Volatility Time Scales
and
Perturbations

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Main Idea

Use perturbation techniques to correct constant volatility models in order to capture the effects of stochastic volatility

Applications to:
Equity, Fixed Income, and Credit Markets

Equity

Perturbations around Black-Scholes

to account for:

- **Distributions of Returns**
  (under physical measure $\mathbb{P}$)
- **Volatility Time Scales**
- **Smile/Skew in Implied Volatilities**
  (under risk neutral measure $\mathbb{P}^\star$)

"Parametrization" of the Implied Volatility Surface $I(t\,;T,K)$

REQUIRED QUALITIES

- **Universal Parsimonous Parameters**: *Model Independence*
- **Stability in Time**: *Predictive Power*
- **Easy Calibration**: *Practical Implementation*
- **Compatibility with Price Dynamics**: *Applicability to Pricing other Derivatives and Hedging*

Extensive Data Ananlysis $\Longrightarrow$
Two-Scale Stochastic Volatility Models

$$\varepsilon << T << 1/\delta$$

Under the physical measure $I\!P$:

$$\begin{aligned} dX_t &= \mu X_t dt + f(Y_t, Z_t) X_t dW_t^{(0)} \\ dY_t &= \frac{1}{\varepsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} dW_t^{(1)} \\ dZ_t &= \delta\ c(Z_t)dt + \sqrt{\delta}\ g(Z_t) dW_t^{(2)} \end{aligned}$$

$$\begin{aligned} d < W^{(0)}, W^{(1)} >_t &= \rho_1 dt \\ d < W^{(0)}, W^{(2)} >_t &= \rho_2 dt \end{aligned}$$

Two-Scale Stochastic Volatility Models

$$\varepsilon << T << 1/\delta$$

Under the **risk neutral** measure $I\!P^\star$ *chosen by the market*:

$$\begin{aligned}
dX_t &= rX_t dt + f(Y_t, Z_t) X_t dW_t^{(0)\star} \\
dY_t &= \left(\frac{1}{\varepsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} \Lambda(Y_t, Z_t) \right) dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} dW_t^{(1)\star} \\
dZ_t &= \left(\delta\ c(Z_t) - \sqrt{\delta}\ g(Z_t)\Gamma(Y_t, Z_t) \right) dt + \sqrt{\delta}\ g(Z_t) dW_t^{(2)\star}
\end{aligned}$$

$$\begin{aligned}
d < W^{(0)\star}, W^{(1)\star} >_t &= \rho_1 dt \\
d < W^{(0)\star}, W^{(2)\star} >_t &= \rho_2 dt
\end{aligned}$$

Λ and Γ: market prices of volatility risk

Pricing Equation

$$P^{\varepsilon,\delta}(t,x,y,z) \quad = \quad \mathbb{E}^{\star}\left\{e^{-r(T-t)}h(X_T)|X_t = x, Y_t = y, Z_t = z\right\}$$

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3\right) P^{\varepsilon,\delta} = 0$$

$$P^{\varepsilon,\delta}(T,x,y,z) = h(x)$$

$$\mathcal{L}_0 = (m-y)\frac{\partial}{\partial y} + \nu^2\frac{\partial^2}{\partial y^2} \qquad \mathcal{M}_1 = -g\Gamma\frac{\partial}{\partial z} + \rho_2 g f x\frac{\partial^2}{\partial x\partial z}$$

$$\mathcal{L}_1 = \nu\sqrt{2}\left(\rho_1 f x\frac{\partial^2}{\partial x\partial y} - \Lambda\frac{\partial}{\partial y}\right) \qquad \mathcal{M}_2 = c\frac{\partial}{\partial z} + \frac{g^2}{2}\frac{\partial^2}{\partial z^2}$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2}f^2x^2\frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right) \qquad \mathcal{M}_3 = \nu\sqrt{2}\,\tilde{\rho}_{12}g\frac{\partial^2}{\partial y\partial z}$$

European Options Approximations

Combination of **singular** and **regular** **perturbations** $\Longrightarrow$

$$P^{\varepsilon,\delta}(t,x,y,z) \approx \mathbf{P_{BS}}(\mathbf{t},\mathbf{x};\mathbf{T},\bar{\sigma}) + (T-t)\left(V_0^\delta \frac{\partial P_{BS}}{\partial \sigma} + V_1^\delta x \frac{\partial^2 P_{BS}}{\partial x \partial \sigma}\right)$$

$$+(T-t)\left(V_2^\varepsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_{BS}}{\partial x^2}\right)\right)$$

Leading order Black-Scholes price $\mathbf{P_{BS}}(\mathbf{t},\mathbf{x};\bar{\sigma}(\mathbf{z}))$:

$$\mathcal{L}_{BS}(\overline{\sigma}(z))P_{BS} = 0$$

$$P_{BS}(T,x;\bar{\sigma}(z)) = h(x)$$

at the z-dependent **effective volatility** $\overline{\sigma}(z)$:

$$\overline{\sigma}^2(z) \quad = \quad \langle f^2(\cdot,z)\rangle$$

where the brackets denote the **average** with respect to the invariant distribution $\mathcal{N}(m_f,\nu_f^2)$.

The **small** parameters $(V_0^\delta, V_1^\delta, V_2^\varepsilon, V_3^\varepsilon)$ are given by

$$V_0^\delta = -\frac{\nu_s\sqrt{\delta}}{\sqrt{2}}\langle \mathbf{\Lambda_s}\rangle\overline{\sigma}' \qquad V_1^\delta = \rho_{\mathbf{2}}\frac{\nu_s\sqrt{\delta}}{\sqrt{2}}\langle f\rangle\overline{\sigma}'$$

$$V_2^\varepsilon = \frac{\nu_f\sqrt{\varepsilon}}{\sqrt{2}}\left\langle \mathbf{\Lambda_f}\frac{\partial\phi}{\partial y}\right\rangle \qquad V_3^\varepsilon = -\rho_{\mathbf{1}}\frac{\nu_f\sqrt{\varepsilon}}{\sqrt{2}}\left\langle f\frac{\partial\phi}{\partial y}\right\rangle$$

$\overline{\sigma}' = d\bar{\sigma}/dz$, and $\phi(y,z)$ is a solution of the **Poisson equation**

$$\mathcal{L}_0\phi(y,z) \quad = \quad f^2(y,z) - \overline{\sigma}^2(z).$$

Accuracy:

- **Smooth** payoffs: error $= \mathcal{O}(\varepsilon + \delta)$
- **Calls** (kinks): error $= \mathcal{O}(\varepsilon \log|\varepsilon| + \delta)$
- **Digitals** (jumps): error $= \mathcal{O}(\varepsilon^{2/3}\log|\varepsilon| + \delta)$

Corrections Equations

$$P_1^\varepsilon(t,x,z) \quad = \quad (T-t)\left(V_2^\varepsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_{BS}}{\partial x^2}\right)\right)$$

solves

$$\mathcal{L}_{BS}(\bar{\sigma})P_1^\varepsilon + \left(V_2^\varepsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_{BS}}{\partial x^2}\right)\right) = 0, \qquad P_1^\varepsilon(T,x,z) = 0$$

$$P_1^\delta(t,x,z) \quad = \quad (T-t)\left(V_0^\delta \frac{\partial P_{BS}}{\partial \sigma} + V_1^\delta x \frac{\partial^2 P_{BS}}{\partial x \partial \sigma}\right)$$

solves

$$\mathcal{L}_{BS}(\bar{\sigma})P_1^\delta + 2\left(V_0^\delta \frac{\partial P_{BS}}{\partial \sigma} + V_1^\delta x \frac{\partial^2 P_{BS}}{\partial x \partial \sigma}\right) = 0, \qquad P_1^\delta(T,x) = 0$$

(for European options: $\frac{\partial P_{BS}}{\partial \sigma} = (T-t)\sigma x^2 \frac{\partial^2 P_{BS}}{\partial x^2}$)

Term Structure of Implied Volatility

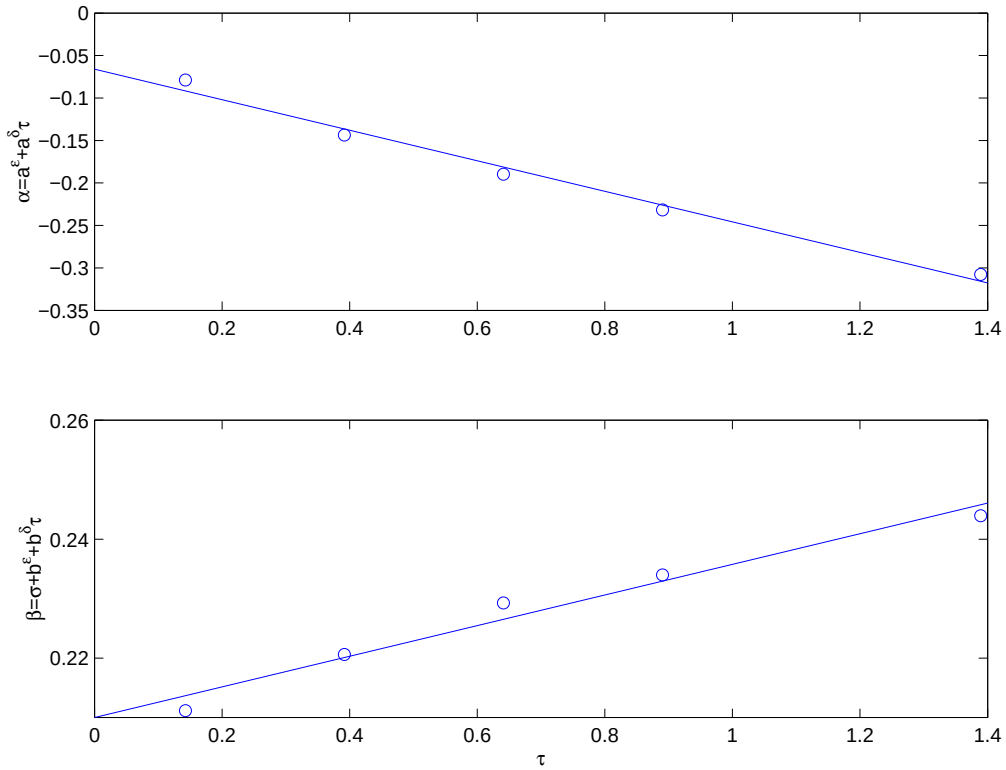
$$I_0 + I_1^\varepsilon + I_1^\delta =$$

$$\bar{\sigma} + [b^\varepsilon + b^\delta(T-t)] + [a^\varepsilon + a^\delta(T-t)]\frac{\log(K/x)}{T-t},$$

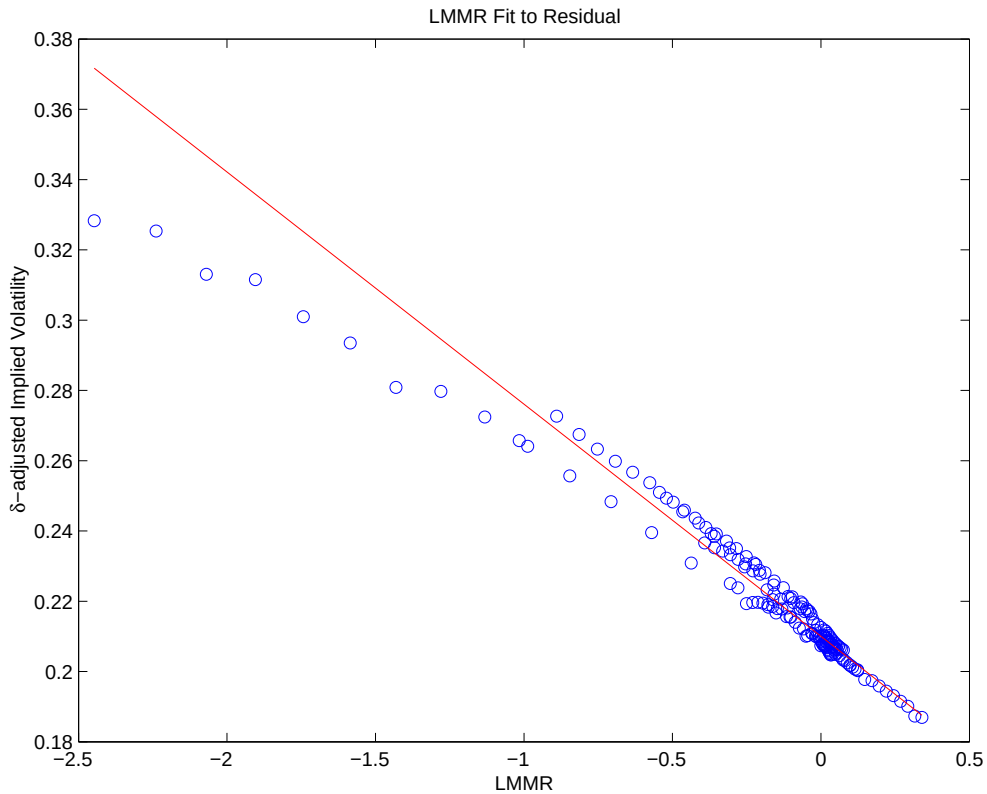
where the parameters $(\bar{\sigma}, a^\varepsilon, a^\delta, b^\varepsilon, b^\delta)$ depend on z and are related to the group parameters $(V_0^\delta, V_1^\delta, V_2^\varepsilon, V_3^\varepsilon)$ by

$$a^\varepsilon = \frac{V_3^\varepsilon}{\bar{\sigma}^3} \quad , \quad b^\varepsilon = \frac{V_2^\varepsilon}{\bar{\sigma}} - \frac{V_3^\varepsilon}{\bar{\sigma}^3}(r - \frac{\bar{\sigma}^2}{2})$$

$$a^\delta = \frac{V_1^\delta}{\bar{\sigma}^2} \quad , \quad b^\delta = V_0^\delta - \frac{V_1^\delta}{\bar{\sigma}^2}(r - \frac{\bar{\sigma}^2}{2})$$

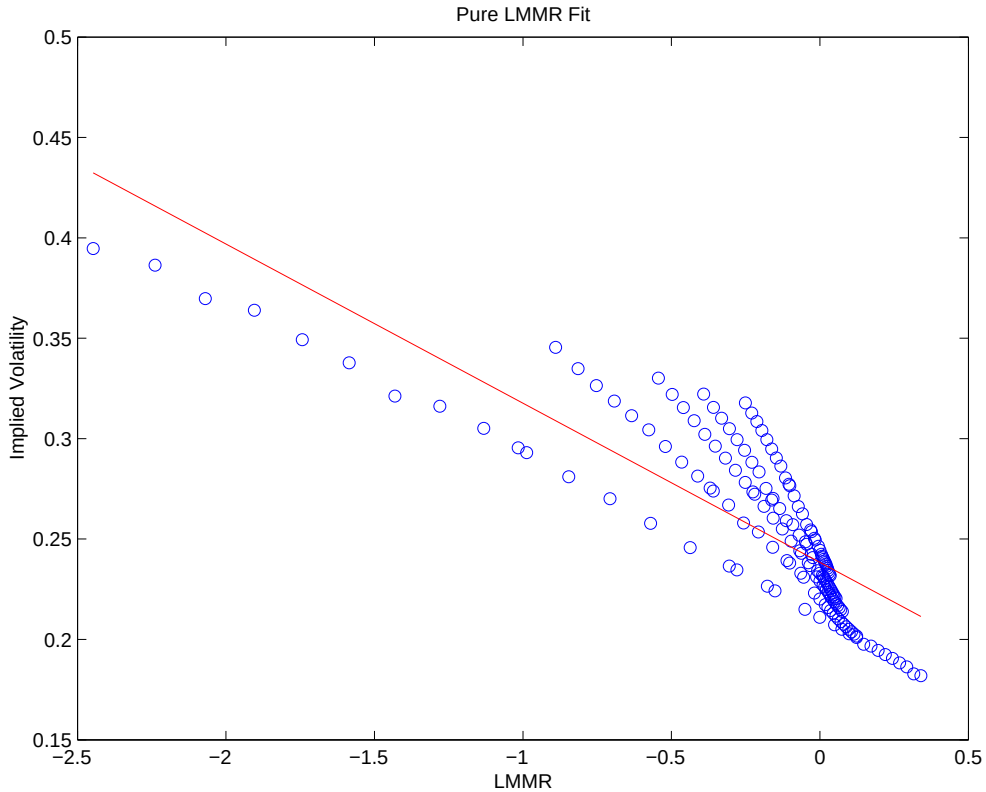


Term-structures fits



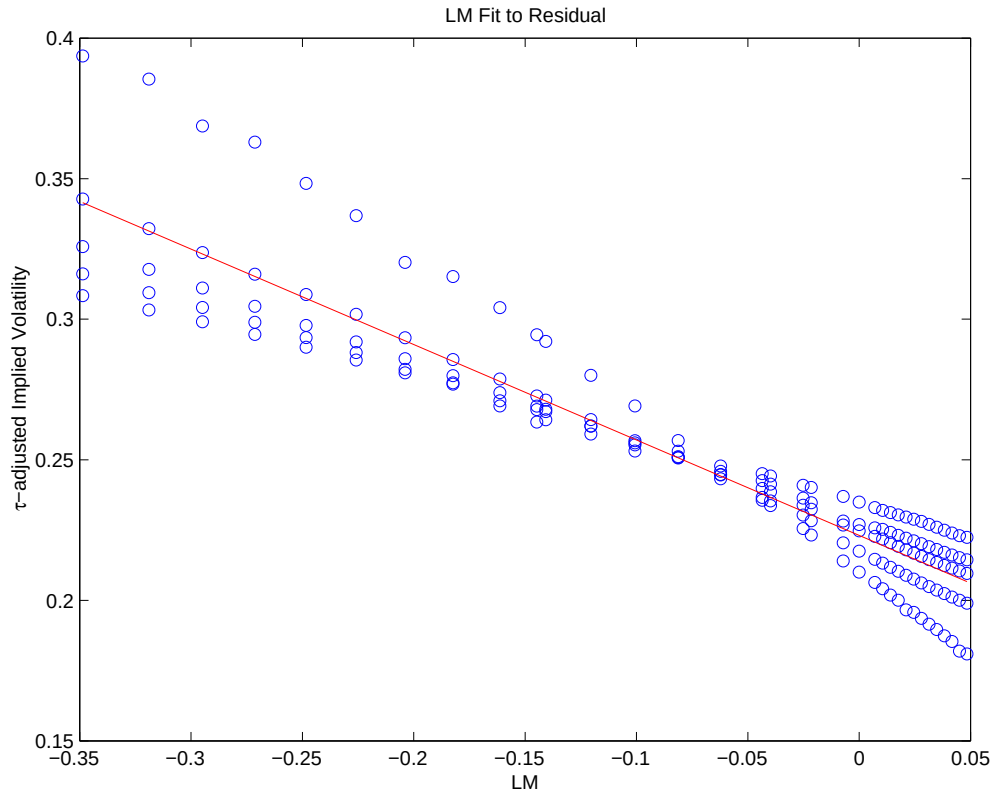
δ-adjusted implied volatility $I - b^{\delta}\tau - a^{\delta}(LM)$ as a function of LMMR. The circles are from S&P 500 data, and the line $R + a^{\varepsilon}(LMMR)$ shows the fit using the estimated parameters.

A slow volatility factor is needed



Implied volatility as a function of LMMR. The circles are from S&P 500 data, and the line $a(LMMR) + b$ shows the fit using maturities up to two years.

A fast volatility factor is needed



The circles are from S&P 500 data, and the line $a^{\delta}(LM) + \bar{\sigma}$ shows the fit using the estimated parameters from only a slow factor fit.

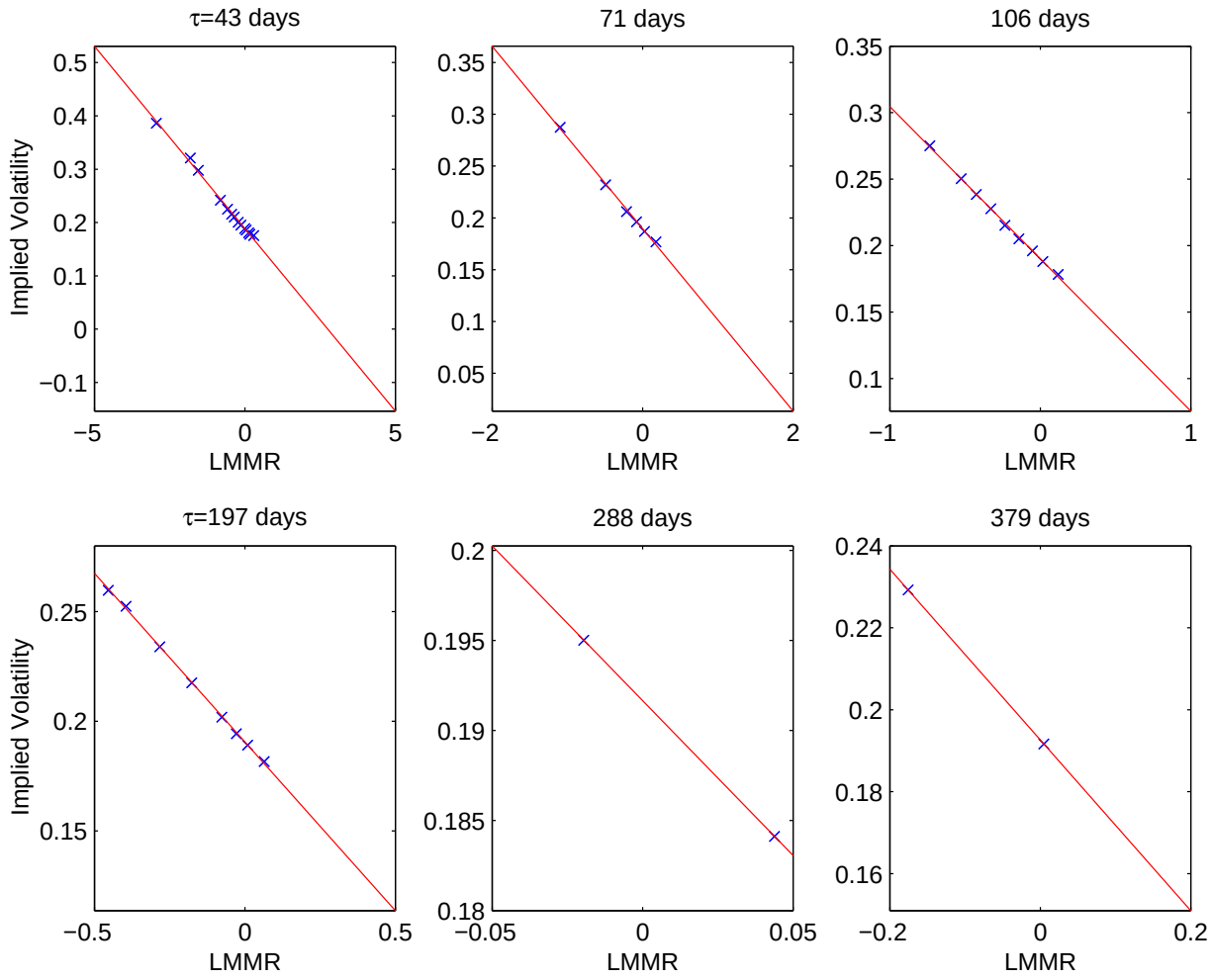


Figure 1: *S&P* 500 *Implied Volatility data on June 5, 2003 and fits to the affine LMMR approximation for six different maturities.*

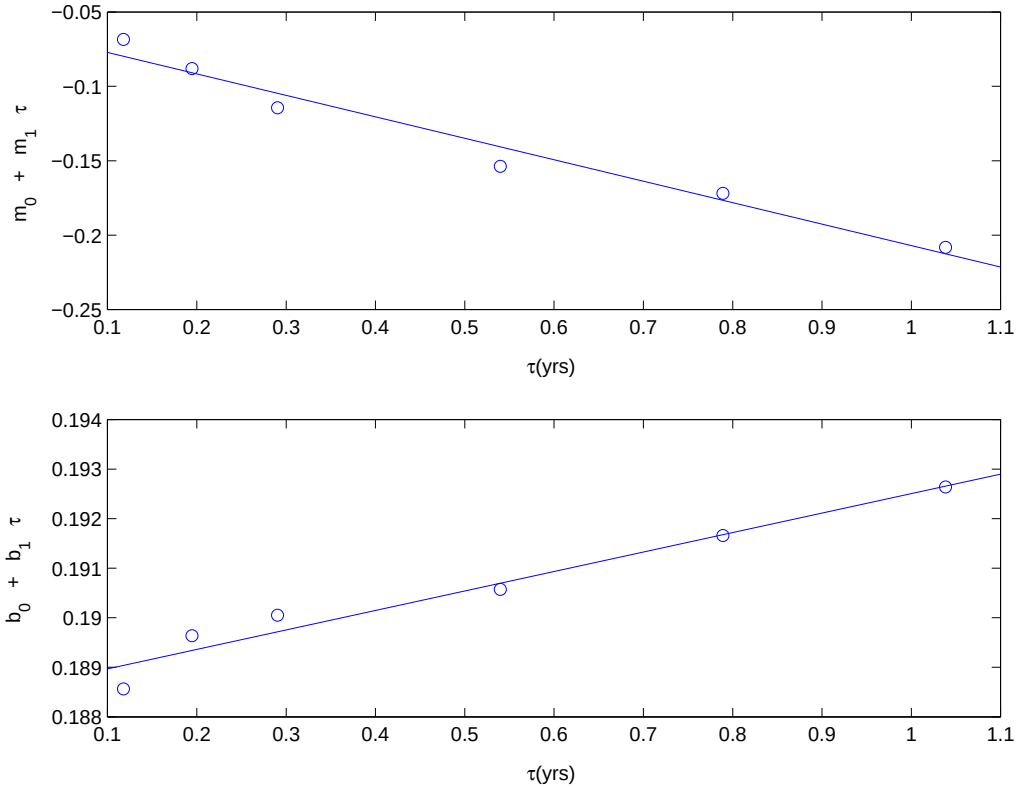


Figure 2: *S&P* 500 *Implied Volatility data on June 5, 2003 and fits to the two-scales asymptotic theory. The bottom (rep. top) figure shows the linear regression of* b *(resp.* a*) with respect to time to maturity* $\tau = T - t$*.*

Higher order terms in ε, δ and $\sqrt{\varepsilon\delta}$

$$I \approx \sum_{j=0}^{4} a_j(\tau)\,(LM)^j + \frac{1}{\tau}\Phi_t,$$

where

τ denotes the time-to maturity $T - t$,

LM denotes the moneyness $\log(K/S)$,

and Φ_t is a rapidly changing component that varies with the fast volatility factor

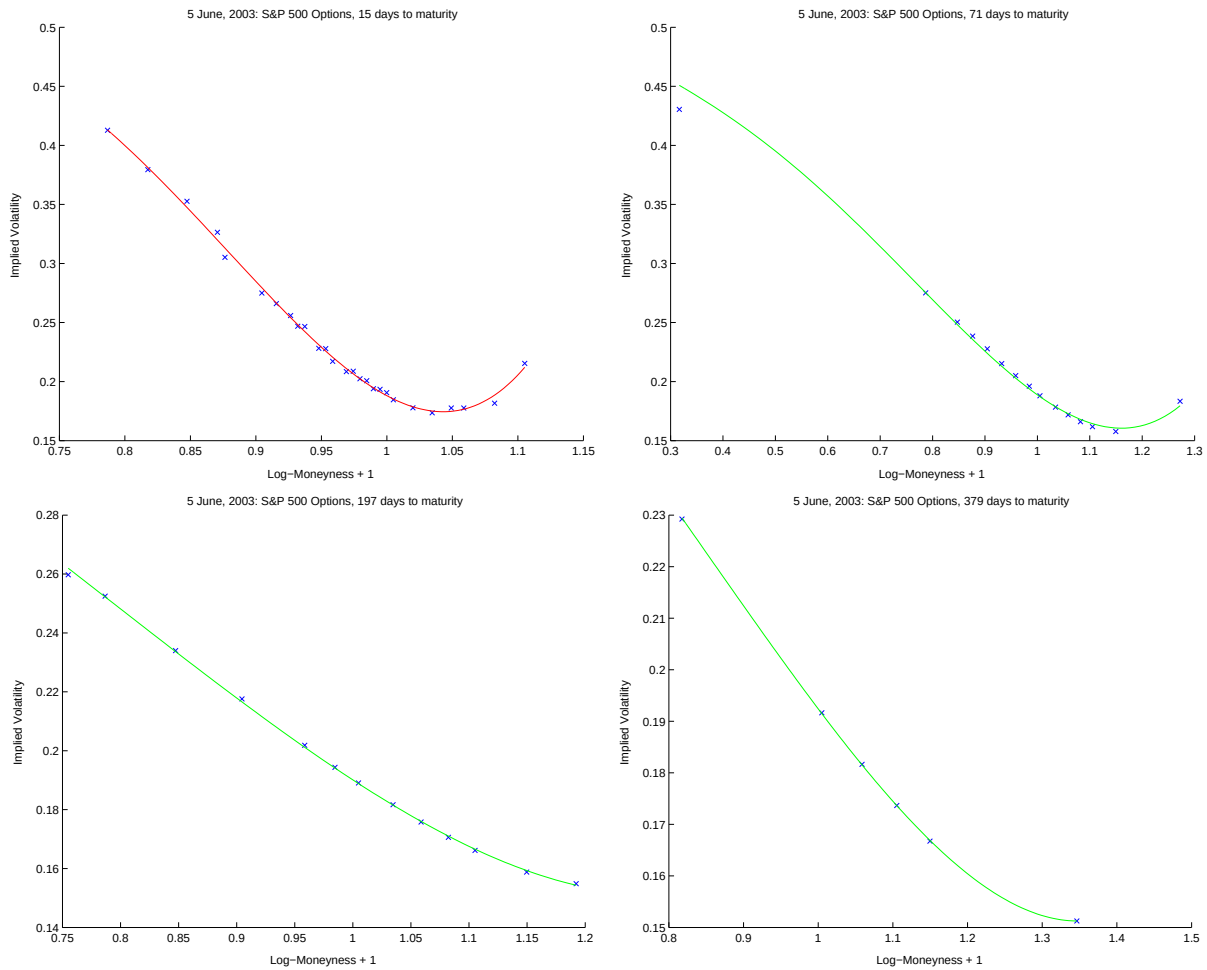


Figure 3: *S&P* 500 *Implied Volatility data on June 5, 2003 and quartic fits to the asymptotic theory for four maturities.*

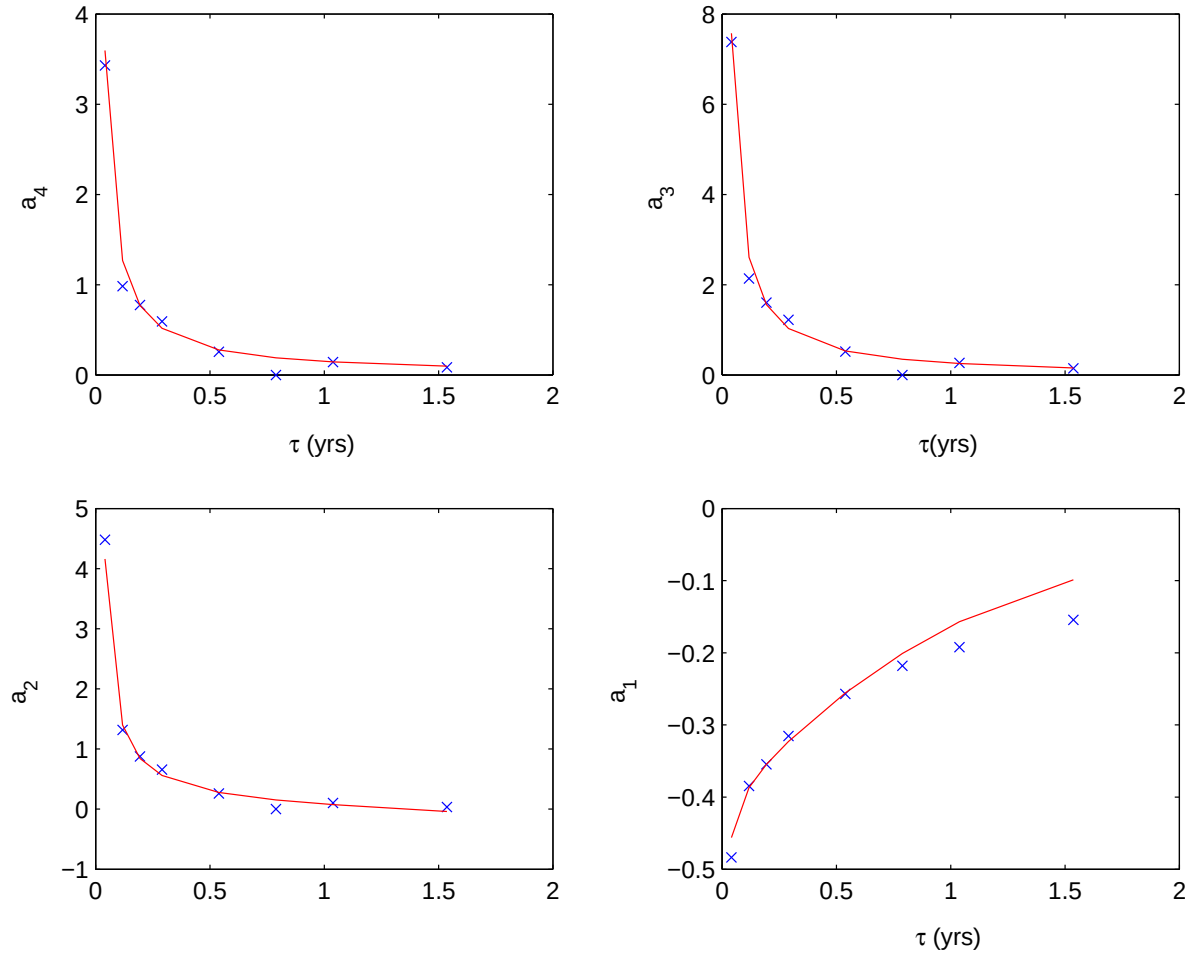


Figure 4: *S&P* 500 *Term-Structure Fit using second order approximation. Data from June 5, 2003.*

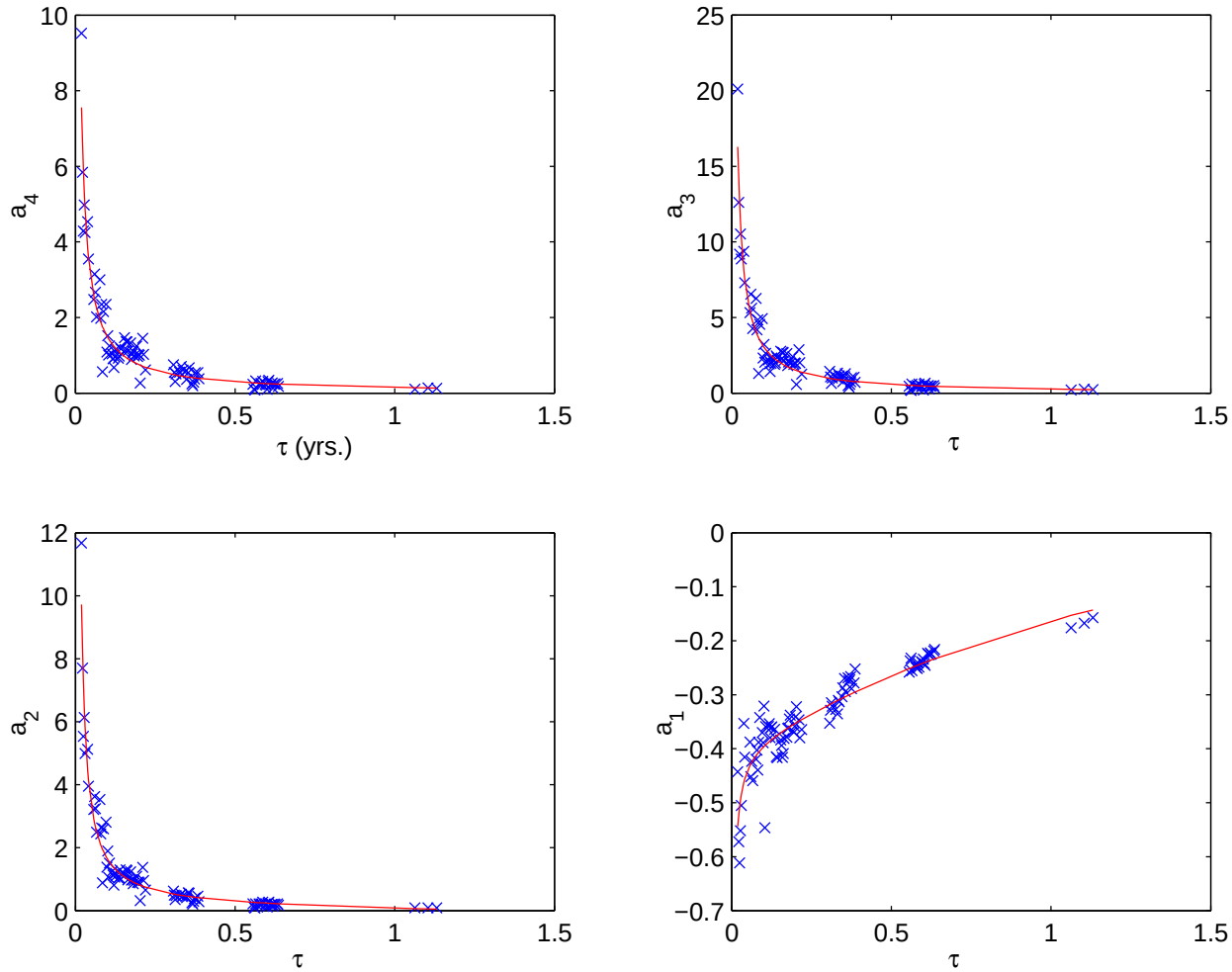


Figure 5: *S&P* 500 *Term-Structure Fit. Data from every trading day in May 2003.*

Parameter Reduction and Direct Calibration

$$\mathcal{L}_{BS}(\bar{\sigma})\left(\tilde{P}_1+\tilde{Q}_1\right) \quad + \quad \left(V_2 x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3 x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_{BS}}{\partial x^2}\right)\right)$$

$$+ \quad 2\left(V_0 \frac{\partial P_{BS}}{\partial \sigma} + V_1 x \frac{\partial^2 P_{BS}}{\partial x \partial \sigma}\right) = 0$$

Set $\sigma^\star = \sqrt{\bar{\sigma}^2 + 2V_2}$. At the same order, the correction is:

$$(T-t)\left(V_0 \frac{\partial P^\star_{BS}}{\partial \sigma} + V_1 x \frac{\partial^2 P^\star_{BS}}{\partial x \partial \sigma} + V_3 x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P^\star_{BS}}{\partial x^2}\right)\right)$$

$I \approx b^\star + \tau b^\delta + \left(a^\varepsilon + \tau a^\delta\right)$ LMMR

$$b^\star = \sigma^\star + \frac{V_3}{2\sigma^\star}\left(1 - \frac{2r}{\sigma^{\star 2}}\right) \quad , \quad a^\varepsilon = \frac{V_3}{\sigma^{\star 3}}$$

$$b^\delta = V_0 + \frac{V_1}{2}\left(1 - \frac{2r}{\sigma^{\star 2}}\right) \quad , \quad a^\delta = \frac{V_1}{\sigma^{\star 2}}$$

Accuracy of Approximation

For European options with **smooth payoffs** $h(x)$:

$$P^{\varepsilon,\delta} = P^{\star}_{BS} + \tilde{P}_1 + \tilde{Q}_1 + \mathcal{O}(\varepsilon + \delta)$$

For European **calls or puts**, $h(x)$ continuous piecewise smooth:

$$P^{\varepsilon,\delta} = P^{\star}_{BS} + \tilde{P}_1 + \tilde{Q}_1 + \mathcal{O}(\varepsilon \log|\varepsilon| + \delta)$$

For European **digital option**, $h(x) = Q\mathbf{1}_{\{x>K\}}$ discontinuous:

$$P^{\varepsilon,\delta} = P^{\star}_{BS} + \tilde{P}_1 + \tilde{Q}_1 + \mathcal{O}(\varepsilon^{\frac{2}{3}} \log|\varepsilon| + \delta)$$

Exotic Derivatives (Binary, Barrier, Asian,...)

- Calibrate $\sigma^\star, V_0, V_1$ and V_3 on the implied volatility surface
- Solve the corresponding problem with **constant volatility** $\sigma^\star$

$$\Longrightarrow P_0 = P_{BS}(\sigma^\star)$$

- Use V_0, V_1 and V_3 to compute the **source**

$$2\left(V_0 \frac{\partial P_{BS}^\star}{\partial \sigma} + V_1 x \frac{\partial^2 P_{BS}^\star}{\partial x \partial \sigma}\right) + V_3 x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_{BS}^\star}{\partial x^2}\right)$$

- Get the **correction** by solving the <u>SAME PROBLEM</u> with **zero boundary conditions** and the **source**.

American Options

- Calibrate $\sigma^\star, V_0, V_1$ and V_3 on the implied volatility surface
- Solve the corresponding problem with **constant volatility** $\sigma^\star$

$$\Longrightarrow P^\star \text{ and the free boundary } x^\star(t)$$

- Use V_0, V_1 and V_3 to compute the **source**

$$2\left(V_0 \frac{\partial P^\star}{\partial \sigma} + V_1 x \frac{\partial^2 P^\star}{\partial x \partial \sigma}\right) + V_3 x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P^\star}{\partial x^2}\right)$$

- Get the **correction** by solving the corresponding problem with **fixed boundary** $x^\star(t)$, **zero boundary conditions** and the **source**.

First Conclusions

- **A short time-scale of order few days is present in volatility dynamics**
- **It cannot be ignored in option pricing and hedging**
- **It can be dealt with by using singular perturbation methods**
- **It is efficient as a parametrization tool for the term structure of implied volatilities when combined with a regular perturbation**

Fixed Income

Perturbations around Vasicek (for instance)

to account for:

- **Volatility Time Scales**
- **Fit to Yield Curves**

Reference:

Stochastic Volatility Corrections for Interest Rate Derivatives

Mathematical Finance 14(2), April 2004

Constant Volatility Vasicek Model

Under the physical probability $I\!P$:

$$d\bar{r}_t = a(\bar{r}_\infty - \bar{r}_t)dt + \bar{\sigma}d\bar{W}_t$$

Under the risk-neutral pricing probability $I\!P^\star$:

$$d\bar{r}_t = a(r^\star - \bar{r}_t)dt + \bar{\sigma}d\bar{W}_t^\star$$

with a constant **market price of interest rate risk** λ:

$$r^\star = \bar{r}_\infty - \frac{\lambda\bar{\sigma}}{a}$$

Bonds Prices

$$\Lambda(t,T) = \mathbb{E}^\star\left\{e^{-\int_t^T \bar{r}_s ds}|\mathcal{F}_t\right\} = \mathbb{E}^\star\left\{e^{-\int_t^T \bar{r}_s ds}|\bar{r}_t\right\} = \bar{P}(t,\bar{r}_t;T)$$

Vasicek PDE:

$$\frac{\partial \bar{P}}{\partial t} + \frac{1}{2}\bar{\sigma}^2\frac{\partial^2 \bar{P}}{\partial x^2} + a(r^\star - x)\frac{\partial \bar{P}}{\partial x} - x\bar{P} = 0$$

with the **terminal condition** $\bar{P}(t,x;T) = 1$.

Introduce the **time-to-maturity** $\tau = T - t$ and seek a solution of the form:

$$\bar{P}(T-\tau,x;T) = A(\tau)e^{-B(\tau)x}$$

by solving linear ODE's with $A(0) = 1$ and $B(0) = 0$.

Affine Yields

$$
\begin{aligned}
B(\tau) &= \frac{1-e^{-a\tau}}{a} \\
A(\tau) &= \exp\left\{-\left[R_\infty \tau - R_\infty \frac{1-e^{-a\tau}}{a} + \frac{\bar{\sigma}^2}{4a^3}\left(1-e^{-a\tau}\right)^2\right]\right\}
\end{aligned}
$$

with

$$
R_\infty = r^\star - \frac{\bar{\sigma}^2}{2a^2} = \bar{r}_\infty - \frac{\lambda\bar{\sigma}}{a} - \frac{\bar{\sigma}^2}{2a^2}
$$

Yield Curve:

$$
\begin{aligned}
R(t,\tau) &= -\frac{1}{\tau}\log\left(\Lambda(t,t+\tau)\right) \\
&= -B(\tau)\bar{r}_t + \log A(\tau) \\
&= R_\infty - (R_\infty - \bar{r}_t)\frac{1-e^{-a\tau}}{a\tau} + \frac{\bar{\sigma}^2}{4a^3\tau}\left(1-e^{-a\tau}\right)^2
\end{aligned}
$$

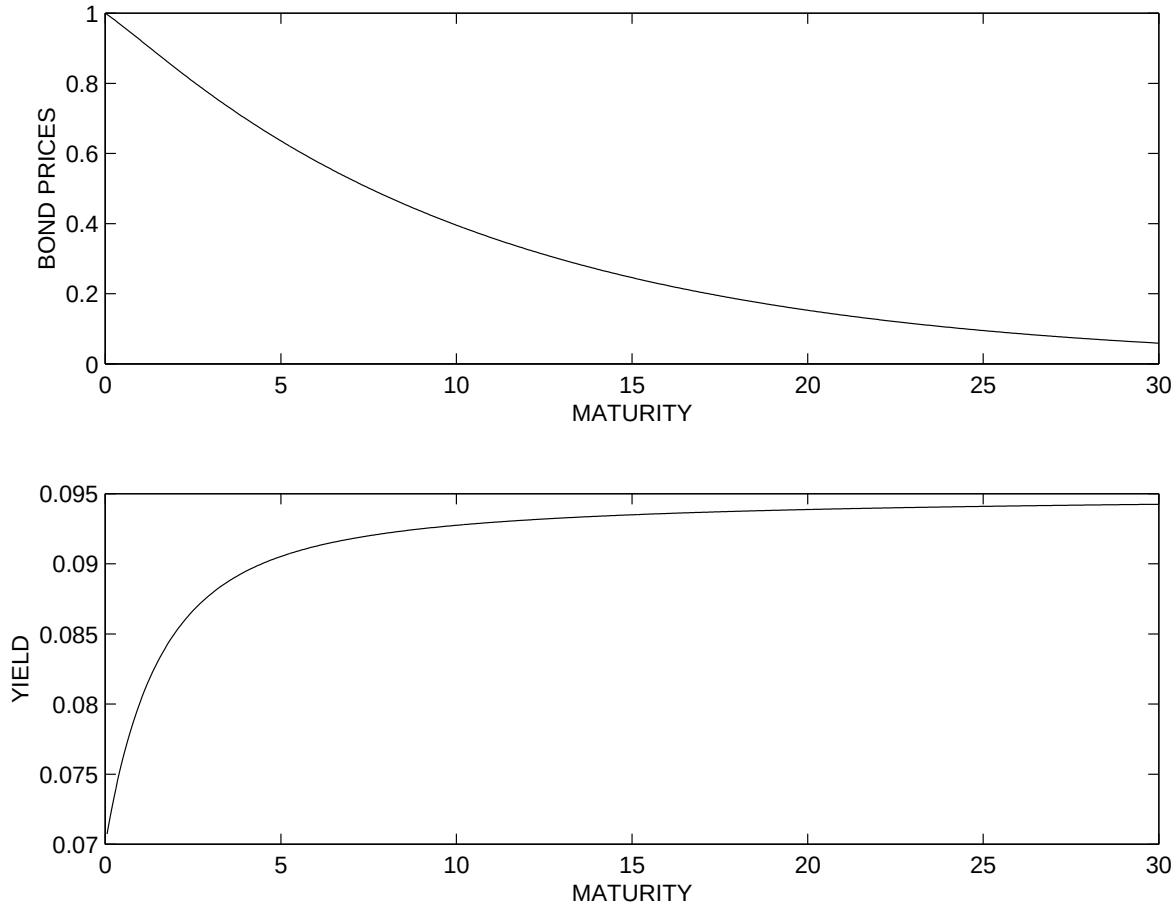


Figure 6: *Bond prices (top) and cblue Yield curve (bottom) in the Vasicek model with* $a = 1$, $r^\star = 0.1$ *and* $\bar{\sigma} = 0.1$. *Maturity* τ *runs from* 0 *to* 30 *years.* $R_\infty = 0.095$ *and the initial rate is* $x = 0.07$.

Bond Options Prices

Example: a **Call Option** with **strike** K and **maturity** T_0 written on a zero-coupon bond with maturity $T > T_0$.

The **payoff**

$$h(\Lambda(T_0, T)) = (\Lambda(T_0, T) - K)^+$$

is a function of $\bar{r}_{T_0}$ since $\Lambda(T_0, T) = \bar{P}(T_0, \bar{r}_{T_0}; T)$

Call Option Price:

$$\bar{C}(t, x; T, T_0) = \mathbb{E}^\star \left\{ e^{-\int_t^{T_0} \bar{r}_s ds} h\left(\Lambda(T_0, T)\right) | \bar{r}_t = x \right\}$$

solution of Vasicek PDE with terminal condition at $t = T_0$:

$$\bar{C}(T_0, x; T, T_0) = (\bar{P}(T_0, x; T) - K)^+$$

$$\bar{C}(t, x; T, T_0) = \bar{P}(t, x; T) N(h_1) - K \bar{P}(t, x; T_0) N(h_2)$$

Stochastic Volatility Vasicek Models

Under the physical measure:

$$dr_t = a(r_\infty - r_t)dt + f(Y_t)dW_t$$

where f is a positive function of a **mean-reverting volatility driving process** Y_t.

Example: Y_t is an OU process:

$$dY_t = \alpha(m - Y_t)dt + \nu\sqrt{2\alpha}d\hat{Z}_t$$

where $\hat{Z}_t$ is a Brownian motion possibly **correlated** to the Brownian motion W_t driving the short rate:

$$\hat{Z}_t = \rho W_t + \sqrt{1-\rho^2}\, Z_t$$

(W_t, Z_t) independent Brownian motions.

Stochastic Volatility Vasicek Pricing Models

Under the risk-neutral pricing probability $I\!P^{\star(\lambda,\gamma)}$:

$$
\begin{aligned}
dr_t &= \left(a(r_\infty - r_t) - \lambda(Y_t) f(Y_t)\right) dt + f(Y_t) dW_t^\star \\
dY_t &= \left(\alpha(m - Y_t) - \nu\sqrt{2\alpha}\left[\rho\lambda(Y_t) + \gamma(Y_t)\sqrt{1-\rho^2}\right]\right) dt \\
&\quad + \nu\sqrt{2\alpha}\left(\rho dW_t^\star + \sqrt{1-\rho^2}\, dZ_t^\star\right)
\end{aligned}
$$

for bounded market prices of risk $\lambda(y)$ and $\gamma(y)$.

Under **fast mean-reversion**:

α is large

Bond Pricing

$$P(t,x,y;T) = \mathbb{E}^{\star(\lambda,\gamma)}\left\{e^{-\int_t^T r_s ds}|r_t = x, Y_t = y\right\}$$

$$\begin{aligned}
\frac{\partial P}{\partial t} &+ \frac{1}{2}f(y)^2\frac{\partial^2 P}{\partial x^2} + (a(r_\infty - x) - \lambda(y)f(y))\frac{\partial P}{\partial x} - xP \\
&+ \alpha\left(\nu^2\frac{\partial^2 P}{\partial y^2} + (m-y)\frac{\partial P}{\partial y}\right) \\
&+ \nu\sqrt{2\alpha}\left(\rho f(y)\frac{\partial^2 P}{\partial x \partial y} - \left[\rho\lambda(y) + \gamma(y)\sqrt{1-\rho^2}\right]\frac{\partial P}{\partial y}\right) = 0
\end{aligned}$$

with the terminal condition $P(T,x,y;T) = 1$ for every x and y.

Expand :

$$P^\varepsilon = P_0 + \sqrt{\varepsilon}P_1 + \varepsilon P_2 + \varepsilon\sqrt{\varepsilon}P_3 + \cdots \qquad \varepsilon = 1/\alpha$$

Leading Order Term

$$\frac{\partial P_0}{\partial t} + \frac{1}{2}\bar{\sigma}^2 x^2 \frac{\partial^2 P_0}{\partial x^2} + a\left(r^\star - x\right)\frac{\partial P_0}{\partial x} - xP_0 = 0$$

Effective volatility $\bar{\sigma}^2 = \left\langle f^2 \right\rangle$ and $r^\star = r_\infty - \langle \lambda f \rangle / a$

The zero order term $P_0(t,x)$ **is the Vasicek bond price**

$$P_0(T-\tau, x; T) = \bar{P}(T-\tau, x; T) = A(\tau)e^{-B(\tau)x}$$

computed with the constant parameters $(a, r^\star, \bar{\sigma})$**.**

The Correction $\tilde{P}_1 = \sqrt{\varepsilon} P_1$

The correction $\tilde{P}_1$ solves the source problem:

$$\mathcal{L}_{Vasicek}(a, r^\star, \bar{\sigma})\tilde{P}_1 = \left(V_1 \frac{\partial}{\partial x} + V_2 \frac{\partial^2}{\partial x^2} + V_3 \frac{\partial^3}{\partial x^3} \right) P_0$$

with the zero terminal condition $\tilde{P}_1(T, x) = 0$.

It involves the **constant** quantities, **small** of order $1/\sqrt{\alpha}$

$$\begin{aligned}
V_3 &= \frac{\nu}{\sqrt{2\alpha}} \rho \langle f \phi' \rangle \\
V_2 &= -\frac{\nu}{\sqrt{2\alpha}} \left(\rho \langle \lambda \phi' \rangle + \sqrt{1-\rho^2} \langle \gamma \phi' \rangle \right) - \nu \rho \sqrt{\frac{2}{\alpha}} \langle f \psi' \rangle \\
V_1 &= \nu \sqrt{\frac{2}{\alpha}} \left(\rho \langle \lambda \psi' \rangle + \sqrt{1-\rho^2} \langle \gamma \psi' \rangle \right)
\end{aligned}$$

The Correction $\tilde{P}_1$: explicit computation

Using the variable $\tau = T - t$ and the explicit form $P_0 = Ae^{-Bx}$:

$$\begin{aligned}\frac{\partial \tilde{P}_1}{\partial \tau} &= \frac{1}{2}\bar{\sigma}^2 \frac{\partial^2 \tilde{P}_1}{\partial x^2} + \hat{a}(r^\star - x)\frac{\partial \tilde{P}_1}{\partial x} - x\tilde{P}_1 \\ &\quad + A(\tau)e^{-B(\tau)x}\left(V_3 B(\tau)^3 - V_2 B(\tau)^2 + V_1 B(\tau)\right)\end{aligned}$$

We seek a solution of the form $\tilde{P}_1(T-\tau, x; T) = D(\tau)A(\tau)e^{-B(\tau)x}$ with the condition $D(0) = 0$ so that $\tilde{P}_1(T, x; T) = 0$

We get:

$$D' = V_3 B^3 - V_2 B^2 + V_1 B$$

and

$$\begin{aligned}D(\tau) &= \frac{V_3}{\hat{a}^3}\left(\tau - B(\tau) - \frac{1}{2}\hat{a}B(\tau)^2 - \frac{1}{3}\hat{a}^2 B(\tau)^3\right) \\ &- \frac{V_2}{\hat{a}^2}\left(\tau - B(\tau) - \frac{1}{2}\hat{a}B(\tau)^2\right) + \frac{V_1}{\hat{a}}\left(\tau - B(\tau)\right)\end{aligned}$$

Summary

The corrected bond price is given by

$$\begin{aligned} P(T-\tau, x, y; T) &\approx P_0(T-\tau, x; T) + \tilde{P}_1(T-\tau, x; T) \\ &= A(\tau)\,(1 + D(\tau))\, e^{-B(\tau)x} \end{aligned}$$

where D is a small factor of order $1/\sqrt{\alpha}$.

The **error**

$$|P^{\varepsilon}(t, x, y; T) - \Big(P_0(t, x : T) + \tilde{P}_1(t, x; T)\Big)|$$

is of order $1/\alpha$.

Corrections for bond options prices are also obtained.

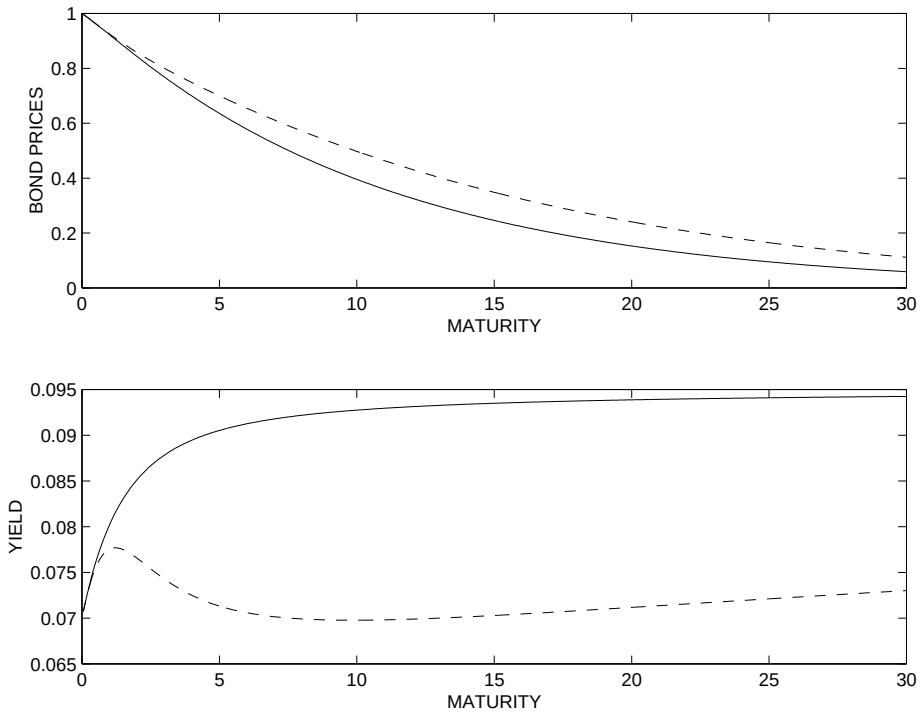


Figure 7: *Top: bond prices and corrected bond prices (dotted curve). Bottom: yield curve and corrected yield curve (dotted curve) in the simulated Vasicek model (constant and stochastic volatility) with:* $a = 1$, $r^\star = 0.1$ *and* $\bar{\sigma} = 0.1$ *as in Figure* 3. *Correction:* $V_3 = 1/\sqrt{\alpha}$ $(\rho \neq 0)$, $\alpha = 10^3$ *and* $\lambda = \gamma = 0$ *implying* $V_1 = 0$ *and* $V_2 = 0$. *Maturity* τ *runs from* 0 *to* 30 *years and the initial rate is* $x = 0.07$.

Model Parameters	**Correction Parameters**
Rate of mean-reversion of short-rate: a	a
Long-run mean under $\mathbb{P}$: $\bar{r}_\infty$	$r^\star$
Specific volatility distribution: $f(\cdot)$	Mean volatility $\bar{\sigma}$
Rate of mean-reversion of volatility : α	Group parameter V_1
Mean-level of (Y_t): m	Group parameter V_2
"V-vol": β	Group parameter V_3
Correlation: ρ	
Interest-rate risk premium: $\lambda(\cdot)$	
Volatility risk premium: $\gamma(\cdot)$	

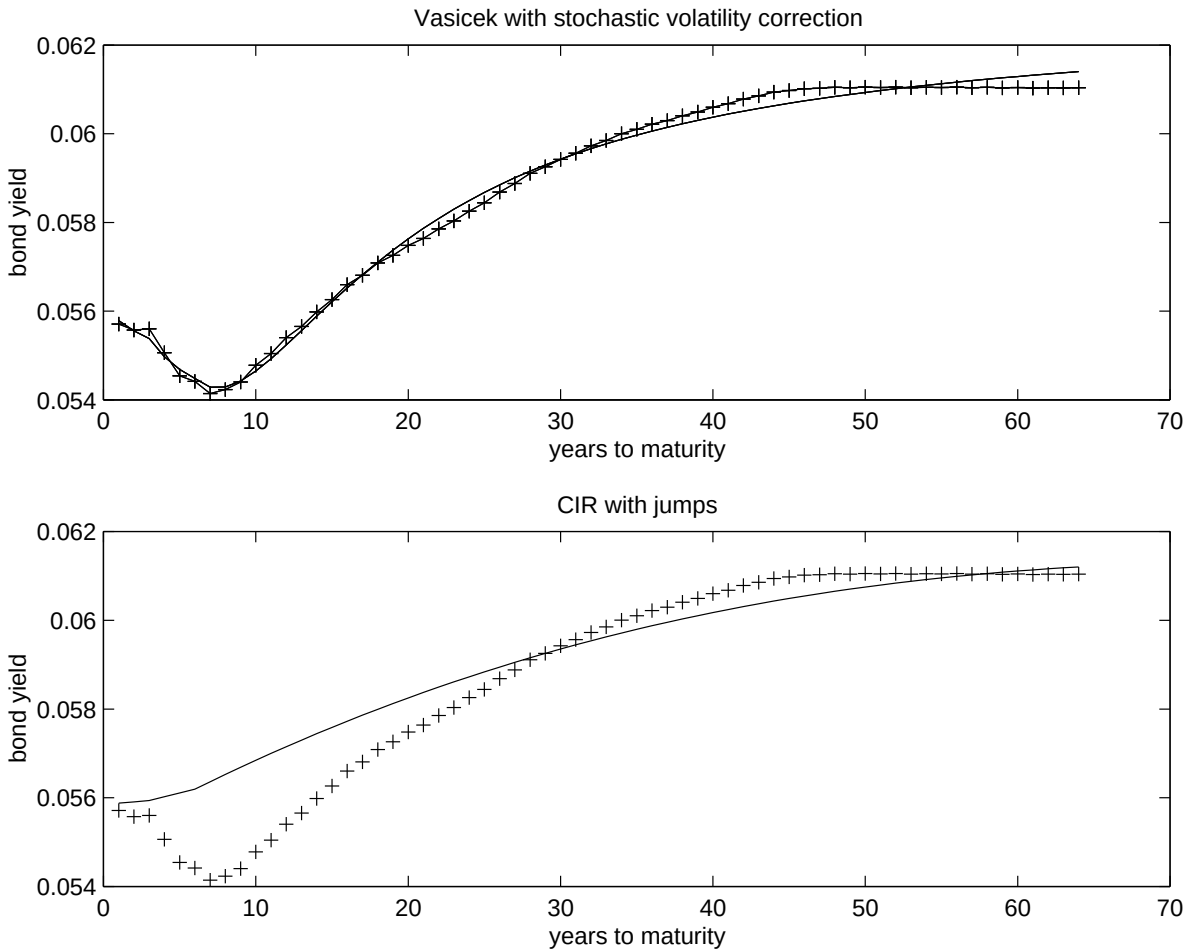


Figure 8: *Snapshot of the yield curve fit with the stochastic volatility corrected Vasicek model (top) and with the single factor CIR model and down jumps (bottom) for September 6, 1998.*

Credit

Perturbations around Merton/Black-Cox

(in the context of the structural approach for instance)

to account for:

- **Volatility Time Scales in Default Times**
- **Fit to Yield Spreads**

References:

Stochastic Volatility Effects on Defaultable Bonds

Submitted 2004

Stochastic Volatility Effects on Default Correlations

In Preparation

Defaultable Bonds

In the first passage structural approach, the payoff of a defaultable zero-coupon bond written on a risky asset X is

$$h(X) \quad = \quad \mathbf{1}_{\{\inf_{0\leq s\leq T} X_s > B\}}.$$

By no-arbitrage, the value of the bond is

$$\begin{aligned} P^B(t,T) &= \mathbb{E}^\star\left\{e^{-r(T-t)}\mathbf{1}_{\{\inf_{0\leq s\leq T} X_s > B\}} \mid \mathcal{F}_t\right\} \\ &= \mathbf{1}_{\{\inf_{0\leq s\leq t} X_s > B\}} e^{-r(T-t)}\mathbb{E}^\star\left\{\mathbf{1}_{\{\inf_{t\leq s\leq T} X_s > B\}} \mid \mathcal{F}_t\right\}, \end{aligned}$$

Using the **predictable stopping time** $\tau_t = \inf\{s \geq t, X_s \leq B\}$:

$$\mathbb{E}^\star\left\{\mathbf{1}_{\{\inf_{t\leq s\leq T} X_s > B\}} \mid \mathcal{F}_t\right\} \quad = \quad \mathbb{P}^\star\{\tau_t > T \mid \mathcal{F}_t\}.$$

This defaultable zero-coupon bond is in fact a binary down-an-out barrier option where the barrier level and the strike price coincide.

Constant Volatility: Merton's Approach

$$
\begin{aligned}
dX_t &= rX_t dt + \sigma X_t dW_t^\star \\
X_t &= X_0 \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma W_t^\star \right).
\end{aligned}
$$

In the Merton's approach, default occurs if $X_T < B$:

Defaultable bond = European digital option

$$
\begin{aligned}
u^d(t,x) &= \mathbb{E}^\star \left\{ e^{-r\tau} \mathbf{1}_{\{X_T > B\}} \mid X_t = x \right\} = e^{-r\tau} \mathbb{P}^\star \left\{ X_T > B \mid X_t = x \right\} \\
&= e^{-r\tau} N(d_2(\tau))
\end{aligned}
$$

with the usual notation $\tau = T - t$ and the *distance to default*:

$$
d_2(\tau) = \frac{\log\left(\frac{x}{B}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}
$$

Constant Volatility: Black-Cox Approach

$$\mathbb{E}^\star\left\{\mathbf{1}_{\{\inf_{t\le s\le T} X_s > B\}} \mid \mathcal{F}_t\right\}$$
$$= \mathbb{P}^\star\left\{\inf_{t\le s\le T}\left((r-\frac{\sigma^2}{2})(s-t)+\sigma(W_s^\star - W_t^\star)\right) > \log\left(\frac{B}{x}\right) \mid X_t = x\right\}$$

computed using distribution of minimum, or using PDE's:

$$\mathbb{E}^\star\left\{e^{-r(T-t)}\mathbf{1}_{\{\inf_{t\le s\le T} X_s > B\}} \mid \mathcal{F}_t\right\} \quad = \quad u(t, X_t)$$

where $u(t,x)$ is the solution of the following problem

$$\mathcal{L}_{BS}(\sigma)u = 0 \text{ on } x > B,\, t < T$$
$$u(t,B) = 0 \quad \text{for any } t \le T$$
$$u(T,x) = 1 \text{ for } x > B,$$

which is to be solved for $x > B$.

Constant Volatility: Barrier Options

Using the European digital pricing function $u^d(t,x)$

$$\mathcal{L}_{BS}(\sigma)u^d = 0 \text{ on } x > 0,\, t < T$$
$$u^d(T,x) = 1 \text{ for } x > B, \text{ and } 0 \text{ otherwise}$$

By the method of images one has:

$$\begin{aligned} u(t,x) &= u^d(t,x) - \left(\frac{x}{B}\right)^{1-\frac{2r}{\sigma^2}} u^d\left(t, \frac{B^2}{x}\right) \\ &= e^{-r(T-t)}\left(N(d_2^+(T-t)) - \left(\frac{x}{B}\right)^{1-\frac{2r}{\sigma^2}} N(d_2^-(T-t))\right) \end{aligned}$$

where we denote

$$d_2^{\pm}(\tau) = \frac{\pm\log\left(\frac{x}{B}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

Yield Spreads Curve

The *yield spread* $Y(0,T)$ at time zero is defined by

$$e^{-Y(0,T)T} = \frac{P^B(0,T)}{P(0,T)},$$

where $P(0,T)$ is the default free zero-coupon bond price given here, in the case of constant interest rate r, by $P(0,T) = e^{-rT}$, and $P^B(0,T) = u(0,x)$, leading to the formula

$$Y(0,T) = -\frac{1}{T}\log\left(N\left(d_2(T)\right) - \left(\frac{x}{B}\right)^{1-\frac{2r}{\sigma^2}} N\left(d_2^-(T)\right)\right)$$

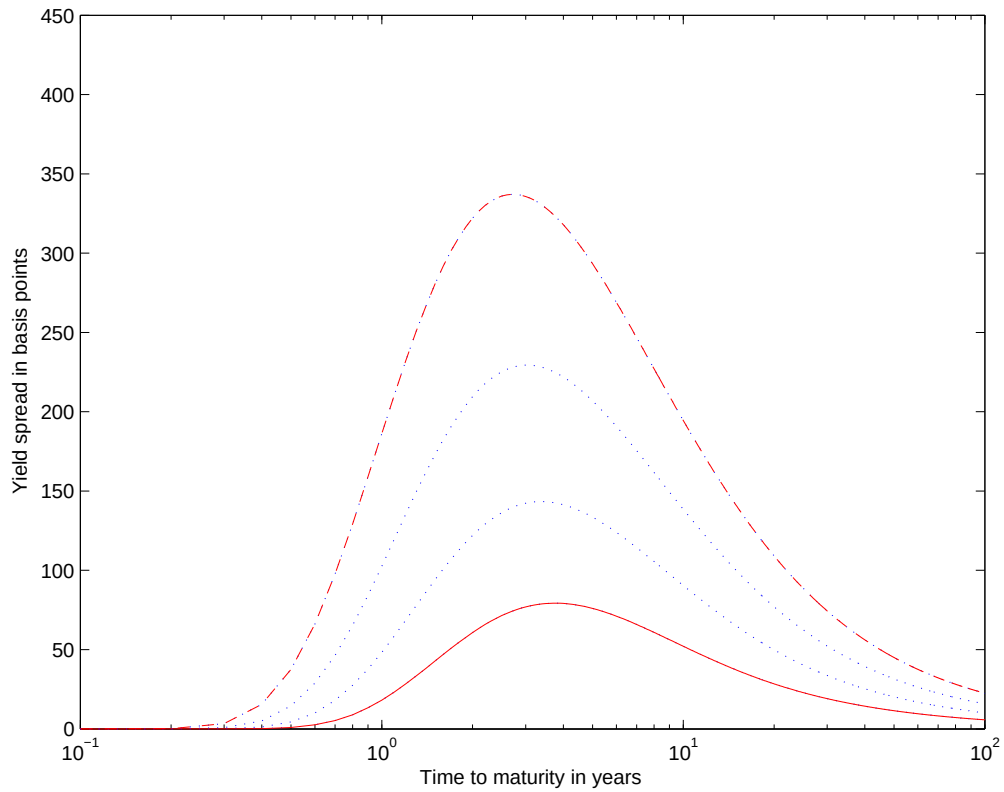


Figure 9: *The figure shows the sensitivity of the yield spread curve to the volatility level. The ratio of the initial value to the default level* x/B *is set to* 1.3*, the interest rate* r *is 6% and the curves increase with the values of* σ*: 10%, 11%, 12% and 13% (time to maturity in unit of years, plotted on the log scale; the yield spread is quoted in basis points)*

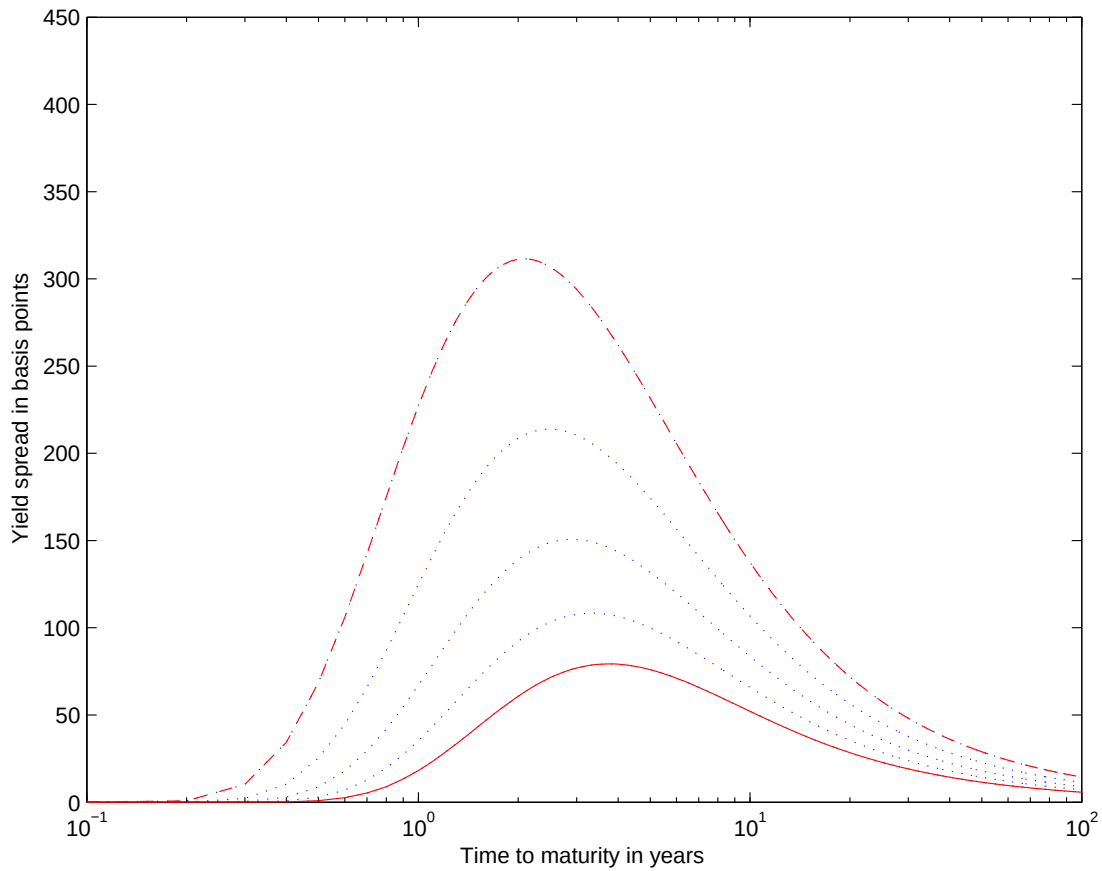


Figure 10: *This figure shows the sensitivity of the yield spread to the leverage level. The volatility level is set to 10%, the interest rate is 6%. The curves increases with the decreasing ratios* x/B: $(1.3, 1.275, 1.25, 1.225, 1.2)$.

Challenge: Yields at Short Maturities

As stated by Eom et.al. (empirical analysis 2001), **the challenge for theoretical pricing models is to raise the average predicted spread relative to crude models such as the constant volatility model, without overstating the risks associated with volatility or leverage.**

Several approaches (**within structural models**) have been proposed that aims at the modeling in this regard. These include

- Introduction of jumps (Zhou,...)
- Stochastic interest rate (Longstaff-Schwartz,...)
- Imperfect information (on X_t) (Duffie-Lando,...)
- Imperfect information (on B) (Giesecke)

Stochastic Volatility Models

$$
\begin{aligned}
dX_t &= \mu X_t dt + f(Y_t) X_t \, dW_t^{(0)} \\
dY_t &= \alpha(m - Y_t) dt + \nu \sqrt{2\alpha} \, dW_t^{(1)}
\end{aligned}
$$

where we assume that

- f non-decreasing, $0 < c_1 \leq f \leq c_2$
- Invariant distribution of Y: $\mathcal{N}(m, \nu^2)$ independent of α
- $\alpha > 0$ is the *rate of mean reversion* of Y
- The standard Brownian motions $W^{(0)}$ and $W^{(1)}$ are correlated

$$
d \left\langle W^{(0)}, W^{(1)} \right\rangle_t = \rho_1 \, dt
$$

Stochastic Volatility Models under $\mathbb{P}^\star$

In order to price defaultable bonds under this model for the underlying we rewrite it under a risk neutral measure $\mathbb{P}^\star$, chosen by the market through the market price of volatility risk Λ_1, as follows

$$\begin{aligned} dX_t &= rX_t dt + f(Y_t)X_t\, dW_t^{(0)\star}, \\ dY_t &= \Big(\alpha(m - Y_t) - \nu\sqrt{2\alpha}\Lambda_1(Y_t)\Big)\, dt + \nu\sqrt{2\alpha}\, dW_t^{(1)\star}. \end{aligned}$$

Here $W^{(0)\star}$ and $W^{(1)\star}$ are standard Brownian motions under $\mathbb{P}^\star$ correlated as $W^{(0)}$ and $W^{(1)}$. We assume that the market price of volatility risk Λ_1 is bounded and a function of y only.

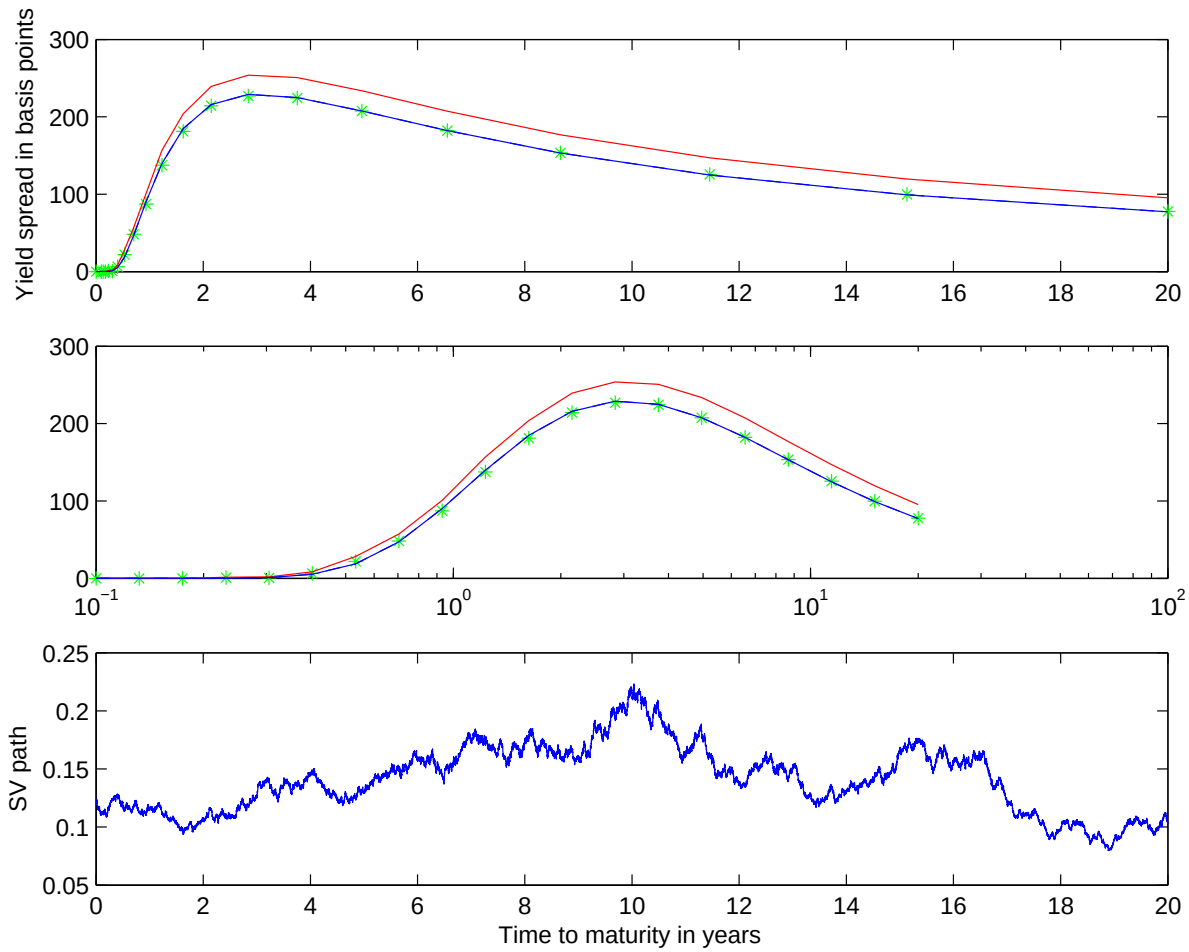


Figure 11: **Uncorrelated slowly mean-reverting stochastic volatility**: $\alpha = 0.05$ *and* $\rho_1 = 0$.

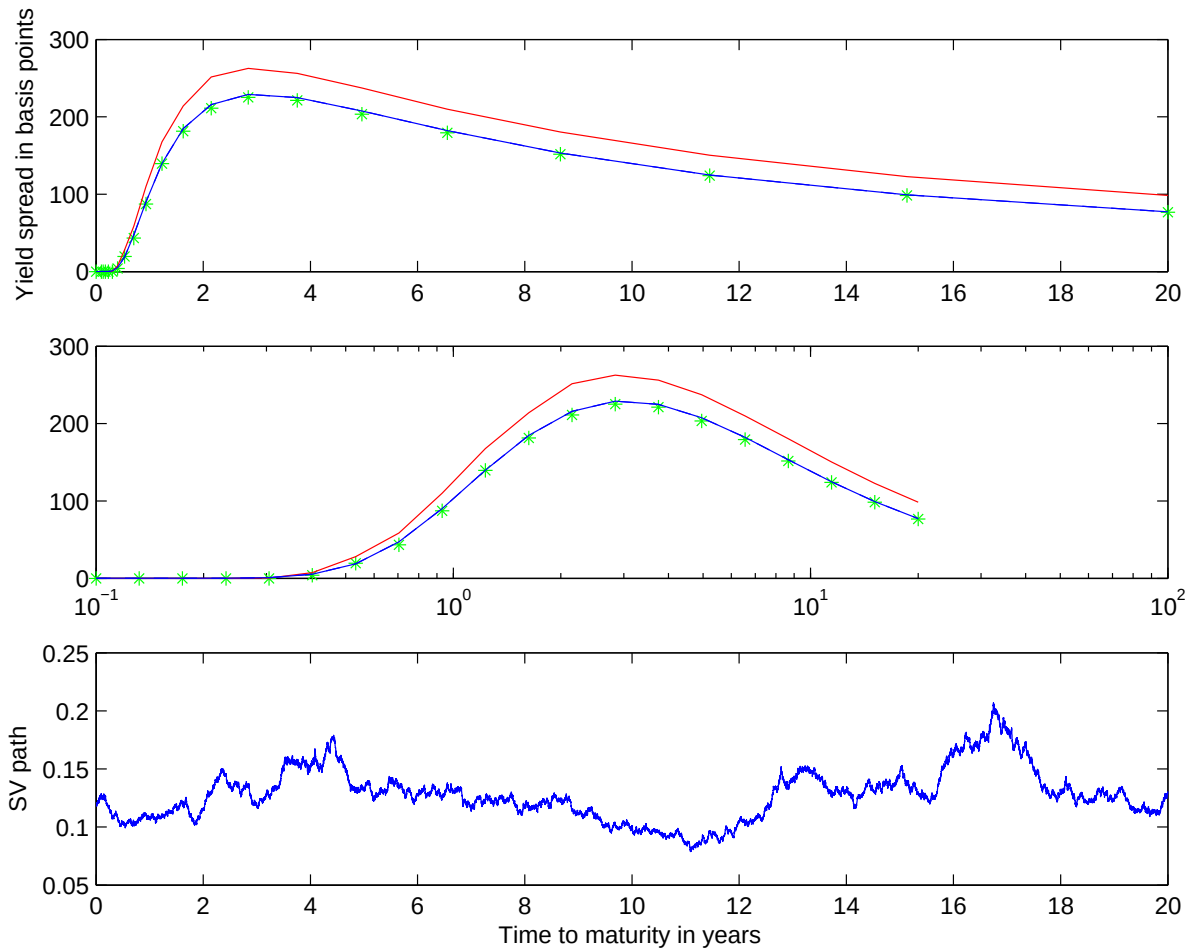


Figure 12: **Correlated slowly mean-reverting stochastic volatility**: $\alpha = 0.05$ *and* $\rho_1 = -0.05$.

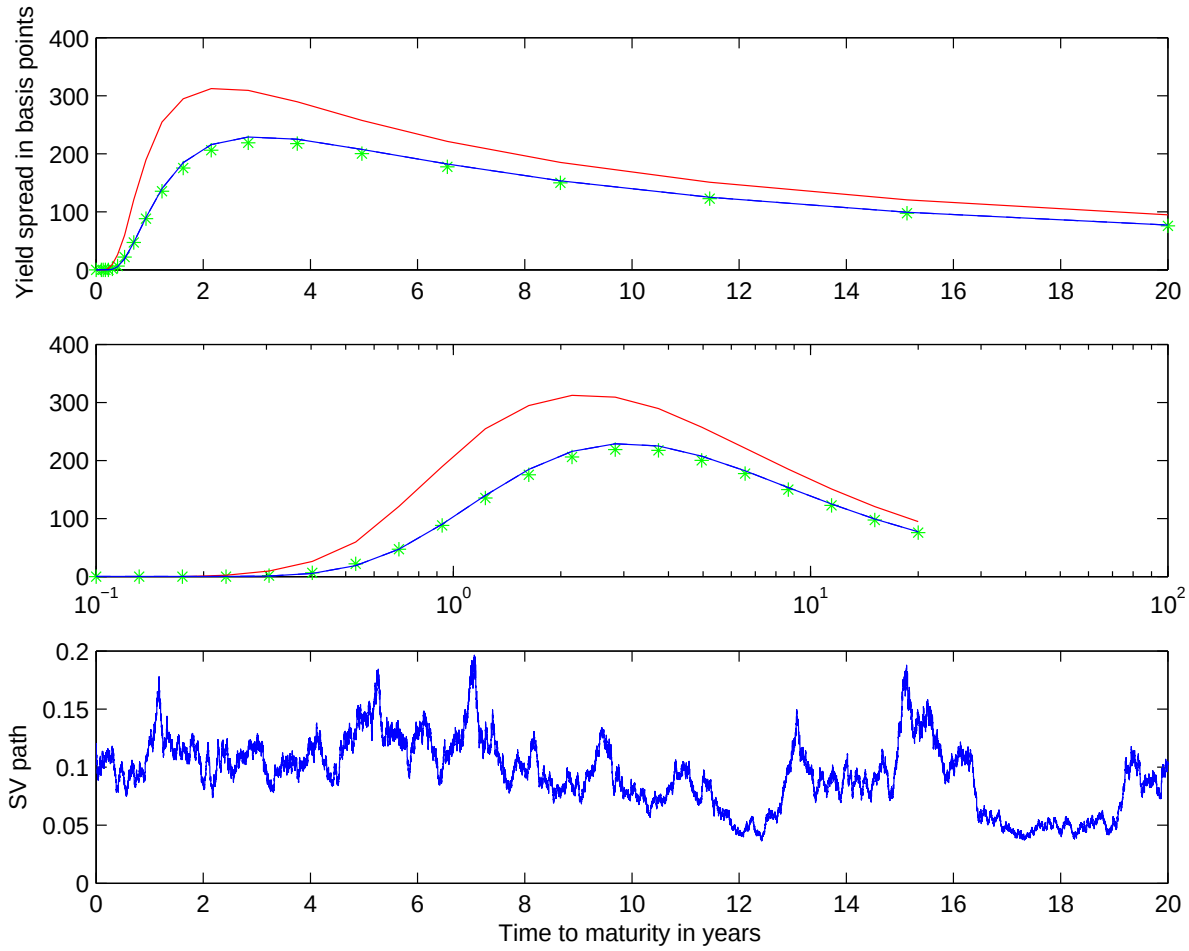


Figure 13: **Uncorrelated stochastic volatility:** $\alpha = 0.5$ **and** $\rho_1 = 0$**.**

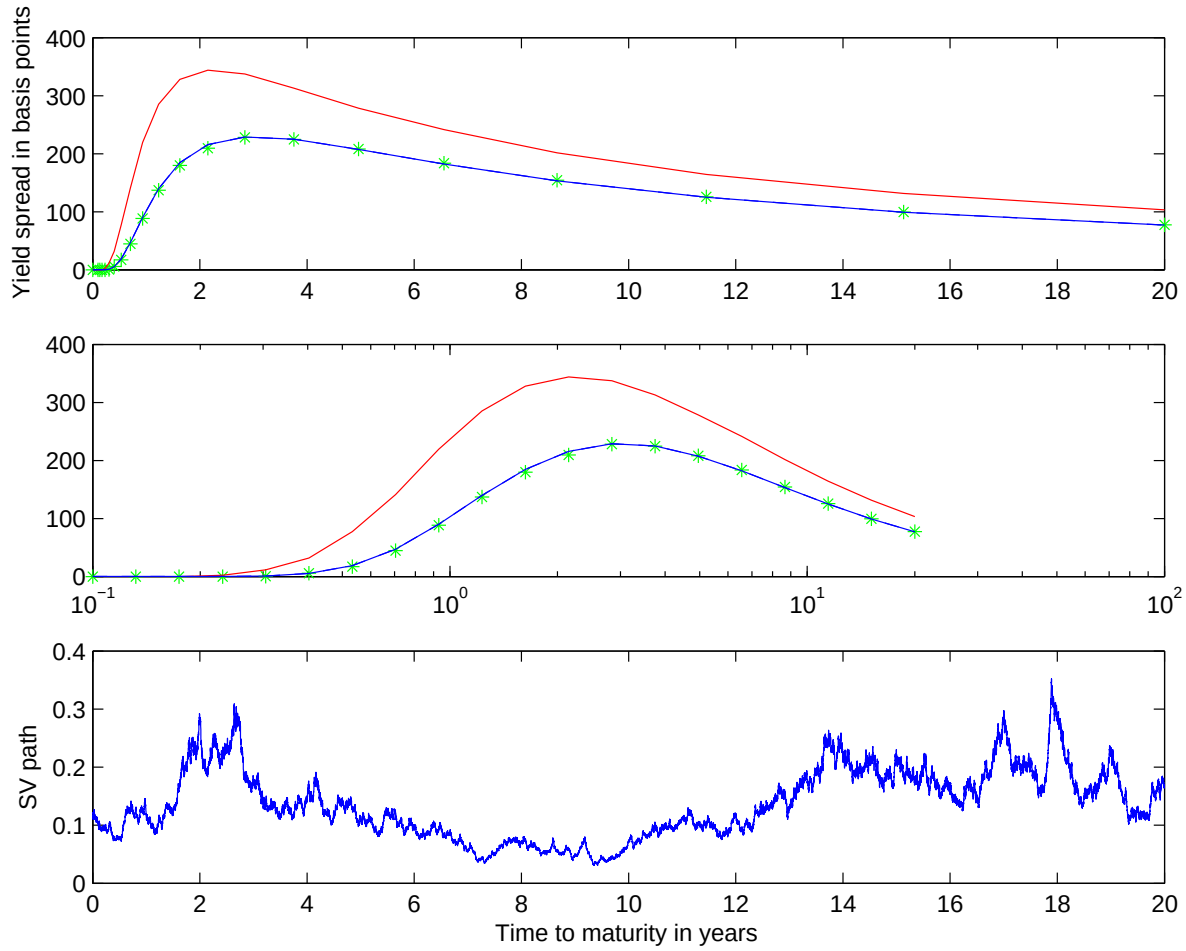


Figure 14: **Correlated stochastic volatility:** $\alpha = 0.5$ **and** $\rho_1 = -0.05$**.**

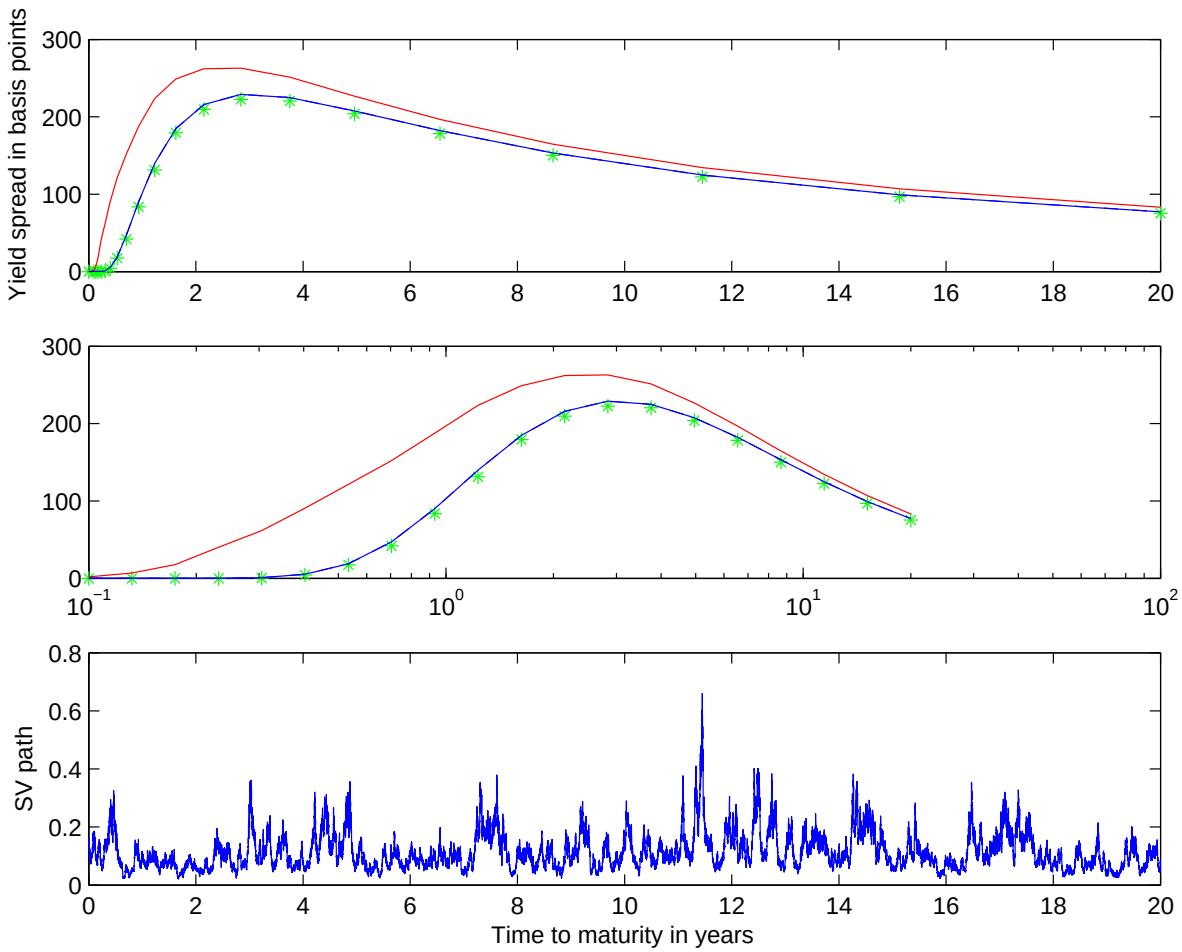


Figure 15: **Uncorrelated fast mean-reverting stochastic volatility:** $\alpha = 10$ **and** $\rho_1 = 0$**.**

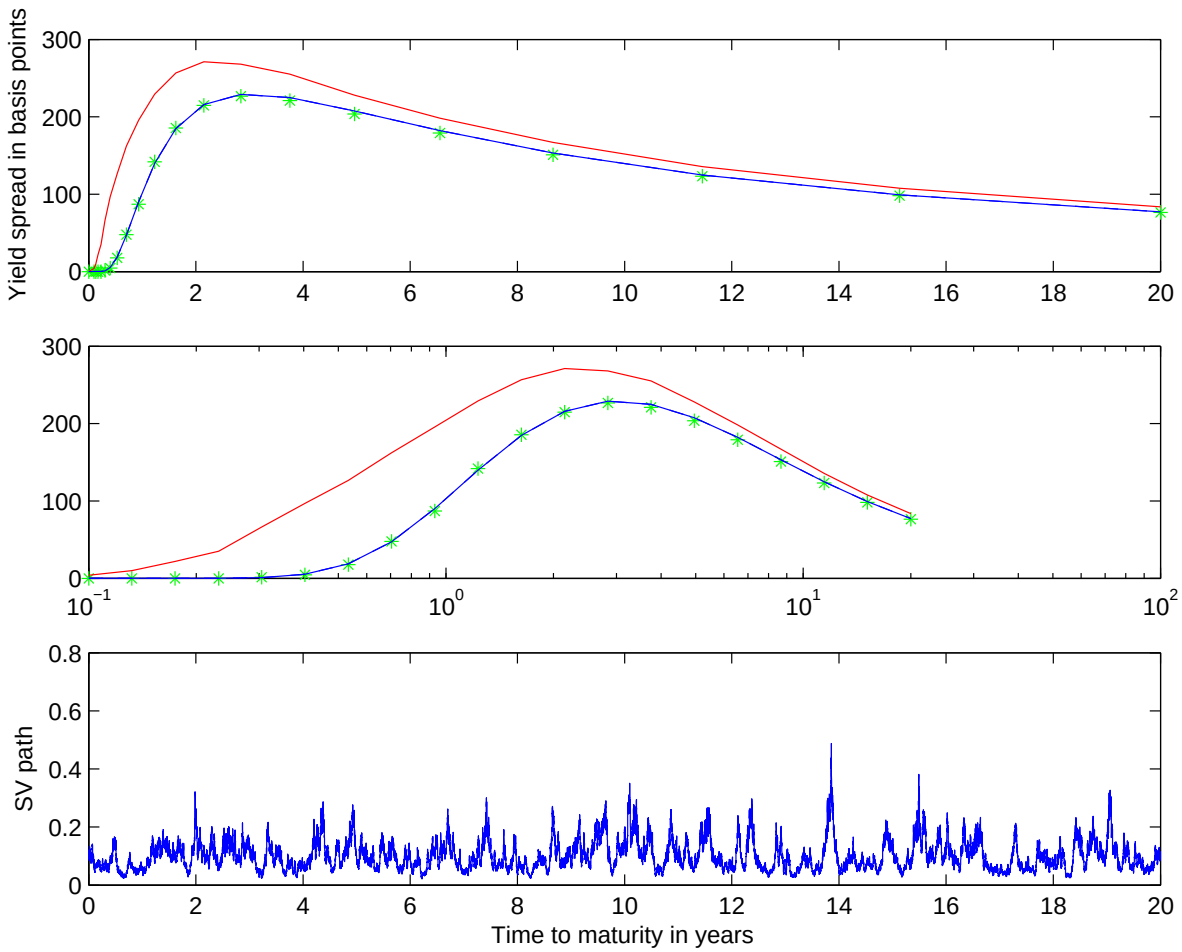


Figure 16: **Correlated fast mean-reverting stochastic volatility:** $\alpha = 10$ **and** $\rho_1 = -0.05$**.**

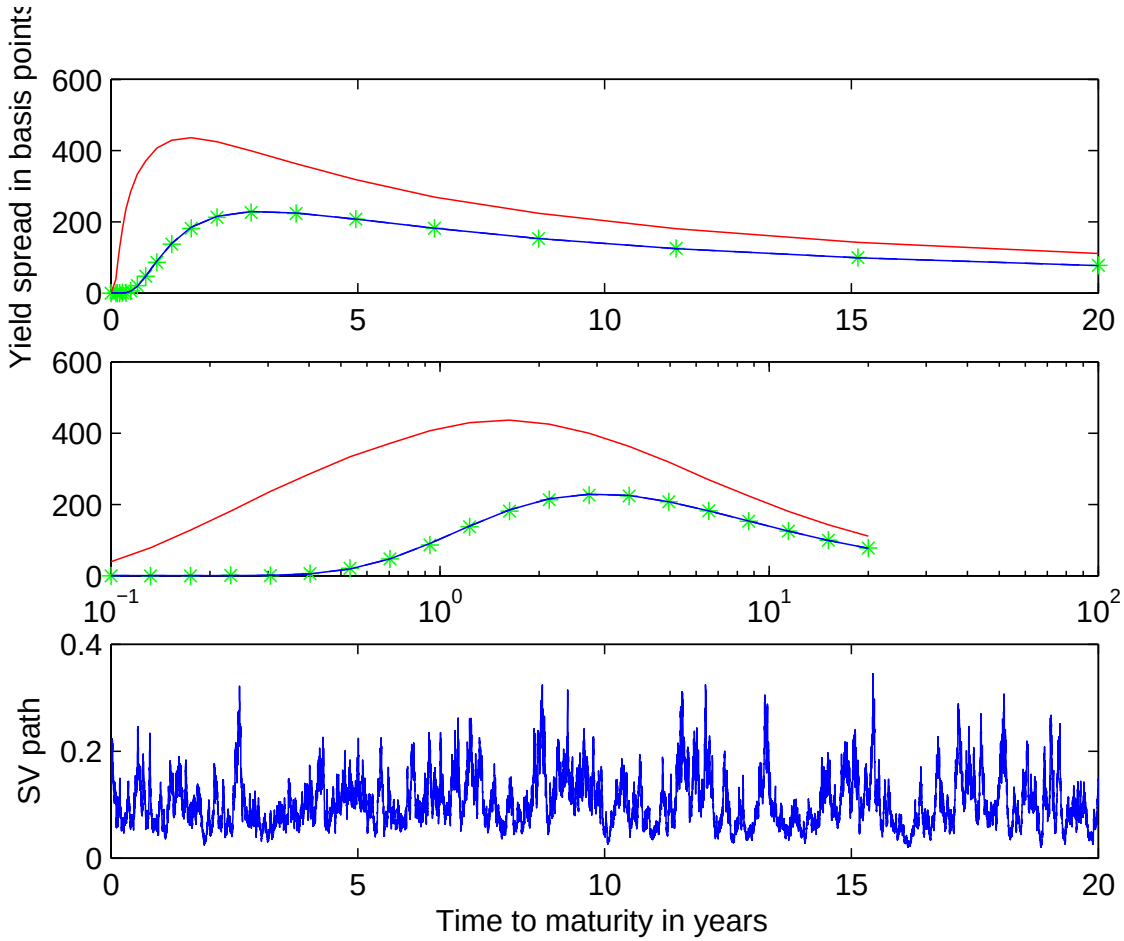


Figure 17: **Highly correlated fast mean-reverting stochastic volatility:** $\alpha = 10$ **and** $\rho_1 = -0.5$**.**

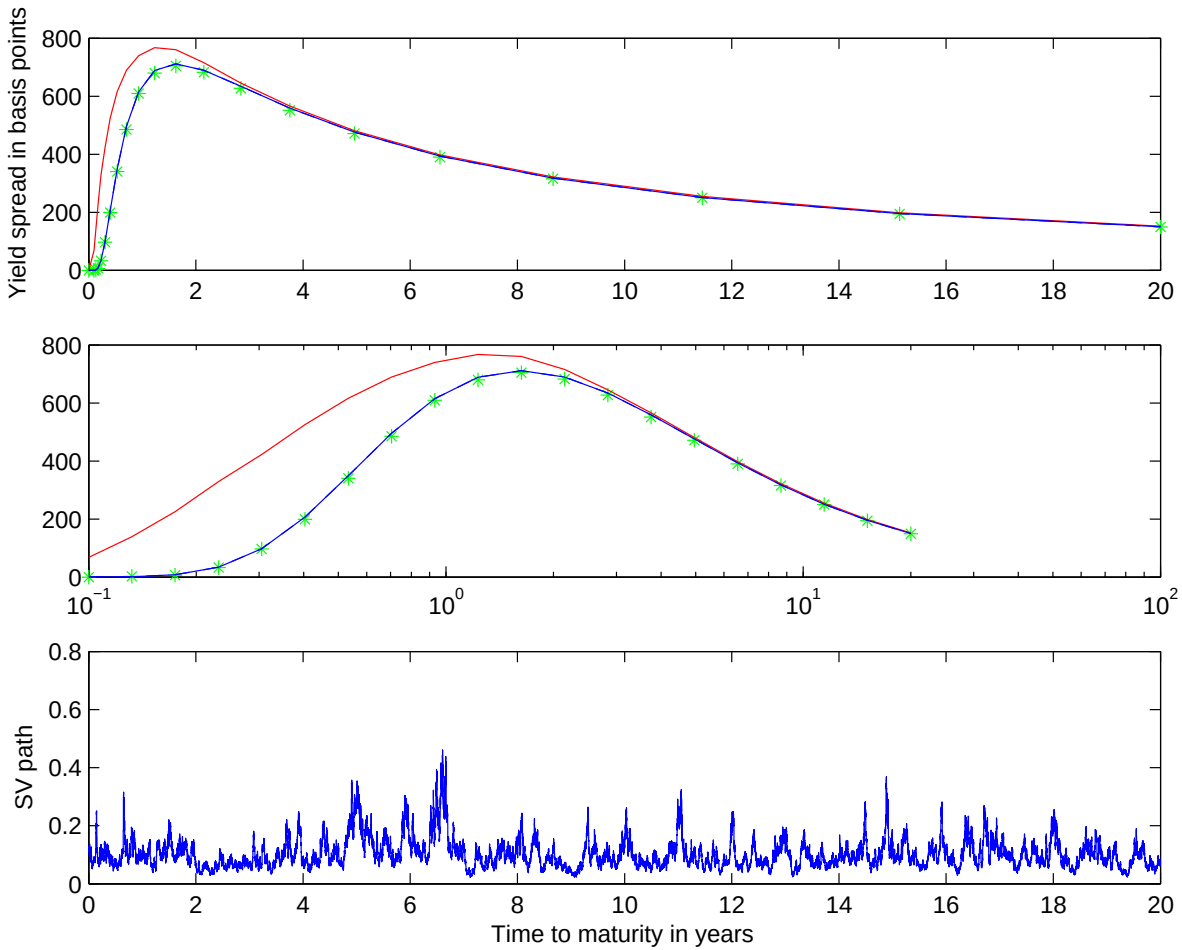


Figure 18: **High leverage correlated fast mean-reverting stochastic volatility:** $x/B = 1.2$, $\alpha = 10$ **and** $\rho_1 = -0.05$.

Barrier Options under Stochastic Volatility

$$u(t,x,y) \quad = \quad e^{-r(T-t)} \mathbb{E}^\star \left\{ h(X_T) \mathbf{1}_{\{\inf_{t \le s \le T} X_s > B\}} \mid X_t = x, Y_t = y \right\} ,$$

$$P^B(t,T) = \mathbf{1}_{\{\inf_{0 \le s \le t} X_s > B\}} u(t, X_t, Y_t).$$

The function $u(t,x,y)$ satisfies for $x \ge B$ the problem

$$\begin{array}{ll} \left(\frac{\partial}{\partial t} + \mathcal{L}_{X,Y} - r\right) u = 0 & \text{on } \ x > B, \, t < T \\ u(t,B) = 0 & \text{for any } \ \ t \le T \\ u(T,x) = h(x) & \text{for } \ \ \ x > B \end{array}$$

where $\mathcal{L}_{X,Y}$ is the infinitestimal generator of the process (X,Y) under $\mathbb{P}^\star$.

Leading Order Term under Stochastic Volatility

In the regime α large, as in the European case, $u(t, x, y)$ is approximated by $u_0^\star(t, x)$ which solves the constant volatility problem

$$\begin{aligned} &\mathcal{L}_{BS}(\sigma^\star)u_0^\star = 0 && \text{on } \ x > B,\, t < T \\ &u_0^\star(t, B) = 0 && \text{for any } \ \ t \leq T \\ &u_0^\star(T, x) = h(x) && \text{for } \ \ \ x > B \end{aligned}$$

where $\sigma^\star$ is the corrected effective volatility.

Stochastic Volatility Correction

Define the correction $u_1^\star(t,x)$ by

$$\begin{aligned} &\mathcal{L}_{BS}(\sigma^\star)u_1^\star = -V_3\, x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 u_0^\star}{\partial x^2}\right) && \text{on } \ x > B,\, t < T \\ &u_1^\star(t,B) = 0 && \text{for any } \ t \le T \\ &u_1^\star(T,x) = 0 && \text{for } \ x > B \end{aligned} \quad .$$

Remarkably, the small parameter V_3 is the same as in the European case (calibrated to implied volatilities).

Computation of the Correction

Define

$$v_1^\star(t,x) = u_1^\star(t,x) - (T-t)V_3\, x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 u_0^\star}{\partial x^2}\right),$$

so that $v_1^\star(t,x)$ solves the simpler problem

$$\begin{aligned}
&\mathcal{L}_{BS}(\sigma^\star)v_1^\star = 0 && \text{on } x > B,\ t < T\\
&v_1^\star(t,B) = g(t) && \text{for any } \ t \le T\\
&v_1^\star(T,x) = 0 && \text{for } \ x > B\\
&g(t) = -V_3\,(T-t)\lim_{x\downarrow B}\left(x\tfrac{\partial}{\partial x}\left(x^2\tfrac{\partial^2 u_0^\star}{\partial x^2}\right)\right)
\end{aligned}$$

To summarize we have

$$u(t,x,y) \quad\approx\quad u_0^\star(t,x) + (T-t)V_3\, x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 u_0^\star}{\partial x^2}\right) + v_1^\star(t,x)$$

with explicit computation in the case $h(x) = 1$.

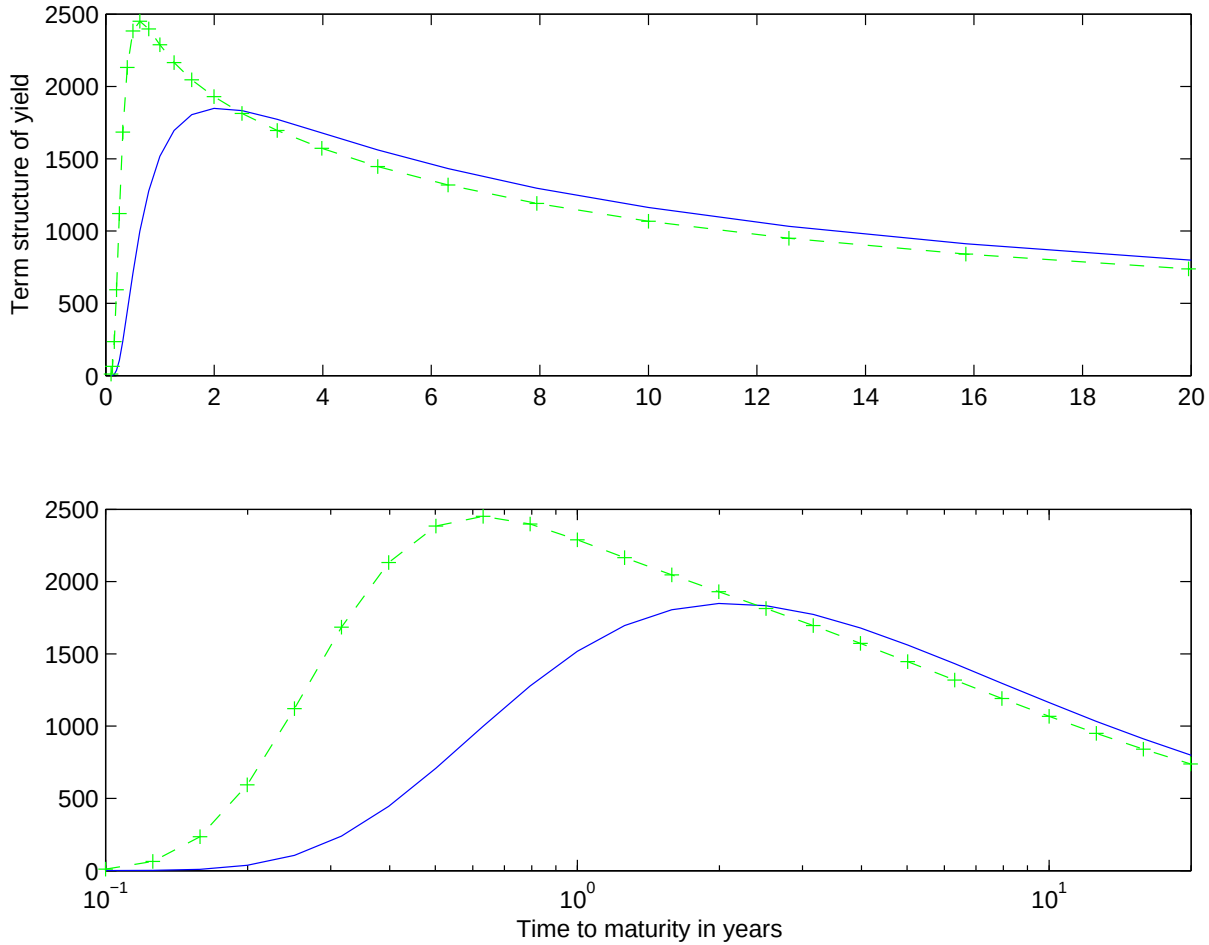


Figure 19: **The price approximation for** $\sigma^\star = 0.12, r = 0.0, V_3 = -0.0003, x/B = 1.2$**.**

Slow Factor Correction

The first correction $u_1^{(z)}(t,x)$ solves the problem

$$\begin{aligned}
&\mathcal{L}_{BS}(\bar{\sigma}(z))u_1^{(z)} = -2\left(V_0(z)\tfrac{\partial u_{BS}}{\partial \sigma} + V_1(z)x\tfrac{\partial}{\partial x}\left(\tfrac{\partial u_{BS}}{\partial \sigma}\right)\right) && \text{on } x > B,\, t < T,\\
&u_1^{(z)}(t,B) = 0 && \text{for } t \le T,\\
&u_1^{(z)}(T,x) = 0 && \text{for } x > B,
\end{aligned}$$

where u_{BS} is evaluated at $(t,x,\bar{\sigma}(z))$, and $V_0(z)$ and $V_1(z)$ are small parameters of order $\sqrt{\delta}$, functions of the model parameters, and depending on the current level z of the slow factor.

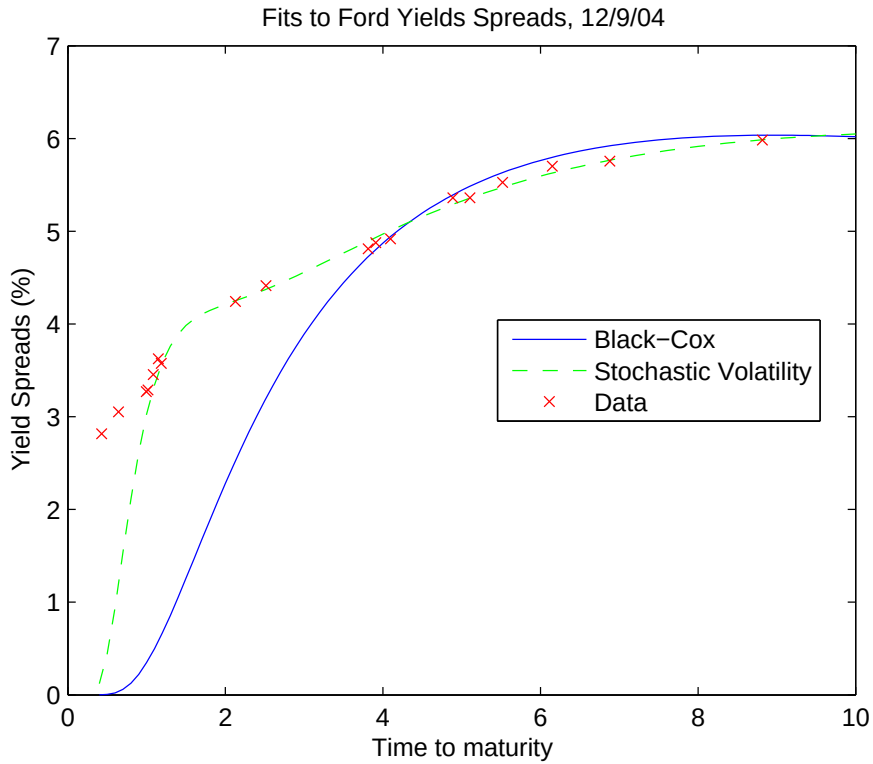
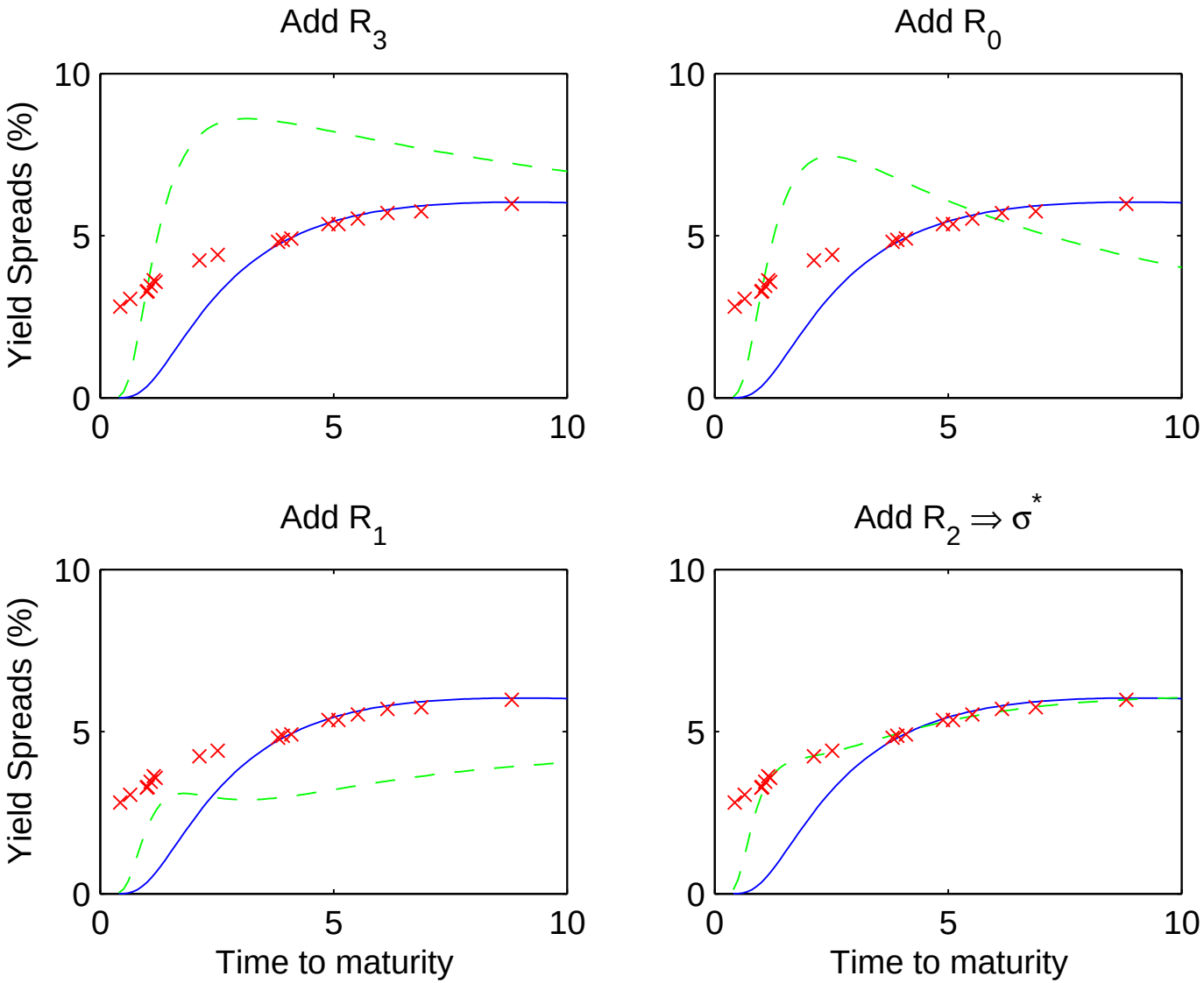


Figure 20: *Black-Cox and two-factor stochastic volatility fits to Ford yield spread data. The short rate is fixed at* $r = 0.025$. *The fitted Black-Cox parameters are* $\bar{\sigma} = 0.35$ *and* $x/B = 2.875$. *The fitted stochastic volatility parameters are* $\sigma^\star = 0.385$, *corresponding to* $R_2 = 0.0129$, $R_3 = -0.012$, $R_1 = 0.016$ *and* $R_0 = -0.008$.


Add R_3
Add R_0
Add R_1
Add $R_2 \Rightarrow \sigma^*$
Yield Spreads (%)
Yield Spreads (%)
Time to maturity
Time to maturity
0
5
10
0
5
10

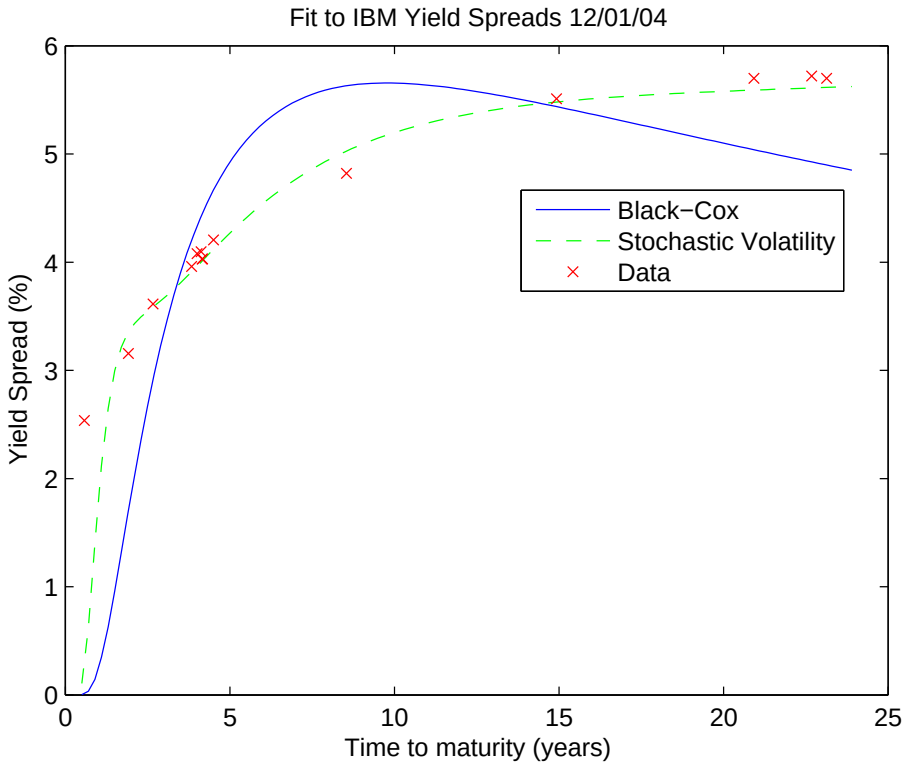


Figure 21: *Black-Cox and two-factor stochastic volatility fits to IBM yield spread data. The short rate is fixed at* $r = 0.025$. *The fitted Black-Cox parameters are* $\bar{\sigma} = 0.35$ *and* $x/B = 3$. *The fitted stochastic volatility parameters are* $\sigma^\star = 0.36$, *corresponding to* $R_2 = 0.00355$, $R_3 = -0.0112$, $R_1 = 0.013$ *and* $R_0 = -0.0045$.