Mean-Reverting Stochastic Volatility
Hedging Strategies

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Hedging Strategies

A perfect hedge is not possible by trading in the underlying asset only.

**Goal:** find an acceptable tradeoff between the risk of a failed hedge and the cost of implementing the hedge.

*Performance: measured by the subjective probability IP*
Black-Scholes Delta Hedging

If the risky asset price is a geometric Brownian motion \( \tilde{X}_t \) with constant parameters \((\mu, \tilde{\sigma})\), a short position in a European derivative which pays \( h(\tilde{X}_T) \) can be perfectly hedged by managing the self-financing portfolio made of, at time \( t \), the Delta

\[
\frac{\partial P_0}{\partial x}(t, \tilde{X}_t)
\]

units of the risky asset, and

\[
e^{-rt} \left( P_0(t, \tilde{X}_t) - \tilde{X}_t \frac{\partial P_0}{\partial x}(t, \tilde{X}_t) \right)
\]

units of the riskless asset because

\[
d \left( P_0(t, \tilde{X}_t) \right) = \frac{\partial P_0}{\partial x}(t, \tilde{X}_t) d\tilde{X}_t + r \left( P_0(t, \tilde{X}_t) - \tilde{X}_t \frac{\partial P_0}{\partial x}(t, \tilde{X}_t) \right) dt
\]

and the Black-Scholes equation satisfied by \( P_0(t, x) \).
The Strategy and its Cost

Use the same strategy under stochastic volatility \((X_t, \sigma_t = Y_t)\):

\[
a_t = \frac{\partial P_0}{\partial x}(t, X_t) \quad \text{stocks}
\]

\[
b_t = e^{-rt} \left( P_0(t, X_t) - X_t \frac{\partial P_0}{\partial x}(t, X_t) \right) \quad \text{bonds}
\]

Its value is \(a_t X_t + b_t e^{rt} = P_0(t, X_t)\) and \(P_0(T, X_T) = h(X_T)\)

**Infinitesimal cost of the strategy:**

\[
dP_0(t, X_t) - a_t dX_t - rb_t e^{rt} dt = \frac{1}{2} \left( f(Y_t)^2 - \bar{\sigma}^2 \right) X_t^2 \frac{\partial^2 P_0}{\partial x^2}(t, X_t) dt
\]

**Cumulative cost:**

\[
E_0(t) = \frac{1}{2} \int_0^t \left( f(Y_s)^2 - \bar{\sigma}^2 \right) X_s^2 \frac{\partial^2 P_0}{\partial x^2}(s, X_s) ds
\]
Averaging Effect

\( \alpha \) large \( \implies \)

\[
E_0(t) = \frac{1}{2} \int_0^t \left( (f(Y_s))^2 - \bar{\sigma}^2 \right) X_s^2 \frac{\partial^2 P_0}{\partial x^2}(s, X_s)ds \quad \text{small}
\]

because \( \bar{\sigma}^2 = \langle f^2 \rangle \) and the centering effect.

One has

\[
E_0(t) = \frac{1}{\sqrt{\alpha}} (B_t + M_t) + O(1/\alpha)
\]

\( B_t \) is a systematic bias and \( (M_t) \) a mean-zero martingale
Averaging Effect: some details

\[ f(Y_s)^2 - \bar{\sigma}^2 = (\mathcal{L}_0 \phi)(Y_s) \]

\[ (\mathcal{L}_0 \phi)(Y_s) ds = \frac{1}{\alpha} \left\{ d(\phi(Y_s)) - \nu \sqrt{2\alpha} \phi'(Y_s) d\hat{Z}_s \right\} \]

\[ E_0(t) = \frac{1}{2\alpha} \int_0^t X_s^2 \frac{\partial^2 P_0}{\partial x^2}(s, X_s) \left\{ d(\phi(Y_s)) - \nu \sqrt{2\alpha} \phi'(Y_s) d\hat{Z}_s \right\} \]

\[ = \frac{1}{2\alpha} \left\{ X_t^2 \frac{\partial^2 P_0}{\partial x^2}(t, X_t) \phi(Y_t) - X_0^2 \frac{\partial^2 P_0}{\partial x^2}(0, X_0) \phi(Y_0) \right. \]

\[ - \int_0^t \phi(Y_s) d \left( X_s^2 \frac{\partial^2 P_0}{\partial x^2} \right) \}

\[ - \frac{\rho \nu}{\sqrt{2\alpha}} \int_0^t f(Y_s) \phi'(Y_s) \left( 2X_s^2 \frac{\partial^2 P_0}{\partial x^2} + X_s^3 \frac{\partial^3 P_0}{\partial x^3} \right) ds \]

\[ - \frac{\nu}{\sqrt{2\alpha}} \int_0^t X_s^2 \frac{\partial^2 P_0}{\partial x^2} \phi'(Y_s) d\hat{Z}_s \]
Mean Self-Financing Hedging Strategy

Correct the hedging strategy:

\[ a_t = \frac{\partial \left( P_0 + \tilde{Q}_1 \right)}{\partial x} (t, X_t) \]

\[ b_t = e^{-rt} \left( P_0(t, X_t) + \tilde{Q}_1(t, X_t) - X_t \frac{\partial \left( P_0 + \tilde{Q}_1 \right)}{\partial x} (t, X_t) \right) \]

where \( \tilde{Q}_1 \) satisfies

\[ \mathcal{L}_{BS}(\bar{\sigma})\tilde{Q}_1 = -V_3 \left( x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_0}{\partial x^2} \right) \right) \quad , \quad \tilde{Q}_1(T, x) = 0 \]

or given explicitly by

\[ \tilde{Q}_1(t, x) = (T - t)V_3 \left( x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_0}{\partial x^2} \right) \right) \]
To summarize

Using the corrected ratio
\[
a_t = \frac{\partial P_0}{\partial x} + \frac{V_3(T-t)}{x} \left(4x^2 \frac{\partial^2 P_0}{\partial x^2} + 5x^3 \frac{\partial^3 P_0}{\partial x^3} + x^4 \frac{\partial^4 P_0}{\partial x^4}\right)
\]

the bias in the total cost is centered:
\[
\frac{\rho \nu}{\sqrt{2\alpha}} \int_0^T \left[2X_t^2 \frac{\partial^2 P_0}{\partial x^2} + X_t^3 \frac{\partial^3 P_0}{\partial x^3}\right] (\langle f \phi' \rangle - f \phi(Y_t)) \, dt
\]

and
\[
E_0(T) = \frac{1}{\sqrt{\alpha}} (B_T + M_T) + \mathcal{O}(1/\alpha)
\]

becomes
\[
E_1^Q(T) = -\frac{\nu}{\sqrt{2\alpha}} \int_0^T X_s^2 \frac{\partial^2 P_0}{\partial x^2} \phi'(Y_s) d\hat{Z}_s + \mathcal{O}(1/\alpha)
\]

Only $\bar{\sigma}$ and $V_3$ (from the skew) are needed
An Alternative: staying close to the price

\[ a_t = \frac{\partial}{\partial x} \left( P_0 + \tilde{P}_1 \right)(t, X_t) \]

\[ b_t = e^{-rt} \left( (P_0 + \tilde{P}_1)(t, X_t) - X_t \frac{\partial}{\partial x} \left( P_0 + \tilde{P}_1 \right)(t, X_t) \right) \]

with the correction

\[ \tilde{P}_1 = (T - t) \left( V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_0}{\partial x^2} \right) \right) \]

For this strategy, the hedging ratio is given by

\[ a_t = \frac{\partial P_0}{\partial x} + \frac{(T - t)}{x} \left( (2V_2 + 4V_3)x^2 \frac{\partial^2 P_0}{\partial x^2} + (V_2 + 5V_3)x^3 \frac{\partial^3 P_0}{\partial x^3} + V_3 x^4 \frac{\partial^4 P_0}{\partial x^4} \right) \]
At any time

$$|P - (P_0 + \widetilde{P}_1)| = O(1/\alpha)$$

but, up to order $1/\alpha$, the total cost becomes

$$E_1^P(T) = \int_0^T V_2 X_t^2 \frac{\partial^2 P_0}{\partial x^2} dt - \frac{\nu}{\sqrt{2\alpha}} \int_0^T X_s^2 \frac{\partial^2 P_0}{\partial x^2} \phi'(Y_s) d\hat{Z}_s$$

with

$$V_2 = \nu \langle \Lambda \phi' \rangle / \sqrt{2\alpha}$$

which reflects the volatility risk premium.