Calibration to Implied Volatility Data

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Calibration Formulas

The implied volatility is an **affine function** of the LMMR:

\[
\text{log-moneyness-to-maturity-ratio} = \log\left(\frac{K}{x}\right)/(T - t)
\]

\[
I = a \left[\text{LMMR}\right] + b + O\left(\frac{1}{\alpha}\right)
\]

with

\[
a = \frac{V_3}{\bar{\sigma}^3}
\]

\[
b = \bar{\sigma} + \frac{V_2}{\bar{\sigma}} - \frac{V_3}{\bar{\sigma}^3} \left( r - \frac{\bar{\sigma}^2}{2} \right)
\]

or for **calibration purpose**:

\[
V_2 = \bar{\sigma} \left( (b - \bar{\sigma}) + a \left( r - \frac{\bar{\sigma}^2}{2} \right) \right)
\]

\[
V_3 = a\bar{\sigma}^3
\]
In sample fit of implied volatilities

\[ I \approx a \text{[LMMR]} + b \]

(Maturities less than 6 months)
A slow volatility factor is needed

Implied volatility as a function of LMMR. The circles are from S&P 500 data, and the line $a(LMMR) + b$ shows the fit using maturities up to two years.
Two-Scale Stochastic Volatility Models

\( \varepsilon << T << 1/\delta \)

\[
dX_t = rX_t dt + f(Y_t, Z_t)X_t dW_t^{(0)*} \\
dY_t = \left( \frac{1}{\varepsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(Y_t, Z_t) \right) dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} dW_t^{(1)*} \\
dZ_t = \left( \delta c(Z_t) - \sqrt{\delta}g(Z_t)\Gamma(Y_t, Z_t) \right) dt + \sqrt{\delta} g(Z_t) dW_t^{(2)*} \\
\]

\[
d < W^{(0)*}, W^{(1)*} >_t = \rho_1 dt \\
d < W^{(0)*}, W^{(2)*} >_t = \rho_2 dt
\]
Pricing Equation

\[ P^{\varepsilon,\delta}(t, x, y, z) = \mathbb{E}^* \left\{ e^{-r(T-t)} h(X_T) \middle| X_t = x, Y_t = y, Z_t = z \right\} \]

\[ \left( \frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta} \mathcal{M}_1 + \delta \mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}} \mathcal{M}_3 \right) P^{\varepsilon,\delta} = 0 \]

\[ P^{\varepsilon,\delta}(T, x, y, z) = h(x) \]

\[
\begin{align*}
\mathcal{L}_0 &= (m - y) \frac{\partial}{\partial y} + \nu^2 \frac{\partial^2}{\partial y^2} \\
\mathcal{L}_1 &= \nu \sqrt{2} \left( \rho_1 f x \frac{\partial^2}{\partial x \partial y} - \Lambda \frac{\partial}{\partial y} \right) \\
\mathcal{L}_2 &= \frac{\partial}{\partial t} + \frac{1}{2} f^2 x^2 \frac{\partial^2}{\partial x^2} + r \left( x \frac{\partial}{\partial x} - \cdot \right) \\
\mathcal{M}_1 &= -g\Gamma \frac{\partial}{\partial z} + \rho_2 g f x \frac{\partial^2}{\partial x \partial z} \\
\mathcal{M}_2 &= c \frac{\partial}{\partial z} + \frac{g^2}{2} \frac{\partial^2}{\partial z^2} \\
\mathcal{M}_3 &= \nu \sqrt{2} \tilde{\rho}_{12} g \frac{\partial^2}{\partial y \partial z}
\end{align*}
\]
Double Expansion

\[ P^{\varepsilon, \delta} = P_0 + \sqrt{\varepsilon} P_{1,0} + \sqrt{\delta} P_{0,1} + \cdots \]
\[ = P_0 + \tilde{P}_1 + \tilde{Q}_1 + \cdots \]

Leading order term: \( P_0(t, x, z) = P_{BS}(t, x; \bar{\sigma}(z)) \)

Correction: \( \tilde{P}_1 = \sqrt{\varepsilon} P_{1,0} \) with \( V_2^\varepsilon, V_3^\varepsilon \) (z-dependent):

\[ \mathcal{L}_{BS}(\bar{\sigma}) \tilde{P}_1 + \left( V_2^\varepsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_{BS}}{\partial x^2} \right) \right) = 0 \]

\[ \tilde{P}_1(T, x, z) = 0 \]

\[ \tilde{P}_1(t, x, z) = (T - t) \left( V_2^\varepsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_{BS}}{\partial x^2} \right) \right) \]
Price Approximation:

\[ P^{\epsilon,\delta}(t, x, y, z) \approx P_{BS}(t, x; T, \bar{\sigma}) \]

\[ + (T - t) \left( V_2^\epsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\epsilon x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_{BS}}{\partial x^2} \right) \right) \]

\[ + (T - t) \left( V_0^\delta \frac{\partial P_{BS}}{\partial \sigma} + V_1^\delta x \frac{\partial^2 P_{BS}}{\partial x \partial \sigma} \right) \]

\[ \mathcal{L}_{BS}(\bar{\sigma}) \tilde{Q}_1 + 2 \left( V_0^\delta \frac{\partial P_{BS}}{\partial \sigma} + V_1^\delta x \frac{\partial^2 P_{BS}}{\partial x \partial \sigma} \right) = 0 \]

\[ \tilde{Q}_1(T, x) = 0 \]

KEY:

\[ \frac{\partial P_{BS}}{\partial \sigma} = (T - t)\sigma x^2 \frac{\partial^2 P_{BS}}{\partial x^2} \]
Term Structure of Implied Volatility

\[ I_0 + I_1^\varepsilon + I_1^\delta = \]
\[ \bar{\sigma} + [b^\varepsilon + b^\delta (T - t)] + [a^\varepsilon + a^\delta (T - t)] \frac{\log(K/x)}{T - t}, \]

where the parameters \((\bar{\sigma}, a^\varepsilon, a^\delta, b^\varepsilon, b^\delta)\) depend on \(z\) and are related to the group parameters \((V_0^\delta, V_1^\delta, V_2^\varepsilon, V_3^\varepsilon)\) by

\[ a^\varepsilon = \frac{V_3^\varepsilon}{\bar{\sigma}^3}, \quad b^\varepsilon = \frac{V_2^\varepsilon}{\bar{\sigma}} - \frac{V_3^\varepsilon}{\bar{\sigma}^3} (r - \frac{\bar{\sigma}^2}{2}) \]
\[ a^\delta = \frac{V_1^\delta}{\bar{\sigma}^2}, \quad b^\delta = V_0^\delta - \frac{V_1^\delta}{\bar{\sigma}^2} (r - \frac{\bar{\sigma}^2}{2}) \]
Term-structures fits
δ-adjusted implied volatility \( I - b^\delta \tau - a^\delta (LM) \) as a function of LMMR. The circles are from S&P 500 data, and the line \( R + a^\varepsilon (LMMR) \) shows the fit using the estimated parameters.
A slow volatility factor is needed

Implied volatility as a function of LMMR. The circles are from S&P 500 data, and the line $a(LMMR) + b$ shows the fit using maturities up to two years.
A fast volatility factor is needed

The circles are from S&P 500 data, and the line $a^\delta(LM) + \bar{\sigma}$ shows the fit using the estimated parameters from only a slow factor fit.
Figure 1: S&P 500 Implied Volatility data on June 5, 2003 and fits to the affine LMMR approximation for six different maturities.
Figure 2: S&P 500 Implied Volatility data on June 5, 2003 and fits to the two-scales asymptotic theory. The bottom (resp. top) figure shows the linear regression of $b$ (resp. $a$) with respect to time to maturity $\tau = T - t$. 
Higher order terms in $\varepsilon$, $\delta$ and $\sqrt{\varepsilon \delta}$

$$I \approx \sum_{j=0}^{4} a_j(\tau) (LM)^j + \frac{1}{\tau} \Phi_t,$$

where

$\tau$ denotes the time-to maturity $T - t$,

LM denotes the moneyness log($K/S$),

and $\Phi_t$ is a rapidly changing component that varies with the fast volatility factor.
Figure 3: S&P 500 Implied Volatility data on June 5, 2003 and quartic fits to the asymptotic theory for four maturities.
Figure 4: S&P 500 Term-Structure Fit using second order approximation. Data from June 5, 2003.
Figure 5: *S&P 500 Term-Structure Fit. Data from every trading day in May 2003.*
Parameter Reduction and Direct Calibration

\[ \mathcal{L}_{BS}(\bar{\sigma}) \left( \tilde{P}_1 + \tilde{Q}_1 \right) + \left( V_2 x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3 x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_{BS}}{\partial x^2} \right) \right) + 2 \left( V_0 \frac{\partial P_{BS}}{\partial \sigma} + V_1 x \frac{\partial^2 P_{BS}}{\partial x \partial \sigma} \right) = 0 \]

Set \( \sigma^* = \sqrt{\bar{\sigma}^2 + 2V_2} \). At the same order, the correction is:

\[ (T - t) \left( V_0 \frac{\partial P^*_{BS}}{\partial \sigma} + V_1 x \frac{\partial^2 P^*_{BS}}{\partial x \partial \sigma} + V_3 x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P^*_{BS}}{\partial x^2} \right) \right) \]

\[ I \approx b^* + \tau b^\delta + \left( a^\varepsilon + \tau a^\delta \right) \text{ LMMR} \]

\[ b^* = \sigma^* + \frac{V_3}{2\sigma^*} \left( 1 - \frac{2r}{\sigma^*} \right) \quad , \quad a^\varepsilon = \frac{V_3}{\sigma^*^3} \]

\[ b^\delta = V_0 + \frac{V_1}{2} \left( 1 - \frac{2r}{\sigma^*} \right) \quad , \quad a^\delta = \frac{V_1}{\sigma^*^2} \]
**Exotic Derivatives** (Binary, Barrier, Asian,...)

- Calibrate $\sigma^*$, $V_0$, $V_1$ and $V_3$ on the implied volatility surface
- Solve the corresponding problem with **constant volatility** $\sigma^*$
  
  \[ \implies P_0 = P_{BS}(\sigma^*) \]

- Use $V_0$, $V_1$ and $V_3$ to compute the source
  
  \[ 2 \left( V_0 \frac{\partial P_{BS}^*}{\partial \sigma} + V_1 x \frac{\partial^2 P_{BS}^*}{\partial x \partial \sigma} \right) + V_3 x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_{BS}^*}{\partial x^2} \right) \]

- Get the correction by solving the **SAME PROBLEM** with zero boundary conditions and the source.
American Options

- Calibrate $\sigma^*$, $V_0$, $V_1$ and $V_3$ on the implied volatility surface
- Solve the corresponding problem with constant volatility $\sigma^*$

$$ \implies P^* \text{ and the free boundary } x^*(t) $$

- Use $V_0$, $V_1$ and $V_3$ to compute the source

$$ 2 \left( V_0 \frac{\partial P^*}{\partial \sigma} + V_1 x \frac{\partial^2 P^*}{\partial x \partial \sigma} \right) + V_3 x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P^*}{\partial x^2} \right) $$

- Get the correction by solving the corresponding problem with fixed boundary $x^*(t)$, zero boundary conditions and the source.
Conclusions

- A short time-scale of order few days is present in volatility dynamics
- It cannot be ignored in option pricing and hedging
- It can be dealt with by using singular perturbation methods
- It is efficient as a parametrization tool for the term structure of implied volatilities when combined with a regular perturbation