Fast Mean-Reverting Stochastic Volatility
Asymptotic Analysis

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Model in the risk-neutral world $\mathcal{P}^{\star}(\gamma)$ in terms of $\varepsilon = 1/\alpha$

Set $\beta = \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}$ so that $\nu^2 = \beta^2/2\alpha$

$$
\begin{align*}
    dX^\varepsilon_t &= rX^\varepsilon_t dt + f(Y^\varepsilon_t)X^\varepsilon_t dW^\star_t \\
    dY^\varepsilon_t &= \left[ \frac{1}{\varepsilon}(m - Y^\varepsilon_t) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} \Lambda(Y^\varepsilon_t) \right] dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} d\hat{Z}^\star_t
\end{align*}
$$

Market price of risks:

$$
\Lambda(y) = \rho \left( \frac{\mu - r}{f(y)} \right) + \gamma(y) \sqrt{1 - \rho^2}
$$

Skew:

$$
\hat{Z}^\star_t = \rho W^\star_t + \sqrt{1 - \rho^2} Z^\star_t , \quad |\rho| < 1
$$
Prices and Pricing PDE’s

\[ P^\varepsilon(t, x, y) = \mathcal{E}^\varepsilon(\gamma) \left\{ e^{-r(T-t)} h(X^\varepsilon_T) \mid X^\varepsilon_t = x, Y^\varepsilon_t = y \right\} \]

\[
\begin{align*}
\frac{\partial P^\varepsilon}{\partial t} + \frac{1}{2} f(y)^2 x^2 \frac{\partial^2 P^\varepsilon}{\partial x^2} + \frac{\rho \nu \sqrt{2}}{\sqrt{\varepsilon}} x f(y) \frac{\partial^2 P^\varepsilon}{\partial x \partial y} + \frac{\nu^2}{\varepsilon} \frac{\partial^2 P^\varepsilon}{\partial y^2} \\
+ r \left( x \frac{\partial P^\varepsilon}{\partial x} - P^\varepsilon \right) + \frac{1}{\varepsilon} (m - y) \frac{\partial P^\varepsilon}{\partial y} - \frac{\nu \sqrt{2}}{\sqrt{\varepsilon}} \Lambda(y) \frac{\partial P^\varepsilon}{\partial y} &= 0
\end{align*}
\]

to be solved for \( t < T \) with the terminal condition

\[ P^\varepsilon(T, x, y) = h(x) \]
Operator Notation

\[
\left( \frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_1 + \mathcal{L}_2 \right) P^\varepsilon = 0
\]

with

\[
\begin{align*}
\mathcal{L}_0 &= \nu^2 \frac{\partial^2}{\partial y^2} + (m - y) \frac{\partial}{\partial y} = \mathcal{L}_{OU} \\
\mathcal{L}_1 &= \sqrt{2}\rho \nu x f(y) \frac{\partial^2}{\partial x \partial y} - \sqrt{2} \nu \Lambda(y) \frac{\partial}{\partial y} \\
\mathcal{L}_2 &= \frac{\partial}{\partial t} + \frac{1}{2} f(y)^2 x^2 \frac{\partial^2}{\partial x^2} + r \left( x \frac{\partial}{\partial x} - \cdot \right) = \mathcal{L}_{BS}(f(y))
\end{align*}
\]
Formal Expansion

Expand:

\[ P^\varepsilon = P_0 + \sqrt{\varepsilon}P_1 + \varepsilon P_2 + \varepsilon \sqrt{\varepsilon}P_3 + \cdots \]

Compute:

\[
\left( \frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_1 + \mathcal{L}_2 \right) \left( P_0 + \sqrt{\varepsilon}P_1 + \varepsilon P_2 + \varepsilon \sqrt{\varepsilon}P_3 + \cdots \right) = 0
\]

Group the terms by powers of \( \varepsilon \):

\[
\frac{1}{\varepsilon} \mathcal{L}_0 P_0 + \frac{1}{\sqrt{\varepsilon}} \left( \mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 \right) + \mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0 + \sqrt{\varepsilon} \left( \mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 \right) + \cdots = 0
\]
Diverging terms

- **Order $1/\varepsilon$:**

  \[ \mathcal{L}_0 P_0 = 0 \]

  \[ \mathcal{L}_0 = \mathcal{L}_{OU}, \text{ acting on } y \implies P_0 = P_0(t, x) \]

  \[ \text{with } P_0(T, x) = h(x) \]

- **Order $1/\sqrt{\varepsilon}$:**

  \[ \mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0 \]

  \[ \mathcal{L}_1 \text{ takes derivatives w.r.t. } y \implies \mathcal{L}_1 P_0 = 0 \]

  \[ \implies \mathcal{L}_0 P_1 = 0 \]

  As for $P_0$:

  \[ P_1 = P_1(t, x) \]

  \[ \text{with } P_1(T, x) = 0 \]

- **Important observation:**

  \[ P_0 + \sqrt{\varepsilon} P_1 \text{ does not depend on } y \]
Zero Order Term

\[ \mathcal{L}_0 P_2 + (\mathcal{L}_1 P_1 = 0) + \mathcal{L}_2 P_0 = 0 \]

**Poisson equation** in \( P_2 \) with respect to \( \mathcal{L}_0 \) and the variable \( y \).

Solution:

\[ P_2 = (-\mathcal{L}_0)^{-1}(\mathcal{L}_2 P_0) \]

Only if \( \mathcal{L}_2 P_0 \) is centered
with respect to the
**invariant distribution** of \( Y \).
Poisson Equations

\[ \mathcal{L}_0 \chi + g = 0 \]

Expectations w.r.t. the invariant distribution of the OU process:

\[
\langle g \rangle = -\langle \mathcal{L}_0 \chi \rangle = - \int (\mathcal{L}_0 \chi(y)) \Phi(y) dy = \int \chi(y) (\mathcal{L}_0^* \Phi(y)) dy = 0
\]

\[
\lim_{t \to +\infty} \mathbb{E} \{ g(Y_t) | Y_0 = y \} = \langle g \rangle = 0 \quad \text{(exponentially fast)}
\]

\[
\chi(y) = \int_0^{+\infty} \mathbb{E} \{ g(Y_t) | Y_0 = y \} dt
\]

checked by applying \( \mathcal{L}_0 \)
Leading Order Term

Centering:

\[ \langle \mathcal{L}_2 P_0 \rangle = \langle \mathcal{L}_2 \rangle P_0 = 0 \]

\[ \langle \mathcal{L}_2 \rangle = \left\langle \frac{\partial}{\partial t} + \frac{1}{2} f(y)^2 x^2 \frac{\partial^2}{\partial x^2} + r \left( x \frac{\partial}{\partial x} - \cdot \right) \right\rangle \]

\[ = \frac{\partial}{\partial t} + \frac{1}{2} \langle f^2 \rangle x^2 \frac{\partial^2}{\partial x^2} + r \left( x \frac{\partial}{\partial x} - \cdot \right) \]

Effective volatility: \( \bar{\sigma}^2 = \langle f^2 \rangle \)

The zero order term \( P_0(t, x) \) is the solution of the Black-Scholes equation

\[ \mathcal{L}_{BS}(\bar{\sigma}) P_0 = 0 \]

with the terminal condition \( P_0(T, x) = h(x) \)
Back to $P_2(t, x, y)$

The centering condition $\langle \mathcal{L}_2 P_0 \rangle = 0$ being satisfied:

$$\mathcal{L}_2 P_0 = \mathcal{L}_2 P_0 - \langle \mathcal{L}_2 P_0 \rangle = \frac{1}{2} (f(y)^2 - \bar{\sigma}^2) x^2 \frac{\partial^2 P_0}{\partial x^2}$$

$$= \frac{1}{2} \mathcal{L}_0 \phi(y) x^2 \frac{\partial^2 P_0}{\partial x^2}$$

for $\phi$ a solution of the Poisson equation:

$$\mathcal{L}_0 \phi = f(y)^2 - \langle f^2 \rangle$$

Then

$$P_2(t, x, y) = -\mathcal{L}_0^{-1} (\mathcal{L}_2 P_0) = -\frac{1}{2} (\phi(y) + c(t, x)) x^2 \frac{\partial^2 P_0}{\partial x^2}$$
Terms of order $\sqrt{\varepsilon}$

Poisson equation in $P_3$:

$$\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0$$

Centering condition:

$$\langle \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 \rangle = 0$$

Equation for $P_1$:

$$\langle \mathcal{L}_2 P_1 \rangle = -\langle \mathcal{L}_1 P_2 \rangle = \frac{1}{2} \left\langle \mathcal{L}_1 \left[ (\phi(y) + c(t, x)) x^2 \frac{\partial^2 P_0}{\partial x^2} \right] \right\rangle$$

$P_1$ independent of $y$ and $\mathcal{L}_1$ takes derivatives w.r.t. $y$

$$\Longrightarrow \quad \mathcal{L}_{BS}(\overline{\sigma}) P_1 = \frac{1}{2} \langle \mathcal{L}_1 \phi(y) \rangle \left[ x^2 \frac{\partial^2 P_0}{\partial x^2} \right]$$

with $P_1(T, x) = 0$
The correction $P_1^\varepsilon(t, x) = \sqrt{\varepsilon} P_1(t, x)$

\[
\mathcal{L}_{BS}(\bar{\sigma}) P_1^\varepsilon - \frac{\nu \sqrt{2\varepsilon}}{2} \left\langle \left( \rho x f(y) \frac{\partial^2}{\partial x \partial y} - \Lambda(y) \frac{\partial}{\partial y} \right) \phi(y) \right\rangle \left[ x^2 \frac{\partial^2 P_0}{\partial x^2} \right] = 0
\]

\[
\mathcal{L}_{BS}(\bar{\sigma}) P_1^\varepsilon + \left( V_2^\varepsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_{BS}}{\partial x^2} \right) \right) = 0
\]

BS equation with source and zero terminal condition

with the two small parameters $V_2^\varepsilon$ and $V_3^\varepsilon$ given by:

\[
V_2^\varepsilon = \frac{\nu}{\sqrt{2\alpha}} \langle \Lambda \phi' \rangle
\]

\[
V_3^\varepsilon = -\rho \nu \frac{1}{\sqrt{2\alpha}} \langle f \phi' \rangle
\]

Recall that $\alpha = 1/\varepsilon$
Explicit Formula for the Corrected Price

\[ P_1^\varepsilon = (T - t) \left( V_2^\varepsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_{BS}}{\partial x^2} \right) \right) \]

where \( V_2^\varepsilon \) and \( V_3^\varepsilon \) are small numbers of order \( \sqrt{\varepsilon} \).

The corrected price is given explicitly by

\[ P_0 + (T - t) \left( V_2^\varepsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_{BS}}{\partial x^2} \right) \right) \]

where \( P_0 \) is the Black-Scholes price with constant volatility \( \bar{\sigma} \).
The small constants $V_2^\varepsilon$ and $V_3^\varepsilon$ are complex functions of the original model parameters $(\mu, m, \nu, \rho, \alpha; f)$ and $\gamma$.

Only $(\bar{\sigma}, V_2^\varepsilon, V_3^\varepsilon)$ are needed to compute the corrected price.

**Probabilistic representation** of $(P_0 + P_1^\varepsilon)(t, x)$:

\[
\bar{E} \left\{ e^{-r(T-t)} h(\bar{X}_T) + \int_t^T e^{-r(s-t)} H(s, \bar{X}_s) ds \mid \bar{X}_t = x \right\}
\]

**Put-Call Parity** is preserved at the order $O(\sqrt{\varepsilon})$.

The $V_2^\varepsilon$ term is a **volatility level correction**

\[
\sigma^* = \sqrt{\bar{\sigma}^2 + 2V_2^\varepsilon}
\]

The $V_3^\varepsilon$ term is the **skew effect**

\[
\rho = 0 \implies V_3^\varepsilon = 0
\]
Accuracy of Approximation

Define

\[ Z^\varepsilon = P_0 + \sqrt{\varepsilon}P_1 + \varepsilon P_2 + \varepsilon \sqrt{\varepsilon}P_3 - P \]

so that

\[ Z^\varepsilon(T, x, y) = \varepsilon (P_2(T, x, y) + \sqrt{\varepsilon}P_3(T, x, y)) \]

Using how \((P_0, P_1, P_2, P_3)\) have been chosen to cancel \(1/\varepsilon, 1/\sqrt{\varepsilon}, O(1)\) and \(\sqrt{\varepsilon}\) terms deduce

\[ \mathcal{L}^\varepsilon Z^\varepsilon = \varepsilon (\mathcal{L}_1 P_3 + \mathcal{L}_2 P_2 + \sqrt{\varepsilon} \mathcal{L}_2 P_3) \]

and conclude that source and terminal condition of order \(\varepsilon\)

\[ \implies Z^\varepsilon = O(\varepsilon) \]

\[ \implies P(t, x, y) = (P_0(t, x) + P_1^\varepsilon(t, x)) + O(\varepsilon) \]
Corrected Call Option Prices

\[ h(x) = (x - K)^+ \]

and

\[ P_0(t, x) = C_{BS}(t, x; K, T; \bar{\sigma}) \]

Compute the **Delta**, the **Gamma** and the **Delta-Gamma**

\[ = \frac{\partial^3 P_0}{\partial x^3} \]

Deduce the source

\[ H = \left( V_2^\varepsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_{BS}}{\partial x^2} \right) \right) \]

and the correction

\[ P_1^\varepsilon(t, x) = (T - t)H(t, x) = \frac{xe^{-d_1^2/2}}{\bar{\sigma} \sqrt{2\pi}} \left( -V_3^\varepsilon \frac{d_1}{\bar{\sigma}} + V_2 \sqrt{T - t} \right) \]
Expansion of Implied Volatilities

Recall

\[ C_{BS}(t, x; K, T; I) = C^{\text{observed}} \]

Expand

\[ I = \bar{\sigma} + \sqrt{\varepsilon}I_1 + \cdots \]

Deduce for given \((K, T)\):

\[ C_{BS}(t, x; \bar{\sigma}) + \sqrt{\varepsilon}I_1 \frac{\partial C_{BS}}{\partial \sigma}(t, x; \bar{\sigma}) + \cdots = P_0(t, x) + P_1^\varepsilon(t, x) + \cdots \]

\[ \implies \sqrt{\varepsilon}I_1 = P_1^\varepsilon(t, x) [\text{Vega}(\bar{\sigma})]^{-1} \]

Compute the \textbf{Vega} = \frac{\partial C_{BS}}{\partial \sigma} = xe^{-d_1^2/2}\sqrt{T-t}/\sqrt{2\pi}

and deduce
Calibration Formulas

The implied volatility is an affine function of the LMMR:

\[
\text{log-moneyness-to-maturity-ratio} = \log(K/x)/(T - t)
\]

\[
I = a [\text{LMMR}] + b + O(1/\alpha)
\]

with

\[
a = \frac{V_3^{\varepsilon}}{\bar{\sigma}^3}
\]

\[
b = \bar{\sigma} - \frac{V_3^{\varepsilon}}{\bar{\sigma}^3} \left( r - \frac{1}{2} \bar{\sigma}^2 \right) + \frac{V_2^{\varepsilon}}{\bar{\sigma}}
\]

or for calibration purpose:

\[
V_2^{\varepsilon} = \bar{\sigma} \left( (b - \bar{\sigma}) + a(r - \frac{1}{2} \bar{\sigma}^2) \right)
\]

\[
V_3^{\varepsilon} = a\bar{\sigma}^3
\]
Figure 1: A typical implied volatility surface predicted by the asymptotic analysis. It is linear in the composite variable LMMR with slope $a = -0.154$ and intercept $b = 0.149$ estimated from S&P 500 options data. We take $t = 0$ and current asset price $x = 460$. 


Figure 2: Ratio of correction $\tilde{P}_1$ to corrected price $\tilde{P}$ for a European call option using parameter values calibrated from the observed S&P 500 implied volatility surface: $a = -0.154, b = 0.149$ and $\bar{\sigma} = 0.1, r = 0.02$. These give $V_2 = -0.0044$ and $V_3 = 0.000154$. 
Liquid Slope Estimates: Mean = -0.154, Std = 0.032

Liquid Intercept Estimates: Mean = 0.149, Std = 0.007

Trading Day Number: 9/20/94 - 12/19/94

Figure 3: Daily fits of S&P 500 European call option implied volatilities to a straight line in LMMR, excluding days when there is insufficient liquidity (16 days out of 60).
Exotic Derivatives (Binary, Barrier, Asian, ...)

- Solve the corresponding problem with constant volatility $\bar{\sigma}$
  $$\Rightarrow P_0$$

- Use $V_2$ and $V_3$ calibrated on the smile to compute the source
  $$V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_0}{\partial x^2} \right)$$

- Get the correction $P_1$ by solving the SAME PROBLEM with zero boundary conditions and the source.
American Options

- Solve the corresponding problem with constant volatility $\bar{\sigma}$

$$\Rightarrow P_0 \text{ and the free boundary } x_0(t)$$

- Use $V_2$ and $V_3$ calibrated on the smile to compute the source

$$V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_0}{\partial x^2} \right)$$

- Get the correction $P_1$ by solving the corresponding problem with fixed boundary $x_0(t)$, zero boundary conditions and the source.