Mean reverting Stochastic Volatility
Scales, Data and Price Dynamics

Jean-Pierre Fouque
University of California Santa Barbara

CHIOS, GREECE
July 2006
• Macro-scale: $\mathcal{O}(1) \sim \text{time to maturity scale } (T)$.
• Intermediate-scale: $\mathcal{O}(1/\alpha) = \mathcal{O}(\epsilon) \sim \text{mean-reversion scale}$.
• Micro-scale: $\sim \text{‘tick by tick’ scale}$.
Outline

→ Return fluctuations and averaging.
→ The S&P 500 data.
→ Estimation of scale of mean reversion.
→ Model specific estimation.
→ Comments on analysis.
Stochastic volatility and return processes

→ Stochastic volatility model: \( dX_t = \mu X_t \, dt + f(Y_t) X_t \, dW_t \).

- Volatility: \( f(Y_t) \).
- Returns: \( f(Y_t) \, dW_t \).
- Fast mean reverting ergodic driving process: \( Y_t \).

→ Rate of mean reversion \( \alpha \):

\[
\text{Covariance}(Y_{t+\text{lag}}, Y_t) \sim \exp(-\alpha \text{lag})
\]

→ Illustration:

- \( Y_t \): Ornstein-Uhlenbeck process
- \( f(x) = \exp(x) \).
Ornstein-Uhlenbeck driving process

Volatility and return realization for $\alpha = 1$:
Ornstein-Uhlenbeck driving process

Volatility and return realization for $\alpha = 1000$:
Ornstein-Uhlenbeck driving process

Volatility and return realization for $\alpha = 100$:

Illustration
Ergodic Theorem

\[
\lim_{N \to \infty} \frac{1}{N/\alpha} \int_0^{N/\alpha} g(Y_s) \, ds = \langle g \rangle \quad \text{almost surely}
\]

In particular

\[
\frac{1}{T-t} \int_t^T f(Y_s)^2 \, ds \approx \langle f^2 \rangle = \bar{\sigma}^2
\]

for \( T - t >> 1/\alpha \).
Averaging

Plot shows: \( \frac{1}{N/\alpha} \int_0^{N/\alpha} f(Y_s)^2 \, ds \).

- \( Y_s \): Ornstein-Uhlenbeck driving process.
- \( f(x) = \exp(x) \)
Exponential decorrelation

Variogram: \[ \mathbb{E}[f(Y_{t+\text{lag}}) - f(Y_t)]^2 \approx 2 \bar{\sigma}^2 (1 - e^{-\alpha \text{lag}}) \]

- \( Y_s \): Ornstein-Uhlenbeck driving process.
- \( f(x) = \exp(x) \)
Exponential decorrelation for returns

Variograms for raw and median filtered returns:

Illustration
**S&P 500 data**

- Average data over five-minute intervals: **72 points per day** × 251 trading days = **18,072 points**.
- Structure function \( \sum_{i=1}^{N} \left[ X_{i+\text{lag}} - X_i \right]^2 / N \) of index:
Estimation

• Stochastic volatility model:

\[ dX_t = \mu X_t dt + f(Y_t)X_t dW_t \]

• Volatility hidden, only see returns:

\[ \frac{dX_t}{X_t} - \mu dt = f(Y_t)dW_t. \]

• The de-meaned return process:

\[ D_{t_i} = \frac{1}{\sqrt{\Delta t}} \left( \frac{\Delta X_{t_i}}{X_{t_i}} - \text{mean} \right) \sim f(Y_{t_i})\Delta W_{t_i} = f(Y_{t_i})\epsilon_{t_i} \]

where \( \epsilon_{t_i} \) is an i.i.d “white noise” sequence.
Variogram of log-returns

- Log-returns:

\[ L_{t_i} = \log |D_{t_i}| = \log(f(Y_{t_i})) + \log|\epsilon_{t_i}| = \tilde{f}(Y_{t_i}) + \tilde{\epsilon}_{t_i}. \]

- Variogram is like an exponential:

\[
\frac{1}{N} \sum_{i=1}^{N} \left[ L_{t_i + \text{lag}} - L_{t_i} \right]^2 \approx c_1(1 - e^{-\alpha \text{lag}}) + c_2.
\]
Notice the **day effect** removed by the fitted exponential.

The **curvature** determines $1/\alpha : 1.5 \pm 0.4$ trading days

annualized $\alpha \approx 130 - 230$: large

**Estimation**
Spectrum of log-returns

Energy spectrum of log returns:

\[ \int \mathbb{E}[L_{\bar{t}+t}L_{\bar{t}}] \cos(\omega t) \, dt \approx d_1 + d_2 \frac{\alpha}{\alpha^2 + \omega^2} + d_3 \delta(\omega - \omega_1) : \]

S&P500 and Lorentzian spectra

Simulated Lorentzian spectra

Estimation
Model-specific aspects

• Mean-reverting Stochastic Volatility:

\[ dX_t = \mu X_t dt + f(Y_t, t) X_t dW_t, \]

\[ dY_t = \alpha (m - Y_t) dt + \beta d\hat{Z}_t, \]

\[ \hat{Z}_t := \rho W_t + \sqrt{1 - \rho^2} Z_t. \]

• Lunch moderated exponential burstiness:

\[ f(Y_t, t) = \exp(Y_t) g(t) \]

(period of \( g \) one trading-day).

• skewness parameter: \( \rho \).
Day variability envelope $g$: 

\[ f(Y_t, t) = \exp(Y_t) \ g(t) \]
Components of returns

Use scale separation in time to separate:

$E[X] \times \delta_{X} \in \mathcal{D}$

Estimation

Synthetic $e_{Y}g_{j}$

Synthetic $\epsilon_{j}$

S&P500 $e_{Y}g_{j}$

S&P500 $\epsilon_{j}$

Components of returns
Synthetic and measured fine scale structure

Estimation
Variability of $\alpha$ estimate

- Repeat variogram analysis with simulated data for $1/\alpha = 1.5$ days. (use model-specific parameter values)

- Notice that even with “perfect” data, estimates of $\alpha$ are of the right order, but highly variable.
Aspects of analysis

\[ P^\epsilon = P^\epsilon(t, \tau, x, y) = P^\epsilon(t, t/\epsilon, x, y) \]

\[ \frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + \frac{1}{\epsilon} \frac{\partial}{\partial \tau} \]

\[ < f^2 > \equiv \int_0^1 \int_{-\infty}^{\infty} f(\tau, y)^2 \Phi(y) \, dy \, d\tau \]

→ Modified averaging!
Conclusions

• Mean Reverting Stochastic Volatility:
  Parsimonious extension to Black-Scholes.

• Evidence for fast mean reversion in S&P 500.

• Exploit separation of time-scales in analysis.

• Asymptotics give robust price correction (next).

• Model specific estimation allows Monte Carlo.

• Effects of \textit{strong} volatility fluctuations incorporated.