Stochastic Volatility Modeling

Jean-Pierre Fouque
University of California Santa Barbara

2008 Daiwa Lecture Series
July 29 - August 1, 2008
Kyoto University, Kyoto
References:

**Derivatives in Financial Markets with Stochastic Volatility**
*Cambridge University Press, 2000*

**Stochastic Volatility Asymptotics**
*SIAM Journal on Multiscale Modeling and Simulation, 2(1), 2003*

Collaborators:

G. Papanicolaou (Stanford), R. Sircar (Princeton), K. Sølna (UCI)

http://www.pstat.ucsb.edu/faculty/fouque
What is Volatility?

Several notions of volatility
Model dependent or not, Data dependent or not

- Realized Volatility (historical data)
- Model Volatility:
  - Local Volatility
  - Stochastic Volatility
- Implied Volatility (option data)
Realized Volatility

$t_0 < t_1 < \cdots < t_N = t$ (present time)

\[
\frac{1}{t-t_0} \int_{t_0}^{t} \sigma_s^2 \, ds \sim \frac{1}{N} \sum_{i=1}^{N} \frac{(\log S_{t_i} - \log S_{t_{i-1}})^2}{t_i - t_{i-1}}
\]

depends on the choice of $t_0$ and on the number of increments $N$ (assuming $t_i - t_{i-1}$ constant).

More details:


http://galton.uchicago.edu/~mykland/publ.html
Volatility Models

\[ dS_t = S_t (\mu dt + \sigma_t dW_t) \]

- **Local Volatility:**
  \[ \sigma_t = \sigma(t, S_t) \]
  where \( \sigma(t, x) \) is a deterministic function.

- **Stochastic Volatility:**
  \[ \sigma_t = f(Y_t) \]
  where \( Y_t \) contains an additional source of randomness.
Implied Volatility

\[ I(t, T, K) = \sigma_{\text{implied}}(t, T, K) \]

where \( \sigma_{\text{implied}}(t, T, K) \) is uniquely defined by inverting Black-Scholes formula:

\[ C_{\text{observed}}(t, T, K) = C_{BS}(t, S_t; T, K; \sigma_{\text{implied}}(t, T, K)) \]

given the call-option data.

t is present time, \( T \) is the option maturity date, and \( K \) is the strike price.
Figure 1: *S&P 500 Implied Volatility Curve* as a function of moneyness from *S&P 500 index options on February 9, 2000*. The current index value is $x = 1411.71$ and the options have over two months to maturity. This is typically described as a downward sloping skew.
“Parametrization” of the 
Implied Volatility Surface $I(t; T, K)$

REQUIRED QUALITIES

- Universal Parsimonous Parameters: Model Independence
- Stability in Time: Predictive Power
- Easy Calibration: Practical Implementation
- Compatibility with Price Dynamics: Applicability to Pricing other Derivatives and Hedging
At least three approaches:

- **Local Volatility Models:** \( \sigma_t = \sigma(t, S_t) \)
  
  +’s: market is complete (no additional randomness), Dupire formula
  
  \[
  \sigma^2(T, K) = 2 \frac{\partial C}{\partial T} + rK \frac{\partial C}{\partial K} \frac{K^2 \frac{\partial^2 C}{\partial K^2}}{K^2}
  \]

  -’s: stability of calibration

- **Implied Volatility Surface Models:** \( dI_t(T, K) = \cdots \)
  
  +’s: predictive power
  
  -’s: no-arbitrage conditions not easy. Which underlying?

- **Stochastic Volatility Models:** \( \sigma_t = f(Y_t) \)
Stochastic Volatility Framework

WHY?

- Distributions of returns are **not** log-normal
- Smile (Skew) effect observed in implied volatilities

HOW?

\[
dS_t = \mu S_t dt + \sigma_t S_t dW_t
\]

with, for instance:

\[
\sigma_t = f(Y_t)
\]

\[
dY_t = \alpha(m - Y_t) dt + \nu \sqrt{2\alpha} dW_t^{(1)}
\]

\[
d\langle W, W^{(1)} \rangle_t = \rho dt
\]
The Popular Heston Model

\[ dS_t = \mu S_t dt + \sigma_t S_t dW_t^{(1)} \]
\[ \sigma_t = \sqrt{Y_t} \]
\[ dY_t = \alpha (m - Y_t) dt + \nu \sqrt{2\alpha Y_t} dW_t^{(2)} \]
\[ d\langle W^{(1)}, W^{(2)} \rangle_t = \rho dt \]

\( Y_t \) is a CIR (Cox-Ingersoll-Ross) process.

The condition \( m \geq \nu^2 \) ensures that the process \( Y_t \) stays strictly positive at all time.
Mean-Reverting Stochastic Volatility Models

\[ dX_t = X_t (\mu dt + \sigma_t dW_t) \]

\[ \sigma_t = f(Y_t) \]

For instance: \[ 0 < \sigma_1 \leq f(y) \leq \sigma_2 \] for every \( y \)

\[ dY_t = \alpha(m - Y_t)dt + \beta(\cdots)d\hat{Z}_t \]

Brownian motion \( \hat{Z} \) correlated to \( W \):

\[ \hat{Z}_t = \rho W_t + \sqrt{1 - \rho^2} Z_t, \quad |\rho| < 1 \]

so that

\[ d\langle W, \hat{Z}\rangle_t = \rho dt \]
Pricing under Stochastic Volatility

Risk-neutral probability chosen by the market: $\mathbb{P}^*(\gamma)$

\[ dX_t = rX_t dt + f(Y_t)X_t dW^*_t \]

\[ dY_t = \left[ \alpha(m - Y_t) - \beta \left( \rho \frac{\mu - r}{f(Y_t)} + \gamma \sqrt{1 - \rho^2} \right) \right] dt + \beta d\hat{Z}^*_t \]

\[ \hat{Z}^*_t = \rho W^*_t + \sqrt{1 - \rho^2} Z^*_t \]

Market price of volatility risk: $\gamma = \gamma(y)$

\[ P_t = \mathbb{E}^*(\gamma) \{ e^{-r(T-t)} h(X_T) | \mathcal{F}_t \} \]

Markovian case:

\[ P(t, x, y) = \mathbb{E}^*(\gamma) \{ e^{-r(T-t)} h(X_T) | X_t = x, Y_t = y \} \]

but $y$ (or $f(y)$) is not directly observable!
Stochastic Volatility Pricing PDE

\[
\frac{\partial P}{\partial t} + \frac{1}{2} f(y)^2 x^2 \frac{\partial^2 P}{\partial x^2} + \rho \beta x f(y) \frac{\partial^2 P}{\partial x \partial y} + \frac{1}{2} \beta^2 \frac{\partial^2 P}{\partial y^2} + r \left( x \frac{\partial P}{\partial x} - P \right) + \alpha (m - y) \frac{\partial P}{\partial y} - \beta \Lambda \frac{\partial P}{\partial y} = 0
\]

where

\[
\Lambda = \rho \frac{(\mu - r)}{f(y)} + \gamma \sqrt{1 - \rho^2}
\]

Terminal condition: \( P(T, x, y) = h(x) \)

No perfect hedge!
Summary of the stochastic volatility approach

Positive aspects:

• More realistic returns distributions (fat tails and asymmetry)
• Smile effect with skew controlled by $\rho$

Difficulties:

• Volatility not directly observed, parameter estimation difficult
• No canonical model. Relevance of explicit formulas?
• Incomplete markets, no perfect hedge
• Volatility risk premium to be estimated from option prices
• Numerical difficulties due to higher dimension
Driving process $Y_t$ and intrinsic time scale

Markovian case: infinitesimal generator $\mathcal{L}$

- Two-state Markov chain:
  \[
  \mathcal{L} = \alpha \begin{pmatrix}
  -1 & 1 \\
  1 & -1
  \end{pmatrix}
  \]

- Pure jump Markov process
  \[
  \mathcal{L}g(y) = \alpha \int (g(z) - g(y)) p(z) dz , \quad p(z) = \frac{1}{2} 1_{(-1,1)}(z)
  \]

- Diffusion: Ornstein-Uhlenbeck Process
  \[
  \alpha \mathcal{L}_{OU} = \alpha \left( (m - y) \frac{\partial}{\partial y} + \nu^2 \frac{\partial^2}{\partial^2 y} \right)
  \]
Invariant probability distribution

\[ \mathcal{L}^* \Phi = 0 \]

- Two-state Markov chain: **linear system**
  \[ \Phi = \left\{ \frac{1}{2}, \frac{1}{2} \right\} \]

- Pure jump Markov process: **integral equation**
  \[ \Phi(y) = \frac{1}{2} \mathbf{1}_{[-1,1]}(y) \]

- OU diffusion process: **differential equation**
  \[ \Phi(y) = \frac{1}{\sqrt{2\pi \nu}} \exp \left( -\frac{(y - m)^2}{2\nu^2} \right) \quad , \quad \nu^2 = \beta^2 / 2\alpha \]
Convergence to Equilibrium

• Equilibrium:

\[ \langle g \rangle = \int g(y)\Phi(y)dy \]

• Exponential convergence to equilibrium with rate \( \alpha \):

\[ |IE\{g(Y_t)|Y_0 = y\} - \langle g \rangle| \leq Ce^{-\alpha t} \]

• Exponential decorrelation with rate \( \alpha \):

\[ |IE_\Phi\{g(Y_s)h(Y_t)\} - \langle g \rangle\langle h \rangle| \leq Ce^{-\alpha|t-s|} \]

• Intrinsic time scale: \( 1/\alpha \)
Figure 2: *Simulated paths of $\sigma_t = f(Y_t)$, with $(Y_t)$ a two-state Markov chain, showing the relation between burstiness and the mean holding time $1/\alpha$.***
Ergodic Theorem

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t g(Y_s) ds = \langle g \rangle \quad \text{almost surely (a.s.)} \]

or \( t > 0 \) fixed and \( \alpha \to +\infty \):

\[ \frac{1}{t} \int_0^t g(Y_s) \approx \langle g \rangle \]

In particular for \( \alpha \) large:

\[ \overline{\sigma^2} = \frac{1}{T-t} \int_t^T f(Y_s)^2 ds \approx \langle f^2 \rangle = \bar{\sigma}^2 \]

LHS random  \quad RHS deterministic
Figure 3: Simulated paths of $\sigma_t = f(Y_t)$, with $(Y_t)$ a pure jump Markov process taking values in $(0.05, 0.4)$, for increasing mean-reversion rates $\alpha$. The mean level $\langle f \rangle = 0.225$. 
Figure 4: Simulated paths of $\sigma_t = f(Y_t)$, with $(Y_t)$ a mean-reverting OU process and $f(y) = 0.35(\arctan y + \pi/2)/\pi + 0.05$, chosen so that $\sigma_t \in (0.05, 0.4)$. Notice how the mean-reversion rates $\alpha$ correspond to the duration of the bursts.
Figure 5: 1996 S&P 500 returns computed from half-hourly data.
Figure 6: Simulated volatility and corresponding returns paths for small and large rates of mean-reversion for the jump volatility model.
Figure 7: Simulated volatility and corresponding returns paths for small and large rates of mean-reversion for the OU model with $f(y) = e^y$. 

\[ \alpha = 1 \quad \alpha = 200 \]
Exponential decorrelation – Variogram

Variogram: $IE[f(Y_{t+\text{lag}}) - f(Y_t)]^2 \approx 2 \bar{\sigma}^2 (1 - e^{-\alpha\text{lag}})$

- $Y_s$: Ornstein-Uhlenbeck driving process.
- $f(x) = \exp(x)$
Exponential decorrelation for returns

Variograms for raw and median filtered returns:
Estimation

- Stochastic volatility model:

\[ dX_t = \mu X_t dt + f(Y_t) X_t dW_t \]

- Volatility hidden, only see returns:

\[ \frac{dX_t}{X_t} - \mu dt = f(Y_t) dW_t. \]

- The de-meaned return process:

\[ D_{t_i} = \frac{1}{\sqrt{\Delta t}} \left( \frac{\Delta X_{t_i}}{X_{t_i}} - \text{mean} \right) \sim f(Y_{t_i}) \Delta W_{t_i} = f(Y_{t_i}) \epsilon_{t_i} \]

where \( \epsilon_{t_i} \) is an i.i.d “white noise” sequence.
Variogram of log-returns

• Log-returns:

\[ L_{t_i} = \log |D_{t_i}| = \log(f(Y_{t_i})) + \log |\epsilon_{t_i}| = \tilde{f}(Y_{t_i}) + \tilde{\epsilon}_{t_i}. \]

• Variogram is like an exponential:

\[ \frac{1}{N} \sum_{i=1}^{N} \left[ L_{t_i+\text{lag}} - L_{t_i} \right]^2 \approx c_1(1 - e^{-\alpha \text{lag}}) + c_2. \]
Median filtered S&P 500 variogram

Notice the day effect removed by the fitted exponential.

The curvature determines $1/\alpha : 1.5 \pm 0.4$ trading days annualized $\alpha \approx 130 - 230$: large
Energy spectrum of log returns:

\[ \int E[L_{\bar{t}+t}L_{\bar{t}}] \cos(\omega t) \, dt \approx d_1 + d_2 \frac{\alpha}{\sigma^2 + \omega^2} + d_3 \delta(\omega - \omega_1) : \]

S&P500 and Lorentzian spectra

Simulated Lorentzian spectra
Volatility Time Scales

Rescale the time of a diffusion process \( Y_t^1 \):

\[
Y_t^\alpha = Y_{\alpha t}^1
\]

\( \alpha \) large \quad \rightarrow \quad “speeding up” \quad the \ process \ \( Y_t^1 \)

\( \alpha \) small \quad \rightarrow \quad “slowing down” \quad the \ process \ \( Y_t^1 \)

\( 1/\alpha \) is the characteristic time scale of the process \( Y_t^\alpha \).

Averaged square volatility:

\[
\overline{\sigma^2}(0, T) = \frac{1}{T} \int_0^T f^2(Y_t^\alpha)dt
\]
Slowing Down the Time

\[ dY_t^1 = c(Y_t^1)dt + g(Y_t^1)dW_t, \quad Y_0^1 = y \]

\[ dY_t^\alpha = c(Y_t^\alpha)d(\alpha t) + g(Y_t^\alpha)dW_{\alpha t} \]

\[ \equiv \alpha c(Y_t^\alpha)dt + \sqrt{\alpha} g(Y_t^\alpha)dW_t, \quad Y_0^\alpha = y \]

Assuming that \( f \) is continuous,

\[ \overline{\sigma^2}(0, T) = \frac{1}{T} \int_0^T f^2(Y_t^\alpha)dt \to f^2(y) \quad \text{as} \quad \alpha \to 0 \]

Volatility is “frozen” at its starting level \( f(y) \)
Rate of Convergence in the Slow Scale Limit

\[
\sigma^2(0, T) - f^2(y) = \frac{1}{T} \int_0^T \left[ f^2 \left( y + \alpha \int_0^t c(Y_s^\alpha) ds + \sqrt{\alpha} \int_0^t g(Y_s^\alpha) dW_s \right) - f^2(y) \right] dt
\]

\[
= 2 \sqrt{\alpha} f(y) f'(y) g(y) \left( \frac{1}{T} \int_0^T W_t dt \right) + O(\alpha)
\]

(smoothness of \( f \) and \( g \) is assumed)

**Risk neutral** \( \longrightarrow \) a market price of volatility risk

\( \longrightarrow \) term: \( -\sqrt{\alpha} g(Y_s^\alpha) \Lambda(Y_s^\alpha) ds \) in the drift of \( Y_t^\alpha \)

\( \longrightarrow \) additional term: \( -2 \sqrt{\alpha} f(y) f'(y) g(y) \Lambda(y) (T/2) \)

**Correlation** with the BM driving the underlying will also come into play at the order \( \sqrt{\alpha} \)
Speeding Up the Time

\[
\overline{\sigma^2}(0, T) = \frac{1}{T} \int_0^T f^2(Y^\alpha_t) dt = \frac{1}{\alpha T} \int_0^{\alpha T} f^2(Y^1_s) ds, \quad \alpha \to +\infty
\]

\[
= \frac{1}{T'} \int_0^{T'} f^2(Y^1_s) ds, \quad T' \equiv \alpha T \to +\infty
\]

Assuming that \( Y^1 \) is ergodic with invariant distribution \( \Phi \) then:

\[
\lim_{T' \to +\infty} \frac{1}{T'} \int_0^{T'} f^2(Y^1_s) ds = \int f^2(y) \Phi(dy) \equiv \langle f^2 \rangle_\Phi.
\]

**Effective volatility:** \( \bar{\sigma} \equiv \sqrt{\langle f^2 \rangle_\Phi} \)

\[
\lim_{\alpha \to \infty} \overline{\sigma^2}(0, T) = \bar{\sigma}^2
\]
The Averaging Principle

Fast “oscillating” integral:

\[
\overline{\sigma^2}(0, T) - \bar{\sigma}^2 = \frac{1}{T} \int_0^T \left( f^2(Y_{\alpha}^s) - \bar{\sigma}^2 \right) ds, \quad \alpha \to +\infty
\]

Observe that \( f^2(Y_{\alpha}^s) \) does not converge for fixed \( s \).

Introduce the Poisson equation:

\[
\mathcal{L}_{Y_1} \phi(y) = f^2(y) - \bar{\sigma}^2
\]

so that

\[
\overline{\sigma^2}(0, T) - \bar{\sigma}^2 = \frac{1}{T} \int_0^T \mathcal{L}_{Y_1} \phi(Y_{\alpha}^s) ds
\]
The Averaging Principle (continued)

Using Ito’s formula:

\[ d\phi(Y_s^\alpha) = \mathcal{L}_{Y_s^\alpha} \phi(Y_s^\alpha) ds + \sqrt{\alpha} \phi'(Y_s^\alpha) g(Y_s^\alpha) dW_s \]

\[ = \alpha \mathcal{L}_{Y_s^1} \phi(Y_s^\alpha) ds + \sqrt{\alpha} \phi'(Y_s^\alpha) g(Y_s^\alpha) dW_s \]

Therefore

\[
\overline{\sigma^2}(0, T) - \bar{\sigma}^2 = \frac{1}{T} \int_0^T \mathcal{L}_{Y_s^1} \phi(Y_s^\alpha) ds \\
= \frac{1}{\alpha T} \int_0^T d\phi(Y_s^\alpha) - \frac{1}{\sqrt{\alpha T}} \int_0^T \phi'(Y_s^\alpha) g(Y_s^\alpha) dW_s \\
= -\frac{1}{\sqrt{\alpha}} \left( \frac{1}{T} \int_0^T \phi'(Y_s^\alpha) g(Y_s^\alpha) dW_s \right) + O \left( \frac{1}{\alpha} \right)
\]

Risk neutral \[ \rightarrow -\sqrt{\alpha} g(Y_s^\alpha) \Lambda(Y_s^\alpha) ds \text{ in the drift of } Y_t^\alpha \]

\[ \rightarrow \text{additional term: } \frac{1}{\sqrt{\alpha}} \left( \frac{1}{T} \int_0^T g(Y_s^\alpha) \Lambda(Y_s^\alpha) ds \right) \]
Multiscale Stochastic Volatility Models

The spot volatility is a function of two factors:

\[ \sigma_t = f(Y_t, Z_t) \]

- \( Y_t \) is fast mean-reverting (ergodic on a fast time scale):
  \[
  dY_t = \frac{1}{\varepsilon} \alpha(Y_t) dt + \frac{1}{\sqrt{\varepsilon}} \beta(Y_t) dW_t^{(1)}, \quad 0 < \varepsilon \ll 1
  \]

- \( Z_t \) is slowly varying:
  \[
  dZ_t = \delta c(Z_t) dt + \sqrt{\delta} g(Z_t) dW_t^{(2)}, \quad 0 < \delta \ll 1
  \]

\((y, z)\) will denote the initial point for \((Y, Z)\)
f is continuous with respect to \(z\)

**Local Effective Volatility:**

\[
\bar{\sigma}^2(z) \equiv \langle f^2(\cdot, z) \rangle_{\Phi_Y}
\]
\[ \varepsilon \ll T \ll 1/\delta \]

Under the risk neutral measure \( \mathbb{IP}^\ast \) chosen by the market:

\[
\begin{align*}
dX_t &= rX_t dt + f(Y_t, Z_t)X_t dW_t^{(0)\ast} \\
dY_t &= \left( \frac{1}{\varepsilon} \alpha(Y_t) - \frac{1}{\sqrt{\varepsilon}} \beta(Y_t) \Lambda(Y_t, Z_t) \right) dt + \frac{1}{\sqrt{\varepsilon}} \beta(Y_t) dW_t^{(1)\ast} \\
dZ_t &= \left( \delta c(Z_t) - \sqrt{\delta} g(Z_t) \Gamma(Y_t, Z_t) \right) dt + \sqrt{\delta} g(Z_t) dW_t^{(2)\ast}
\end{align*}
\]

\[
\begin{align*}
d < W^{(0)\ast}, W^{(1)\ast} >_t &= \rho_1 dt \\
d < W^{(0)\ast}, W^{(2)\ast} >_t &= \rho_2 dt
\end{align*}
\]

\( \Lambda \) and \( \Gamma \): market prices of volatility risk
Pricing Equation

\[ P^{\varepsilon,\delta}(t, x, y, z) = IE^* \left\{ e^{-r(T-t)} h(X_T) | X_t = x, Y_t = y, Z_t = z \right\} \]

\[ \left( \frac{1}{\varepsilon} L_0 + \frac{1}{\sqrt{\varepsilon}} L_1 + L_2 + \sqrt{\delta} M_1 + \delta M_2 + \sqrt{\frac{\delta}{\varepsilon}} M_3 \right) P^{\varepsilon,\delta} = 0 \]

\[ P^{\varepsilon,\delta}(T, x, y, z) = h(x) \]

\[ L_0 = \frac{\alpha}{\partial y} + \frac{1}{2} \beta^2 \frac{\partial^2}{\partial y^2} \]

\[ L_1 = \beta \left( \rho_1 f x \frac{\partial^2}{\partial x \partial y} - \Lambda \frac{\partial}{\partial y} \right) \]

\[ L_2 = \frac{\partial}{\partial t} + \frac{1}{2} f^2 x^2 \frac{\partial^2}{\partial x^2} + r \left( x \frac{\partial}{\partial x} - \cdot \right) \]

\[ M_1 = \frac{g}{\partial} \left( \rho_2 f x \frac{\partial^2}{\partial x \partial z} - \Gamma \frac{\partial}{\partial z} \right) \]

\[ M_2 = c \frac{\partial}{\partial z} + \frac{g^2}{2} \frac{\partial^2}{\partial z^2} \]

\[ M_3 = \rho_{12} \beta g \frac{\partial^2}{\partial y \partial z} \]