Mean reverting Stochastic Volatility

Implied Vol, Data and Price Dynamics

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Implied Volatility under fast mean-reversion

\[ I \approx a \frac{\log(K/x)}{(T - t)} + b, \]

→ is linear (affine) in \( \log \text{moneyness/maturity} \) (LMMR).

What does the data tell?

- Sure it is!
- Only if maturity date is fixed!
- No, it is affine in \( \frac{\log(K/x)}{\sqrt{T-t}} \).
Outline

→ A look at Options data.
→ Time dependent price dynamics.
→ Options data revisited.
→ Comments on analysis.
Options data

- Provided by: Y. Ait-Sahalia and A. Lo
- # option prices: \( \approx 12,000 \)
- Time to maturities: \( \approx 0 \) to 100 days
Parameter estimation

For each day $t$:

- Collect all implied volatility records $\{I_i, K_i, x_i, T_i\}_{i=1}^{N_t}$.
- Fit $a_t$ and $b_t$ in:

$$I_i \approx a_t \frac{\log(K_i/x_i)}{(T_i - t)} + b_t,$$

by least squares.

Theory: $a_t \equiv a$ and $b_t \equiv b$.

Plots:

→ In sample fit: $I_i (\text{LMMR}_i)$ versus $a_t \times \text{LMMR} + b_t$.

$(\text{LMMR} = \frac{\log(K/x)}{(T-t)})$

→ Time series of $a_t$ and $b_t$.  

Options data
In sample fit

Not so good!
Wrong slope?

Options data
Parameter stability

Not bad in $b$ but very strange in $a$ !!!
Periodic behavior?

Options data
Conditioning on maturity date

For each day $t$ and maturity date $T_j$:

- Collect all implied volatility records \{${I_i, K_i, x_i, T_j}_{i=1}^{N(t,j)}$\}.

- Fit $a(t,j)$ and $b(t,j)$ in:

\[
I_i \approx a(t,j) \frac{\log(K_i/x_i)}{(T_j - t)} + b(t,j),
\]

by least squares.

Theory: $a(t,j) \equiv a$ and $b(t,j) \equiv b$.

Plot:

\[ a(t,j) \text{ as function of } T_j - t. \]
Maturity dependence

Not very good “close” to maturity !!!
Square root behavior ?

Options data
Time dependent dynamics

- Mean-reverting Stochastic Volatility:

\[
\begin{align*}
    dX_t &= \mu X_t dt + f(Y_t, t) X_t dW_t, \\
    dY_t &= \alpha (m - Y_t) dt + \beta d\hat{Z}_t, \\
    \hat{Z}_t &= \rho(t) W_t + \sqrt{1 - \rho(t)^2} Z_t.
\end{align*}
\]

- Day effect:

\( f(y, t) \) periodic in \( t \), period 1 (trading) day.

- Maturity date modulated leverage:

\( \rho(t) \) “periodic” in \( t \), period \( T_0 \) (time between maturity dates).
**Generalized LMMR formula**

- Under fast mean-reversion implied volatility determined as

\[ I \approx a \frac{\log(K/x)}{\mathcal{M}(T - t)} + b, \]

- it is linear in log moneyness/effective maturity.

Example:

\[ T - t = T_0(N + \delta) \quad (0 \leq \delta < 1) \]
\[ \rho(t) \text{ proportional to } -\sqrt{\delta} \]
\[ \rightarrow \mathcal{M}(T - t) = (T - t) \frac{N+\delta}{N+\delta^{3/2}} = T_0(N + \delta) \frac{N+\delta}{N+\delta^{3/2}} \]

behaves like \( T - t \) far from maturity (\( N \) “large”) and like \( \sqrt{T - t} \) close to maturity (\( N = 0 \)).

In practice: “calendarize” \( \rho \) and use \( \rho(t) \) proportional to \( -\delta^p \)

**Time dependence**
Effective time to maturity

Effective time to maturity for above example ($T_0 = 25$ days):

![Graph showing effective time to maturity over trading days.](image)

Note the square root behavior

Time dependence
In sample fit

Much better!

Options data II
Time stability

Much better!
Will improve with the “calendarization” of \( \rho \)

Options data II
Stability with respect to maturity

The square root effect!

Options data II
Aspects of analysis

The correction to the price now satisfies:

\[ \langle \mathcal{L}_2 \rangle \tilde{P}_1 = V_2(t)x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3(t)x^3 \frac{\partial^3 P_0}{\partial x^3} \]

Note that the average is with respect to the same time-independent invariant distribution.

\[ \tilde{P}_1 = -(T - t) \left[ \overline{V_2} x^2 \frac{\partial^2 P_0}{\partial x^2} + \overline{V_3} x^3 \frac{\partial^3 P_0}{\partial x^3} \right] \]

where

\[ \overline{V_3} = \frac{1}{N + \delta} \int_{\delta}^{1} V_3(T_0 s) ds + \frac{N}{N + \delta} \int_{0}^{1} V_3(T_0 s) ds \]
\[ \overline{V_2} = \text{similar.} \]
Conclusions

**Stochastic Volatility under separation of scales:**

- Flexible and parsimonious extension to Black-Scholes.
- A consistent framework honoring underlying price dynamics.
- Robust to modelling assumptions.
- Provides an environment/tool for data analysis.
- Provides leading order corrections to pricing formulas, hedging strategies etc.