LECTURE 13: Sample PACF; Durbin-Levinson algorithm.

7. **Sample ACF and PACF.** ([BD] 3.2.3 and 3.2.4. D-L in 2.5.1)

Last time: Assuming stationary sample $X_1, \ldots, X_n$, we defined:

$\bar{X}_n = (1/n)\sum_{t=1}^{n} X_t$; $\hat{\sigma}^2 \equiv \hat{\gamma}(0) = (1/n)\sum_{t=1}^{n}(X_t - \bar{X}_n)^2$.

$\hat{\gamma}(h) = (1/n)\sum_{t=1}^{n-h}(X_t - \bar{X}_n)(X_{t+h} - \bar{X}_n)$; $\hat{\rho}(h) = \hat{\gamma}(h)/\hat{\gamma}(0)$.

Bartlett's formula: Assume that n observations $X_1, \ldots, X_n$ come from a stationary TS with IID innovations Z_t's, and that n is large. Then:
If $\rho(q+j) = 0$, $j = 1, 2, \ldots$ then

$$Var(\hat{\rho}(h) \approx (1/n)(1 + 2\sum_{k=1}^{q} \rho^2(k)),\ h > q$$

and asymptotic distribution of $\hat{\rho}(h)$ is Normal with mean $\rho(h)$ and variance given above. Use $n \geq 50$, $h \leq n/4$.

. . . .

Sample PACF ([BD], §3.2.3) $\hat{\alpha}_h \equiv \hat{\phi}_{hh}$ is defined as the last component of vector $\hat{\phi}_h$ which is the solution of the system of equations ($\hat{\phi}_h = \hat{R}_h^{-1}\hat{\rho}_h$).

[Solution of Yule-Walker equations where we substitute sample ACF $\hat{\rho}$ for theoretical ACF ρ].

Result: For AR(p) process, the sample PACF at lags greater than p are approximately independent Normal r.v.s with mean zero and variance $Var\hat{\phi}_{hh} \sim 1/n$. Here n = sample size, large.

To decide that the value of the PACF is zero, compare it with the standard deviation. If $|\hat{\alpha}(h)| > 1.96n^{-1/2}$ for $h = p$ and $|\hat{\alpha}(h)| \leq 1.96n^{-1/2}$ for $h > p$, then the model is AR(p).

About calculation of sample PACF:

• to calculate $\hat{\phi}_{hh}$ at a specific lag h one need to solve the system of linear equations: $\hat{R}_h\hat{\phi}_h = \hat{\rho}_h$. When h runs as $h = 1, 2, \ldots, K$ with K large ($= 40$ in ITSM) this means the evaluation of many determinants of high dimension.

• Alternatively, we can fit AR models of increasing orders; then the estimate of the last coefficient in each model is the sample PACF. Specifically, the estimates $\hat{\phi}_{hj}, j = 1, \ldots, h$ in

$$X_t = \phi_{h1}X_{t-1} + \ldots + \phi_{hh}X_{t-h} + Z_t$$

can be updated recursively from the estimates $\phi_{h-1,j}$ at step $h-1$ corresponding to

$$X_t = \phi_{h-1,1}X_{t-1} + \ldots + \phi_{h-1,h-1}X_{t-h+1} + Z_t$$

The updating recursive equations (*Durbin-Levinson Algorithm*) are given by

$$\hat{\phi}_{hh} = \frac{\hat{\rho}(h) - \sum_{j=1}^{h-1}\hat{\phi}_{h-1,j}\hat{\rho}(h-j)}{1 - \sum_{j=1}^{h-1}\hat{\phi}_{h-1,j}\hat{\rho}(j)}$$

$$\hat{\phi}_{h,j} = \hat{\phi}_{h-1,j} - \hat{\phi}_{hh}\hat{\phi}_{h-1,h-j},\ j = 1, \ldots, h-1$$

These recursive equations avoid the inversion of $h \times h$ matrices.

Example. Assume that sample ACF $\hat{\rho}_h$ were calculated previously.

$\hat{\phi}_{11} = \hat{\rho}(1)$

$\hat{\phi}_{22} = (\hat{\rho}(2) - \hat{\phi}_{11}\hat{\rho}(1))/(1 - \hat{\phi}_{11}\hat{\rho}(1)) = (\hat{\rho}(2) - \hat{\rho}^2(1))/(1 - \hat{\rho}^2(1))$
$\hat{\phi}_{21} = \hat{\phi}_{11} - \hat{\phi}_{22}\hat{\phi}_{11}$

$\hat{\phi}_{33} = (\hat{\rho}(3) - \hat{\phi}_{2,1}\hat{\rho}(2) - \hat{\phi}_{22}\hat{\rho}(1))/(1 - \hat{\phi}_{2,1}\hat{\rho}(1) - \hat{\phi}_{2,2}\hat{\rho}(2))$
$\hat{\phi}_{31} = \hat{\phi}_{21} - \hat{\phi}_{33}\hat{\phi}_{22}$
$\hat{\phi}_{32} = \hat{\phi}_{22} - \hat{\phi}_{33}\hat{\phi}_{21}$

etc.

8. Model Building. Preliminary identification

DIAGRAM

Identification.

(a) plot your data and look at it. If it looks non-stationary, try to guess trend and do differencing to remove it. If the series is seasonal, it requires special treatment (later in the class).

(b) compute and look at ACF. The idea is to match the estimated autocorrelations with the theoretical autocorrelations, produced by a particular TS model. A few points here:

1. If large values of $\hat{\rho}(h)$ persist, one should suspect non-stationarity. Go to (a), i.e. look at your data, difference.
2. Usually one differences 0, 1, or 2 times. The first available stationary difference should be used. Beware of overdifferencing:
   $*$ increasing variance after differencing
   $*$ increasing order of MA after differencing
   $*$ noninvertible MA (unit root)
3. If the values of $\hat{\rho}(h)$ can be taken zero's after some q, then assume MA(q).
4. If ACF decays exponentially, oscillates or has sinusoidal behavior, suspect AR or ARMA. To find order p of AR, look at estimated PACF. If ϕ_{nn} can be taken zeros after some value p, then assume AR(p).

5. In order to look at ACF, usually assume 50 or more observations.

6. Many times you can't say definitely which model you have. For example, AR and ARMA could both have exponential decay of ACF. Then the selection of the model should be influenced by the principle of parsimony: take the one with the fewest parameters. Your other option,– to do full TS analysis for more than one models and take the one which has best results in diagnostic checking.

PREPARE TRANSPARENCY WITH REFREGIRATOR DATA