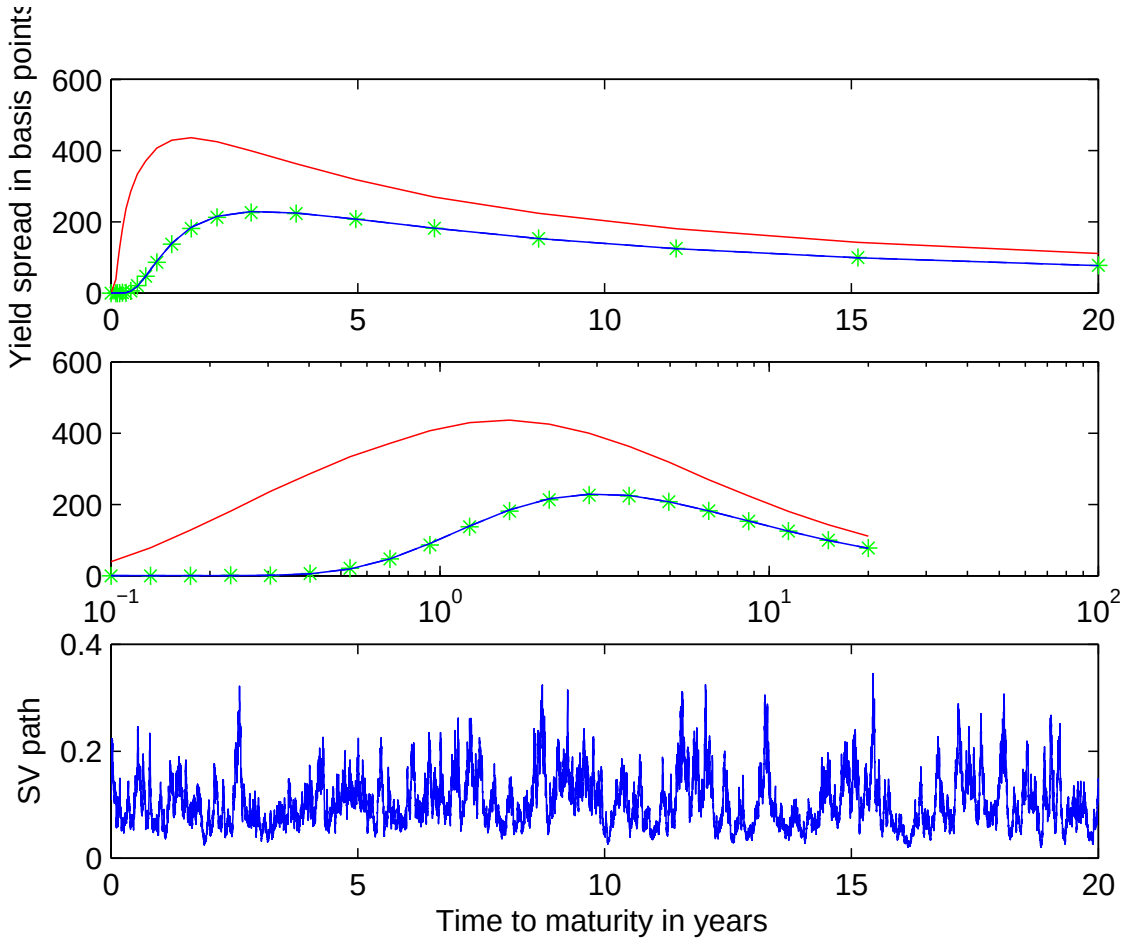


Figure 12: **Highly correlated fast mean-reverting stochastic volatility:** $\alpha = 10$ **and** $\rho_1 = -0.5$**.**

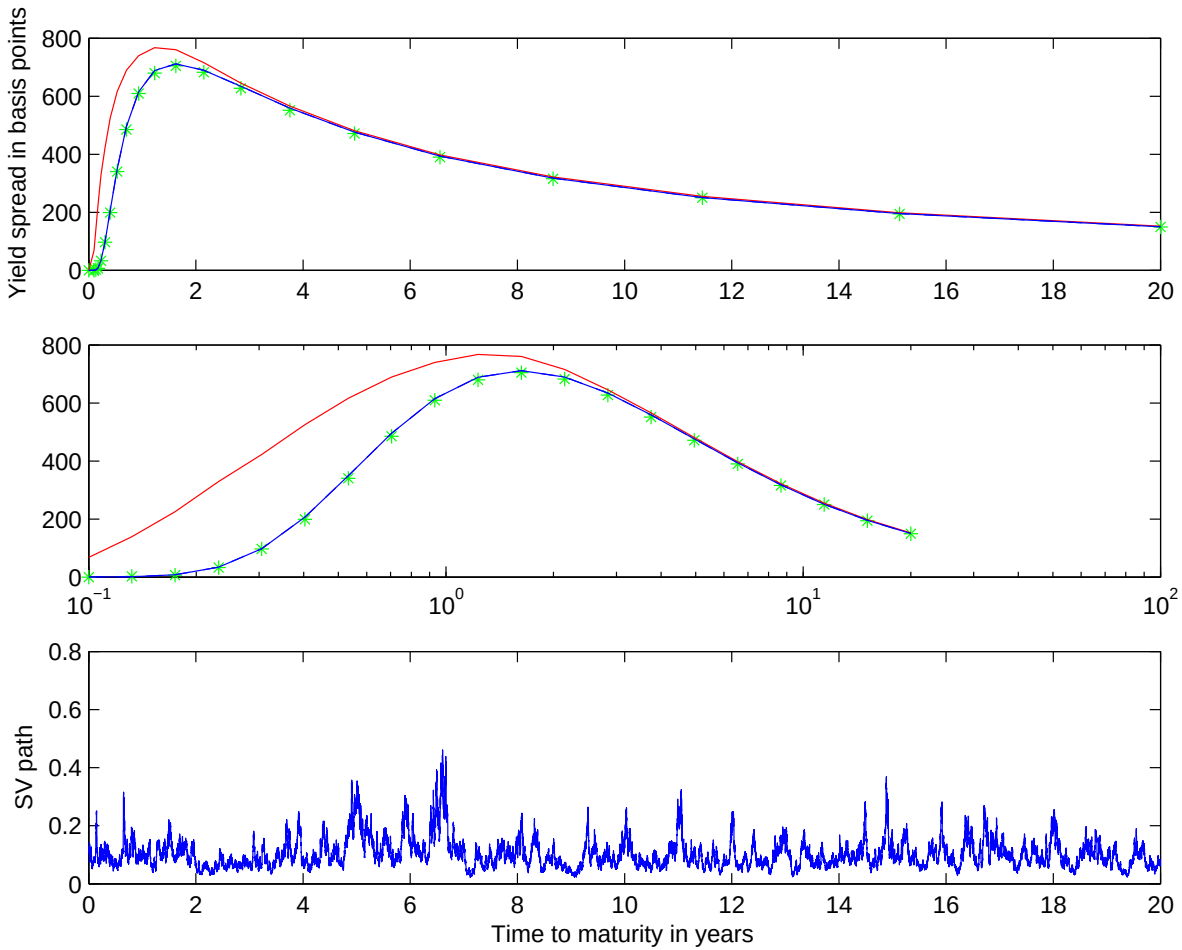


Figure 13: **High leverage correlated fast mean-reverting stochastic volatility:** $x/B = 1.2$, $\alpha = 10$ **and** $\rho_1 = -0.05$.

Barrier Options under Stochastic Volatility

$$u(t,x,y) \quad = \quad e^{-r(T-t)} \mathbb{E}^\star \left\{ h(X_T) \mathbf{1}_{\{\inf_{t \le s \le T} X_s > B\}} \mid X_t = x, Y_t = y \right\} ,$$

$$P^B(t,T) = \mathbf{1}_{\{\inf_{0 \le s \le t} X_s > B\}} u(t, X_t, Y_t).$$

The function $u(t,x,y)$ satisfies for $x \ge B$ the problem

$$\begin{array}{ll} \left(\frac{\partial}{\partial t} + \mathcal{L}_{X,Y} - r\right) u = 0 & \text{on } \ x > B, \, t < T \\ u(t,B) = 0 & \text{for any } \ \ t \le T \\ u(T,x) = h(x) & \text{for } \ \ \ x > B \end{array}$$

where $\mathcal{L}_{X,Y}$ is the infinitestimal generator of the process (X,Y) under $\mathbb{P}^\star$.

Leading Order Term under Stochastic Volatility

In the regime α large, as in the European case, $u(t, x, y)$ is approximated by $u_0^\star(t, x)$ which solves the constant volatility problem

$$\begin{aligned} &\mathcal{L}_{BS}(\sigma^\star)u_0^\star = 0 && \text{on } \ x > B,\, t < T \\ &u_0^\star(t, B) = 0 && \text{for any } \ \ t \leq T \\ &u_0^\star(T, x) = h(x) && \text{for } \ \ \ x > B \end{aligned}$$

where $\sigma^\star$ is the corrected effective volatility.

Stochastic Volatility Correction

Define the correction $u_1^\star(t,x)$ by

$$\begin{aligned} &\mathcal{L}_{BS}(\sigma^\star)u_1^\star = -V_3\, x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 u_0^\star}{\partial x^2}\right) && \text{on } \; x > B,\, t < T \\ &u_1^\star(t,B) = 0 && \text{for any } \; t \le T \\ &u_1^\star(T,x) = 0 && \text{for } \; x > B \end{aligned} \quad .$$

Remarkably, the small parameter V_3 is the same as in the European case (calibrated to implied volatilities).

Computation of the Correction

Define

$$v_1^\star(t,x) = u_1^\star(t,x) - (T-t)V_3\, x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 u_0^\star}{\partial x^2}\right),$$

so that $v_1^\star(t,x)$ solves the simpler problem

$$\begin{aligned}
&\mathcal{L}_{BS}(\sigma^\star)v_1^\star = 0 && \text{on } x > B,\ t < T\\
&v_1^\star(t,B) = g(t) && \text{for any } \ t \le T\\
&v_1^\star(T,x) = 0 && \text{for } \ x > B\\
&g(t) = -V_3\,(T-t)\lim_{x\downarrow B}\left(x\tfrac{\partial}{\partial x}\left(x^2\tfrac{\partial^2 u_0^\star}{\partial x^2}\right)\right)
\end{aligned}$$

To summarize we have

$$u(t,x,y) \quad\approx\quad u_0^\star(t,x) + (T-t)V_3\, x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 u_0^\star}{\partial x^2}\right) + v_1^\star(t,x)$$

with explicit computation in the case $h(x) = 1$.

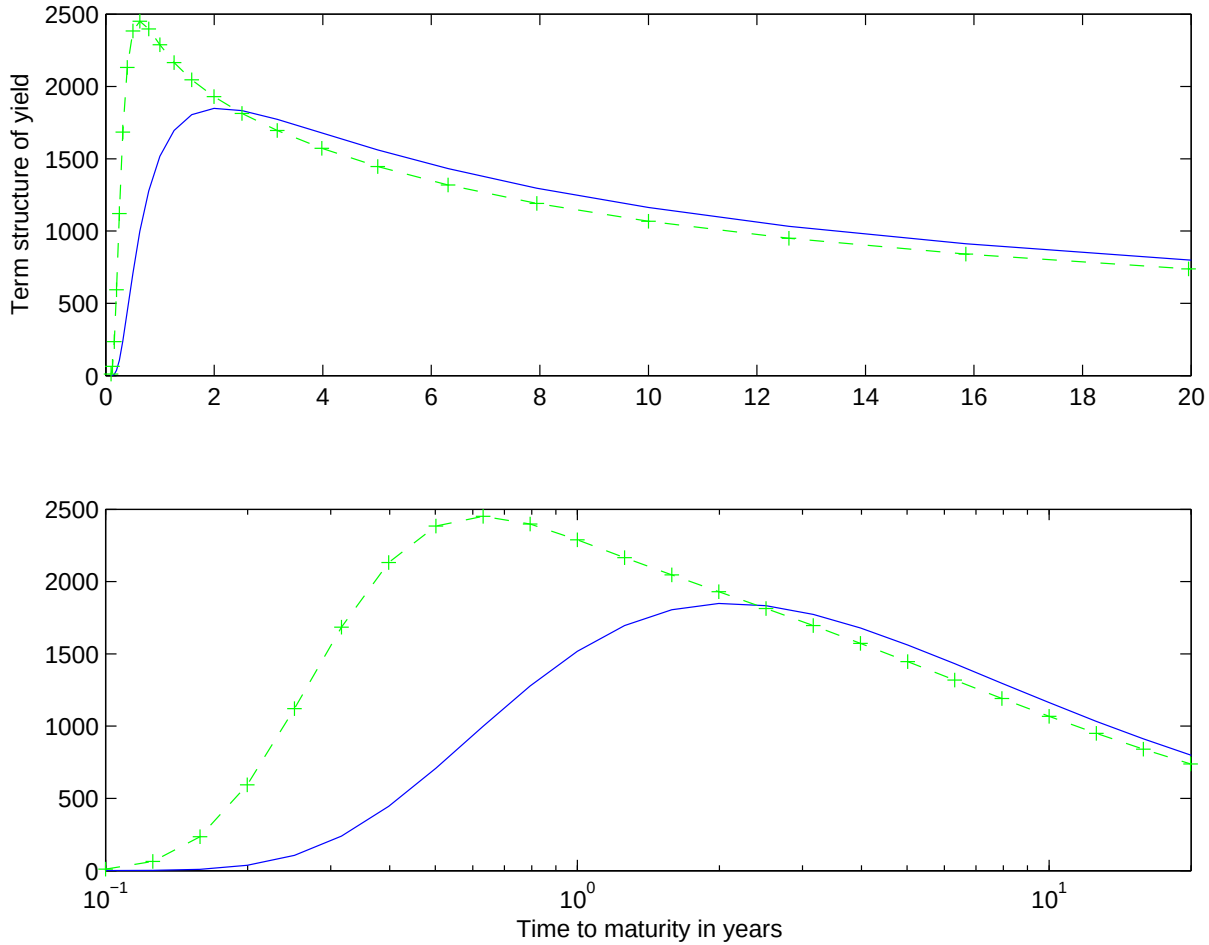


Figure 14: **The price approximation for** $\sigma^\star = 0.12, r = 0.0, V_3 = -0.0003, x/B = 1.2$**.**

Slow Factor Correction

The first correction $u_1^{(z)}(t,x)$ solves the problem

$$\begin{aligned}
&\mathcal{L}_{BS}(\bar{\sigma}(z))u_1^{(z)} = -2\left(V_0(z)\tfrac{\partial u_{BS}}{\partial \sigma} + V_1(z)x\tfrac{\partial}{\partial x}\left(\tfrac{\partial u_{BS}}{\partial \sigma}\right)\right) && \text{on } x > B,\, t < T,\\
&u_1^{(z)}(t,B) = 0 && \text{for } t \le T,\\
&u_1^{(z)}(T,x) = 0 && \text{for } x > B,
\end{aligned}$$

where u_{BS} is evaluated at $(t,x,\bar{\sigma}(z))$, and $V_0(z)$ and $V_1(z)$ are small parameters of order $\sqrt{\delta}$, functions of the model parameters, and depending on the current level z of the slow factor.

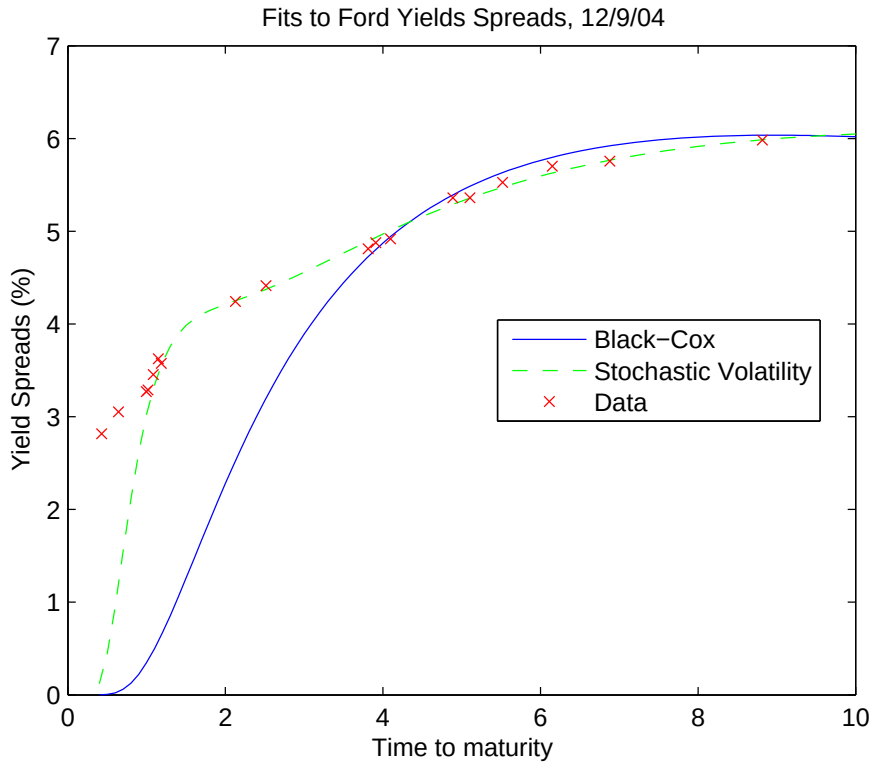
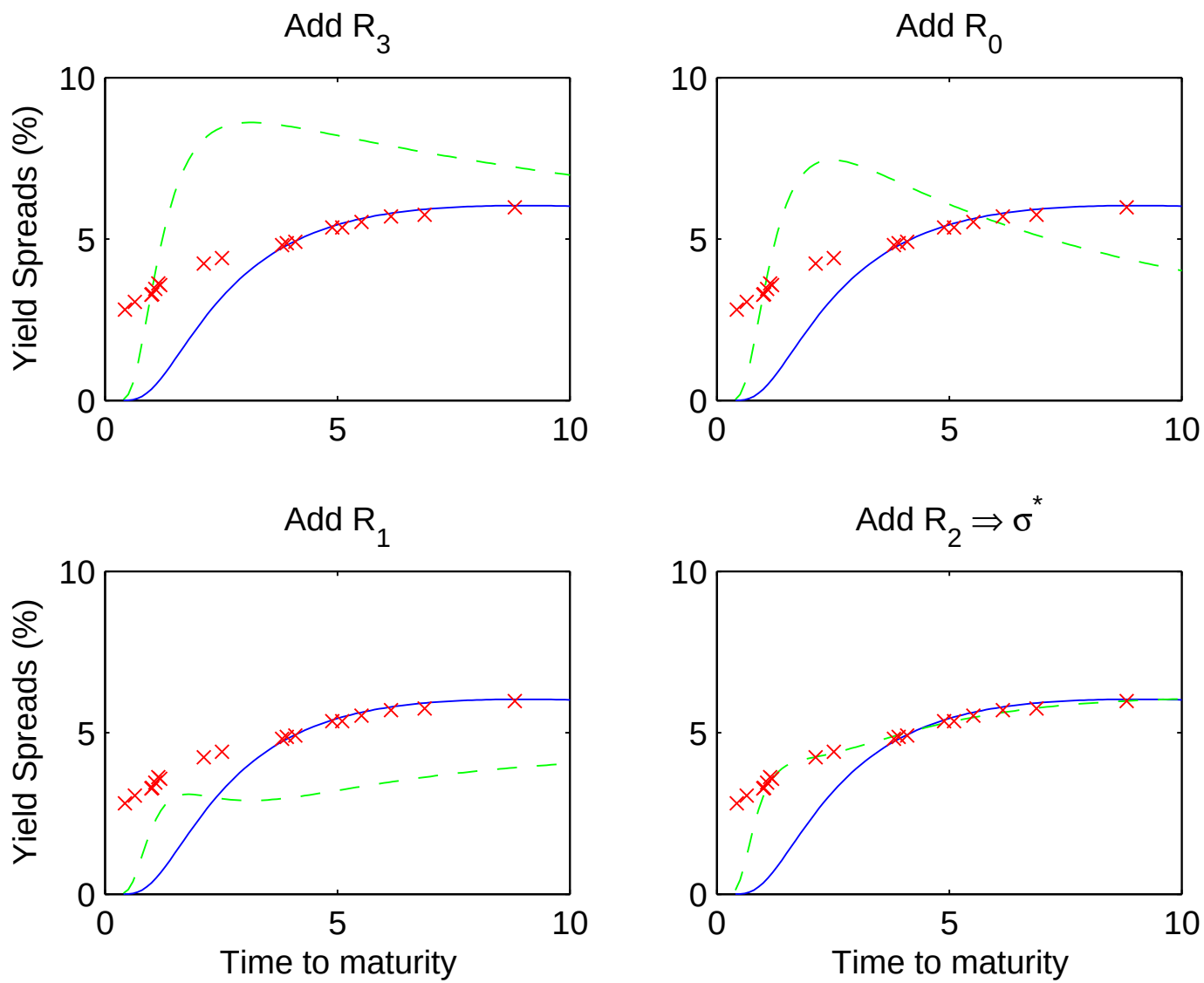


Figure 15: *Black-Cox and two-factor stochastic volatility fits to Ford yield spread data. The short rate is fixed at* $r = 0.025$. *The fitted Black-Cox parameters are* $\bar{\sigma} = 0.35$ *and* $x/B = 2.875$. *The fitted stochastic volatility parameters are* $\sigma^\star = 0.385$, *corresponding to* $R_2 = 0.0129$, $R_3 = -0.012$, $R_1 = 0.016$ *and* $R_0 = -0.008$.


Add R_3
Add R_0
Add R_1
Add $R_2 \Rightarrow \sigma^*$
Yield Spreads (%)
Yield Spreads (%)
Time to maturity
Time to maturity
0
5
10
0
5
10

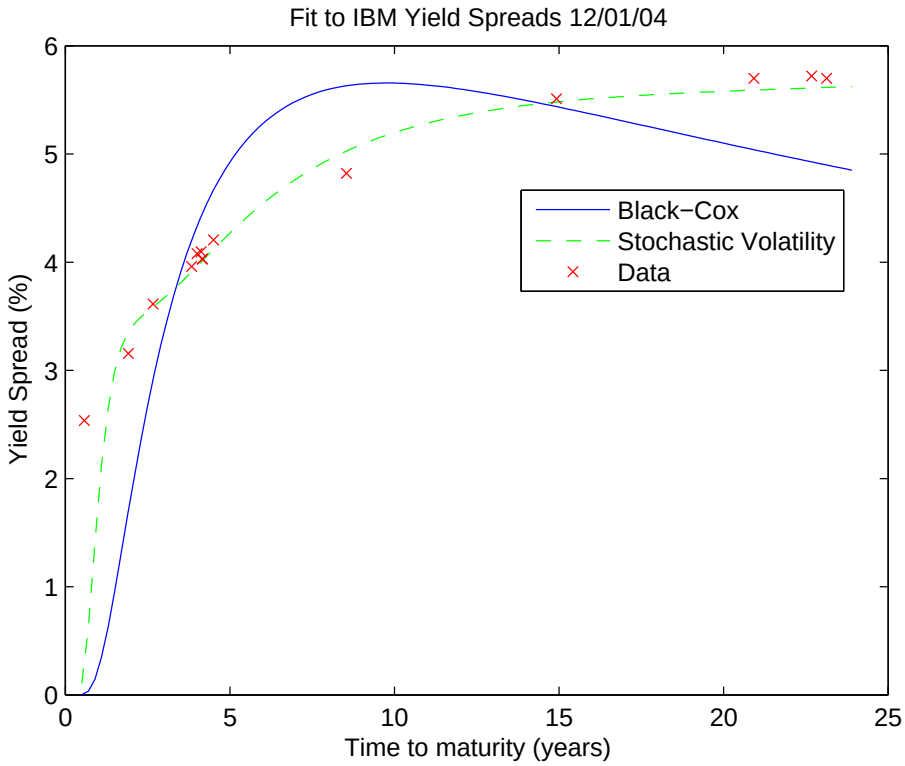


Figure 16: *Black-Cox and two-factor stochastic volatility fits to IBM yield spread data. The short rate is fixed at* $r = 0.025$. *The fitted Black-Cox parameters are* $\bar{\sigma} = 0.35$ *and* $x/B = 3$. *The fitted stochastic volatility parameters are* $\sigma^\star = 0.36$, *corresponding to* $R_2 = 0.00355$, $R_3 = -0.0112$, $R_1 = 0.013$ *and* $R_0 = -0.0045$.

Multiname Model Setup

Under risk neutral pricing probability:

$$
\begin{aligned}
\mathrm{d}X_t^{(1)} &= rX_t^{(1)}\mathrm{d}t + f_1(Y_t, Z_t)X_t^{(1)}\mathrm{d}W_t^{(1)}, \\
\mathrm{d}X_t^{(2)} &= rX_t^{(2)}\mathrm{d}t + f_2(Y_t, Z_t)X_t^{(2)}\mathrm{d}W_t^{(2)}, \\
\cdots\cdots &\cdots \quad \cdots\cdots\cdots\cdots\cdots\cdots \\
\mathrm{d}X_t^{(n)} &= rX_t^{(n)}\mathrm{d}t + f_n(Y_t, Z_t)X_t^{(n)}\mathrm{d}W_t^{(n)}, \\
\mathrm{d}Y_t &= \left[\frac{1}{\varepsilon}(m_Y - Y_t) - \frac{\nu_Y\sqrt{2}}{\sqrt{\varepsilon}}\Lambda_1(Y_t, Z_t)\right]\mathrm{d}t + \frac{\nu_Y\sqrt{2}}{\sqrt{\varepsilon}}\mathrm{d}W_t^{(Y)}, \\
\mathrm{d}Z_t &= \left[\delta(m_Z - Z_t) - \nu_Z\sqrt{2\delta}\Lambda_2(Y_t, Z_t)\right]\mathrm{d}t + \nu_Z\sqrt{2\delta}\mathrm{d}W_t^{(Z)},
\end{aligned}
$$

where the $W_t^{(i)}$'s are **independent** standard Brownian motions and

$\mathrm{d}\langle W^{(Y)}, W^{(i)}\rangle_t = \rho_{iY}\mathrm{d}t$, $\mathrm{d}\langle W^{(Z)}, W^{(i)}\rangle_t = \rho_{iZ}\mathrm{d}t$, $\mathrm{d}\langle W^{(Y)}, W^{(Z)}\rangle_t = \rho_{YZ}\mathrm{d}t$.

with $\sum_{i=1}^n \rho_{iY}^2 \leq 1$ and $\sum_{i=1}^n \rho_{iZ}^2 \leq 1$.

Objective

Find the joint (risk-neutral) survival probabilities

$$\begin{aligned} u^{\varepsilon,\delta} &\equiv u^{\varepsilon,\delta}(t,\mathbf{x},y,z) \\ &\equiv I\!P^{\star}\left\{\tau_t^{(1)} > T,\ldots,\tau_t^{(n)} > T \middle| \mathbf{X}_t = \mathbf{x}, Y_t = y, Z_t = z\right\}, \end{aligned}$$

where $t < T$, $\mathbf{X}_t \equiv (X_t^{(1)},\ldots,X_t^{(n)})$, $\mathbf{x} \equiv (x_1,\ldots,x_n)$, and $\tau_t^{(i)}$ is the default time of firm i:

$$\tau_t^{(i)} = \inf\left\{s \geq t |\, X_s^{(i)} \leq B_i(s)\right\},$$

where $B_i(t)$ is the exogenously pre-specified default threshold at time t for firm i. Following Black and Cox (1976) we assume

$$B_i(t) = K_i \mathrm{e}^{\eta_i t},$$

with constant parameters $K_i > 0$ and $\eta_i \geq 0$.

PDE Formulation

$$\mathcal{L}^{\varepsilon,\delta} u^{\varepsilon,\delta}(t, \mathbf{x}, y, z) \quad = \quad 0, \quad x_i > B_i(t), \text{ for all } i,\ t < T$$

$$\mathcal{L}^{\varepsilon,\delta} = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3$$

Boundary conditions:

$$u^{\epsilon,\delta}(t, x_1, x_2, \ldots, x_n, y, z) \quad = \quad 0, \quad \exists i \in \{1, \cdots, n\}, x_i = B_i(t), t \leq T,$$

Terminal condition:

$$u^{\varepsilon,\delta}(T, x_1, x_2, \ldots, x_n, y, z) \quad = \quad 1, \quad x_i > B_i(t), \text{ for all } i$$

Expansion and Approximation

$$u^{\varepsilon,\delta} = \underbrace{\mathbf{u_0} + \sqrt{\varepsilon}\mathbf{u_{1,0}} + \sqrt{\delta}\mathbf{u_{0,1}}} + \varepsilon u_{2,0} + \sqrt{\varepsilon\delta}u_{1,1} + \delta u_{0,2} + \cdots$$

Leading Order Term $\mathbf{u_0}$:

$$\begin{aligned}
\langle \mathcal{L}_2 \rangle u_0 &= 0, \quad x_i > B_i(t), \text{ for all } i,\, t < T \\
u_0(t, x_1, x_2, \ldots, x_n) &= 0, \quad \exists i \in \{1, \cdots, n\}, x_i = B_i(t), t \leq T, \\
u_0(T, x_1, x_2, \ldots, x_n) &= 1, \quad x_i > B_i(t), \text{ for all } i
\end{aligned}$$

$$\begin{aligned}
\langle \mathcal{L}_2 \rangle &= \frac{\partial}{\partial t} + \sum_{i=1}^{n} \left(\frac{1}{2}\sigma_i(z)^2 x_i^2 \frac{\partial^2}{\partial x_i^2} + r x_i \frac{\partial}{\partial x_i} \right) \\
\sigma_i(z) &= \sqrt{\langle f_i^2(\cdot, z) \rangle}, \quad \langle \cdot \rangle : \, average\, w.r.t.\, \mathcal{N}(m_Y, \nu_Y^2)
\end{aligned}$$

A Formula for $\mathbf{u_0}$

$$\mathbf{u_0} = \prod_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{Q_i} \equiv \prod_{\mathbf{i=1}}^{\mathbf{n}} \left[\mathrm{N}\left(\mathbf{d^+_{2(i)}}\right) - \left(\frac{\mathbf{x_i}}{\mathbf{B_i(t)}}\right)^{\mathbf{p_i}} \mathrm{N}\left(\mathbf{d^-_{2(i)}}\right) \right],$$

where $\mathrm{N}(\cdot)$ is the standard normal distribution function,

$$\begin{aligned} d^{\pm}_{2(i)} &= \frac{\pm \ln \frac{x_i}{B_i(t)} + \left(r - \eta_i - \frac{\sigma_i^2(z)}{2}\right)(T-t)}{\sigma_i(z)\sqrt{T-t}}, \\ \sigma_{\mathbf{i}}(\mathbf{z}) &= \sqrt{\langle \mathbf{f_i^2}(\cdot, \mathbf{z}) \rangle}, \\ p_i &= 1 - \frac{2(r-\eta_i)}{\sigma_i^2(z)}. \end{aligned}$$

Correction Term $\sqrt{\varepsilon}\,\mathbf{u_{1,0}}$

$$
\begin{aligned}
\langle \mathcal{L}_2 \rangle u_{1,0} &= \mathcal{A} u_0, \quad x_i > B_i(t), \text{ for all } i,\ t < T \\
u_{1,0}(t, x_1, x_2, \ldots, x_n) &= 0, \quad \exists i \in \{1, \cdots, n\}, x_i = B_i(t), t \leq T, \\
u_{1,0}(T, x_1, x_2, \ldots, x_n) &= 0, \quad x_i > B_i(t), \text{ for all } i
\end{aligned}
$$

$$
\mathcal{A} = \langle \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle =
$$

$$
\frac{\nu_Y}{\sqrt{2}} \left[\sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{iY} \left\langle f_i(\cdot, z) \frac{\partial \phi_j}{\partial y} \right\rangle x_i \frac{\partial}{\partial x_i} \left(x_j^2 \frac{\partial^2}{\partial x_j^2} \right) - \sum_{j=1}^{n} \left\langle \Lambda_1(\cdot, z) \frac{\partial \phi_j}{\partial y} \right\rangle x_j^2 \frac{\partial^2}{\partial x_j^2} \right]
$$

where the ϕ_i's are given by the Poisson equations w.r.t. y:

$$
\mathcal{L}_0 \phi_i(y, z) = f_i^2(y, z) - \langle f_i^2(\cdot, z) \rangle.
$$

Then use $u_0(t, x_1, \cdots, x_n) = \prod_{i=1}^{n} Q_i(t, x_i)$.

Correction Term $\sqrt{\delta}\,\mathbf{u_{0,1}}$

$$
\begin{aligned}
\langle \mathcal{L}_2 \rangle u_{0,1} &= -\langle \mathcal{M}_1 \rangle u_0, \quad x_i > B_i(t), \text{for all } i,\ t < T \\
u_{0,1}(t, x_1, x_2, \ldots, x_n) &= 0, \quad \exists i \in \{1, \cdots, n\}, x_i = B_i(t), t \leq T, \\
u_{0,1}(T, x_1, x_2, \ldots, x_n) &= 0, \quad x_i > B_i(t), \text{ for all } i.
\end{aligned}
$$

$$
\langle \mathcal{M}_1 \rangle = \nu_Z \sqrt{2} \left[\sum_{i=1}^{n} \rho_{iZ} \langle f(\cdot, z) \rangle x_i \frac{\partial^2}{\partial x_i \partial z} - \langle \Lambda_2(\cdot, z) \rangle \frac{\partial}{\partial z} \right] =
$$

$$
\nu_Z \sqrt{2} \left[\sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{iZ} \langle f(\cdot, z) \rangle \sigma_j'(z) x_i \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial \sigma_j} \right) - \langle \Lambda_2(\cdot, z) \rangle \sum_{i=1}^{n} \sigma_i'(z) \frac{\partial}{\partial \sigma_i} \right]
$$

Then use $u_0(t, x_1, \cdots, x_n) = \prod_{i=1}^{n} Q_i(t, x_i)$.

Homogeneous Portfolio Case

$$
\begin{aligned}
\mathbf{u_0}(t,x,\cdots,x) &= \prod_{i=1}^{n} Q_i(t,x) = Q(t,x)^n \equiv \mathbf{q^n} \\
\sqrt{\varepsilon}\,\mathbf{u_{1,0}} &= n\left(R_1^{(2)} w_1^{(2)}(t,x) + R_1^{(3)} w_1^{(3)}(t,x)\right) q^{n-1} \\
&\quad + n(n-1) R_{12}^{(3)} w_{12}^{(3)}(t,x,x) q^{n-2} \\
\sqrt{\delta}\,\mathbf{u_{0,1}} &= n\left(R_1^{(0)} w_1^{(0)}(t,x) + R_1^{(1)} w_1^{(1)}(t,x)\right) q^{n-1} \\
&\quad + n(n-1) R_{12}^{(1)} w_{12}^{(1)}(t,x,x) q^{n-2}
\end{aligned}
$$

Joint survival probabilities: $S_n \approx \tilde{u} \equiv u_0 + \sqrt{\epsilon}\, u_{1,0} + \sqrt{\delta}\, u_{0,1}$

$$
= \mathbf{q^n} + \mathbf{Anq^{n-1}} + \mathbf{Bn(n-1)q^{n-2}}
$$

$$
A = R_1^{(0)} w_1^{(0)}(t,x) + R_1^{(1)} w_1^{(1)}(t,x) + R_1^{(2)} w_1^{(2)}(t,x) + R_1^{(3)} w_1^{(3)}(t,x)
$$

$$
B = R_{12}^{(1)} w_{12}^{(1)}(t,x,x) + R_{12}^{(3)} w_{12}^{(3)}(t,x,x)
$$

Loss Distribution

For N names **perfectly symmetric**, if L is the **number of defaults by time T**, then

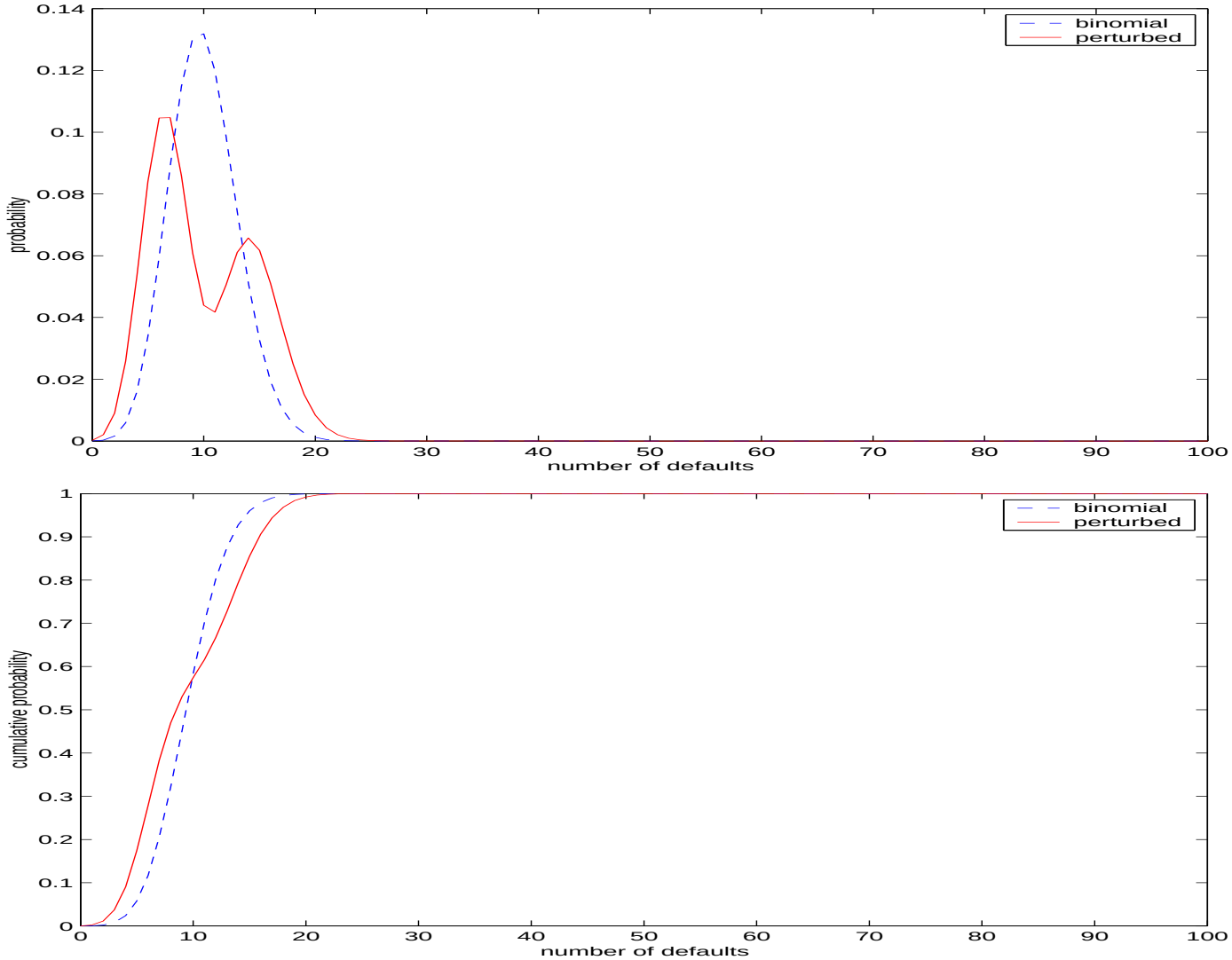
$$
\begin{aligned}
\mathbb{P}^\star(L=k) &= \binom{N}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j S_{N+j-k} \\
&\approx \binom{N}{k}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i} q^{N-i} \\
&\quad + A\binom{N}{k}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}(N-i)q^{N-i-1} \\
&\quad + B\binom{N}{k}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}(N-i)(N-i-1)q^{N-i-2} \\
&\equiv I_0 + AI_1 + BI_2
\end{aligned}
$$

Loss Distribution Formulas

$$\mathbb{P}^\star(L = k) \quad \approx \quad I_0 + A I_1 + B I_2$$

with

$$\begin{aligned}
I_0 &= \binom{N}{k}(1-q)^k q^{N-k} \\
I_1 &= \left[\frac{N-k}{q} - \frac{k}{1-q}\right] I_0 \\
I_2 &= \left[\frac{(N-k)(N-k-1)}{q^2} - \frac{2k(N-k)}{q(1-q)} + \frac{k(k-1)}{(1-q)^2}\right] I_0
\end{aligned}$$



$$N = 100, \quad q = 0.9, \quad A = 0.00, \quad B = 0.0006$$

Models with Name-Name Correlation

$$\mathrm{d}\langle W^{(i)}, W^{(j)}\rangle_t = \rho_{ij}\,dt, \qquad |\rho_{ij}| < 1 \text{ for } i \neq j$$

$$\mathcal{L}^{\epsilon,\delta,\rho} = \mathcal{L}^{\epsilon,\delta} + \sum_{i<j}^{n} \rho_{ij}\mathcal{L}_{\rho}^{(ij)}$$

$$\text{with} \quad \mathcal{L}_{\rho}^{(ij)} = f_i(y,z)f_j(y,z)x_i x_j \frac{\partial^2}{\partial x_i \partial x_j}$$

Expand

$$u^{\epsilon,\delta,\rho} = u^{\epsilon,\delta} + \sum_{i<j}^{n} \rho_{ij}\left(u_{0,0,1}^{(ij)} + \sqrt{\epsilon}u_{1,0,1}^{(ij)} + \sqrt{\delta}u_{0,1,1}^{(ij)} + \cdots\right) + \cdots$$

and retain the first corrections,

$$\tilde{u} \equiv u_0 + \sqrt{\epsilon}\,u_{1,0} + \sqrt{\delta}\,u_{0,1} + \sum_{i<j}^{n} \rho_{ij}\,u_{0,0,1}^{(ij)}$$

Correction Terms $\rho_{ij}\, u^{(ij)}_{0,0,1}$

$$
\begin{aligned}
\langle \mathcal{L}_2 \rangle u^{(ij)}_{0,0,1} &= -\langle \mathcal{L}^{(ij)}_{\rho} \rangle u_0, \quad x_l > B_l(t), \text{ for all } l, t < T \\
u^{(ij)}_{0,0,1}(t, x_1, x_2, \ldots, x_n) &= 0, \quad \exists l \in \{1, \cdots, n\}, x_l = B_l(t), t \leq T \\
u^{(ij)}_{0,0,1}(T, x_1, x_2, \ldots, x_n) &= 0, \quad x_l > B_l(t), \text{ for all } l
\end{aligned}
$$

where

$$
\langle \mathcal{L}^{(ij)}_{\rho} \rangle = \langle f_i(\cdot, z) f_j(\cdot, z) \rangle x_i x_j \frac{\partial^2}{\partial x_i \partial x_j}
$$

Then use $u_0(t, x_1, \cdots, x_n) = \prod_{i=1}^{n} Q_i(t, x_i)$ to deduce

$$
\begin{aligned}
\rho_{ij}\, u^{(ij)}_{0,0,1} &= R^{(4)}_{ij} w^{(4)}_{ij} \prod_{\substack{k=1 \\ k \neq i,j}}^{n} Q_k \\
R^{(4)}_{ij} &= \rho_{ij} \langle f_i(\cdot, z) f_j(\cdot, z) \rangle, \; i \neq j
\end{aligned}
$$

Homogeneous Portfolio Case $\rho_{ij} = \rho$

$$u_0(t, x, \cdots, x) \quad = \quad \prod_{i=1}^{n} Q_i(t, x) = Q(t, x)^n \equiv q^n$$

Joint survival probabilities:

$$\begin{aligned} S_n \approx \tilde{u} \quad &\equiv \quad u_0 + \sqrt{\epsilon}\, u_{1,0} + \sqrt{\delta}\, u_{0,1} + \sum_{i<j}^{n} \rho_{ij}\, u_{0,0,1}^{(ij)} \\ &= \quad q^n + Anq^{n-1} + (B + B_\rho) n(n-1) q^{n-2} \end{aligned}$$

$$\begin{aligned} A \quad &= \quad R_1^{(0)} w_1^{(0)}(t, x) + R_1^{(1)} w_1^{(1)}(t, x) + R_1^{(2)} w_1^{(2)}(t, x) + R_1^{(3)} w_1^{(3)}(t, x) \\ B \quad &= \quad R_{12}^{(1)} w_{12}^{(1)}(t, x, x) + R_{12}^{(3)} w_{12}^{(3)}(t, x, x) \\ B_\rho \quad &= \quad \frac{1}{2} R_{12}^{(4)} w_{12}^{(4)}(t, x, x), \qquad R_{12}^{(4)} = \rho\sigma^2(z) \end{aligned}$$

Comparison of the Two Sources of Correlation

For a single maturity T:

the correlations generated by stochastic volatility and name-name correlation are of the same form to leading order.

Term structure of correlation across several maturities:

the shape of the function $w_{12}^{(4)}$ is different from the shapes of $w_{12}^{(1)}$ and $w_{12}^{(3)}$ and therefore the nature of the correlation plays a role.