Mean reverting Stochastic Volatility
Hedging Strategies

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Hedging Strategies

A perfect hedge is not possible by trading in the underlying asset only.

Goal: find an acceptable tradeoff between the risk of a failed hedge and the cost of implementing the hedge.

Performance: measured by the subjective probability $IP$
Black-Scholes Delta Hedging

If the risky asset price is a geometric Brownian motion $\tilde{X}_t$ with constant parameters $(\mu, \tilde{\sigma})$, a short position in a European derivative which pays $h(\tilde{X}_T)$ can be perfectly hedged by managing the self-financing portfolio made of, at time $t$, the Delta

$$\frac{\partial P_0}{\partial x}(t, \tilde{X}_t)$$

units of the risky asset, and

$$e^{-rt} \left( P_0(t, \tilde{X}_t) - \tilde{X}_t \frac{\partial P_0}{\partial x}(t, \tilde{X}_t) \right)$$

units of the riskless asset because

$$d \left( P_0(t, \tilde{X}_t) \right) = \frac{\partial P_0}{\partial x}(t, \tilde{X}_t) d\tilde{X}_t + r \left( P_0(t, \tilde{X}_t) - \tilde{X}_t \frac{\partial P_0}{\partial x}(t, \tilde{X}_t) \right) dt$$

and the Black-Scholes equation satisfied by $P_0(t, x)$. 
The Strategy and its Cost

Use the same strategy under stochastic volatility \((X_t, \sigma_t = Y_t)\):

\[
a_t = \frac{\partial P_0}{\partial x}(t, X_t) \quad \text{stocks}
\]

\[
b_t = e^{-rt} \left( P_0(t, X_t) - X_t \frac{\partial P_0}{\partial x}(t, X_t) \right) \quad \text{bonds}
\]

Its value is \(a_t X_t + b_t e^{rt} = P_0(t, X_t)\) and \(P_0(T, X_T) = h(X_T)\)

**Infinitesimal cost of the strategy:**

\[
dP_0(t, X_t) - a_t dX_t - r b_t e^{rt} dt = \frac{1}{2} \left( f(Y_t)^2 - \bar{\sigma}^2 \right) X_t^2 \frac{\partial^2 P_0}{\partial x^2}(t, X_t) dt
\]

**Cumulative cost:**

\[
E_0(t) = \frac{1}{2} \int_0^t \left( f(Y_s)^2 - \bar{\sigma}^2 \right) X_s^2 \frac{\partial^2 P_0}{\partial x^2}(s, X_s) ds
\]
Averaging Effect

\[ \alpha \text{ large } \implies \]

\[ E_0(t) = \frac{1}{2} \int_0^t \left( f(Y_s)^2 - \bar{\sigma}^2 \right) X_s^2 \frac{\partial^2 P_0}{\partial x^2}(s, X_s) ds \quad \text{small} \]

because \( \bar{\sigma}^2 = \langle f^2 \rangle \) and the centering effect.

One has

\[ E_0(t) = \frac{1}{\sqrt{\alpha}} (B_t + M_t) + \mathcal{O}(1/\alpha) \]

\( B_t \) is a systematic bias and \( (M_t) \) a mean-zero martingale.
Averaging Effect: some details

\[ f(Y_s)^2 - \sigma^2 = (\mathcal{L}_0 \phi)(Y_s) \]

\[
(\mathcal{L}_0 \phi)(Y_s)ds = \frac{1}{\alpha} \left\{ d(\phi(Y_s)) - \nu \sqrt{2\alpha} \phi'(Y_s)d\hat{Z}_s \right\}
\]

\[
E_0(t) = \frac{1}{2\alpha} \int_0^t X_s^2 \frac{\partial^2 P_0}{\partial x^2} (s, X_s) \left\{ d(\phi(Y_s)) - \nu \sqrt{2\alpha} \phi'(Y_s)d\hat{Z}_s \right\}
\]

\[
= \frac{1}{2\alpha} \left\{ X_t^2 \frac{\partial^2 P_0}{\partial x^2} (t, X_t)\phi(Y_t) - X_0^2 \frac{\partial^2 P_0}{\partial x^2} (0, X_0)\phi(Y_0) - \int_0^t \phi(Y_s)d \left( X_s^2 \frac{\partial^2 P_0}{\partial x^2} \right) \right\}
\]

\[
- \frac{\rho \nu}{\sqrt{2\alpha}} \int_0^t f(Y_s)\phi'(Y_s) \left( 2X_s^2 \frac{\partial^2 P_0}{\partial x^2} + X_s^3 \frac{\partial^3 P_0}{\partial x^3} \right) ds
\]

\[
- \frac{\nu}{\sqrt{2\alpha}} \int_0^t X_s^2 \frac{\partial^2 P_0}{\partial x^2} \phi'(Y_s)d\hat{Z}_s
\]
Mean Self-Financing Hedging Strategy

Correct the hedging strategy:

\[ a_t = \frac{\partial \left( P_0 + \widetilde{Q}_1 \right)}{\partial x} (t, X_t) \]

\[ b_t = e^{-rt} \left( P_0(t, X_t) + \widetilde{Q}_1(t, X_t) - X_t \frac{\partial \left( P_0 + \widetilde{Q}_1 \right)}{\partial x} (t, X_t) \right) \]

where \( \widetilde{Q}_1 \) satisfies

\[ \mathcal{L}_{BS}(\bar{\sigma})\widetilde{Q}_1 = -V_3 \left( x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_0}{\partial x^2} \right) \right) , \quad \widetilde{Q}_1(T, x) = 0 \]

or given explicitly by

\[ \widetilde{Q}_1(t, x) = (T - t)V_3 \left( x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_0}{\partial x^2} \right) \right) \]
To summarize

Using the **corrected ratio**

\[ a_t = \frac{\partial P_0}{\partial x} + \frac{V_3(T-t)}{x} \left( 4x^2 \frac{\partial^2 P_0}{\partial x^2} + 5x^3 \frac{\partial^3 P_0}{\partial x^3} + x^4 \frac{\partial^4 P_0}{\partial x^4} \right) \]

the **bias in the total cost** is centered:

\[ \frac{\rho \nu}{\sqrt{2\alpha}} \int_0^T \left[ 2X_t^2 \frac{\partial^2 P_0}{\partial x^2} + X_t^3 \frac{\partial^3 P_0}{\partial x^3} \right] (\langle f \phi' \rangle - f \phi(Y_t)) \, dt \]

and

\[ E_0(T) = \frac{1}{\sqrt{\alpha}} (B_T + M_T) + \mathcal{O}(1/\alpha) \]

becomes

\[ E_1^Q(T) = -\frac{\nu}{\sqrt{2\alpha}} \int_0^T X_s^2 \frac{\partial^2 P_0}{\partial x^2} \phi'(Y_s) d\hat{Z}_s + \mathcal{O}(1/\alpha) \]

**Only** \( \bar{\sigma} \) and \( V_3 \) (from the skew) are needed
An Alternative: staying close to the price

\[ a_t = \frac{\partial \left( P_0 + \tilde{P}_1 \right)}{\partial x} (t, X_t) \]

\[ b_t = e^{-rt} \left( (P_0 + \tilde{P}_1)(t, X_t) - X_t \frac{\partial \left( P_0 + \tilde{P}_1 \right)}{\partial x} (t, X_t) \right) \]

with the correction

\[ \tilde{P}_1 = (T - t) \left( V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_0}{\partial x^2} \right) \right) \]

For this strategy, the hedging ratio is given by

\[ a_t = \frac{\partial P_0}{\partial x} + \frac{(T - t)}{x} \left( (2V_2 + 4V_3)x^2 \frac{\partial^2 P_0}{\partial x^2} + (V_2 + 5V_3)x^3 \frac{\partial^3 P_0}{\partial x^3} + V_3 x^4 \frac{\partial^4 P_0}{\partial x^4} \right) \]
At any time

\[ |P - \left( P_0 + \widetilde{P}_1 \right) | = O(1/\alpha) \]

but, up to order \( 1/\alpha \), the total cost becomes

\[
E_1^P(T) = \int_0^T V_2 X_t^2 \frac{\partial^2 P_0}{\partial x^2} dt - \frac{\nu}{\sqrt{2\alpha}} \int_0^T X_s^2 \frac{\partial^2 P_0}{\partial x^2} \phi'(Y_s) d\hat{Z}_s
\]

with

\[
V_2 = \nu \langle \Lambda \phi' \rangle / \sqrt{2\alpha}
\]

which reflects the \textit{volatility risk premium}. 