Mean-Reverting Stochastic Volatility

Hedging Startegies

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Hedging Strategies

A perfect hedge is not possible by trading in the underlying asset only.

Goal: find an acceptable tradeoff between the risk of a failed hedge and the cost of implementing the hedge.

Performance: measured by the subjective probability $\mathbb{P}$

Black-Scholes Delta Hedging

If the risky asset price is a geometric Brownian motion $\bar{X}_t$ with constant parameters $(\mu, \bar{\sigma})$, a short position in a European derivative which pays $h(\bar{X}_T)$ can be **perfectly hedged** by managing the **self-financing** porfolio made of, at time t, the **Delta**

$$\frac{\partial P_0}{\partial x}(t, \bar{X}_t)$$

units of the risky asset, and

$$e^{-rt}\left(P_0(t, \bar{X}_t) - \bar{X}_t \frac{\partial P_0}{\partial x}(t, \bar{X}_t)\right)$$

units of the riskless asset because

$$d\left(P_0(t, \bar{X}_t)\right) = \frac{\partial P_0}{\partial x}(t, \bar{X}_t) d\bar{X}_t + r\left(P_0(t, \bar{X}_t) - \bar{X}_t \frac{\partial P_0}{\partial x}(t, \bar{X}_t)\right) dt$$

and the Black-Scholes equation satisfied by $P_0(t, x)$.

The Strategy and its Cost

Use the same strategy under stochastic volatility $(X_t, \sigma_t = Y_t)$:

$$a_t = \frac{\partial P_0}{\partial x}(t, X_t) \quad \text{stocks}$$

$$b_t = e^{-rt}\left(P_0(t, X_t) - X_t \frac{\partial P_0}{\partial x}(t, X_t)\right) \quad \text{bonds}$$

Its **value** is $a_t X_t + b_t e^{rt} = P_0(t, X_t)$ and $P_0(T, X_T) = h(X_T)$

Infinitesimal cost of the strategy:

$$dP_0(t, X_t) - a_t dX_t - r b_t e^{rt} dt = \frac{1}{2}\left(f(Y_t)^2 - \bar{\sigma}^2\right) X_t^2 \frac{\partial^2 P_0}{\partial x^2}(t, X_t) dt$$

Cumulative cost:

$$E_0(t) = \frac{1}{2}\int_0^t \left(f(Y_s)^2 - \bar{\sigma}^2\right) X_s^2 \frac{\partial^2 P_0}{\partial x^2}(s, X_s) ds$$

Averaging Effect

α large $\Longrightarrow$

$$E_0(t) = \frac{1}{2}\int_0^t \left(f(Y_s)^2 - \bar{\sigma}^2\right) X_s^2 \frac{\partial^2 P_0}{\partial x^2}(s, X_s)ds \quad \text{small}$$

because $\bar{\sigma}^2 = \langle f^2 \rangle$ and the **centering effect**.

One has

$$E_0(t) = \frac{1}{\sqrt{\alpha}}(B_t + M_t) + \mathcal{O}(1/\alpha)$$

B_t is a **systematic bias** and (M_t) a **mean-zero martingale**

Averaging Effect: some details

$$f(Y_s)^2 - \bar{\sigma}^2 = (\mathcal{L}_0\phi)(Y_s)$$

$$(\mathcal{L}_0\phi)(Y_s)ds = \frac{1}{\alpha}\left\{d(\phi(Y_s)) - \nu\sqrt{2\alpha}\,\phi'(Y_s)d\hat{Z}_s\right\}$$

$$E_0(t) = \frac{1}{2\alpha}\int_0^t X_s^2\frac{\partial^2 P_0}{\partial x^2}(s, X_s)\left\{d(\phi(Y_s)) - \nu\sqrt{2\alpha}\,\phi'(Y_s)d\hat{Z}_s\right\}$$

$$\begin{aligned}
= \quad & \frac{1}{2\alpha}\{X_t^2\frac{\partial^2 P_0}{\partial x^2}(t, X_t)\phi(Y_t) - X_0^2\frac{\partial^2 P_0}{\partial x^2}(0, X_0)\phi(Y_0) \\
& - \int_0^t \phi(Y_s)d\left(X_s^2\frac{\partial^2 P_0}{\partial x^2}\right)\} \\
& - \frac{\rho\nu}{\sqrt{2\alpha}}\int_0^t f(Y_s)\phi'(Y_s)\left(2X_s^2\frac{\partial^2 P_0}{\partial x^2} + X_s^3\frac{\partial^3 P_0}{\partial x^3}\right)ds \\
& - \frac{\nu}{\sqrt{2\alpha}}\int_0^t X_s^2\frac{\partial^2 P_0}{\partial x^2}\phi'(Y_s)d\hat{Z}_s
\end{aligned}$$

Mean Self-Financing Hedging Strategy

Correct the hedging strategy:

$$a_t = \frac{\partial \left(P_0 + \widetilde{Q_1} \right)}{\partial x}(t, X_t)$$

$$b_t = e^{-rt} \left(P_0(t, X_t) + \widetilde{Q_1}(t, X_t) - X_t \frac{\partial \left(P_0 + \widetilde{Q_1} \right)}{\partial x}(t, X_t) \right)$$

where $\widetilde{Q_1}$ satisfies

$$\mathcal{L}_{BS}(\bar{\sigma})\widetilde{Q_1} = -V_3 \left(x \frac{\partial}{\partial x} \left(x^2 \frac{\partial^2 P_0}{\partial x^2} \right) \right) \quad , \qquad \widetilde{Q_1}(T, x) = 0$$

or given explicitly by

$$\widetilde{Q_1}(t, x) = (T - t) V_3 \left(x \frac{\partial}{\partial x} \left(x^2 \frac{\partial^2 P_0}{\partial x^2} \right) \right)$$

To summarize

Using the **corrected ratio**

$$a_t = \frac{\partial P_0}{\partial x} + \frac{V_3(T-t)}{x}\left(4x^2\frac{\partial^2 P_0}{\partial x^2} + 5x^3\frac{\partial^3 P_0}{\partial x^3} + x^4\frac{\partial^4 P_0}{\partial x^4}\right)$$

the **bias in the total cost is centered**:

$$\frac{\rho\nu}{\sqrt{2\alpha}}\int_0^T \left[2X_t^2\frac{\partial^2 P_0}{\partial x^2} + X_t^3\frac{\partial^3 P_0}{\partial x^3}\right]\left(\langle f\phi'\rangle - f\phi(Y_t)\right)dt$$

and

$$E_0(T) = \frac{1}{\sqrt{\alpha}}(B_T + M_T) + \mathcal{O}(1/\alpha)$$

becomes

$$E_1^Q(T) = -\frac{\nu}{\sqrt{2\alpha}}\int_0^T X_s^2\frac{\partial^2 P_0}{\partial x^2}\phi'(Y_s)d\hat{Z}_s + \mathcal{O}(1/\alpha)$$

Only $\bar{\sigma}$ and V_3 (from the skew) are needed

An Alternative: staying close to the price

$$a_t = \frac{\partial \left(P_0 + \widetilde{P_1}\right)}{\partial x}(t, X_t)$$

$$b_t = e^{-rt}\left((P_0 + \widetilde{P_1})(t, X_t) - X_t \frac{\partial \left(P_0 + \widetilde{P_1}\right)}{\partial x}(t, X_t)\right)$$

with the correction

$$\widetilde{P_1} = (T-t)\left(V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_0}{\partial x^2}\right)\right)$$

For this strategy, the **hedging ratio** is given by

$$a_t = \frac{\partial P_0}{\partial x} + \frac{(T-t)}{x}\left((2V_2 + 4V_3)x^2 \frac{\partial^2 P_0}{\partial x^2} + (V_2 + 5V_3)x^3 \frac{\partial^3 P_0}{\partial x^3} + V_3 x^4 \frac{\partial^4 P_0}{\partial x^4}\right)$$

At any time

$$|P - \left(P_0 + \widetilde{P_1}\right)| = \mathcal{O}(1/\alpha)$$

but, up to order $1/\alpha$, the total cost becomes

$$E_1^P(T) = \int_0^T V_2 X_t^2 \frac{\partial^2 P_0}{\partial x^2} dt - \frac{\nu}{\sqrt{2\alpha}} \int_0^T X_s^2 \frac{\partial^2 P_0}{\partial x^2} \phi'(Y_s) d\hat{Z}_s$$

with

$$V_2 = \nu \langle \Lambda \phi' \rangle / \sqrt{2\alpha}$$

which reflects the **volatility risk premium**.