Fast Mean-Reverting Stochastic Volatility

Asymptotic Analysis

Jean-Pierre Fouque
University of California Santa Barbara

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Model in the risk-neutral world $\mathbb{P}^{\star(\gamma)}$ in terms of $\varepsilon = 1/\alpha$

Set $\beta = \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}$ so that $\nu^2 = \beta^2/2\alpha$

$$\begin{aligned} dX_t^\varepsilon &= rX_t^\varepsilon dt + f(Y_t^\varepsilon)X_t^\varepsilon dW_t^\star \\ dY_t^\varepsilon &= \left[\frac{1}{\varepsilon}(m - Y_t^\varepsilon) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(Y_t^\varepsilon)\right] dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} d\hat{Z}_t^\star \end{aligned}$$

Market price of risks:

$$\Lambda(y) = \rho\frac{(\mu - r)}{f\,(y)} + \gamma(y)\sqrt{1-\rho^2}$$

Skew:

$$\hat{Z}_t^\star = \rho W_t^\star + \sqrt{1-\rho^2}Z_t^\star \quad , \quad |\rho| < 1$$

Prices and Pricing PDE's

$$P^\varepsilon(t,x,y) = \mathbb{E}^{\star(\gamma)}\left\{e^{-r(T-t)}h(X_T^\varepsilon)|X_t^\varepsilon = x, Y_t^\varepsilon = y\right\}$$

$$\frac{\partial P^\varepsilon}{\partial t} + \frac{1}{2}f(y)^2x^2\frac{\partial^2 P^\varepsilon}{\partial x^2} + \frac{\rho\nu\sqrt{2}}{\sqrt{\varepsilon}}xf(y)\frac{\partial^2 P^\varepsilon}{\partial x\partial y} + \frac{\nu^2}{\varepsilon}\frac{\partial^2 P^\varepsilon}{\partial y^2}$$
$$+r\left(x\frac{\partial P^\varepsilon}{\partial x} - P^\varepsilon\right) + \frac{1}{\varepsilon}(m-y)\frac{\partial P^\varepsilon}{\partial y} - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(y)\frac{\partial P^\varepsilon}{\partial y} = 0$$

to be solved for $t < T$ with the **terminal condition**

$$P^\varepsilon(T,x,y) = h(x)$$

Operator Notation

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) P^\varepsilon = 0$$

with

$$\begin{aligned}
\mathcal{L}_0 &= \nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y} \qquad = \mathcal{L}_{OU} \\
\mathcal{L}_1 &= \sqrt{2}\rho\nu x f(y) \frac{\partial^2}{\partial x \partial y} - \sqrt{2}\nu\Lambda(y)\frac{\partial}{\partial y} \\
\mathcal{L}_2 &= \frac{\partial}{\partial t} + \frac{1}{2} f(y)^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right) \qquad = \mathcal{L}_{BS}(f(y))
\end{aligned}$$

Formal Expansion

Expand:

$$P^\varepsilon = P_0 + \sqrt{\varepsilon} P_1 + \varepsilon P_2 + \varepsilon\sqrt{\varepsilon} P_3 + \cdots$$

Compute:

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)\left(P_0 + \sqrt{\varepsilon} P_1 + \varepsilon P_2 + \varepsilon\sqrt{\varepsilon} P_3 + \cdots\right) = 0$$

Group the terms by powers of ε:

$$\begin{aligned}
\frac{1}{\varepsilon}\mathcal{L}_0 P_0 \quad &+ \quad \frac{1}{\sqrt{\varepsilon}}\left(\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0\right) \\
&+ \quad \left(\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0\right) \\
&+ \quad \sqrt{\varepsilon}\left(\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1\right) \\
&+ \quad \cdots = 0
\end{aligned}$$

Diverging terms

- Order $1/\varepsilon$:

$$\mathcal{L}_0 P_0 = 0$$

$\mathcal{L}_0 = \mathcal{L}_{OU}$, acting on y $\quad\Longrightarrow\quad$ $\underline{P_0 = P_0(t,x)}$

with $P_0(T,x) = h(x)$

- Order $1/\sqrt{\varepsilon}$:

$$\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0$$

$\mathcal{L}_1$ takes derivatives w.r.t. y $\quad\Longrightarrow\quad$ $\mathcal{L}_1 P_0 = 0$

$$\Longrightarrow \quad \mathcal{L}_0 P_1 = 0$$

As for P_0 : $\quad \underline{P_1 = P_1(t,x)} \quad$ with $\quad P_1(T,x) = 0$

- **Important observation:**

$P_0 + \sqrt{\varepsilon} P_1$ **does not depend on** y

Zero Order Term

$$\mathcal{L}_0 P_2 + (\mathcal{L}_1 P_1 = 0) + \mathcal{L}_2 P_0 = 0$$

Poisson equation in P_2 with respect to $\mathcal{L}_0$ and the variable y.
Solution:

$$P_2 = (-\mathcal{L}_0)^{-1}(\mathcal{L}_2 P_0)$$

<u>Only if</u> $\mathcal{L}_2 P_0$ is **centered**
with respect to the
invariant distribution of Y.

Poisson Equations

$$\mathcal{L}_0\chi + g = 0$$

Expectations w.r.t. the invariant distribution of the OU process:

$$\langle g \rangle = -\langle \mathcal{L}_0\chi \rangle = -\int (\mathcal{L}_0\chi(y))\Phi(y)dy = \int \chi(y)(\mathcal{L}_0^\star\Phi(y))dy = 0$$

$$\lim_{t\to+\infty} \mathbb{E}\left\{g(Y_t)|Y_0 = y\right\} = \langle g \rangle = 0 \quad \text{(exponentially fast)}$$

$$\chi(y) = \int_0^{+\infty} \mathbb{E}\left\{g(Y_t)|Y_0 = y\right\} dt$$

checked by applying $\mathcal{L}_0$

Leading Order Term

Centering:

$$\langle \mathcal{L}_2 P_0 \rangle = \langle \mathcal{L}_2 \rangle P_0 = 0$$

$$\begin{aligned} \langle \mathcal{L}_2 \rangle &= \left\langle \frac{\partial}{\partial t} + \frac{1}{2} f(y)^2 x^2 \frac{\partial^2}{\partial x^2} + r \left(x \frac{\partial}{\partial x} - \cdot \right) \right\rangle \\ &= \frac{\partial}{\partial t} + \frac{1}{2} \langle f^2 \rangle x^2 \frac{\partial^2}{\partial x^2} + r \left(x \frac{\partial}{\partial x} - \cdot \right) \end{aligned}$$

Effective volatility: $\bar{\sigma}^2 = \left\langle f^2 \right\rangle$

The zero order term $P_0(t, x)$ is the solution of the Black-Scholes equation

$$\mathcal{L}_{BS}(\bar{\sigma}) P_0 = 0$$

with the terminal condition $P_0(T, x) = h(x)$

Back to $P_2(t,x,y)$

The **centering condition** $\langle \mathcal{L}_2 P_0 \rangle = 0$ being satisfied:

$$\mathcal{L}_2 P_0 = \mathcal{L}_2 P_0 - \langle \mathcal{L}_2 P_0 \rangle = \frac{1}{2}\left(f(y)^2 - \bar{\sigma}^2\right) x^2 \frac{\partial^2 P_0}{\partial x^2}$$

$$= \frac{1}{2}\mathcal{L}_0 \phi(y) x^2 \frac{\partial^2 P_0}{\partial x^2}$$

for ϕ a solution of the Poisson equation:

$$\mathcal{L}_0 \phi = f(y)^2 - \langle f^2 \rangle$$

Then

$$P_2(t,x,y) = -\mathcal{L}_0^{-1}\left(\mathcal{L}_2 P_0\right) = -\frac{1}{2}\left(\phi(y) + c(t,x)\right) x^2 \frac{\partial^2 P_0}{\partial x^2}$$

Terms of order $\sqrt{\varepsilon}$

Poisson equation in P_3:

$$\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0$$

Centering condition:

$$\langle \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 \rangle = 0$$

Equation for P_1:

$$\langle \mathcal{L}_2 P_1 \rangle = -\langle \mathcal{L}_1 P_2 \rangle = \frac{1}{2} \left\langle \mathcal{L}_1 \left[(\phi(y) + c(t,x))\, x^2 \frac{\partial^2 P_0}{\partial x^2} \right] \right\rangle$$

P_1 independent of y and $\mathcal{L}_1$ takes derivatives w.r.t. y

$$\Longrightarrow \qquad \mathcal{L}_{BS}(\bar{\sigma}) P_1 = \frac{1}{2} \langle \mathcal{L}_1 \phi(y) \rangle \left[x^2 \frac{\partial^2 P_0}{\partial x^2} \right]$$

with $P_1(T,x) = 0$

The correction $P_1^\varepsilon(t,x) = \sqrt{\varepsilon} P_1(t,x)$

$$\begin{aligned}
\mathcal{L}_{BS}(\bar{\sigma})P_1^\varepsilon - \frac{\nu\sqrt{2\varepsilon}}{2}\left\langle\left(\rho x f(y)\frac{\partial^2}{\partial x \partial y} - \Lambda(y)\frac{\partial}{\partial y}\right)\phi(y)\right\rangle\left[x^2\frac{\partial^2 P_0}{\partial x^2}\right] &= 0 \\
\mathcal{L}_{BS}(\bar{\sigma})P_1^\varepsilon + \left(V_2^\varepsilon x^2\frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 P_{BS}}{\partial x^2}\right)\right) &= 0
\end{aligned}$$

BS equation with source and zero terminal condition

with the two **small parameters** V_2^ε and V_3^ε given by:

$$\begin{aligned}
V_2^\varepsilon &= \frac{\nu}{\sqrt{2\alpha}}\langle\Lambda\phi'\rangle \\
V_3^\varepsilon &= \frac{-\rho\nu}{\sqrt{2\alpha}}\langle f\phi'\rangle
\end{aligned}$$

Recall that $\alpha = 1/\varepsilon$

Explicit Formula for the Corrected Price

$$P_1^\varepsilon = (T-t)\left(V_2^\varepsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_{BS}}{\partial x^2}\right)\right)$$

where V_2^ε and V_3^ε are small numbers of order $\sqrt{\varepsilon}$.

The corrected price is given explicitly by

$$P_0 + (T-t)\left(V_2^\varepsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_{BS}}{\partial x^2}\right)\right)$$

where P_0 is the Black-Scholes price with constant volatility $\bar{\sigma}$

Comments

- The small constants V_2^ε and V_3^ε are complex functions of the original model parameters $(\mu, m, \nu, \rho, \alpha; f)$ and γ
- Only $(\bar{\sigma}, V_2^\varepsilon, V_3^\varepsilon)$ are needed to compute the corrected price
- **Probabilistic representation** of $(P_0 + P_1^\varepsilon)(t, x)$:

$$\bar{\mathbb{E}}\left\{e^{-r(T-t)}h(\bar{X}_T) + \int_t^T e^{-r(s-t)}H(s, \bar{X}_s)ds|\bar{X}_t = x\right\}$$

- **Put-Call Parity** is preserved at the order $\mathcal{O}(\sqrt{\varepsilon})$
- The V_2^ε term is a **volatility level correction**

$$\sigma^\star = \sqrt{\bar{\sigma}^2 + 2V_2^\varepsilon}$$

- The V_3^ε term is the **skew effect**

$$\rho = 0 \Longrightarrow V_3^\varepsilon = 0$$

Accuracy of Approximation

Define

$$Z^\varepsilon = P_0 + \sqrt{\varepsilon} P_1 + \varepsilon P_2 + \varepsilon\sqrt{\varepsilon} P_3 - P$$

so that

$$Z^\varepsilon(T,x,y) = \varepsilon\left(P_2(T,x,y) + \sqrt{\varepsilon} P_3(T,x,y)\right)$$

Using how (P_0, P_1, P_2, P_3) have been chosen to cancel $1/\varepsilon$, $1/\sqrt{\varepsilon}$, $\mathcal{O}(1)$ and $\sqrt{\varepsilon}$ terms deduce

$$\mathcal{L}^\varepsilon Z^\varepsilon = \varepsilon\left(\mathcal{L}_1 P_3 + \mathcal{L}_2 P_2 + \sqrt{\varepsilon}\mathcal{L}_2 P_3\right)$$

and conclude that **source and terminal condition of order** ε

$$\Longrightarrow \qquad Z^\varepsilon = \mathcal{O}(\varepsilon)$$

$$\Longrightarrow \qquad P(t,x,y) = \left(P_0(t,x) + P_1^\varepsilon(t,x)\right) + \mathcal{O}(\varepsilon)$$

Corrected Call Option Prices

$$h(x) = (x - K)^+$$

and

$$P_0(t, x) = C_{BS}(t, x; K, T; \bar{\sigma})$$

Compute the **Delta**, the **Gamma** and the **Delta-Gamma** $= \partial^3 P_0 / \partial x^3$

Deduce the **source**

$$H = \left(V_2^\varepsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x} \left(x^2 \frac{\partial^2 P_{BS}}{\partial x^2} \right) \right)$$

and the **correction**

$$P_1^\varepsilon(t, x) = (T - t) H(t, x) = \frac{x e^{-d_1^2/2}}{\bar{\sigma}\sqrt{2\pi}} \left(-V_3 \frac{d_1}{\bar{\sigma}} + V_2 \sqrt{T - t} \right)$$

Expansion of Implied Volatilities

Recall

$$C_{BS}(t, x; K, T; \mathbf{I}) = C^{\text{observed}}$$

Expand

$$\mathbf{I} = \bar{\sigma} + \sqrt{\varepsilon} I_1 + \cdots$$

Deduce for given (K, T):

$$C_{BS}(t, x; \bar{\sigma}) + \sqrt{\varepsilon} I_1 \frac{\partial C_{BS}}{\partial \sigma}(t, x; \bar{\sigma}) + \cdots = P_0(t, x) + P_1^{\varepsilon}(t, x) + \cdots$$

$$\Longrightarrow \qquad \sqrt{\varepsilon} I_1 = P_1^{\varepsilon}(t, x) \left[\mathbf{Vega}(\bar{\sigma})\right]^{-1}$$

Compute the $\mathbf{Vega} = \partial C_{BS}/\partial \sigma = x e^{-d_1^2/2} \sqrt{T - t}/\sqrt{2\pi}$

and deduce

Calibration Formulas

The **implied volatility** is an **affine function** of the **LMMR**:

log-moneyness-to-maturity-ratio $= \log(K/x)/(T-t)$

$$\mathbf{I} = a\,[\mathbf{LMMR}] + b + \mathcal{O}(1/\alpha)$$

with

$$\begin{aligned} a &= \frac{V_3^\varepsilon}{\bar{\sigma}^3} \\ b &= \bar{\sigma} - \frac{V_3^\varepsilon}{\bar{\sigma}^3}\left(r - \frac{1}{2}\bar{\sigma}^2\right) + \frac{V_2^\varepsilon}{\bar{\sigma}} \end{aligned}$$

or for **calibration purpose**:

$$\begin{aligned} V_2^\varepsilon &= \bar{\sigma}\left((b-\bar{\sigma}) + a(r - \frac{1}{2}\bar{\sigma}^2)\right) \\ V_3^\varepsilon &= a\bar{\sigma}^3 \end{aligned}$$

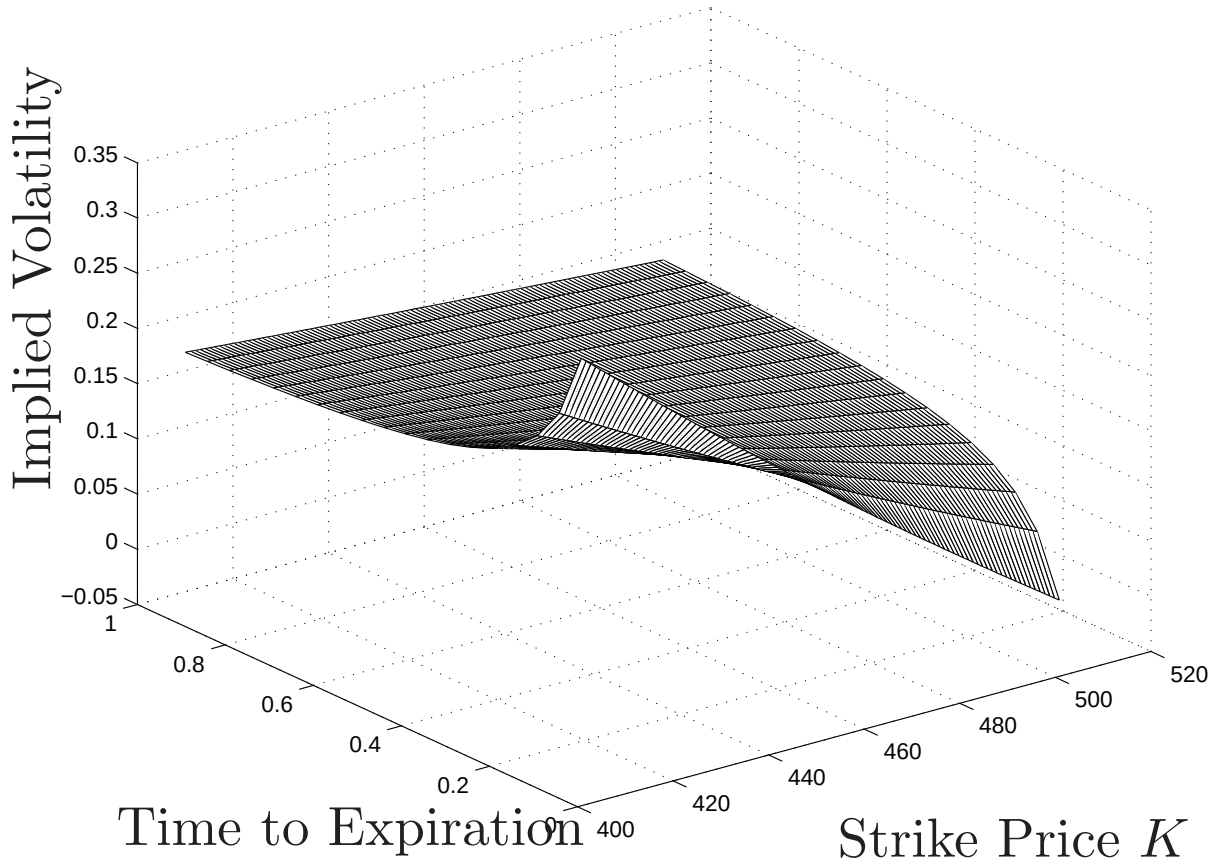

Figure 1: *A typical implied volatility surface predicted by the asymptotic analysis. It is linear in the composite variable LMMR with slope* $a = -0.154$ *and intercept* $b = 0.149$ *estimated from S&P* 500 *options data. We take* $t = 0$ *and current asset price* $x = 460$.

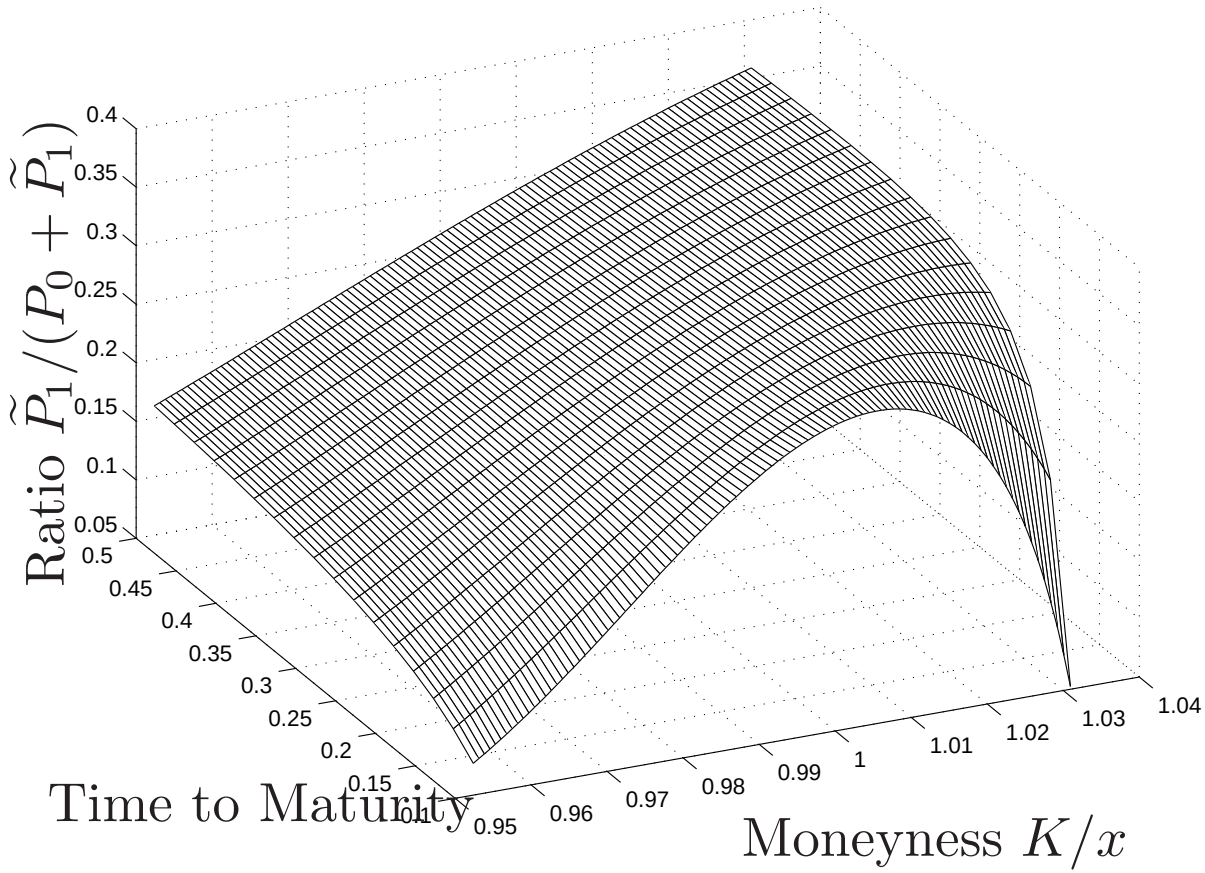

Figure 2: *Ratio of correction $\widetilde{P_1}$ to corrected price $\widetilde{P}$ for a European call option using parameter values calibrated from the observed S&P 500 implied volatility surface: $a = -0.154, b = 0.149$ and $\bar{\sigma} = 0.1, r = 0.02$. These give $V_2 = -0.0044$ and $V_3 = 0.000154$.*

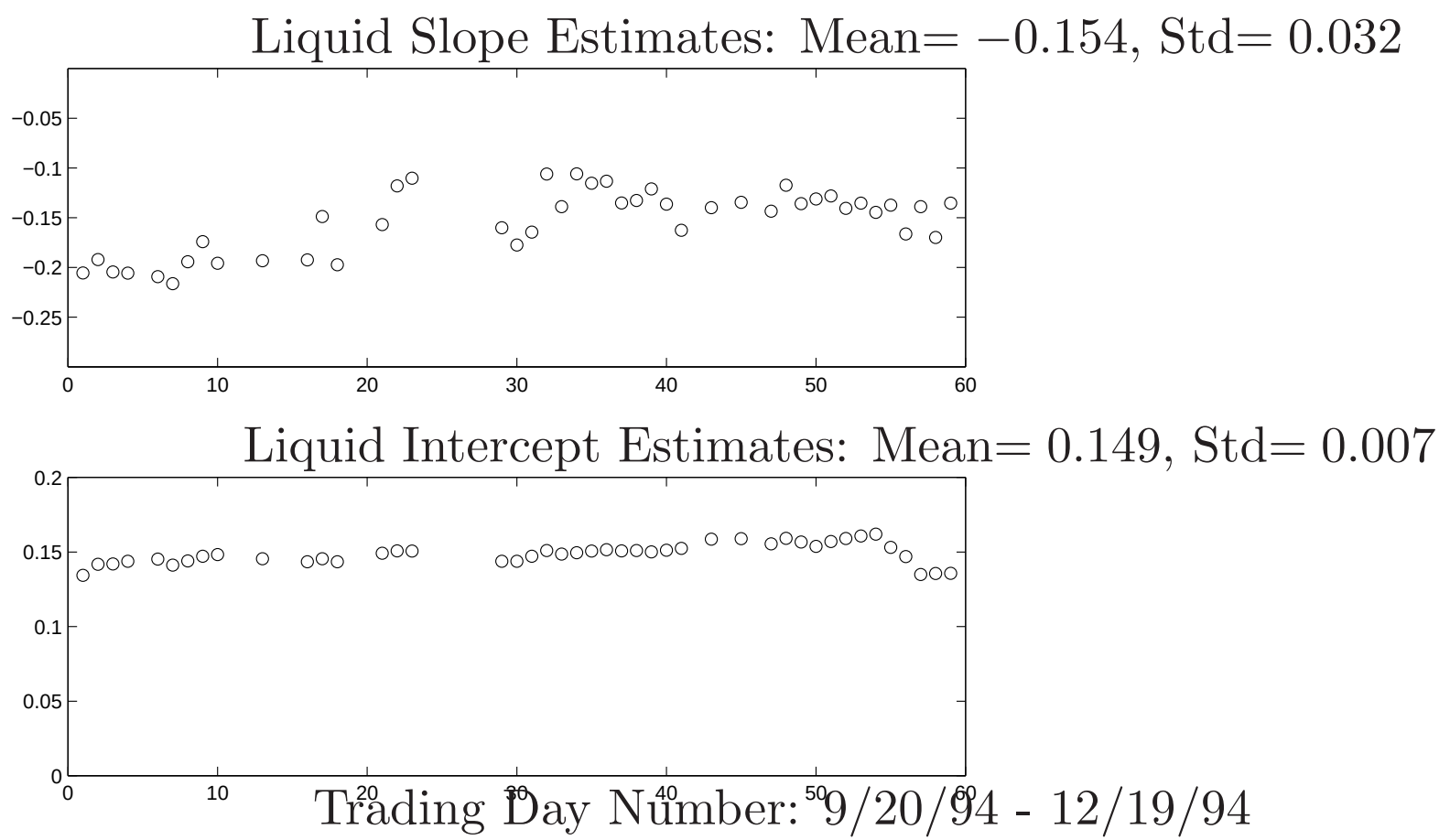

Figure 3: *Daily fits of S&P* 500 *European call option implied volatilties to a straight line in LMMR, excluding days when there is insufficient liquidity (*16 *days out of* 60*).*

Exotic Derivatives (Binary, Barrier,Asian,...)

- Solve the corresponding problem with **constant volatility** $\bar{\sigma}$

$$\Longrightarrow P_0$$

- Use V_2 and V_3 **calibrated on the smile** to compute the **source**

$$V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_0}{\partial x^2}\right)$$

- Get the **correction** P_1 by solving the <u>SAME PROBLEM</u> with **zero boundary conditions** and the **source**.

American Options

- Solve the corresponding problem with **constant volatility** $\bar{\sigma}$

$$\Longrightarrow P_0 \textbf{ and the free boundary } x_0(t)$$

- Use V_2 and V_3 **calibrated on the smile** to compute the **source**

$$V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_0}{\partial x^2}\right)$$

- Get the **correction** P_1 by solving the corresponding problem with **fixed boundary** $x_0(t)$, **zero boundary conditions** and the **source**.