THE CENTER OF MASS FOR SPATIAL BRANCHING PROCESSES AND AN APPLICATION FOR SELF-INTERACTION

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Abstract. Consider the center of mass of a supercritical branching-Brownian motion, or that of a supercritical super-Brownian motion. In this paper we prove that it is a Brownian motion being slowed down such that it tends to a limiting position almost surely, which, in a sense complements a result of Tribe on the final behavior of a critical super-Brownian motion. This is shown to be true also for a model where branching Brownian motion is modified by attraction/repulsion between particles.

We then put this observation together with the description of the interacting system as viewed from its center of mass, and get the following asymptotic behavior: the system asymptotically becomes a branching Ornstein-Uhlenbeck process (inward for attraction and outward for repulsion), but

(1) the origin is shifted to a random point which has normal distribution, and

(2) the Ornstein-Uhlenbeck particles are not independent but constitute a system with a degree of freedom which is less than their number by precisely one.

Finally, in the attractive case, we prove a scaling limit theorem for the local mass.

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1. Introduction

We start with some basic notation.

Notation 1. In this paper $\mathcal{M}_f(\mathbb{R}^d)$ and $\mathcal{M}_1(\mathbb{R}^d)$ denote the space of finite measures and the space of probability measures, respectively, on $\mathbb{R}^d$. For $\mu \in \mathcal{M}_f(\mathbb{R}^d)$, we define $\|\mu\| := \mu(\mathbb{R}^d)$. Finally $X \oplus Y$ will denote the independent sum of the random variables $X$ and $Y$, and $\mathbf{I}_d$ will denote the $d$-dimensional unit matrix.

1.1. A model with self-interaction. Consider a dyadic (i.e. precisely two offspring replaces the parent) branching Brownian motion (BBM) in $\mathbb{R}^d$ with unit time branching and with the following interaction between particles: if $Z$ denotes the process and $Z_i^t$ is the $i$th particle, then $Z_i^t$ feels the drift

$$\frac{1}{nt} \sum_{1 \leq j \leq n_t} \gamma \left( Z_j^t - x \right),$$

where $\gamma \neq 0$, that is the particle’s infinitesimal generator is

$$\frac{1}{2} \Delta + \frac{1}{nt} \sum_{1 \leq j \leq n_t} \gamma \left( Z_j^t - x \right) \cdot \nabla. \tag{1.1}$$

(Here and in the sequel, $n_t$ is a shorthand for $2^\lfloor t \rfloor$, where $\lfloor t \rfloor$ is the integer part of $t$.) If $\gamma > 0$, then this means attraction, if $\gamma < 0$, then it means repulsion.

To be a bit more precise, we can define the process by induction as follows. $Z_0$ is a single particle at the origin. In the time interval $[m, m+1)$ we define a system of $2^m$ interacting diffusions, starting at the position of their parents at the end of the previous step (at time $m-0$) by the following system of SDE's:

$$dZ_i^t = dW_i^t + \frac{\gamma}{2m} \sum_{1 \leq j \leq 2^m} (Z_j^t - Z_i^t) \, dt; \quad i = 1, 2, \ldots, 2^m, \tag{1.2}$$

where $W^i, i = 1, 2, \ldots, 2^m$ are independent Brownian motions.

Remark 2 (Attractive interaction). If there were no branching and the interval $[m, m+1)$ were extended to $[0, \infty)$, then for $\gamma > 0$ the interaction (1.2) would describe the ferromagnetic Curie-Weiss model, a model appearing in the microscopic statistical description of a spatially homogeneous gas in a granular medium. It is known that as $m \to \infty$, a Law of Large Numbers, the McKean-Vlasov limit holds and the normalized empirical measure

$$\rho_m(t) := 2^{-m} \sum_{i=1}^{2^m} \delta_{Z_i^t}$$

tends to a probability measure-valued solution of

$$\frac{\partial}{\partial t} \rho = \frac{1}{2} \Delta \rho + \frac{\gamma}{2} \nabla \cdot (\rho \nabla f^\rho),$$

where $f^\rho(x) := \int_{\mathbb{R}^d} |x-y|^2 \rho(dy)$. (See p. 24 in [1] and the references therein.)

It may also seem natural to replace the interaction we defined by the gravitational force between particles\footnote{I.e. the forces are given by Newton’s law of universal gravitation: they vary as the inverse square of the distance between the particles. Using electrostatic force instead gravitational (between electric charges) would extend the model for the repulsive case.}, however this would lead to a randomized version of the (notoriously difficult) ‘n-body problem.’
1.2. Existence and uniqueness. Notice that the $2^m$ interacting diffusions can be considered as a single $2^m d$-dimensional Brownian motion with linear (and therefore Lipschitz) drift $b : \mathbb{R}^{2^m d} \to \mathbb{R}^{2^m d}$:

$$b(x_1, x_2, \ldots, x_d, x_{1+d}, x_{2+d}, \ldots, x_{2d}, \ldots, x_{1+(2^m-1)d}, x_{2+(2^m-1)d}, \ldots, x_{2^m d}) =: \gamma(\beta_1, \beta_2, \ldots, \beta_{2^m d})^T,$$

where

$$\beta_k = 2^{-m} (x_{k} + x_{k+d} + \ldots + x_{k+(2^m-1)d} - x_k), \quad 1 \leq k \leq 2^m d,$$

and $\widehat{k} \equiv k \text{ (mod } d), \quad 1 \leq \widehat{k} \leq d$. This yields existence and uniqueness for our model.

1.3. Results on the self-interacting model. We are interested in the long time behavior of $Z$, and also whether we can say something about the number of particles in a given compact set for $n$ large. In the sequel we will use the standard notation $\langle Z_t, g \rangle := \sum_{i=1}^{n_t} g(Z_{t_i})$.

In this paper we will show (Theorem 14) that $Z$ asymptotically becomes a branching Ornstein-Uhlenbeck process (inward for attraction and outward for repulsion), but

(1) the origin is shifted to a random point which has $d$-dimensional normal distribution $N(0, 2I_d)$, and

(2) the Ornstein-Uhlenbeck particles are not independent but constitute a system with a degree of freedom which is less than their number by precisely one.

For the local behavior we formulate and motivate a conjecture (Conjecture 18).

1.4. An extension of Tribe’s result on critical super-Brownian motion. In the proof of Theorem 14 we will first show that $Z_t = \frac{1}{n_t} \sum_{i=1}^{n_t} Z_{t_i}$, the center of mass for $Z$ satisfies $\lim_{t \to \infty} Z_t = N$, where $N \sim N(0, 2I_d)$. In fact, the proof will reveal that $Z$ moves like a Brownian motion, which is nevertheless slowed down tending to a final limiting location (see Lemma 5 and its proof).

Since this is also true for $\gamma = 0$ (BBM with unit time branching and no self-interaction), our first natural question is whether we can prove a similar result for the supercritical super-Brownian motion.

Another motivation for the same goal is as follows. Tribe [8] proved that a critical super-Brownian motion near its extinction time $\xi$ behaves like a single Brownian path stopped at $\xi$. More precisely, $X_t \to \delta_F$ as $t \to \xi$ a.s. in the weak topology, where $F$ is a $d$-dimensional random variable and its distribution is the same as that of a Brownian motion at time $\xi$.

We would like to extend Tribe’s result to the supercritical super-Brownian motion $X$. We are interested in whether we can obtain a similar result on the survival set. Of course, then $X_t$ does not shrink to a point in any sense, however, we may hope to get an analogous result regarding the center of mass, defined as

$$X_t := \frac{1}{\|X_t\|} \int_{\mathbb{R}^d} x X_t(dx) = \langle x/\|X_t\|, X_t \rangle.$$

(Since $f(x) = x$ is not a bounded function we may not hope to use Tribe’s techniques though.)

In the sequel we will prove that $X$ will be a time changed Brownian motion on a finite (but random) time interval.
At the beginning of this subsection we referred to a result on $Z$. In fact we now see that the results on $X$ and $Z$ are analogous: in both cases the center of mass is a Brownian motion, slowed down in such a way that the time interval $[0, \infty)$ is compressed into a finite one. A slight difference is that, in case of $Z$, the terminal time is deterministic, because so is the offspring distribution. (The terminal time is $t = 2$.)

Let $X$ be the $\left(\frac{1}{2}\Delta, \beta, \alpha; \mathbb{R}^d\right)$-superdiffusion with $\alpha, \beta > 0$ (supercritical super-Brownian motion). Here $\beta$ is the ‘mass creation parameter’ or ‘mass drift’, while $\alpha > 0$ is the ‘variance (or intensity) parameter’ of the branching — see [4] for more elaboration and for a more general setting.

Let $P$ denote the corresponding probability. Let us restrict $\Omega$ to the survival set $S := \{\omega \in \Omega | X_t(\omega) > 0, \forall t > 0\}$.

Since $\beta > 0$, $P_{\mu}(S) > 0$ for all $\mu \neq 0$.

Our main result is that the center of mass for $X$ stabilizes as $t \to \infty$, and furthermore the path of the center of mass is a finite piece of a Brownian path (with a different time parametrization).

**Theorem 3.** Let $\alpha, \beta > 0$ and let $X$ denote the center of mass process for the $\left(\frac{1}{2}\Delta, \beta, \alpha; \mathbb{R}^d\right)$-superdiffusion $X$. Then, on $S$,

(i) $X_t$ converges $P_{\delta_x}$-almost surely as $t \to \infty$.

(ii) In fact, the finite path $\{X_1\}_{t \geq 0}$ is the same as the path of a time changed Brownian motion on $[0, T_\infty)$, where $T_\infty$ is a positive and finite random variable. More precisely, if $X_0 = \delta_x$, $x \in \mathbb{R}^d$, then there exists a $d$-dimensional Brownian motion $B$ (on an enlarged space) starting at $x$, such that $X = B \circ T$ where $T : \mathbb{R}_+ \to \mathbb{R}_+$ is a random time change satisfying $\lim_{t \to \infty} T(t) < \infty$ $P_{\delta_x}$-a.s.

**Remark 4.**

(a) A heuristic argument for (i) is as follows. Obviously, the center of mass is invariant under $H$-transforms whenever $H$ is spatially (but not temporally) constant. Let $H(t) := e^{-\beta t}$. Then $X^H$ is a $\left(\frac{1}{2}\Delta, 0, e^{-\beta t}\alpha; \mathbb{R}^d\right)$-superdiffusion, that is, a critical super-Brownian motion with a clock that is slowing down. Therefore, heuristically it seems plausible that $X^H$, the center of mass for the transformed process stabilizes, because this is obviously true in case of extinction, and otherwise the center of mass under the heat flow does not move.

(b) From (ii) one can easily conclude some of the a.s. path properties of $X$.

For example, since the $p$-variation ($p > 0$) is invariant under changing the parametrization, we know that any segment of the path has infinite total variation, but the whole path has finite quadratic variation a.s. ☋

2. The mass center stabilizes

Notice that

\[
\frac{1}{n_t} \sum_{1 \leq j \leq n_t} \left( Z_i^j - Z_i^{j+1} \right) = \frac{1}{n_t} \left( \sum_{1 \leq j \leq n_t} Z_i^j - n_t Z_i \right) = Z_i - Z_i^1,
\]

and so the net attraction pulls the particle towards the center of mass (net repulsion pushes it away from the center of mass). Thus the following lemma is relevant:
Lemma 5 (Mass center stabilizes). Let \( Z_t := \frac{1}{\alpha} \sum_{i=1}^{n_t} Z_i^t \), that is, \( Z \) is the center of mass for \( Z \). Then \( \lim_{t \to \infty} Z_t = N \) a.s., where \( N \sim \mathcal{N}(0, 2I_d) \).

Proof. For \( t \in [m, m+1) \) there are \( 2^n \) particles moving around and particle \( Z_i^t \)'s motion is governed by

\[
dZ_i^t = dB^0_i(t) + \gamma(Z_t - Z_i^t) \, dt,
\]

where \( B^0_i, \ i = 1, 2, \ldots, 2^m \) are independent Brownian motions. Since \( 2^n Z_t = \sum_{i=1}^{2^m} Z_i^t \), we obtain that

\[
dZ_t = 2^{-m} \sum_{i=1}^{2^m} dZ_i^t = 2^{-m} \sum_{i=1}^{2^m} dB^0_i(t) + \gamma 2^m (Z_t - \sum_{i=1}^{2^m} Z_i^t) \, dt = 2^{-m} \sum_{i=1}^{2^m} dB^0_i(t).
\]

So, for \( 0 \leq \tau < 1 \),

\[
(2.2) \quad Z_{m+\tau} = Z_m + 2^{-m} \bigoplus_{i=1}^{2^m} B^0_i(\tau).
\]

Let \( \hat{B}^{(m)}(\tau) := 2^{-m/2} \bigoplus_{i=1}^{2^m} B^0_i(\tau) \). Denote \( \tau := t - [t] \). Using induction, we obtain that

\[
(2.3) \quad Z_t = \hat{B}^{(0)}(1) \bigoplus \frac{1}{\sqrt{2}} \hat{B}^{(1)}(1) \bigoplus \ldots \bigoplus \frac{1}{\sqrt{2^{(t-1)}}} \hat{B}^{(2^{(t-1)-1})}(1) \bigoplus \frac{1}{\sqrt{m_t}} \hat{B}^{(n_t)}(\tau),
\]

which, using Kolmogorov’s Three Series Theorem converges almost surely; we will denote the a.s. limit by \( N \). On the other hand, since \( \hat{B}^{(m)} \) is a Brownian motion, we can apply Brownian scaling and get \( 2^{-m/2} \hat{B}^{(m)}(\tau) \overset{d}{=} W^{(m)} \left( \frac{\tau}{2^m} \right) \), where \( W^{(m)}, \ m \geq 1 \) are independent Brownian motions. We have

\[
Z_{m+\tau} \overset{d}{=} Z_m \oplus W^{(m)} \left( \frac{\tau}{2^m} \right),
\]

and so,

\[
\hat{B}^{(0)}(1) \bigoplus \frac{1}{\sqrt{2}} \hat{B}^{(1)}(1) \bigoplus \ldots \bigoplus \frac{1}{\sqrt{2^{(t-1)}}} \hat{B}^{(2^{(t-1)-1})}(1) \bigoplus \frac{1}{\sqrt{m_t}} \hat{B}^{(n_t)}(1)
\]

\[
\overset{d}{=} W^{(1)}(1) \bigoplus W^{(2)} \left( \frac{1}{2} \right) \bigoplus \ldots \bigoplus W^{(2^{(t-1)-1})} \left( \frac{1}{2^{(t-1)}} \right) \bigoplus W^{(n_t)} \left( \frac{\tau}{m_t} \right).
\]

Since the summands are independent on the right hand side, \( N \) is distributed as a Brownian motion at \( t = 2 \), that is, \( N \sim \mathcal{N}(0, 2I_d) \).

\[ \square \]

For another proof see the remark after Lemma 8.

Remark 6. It is interesting to note that \( Z \) is in fact a Markov process. To see this, let \( \{\mathcal{F}_t\}_{t \geq 0} \) be the canonical filtration for \( Z \). Since \( \sigma(Z_s) \subset \mathcal{F}_s \), it is enough to check the Markov property with \( \sigma(Z_s) \) replaced by \( \mathcal{F}_s \). Assume first \( 0 \leq s < t, \ \lfloor s \rfloor = \lfloor t \rfloor =: m \). Then the distribution of \( Z_t \) conditional on \( \mathcal{F}_s \) is the same as conditional on \( Z_s \), because \( Z_t \) itself is a Markov process. But the distribution of \( Z_t \) only depends on \( Z_s \) through \( Z_s \), as

\[
Z_t \overset{d}{=} Z_s \oplus W^{(2^m)} \left( \frac{t-s}{2^m} \right).
\]

\[ \text{It is easy to check that, as the notation suggests, the summands are independent.} \]
The coordinate processes of $Z_t$ in the direction of $v$ are independent.

We will leave the simple proof to the reader.

3. Normality via Decomposition

We will need the following result. The decomposition appearing in the proof will also be useful. Recall that $m = \lfloor t \rfloor$.

**Lemma 7.** The coordinate processes of $Z$ are independent.

We leave the simple proof to the reader.

**Lemma 8.** In the time interval $[m, m+1)$ the process $(Z^1_t, Z^2_t, \ldots, Z^m_t)$ can be decomposed into two components: a one-dimensional Brownian motion and an independent $(2^m - 1)$-dimensional Ornstein-Uhlenbeck process with parameter $\gamma$. In particular, $(Z^1_t, Z^2_t, \ldots, Z^m_t)$ is joint normal for all $t \geq 0$.

**Proof of Lemma.** By Lemma 7, we may assume that $d = 1$. We prove the statement by induction.

For $m = 1$ it is trivial.

Suppose that the statement is true for $m - 1$. Since $(Z^1_{m-1}, Z^2_{m-1}, \ldots, Z^m_{m-1})$ is normal, we can consider it just as well as a 2-dimensional Brownian motion and an independent $(2^{m-1})$-dimensional Ornstein-Uhlenbeck process with parameter $\gamma$. In particular, $(Z^1_0, Z^2_0, \ldots, Z^m_0)$ is joint normal for all $t \geq 0$.

Define the 2-dimensional process $Z^*$ on the time interval $t \in [0, 1)$ by

$$Z^*_t := (Z^1_t, Z^2_t, \ldots, Z^m_t),$$

starting at the origin. Because of the interaction between the particles the process is a Brownian motion with drift

$$\gamma \left[ (Z^*_t, Z^*_t, \ldots, Z^*_t) - (Z^1_t, Z^2_t, \ldots, Z^m_t) \right].$$

Notice that this drift is orthogonal to the vector $v := (1, 1, \ldots, 1)$, that is, the vector $(Z^*_t, Z^*_t, \ldots, Z^*_t)$ is nothing but the orthogonal projection of $(Z^1_t, Z^2_t, \ldots, Z^m_t)$ to the line of $v$. This observation immediately leads to the following decomposition. The process $Z^*$ can be decomposed into two components:

- the component in the direction of $v$ is a Brownian motion

\[ \text{For simplicity, we use row vectors in this proof.} \]
Remark 9. Consider the Brownian component in the decomposition appearing in the proof. Since, on the other hand, this coordinate is $2^m/2Z_t$, using Brownian scaling, one obtains another way of seeing that $Z_t$ stabilizes at a position which is distributed as the time $1 + 2^{-1} + 2^{-2} + \ldots + 2^{-m} + \ldots = 2$ value of a Brownian motion. (The decomposition shows this for $d = 1$ and then it is immediately upgraded to general $d$ by independence.)

Corollary 10 (Asymptotics for finite subsystem). Let $k \geq 1$ and consider the subsystem $(Z^1_t, Z^2_t, \ldots, Z^k_t)$, $t \geq m_0$ for $m_0 := ([\log k]) + 1$. (This means that at time $m_0$ we pick $k$ particles and at every fission replace the parent particle by randomly picking one of its two descendants.) Let the real numbers $c_1, \ldots, c_k$ satisfy

\begin{equation}
\sum_{i=1}^{k} c_i = 0, \quad \sum_{i=1}^{k} c_i^2 = 1.
\end{equation}

Define $\Psi^{(c_1, \ldots, c_k)} := \sum_{i=1}^{k} c_i Z_i^t$ and note that $\Psi_1$ is invariant under the translations of the coordinate system. Let $\mathcal{L}_t$ denote its law.

For every $k \geq 1$ and $c_1, \ldots, c_k$ satisfying (3.1), $\Psi^{(c_1, \ldots, c_k)}$ is the same $d$-dimensional Ornstein-Uhlenbeck process corresponding to the operator $1/2\Delta - \gamma \nabla \cdot x$, and in particular, for $\gamma > 0$,

$$
\lim_{t \to \infty} \mathcal{L}_t = \mathcal{N}\left(0, d^{2\gamma} \right).
$$

For example, taking $c_1 = 1/\sqrt{2}, c_2 = -1/\sqrt{2}$, we obtain that when viewed from a tagged particle's position, any given other particle moves as $\sqrt{2}$ times the above Ornstein-Uhlenbeck process.

Proof. By independence (Lemma 7) it is enough to consider $d = 1$. For $m$ fixed, consider the decomposition appearing in the proof of Lemma 8 and recall the notation. By (3.1), whatever $m \geq m_0$ is, the $2^m$ dimensional unit vector

$$(c_1, c_2, \ldots, c_k, 0, 0, \ldots, 0)$$

is orthogonal to the $2^m$ dimensional vector $v$. This means that $\Psi^{(c_1, \ldots, c_k)}$ is a one dimensional projection of the Ornstein-Uhlenbeck component of $Z^*$, and thus it is itself a one dimensional Ornstein-Uhlenbeck process (with parameter $\gamma$) on the unit time interval.

Now, although as $m$ grows, the Ornstein-Uhlenbeck components of $Z^*$ are defined on larger and larger spaces ($S \subset \mathbb{R}^{2^m}$ is a $2^{m-1}$ dimensional linear subspace), the projection onto the direction of $(c_1, c_2, \ldots, c_k, 0, 0, \ldots, 0)$ is always the same one dimensional Ornstein-Uhlenbeck process, i.e. the different unit time ‘pieces’ of $\Psi^{(c_1, \ldots, c_k)}$ obtained by those projections may be concatenated. \hfill \Box

4. The Interacting System as Viewed from the Center of Mass

Recall that by (2.2) the interaction has no effect on the motion of $Z$. Let us see now how the interacting system looks like when viewed from $Z$. 
4.1. The description of a single particle. Using our usual notation, assume that \( t \in [m, m + 1) \) and let \( \tau := t - [t] \). When viewed from \( Z_t \), the relocation\(^4\) of a particle is governed by

\[
d(Z_t^1 - Z_t) = dZ_t^1 - dZ_t = dB_0^1(t) - 2^{-m} \sum_{i=1}^{2^m} dB_i^1(t) - \gamma(Z_t^1 - Z_t) dt.
\]

So if \( Y_t^1 := Z_t^1 - Z_t \), then

\[
dY_t^1 = dB_0^1(t) - 2^{-m} \sum_{i=1}^{2^m} dB_i^1(t) - \gamma Y_t^1 dt.
\]

Clearly,

\[
B_0^1(\tau) - 2^{-m} \bigoplus_{i=1}^{2^m} B_i^1(\tau) = \bigoplus_{i=2}^{2^m} 2^{-m} B_i^1(\tau) \oplus (1 - 2^{-m}) B_0^1(\tau),
\]

and thus the right hand side is a Brownian motion with mean zero and variance \((1 - 2^{-m}) \gamma I_d := \sigma_m^2 \gamma \mathbf{I}_d\). That is,

\[
dY_t^1 = \sigma_m dW^1(t) - \gamma Y_t^1 dt,
\]

where \( W^1 \) is a standard Brownian motion.

We have thus obtained that on the time interval \([m, m + 1)\), \( Y^1 \) corresponds to the Ornstein-Uhlenbeck operator

\[
\frac{1}{2} \sigma_m \Delta - \gamma x \cdot \nabla.
\]

Since for \( m \) large \( \sigma_m \) is close to one, the relocation viewed from the center of mass is asymptotically governed by an O-U process corresponding to \( \frac{1}{2} \Delta - \gamma x \cdot \nabla \).

Remark 11 (Asymptotically vanishing correlation between driving BM’s). Let \( W^{i,k} \) be the \( k^{th} \) coordinate of the \( i^{th} \) Brownian motion: \( W^i = (W^{i,k}, k = 1, 2, \ldots, d) \). For \( 1 \leq i \neq j \leq 2^m \), we have

\[
E[\sigma_m W^{i,k}(\tau) \cdot \sigma_m W^{j,k}(\tau)] = E\left(\left(B_{0}^{i,k}(\tau) - 2^{-m} \bigoplus_{r=1}^{2^m} B_{0}^{r,k}(\tau)\right)\left(B_{0}^{j,k}(\tau) - 2^{-m} \bigoplus_{r=1}^{2^m} B_{0}^{r,k}(\tau)\right)\right) = 2^{-m} \text{Var}(B_{0}^{i,k}(\tau)) + \text{Var}(B_{0}^{j,k}(\tau)) - 2^{-2m} 2^m \tau = 2^{-m} \tau,
\]

that is, for \( i \neq j \),

\[
E[W^{i,k}(\tau)W^{j,l}(\tau)] = \delta_{kl} \cdot \frac{2^{-m}}{(1 - 2^{-m})^2} \tau.
\]

Hence the pairwise correlation decays to zero as \( t \to \infty \) (recall that \( m = [t] \) and \( \tau = t - m \in [0,1) \)).

And of course, for the variances we have

\[
E[W^{i,k}(\tau)W^{i,\ell}(\tau)] = \delta_{k\ell} \cdot \tau, \text{ for } 1 \leq i \leq 2^m.
\]

\(^4\)i.e. the relocation between time \( m \) and time \( t \).
4.2. The description of the system; the ‘degree of freedom’. Fix \( m \geq 1 \) and for \( t \in [m, m + 1) \) let \( Y_t := (Y_1^t, ..., Y_{2^m}^t)^T \), where \((\cdot)^T\) denotes transposed. (This is a vector of length \( 2^m \) where each component itself is a \( d \) dimensional vector; one can actually look at it as a \( 2^m \times d \) matrix too.) We then have
\[
dY_t = \sigma_m dW_t - \gamma Y_t dt,
\]
where
\[
W = \left( W^1, ..., W^{2^m} \right)^T
\]
and
\[
W^i(\tau) := B^i_0(\tau) - 2^{-m} \sum_{j=1}^{2^m} B^j_0(\tau), \quad i = 1, 2, ..., 2^m
\]
are mean zero Brownian motions with correlation structure given by (4.1)-(4.2).

Just like at the end of subsection 1.1, we can consider \( Y \) as a single \( 2^m \times d \)-dimensional diffusion. Each of its components is an Ornstein-Uhlenbeck process with asymptotically unit diffusion coefficient.

By independence, it is enough to consider the \( d = 1 \) case, and so from now on, in this subsection we assume that \( d = 1 \).

Let us first describe the distribution of \( W_t \) for \( t \geq 0 \) fixed. Recall that \( \{B^i_0(s), s \geq 0; \quad i = 1, 2, ..., 2^m\} \) are independent Brownian motions starting at the origin. By definition, \( W_t \) is a \( 2^m \)-dimensional multivariate normal:
\[
W_t = \sigma_m^{-1} \cdot \begin{pmatrix}
1 - 2^{-m} & -2^{-m} & ... & -2^{-m} \\
-2^{-m} & 1 - 2^{-m} & ... & -2^{-m} \\
& & \ddots & \\
-2^{-m} & -2^{-m} & ... & 1 - 2^{-m}
\end{pmatrix} B_0(t) =: \sigma_m^{-1} A^{(m)} B_0(t),
\]
yielding
\[
dY_t = A^{(m)} dB_0(t) - \gamma Y_t dt.
\]
Since we are viewing the system from the center of mass, \( W_t \) is a singular multivariate normal and thus \( Y \) is a degenerate diffusion. The ‘true’ dimension of \( W_t \) is the rank of the matrix \( A^{(m)} \).

**Lemma 12.** \( \text{rank}(A^{(m)}) = 2^m - 1 \).

**Proof.** We will simply write \( A \) instead of \( A^{(m)} \). Since the columns of \( A \) add up to zero, the matrix \( A \) is not of full rank: \( r(A) \leq 2^m - 1 \). On the other hand,
\[
2^m A + \begin{pmatrix}
1 & 1 & ... & 1 \\
1 & 1 & ... & 1 \\
& & \ddots & \\
1 & 1 & ... & 1
\end{pmatrix} = 2^m I,
\]
where \( I \) is the \( 2^m \)-dimensional unit matrix, and so by subadditivity,
\[
r(A) + 1 = r(2^m A) + 1 \geq 2^m. \quad \Box
\]
By Lemma 12, \( W_t \) is concentrated on the \( (2^m - 1) \)-dimensional linear subspace given by the orthogonal complement of the vector \((1, 1, ..., 1)^T\); in this \( 2^m - 1 \) dimensional
subspace $W_t$ has non-singular multivariate normal distribution. What this means is that even though $W_1, W_2, \ldots, W_n$ are not independent, their ‘degree of freedom’ is $2^n - 1$, i.e. the $2^n$-dimensional vector $W$ is determined by $2^n - 1$ independent components (corresponding to $2^n - 1$ principal axes).

5. Asymptotic Behavior

How can we put together that $Z_t$ tends to a random final position a.s. with the description of the system ‘as viewed from $Z_t$’?

**Lemma 13.** For $t \geq 0$, the random vector $Y_t$ is independent of the path $\{Z_s\}_{s \geq t}$.

**Proof.** First, for any $t > 0$, $Y_t$ is independent of $Z_t$, because (assuming $d = 1$) the vector

$$(Z_t, Z_1^t - Z_t, Z_2^t - Z_t, \ldots, Z_{2^n}^t - Z_t)^T$$

is normal (since it is a linear transformation of the vector $(Z_1^t, Z_2^t, \ldots, Z_{2^n}^t)^T$, which is normal by Lemma 8), and so it is sufficient to show that $Z_t$ and $Z_i - Z_t$ are uncorrelated for $1 \leq i \leq 2^n$. But this is obvious, because the random variables $Z_1^t, Z_2^t, \ldots, Z_{2^n}^t$ are exchangeable and thus, denoting $n = 2^m$,

$$E[Z_i(Z_i - Z_t)] = E \left[ \frac{Z_1^t + Z_2^t + \ldots + Z_i^t}{n} \left( \frac{Z_i^t - Z_1^t + Z_2^t + \ldots + Z_i^t}{n} \right) \right] = \frac{1}{n} E \left[ (Z_i^t)^2 \right] + \frac{n-1}{n} E(Z_i^t Z_t^t) = 0.$$ 

To complete the proof of the lemma, recall (2.2) and (2.3) and notice that the distribution of $\{Z_s\}_{s \geq t}$ only depends on its starting point $Z_t$, as it is that of a Brownian path appropriately slowed down, whatever $Y_t$ (or, equivalently, whatever $Z_t = Y_t + Z_t$) is. Since, as we have seen, $Y_t$ is independent of $Z_t$, we are done. □

Putting Lemma 13 together with the description of $Y$ in sections 4.1 and 4.2, we immediately obtain the following result on the asymptotic behavior of the system.

**Theorem 14.** If $Z$ is the interacting branching Brownian motion of Section 1.1 with $\gamma \neq 0$, then the asymptotic behavior of $Z$ is that of a weakly coupled branching Ornstein-Uhlenbeck process, but with the origin shifted by a random, normally distributed vector.

More precisely, recall from Lemma 5 that $\overline{Z}_t := \frac{1}{m} \sum_{i=1}^{m} Z_i^t$ has an almost sure Gaussian limit, and let $Y_t := Z_i^t - \overline{Z}_t$, $i = 1, 2, \ldots, 2^m$, $t \in [m, m + 1]$. Then the two components are independent: $Z_t = \overline{Z}_t + Y_t$, and furthermore, $Y$ satisfies that

- (a) each particle of $Y$ is asymptotically Ornstein-Uhlenbeck \(^5\) corresponding to the operator $\frac{1}{2} \Delta - \gamma x \cdot \nabla$,
- (b) the driving Brownian motions have asymptotically vanishing pairwise correlations, and
- (c) they are obtained from independent Brownian motions by a linear transformation with a kernel of dimension one.

**Remark 15** (Conditioning on the final position of $Z$). Let $N := \lim_{t \to \infty} Z_t$ (recall that $N \sim N(0, 2I_d)$) and

$$P^x(\cdot) := P(\cdot \mid N = x).$$

\(^5\)If $\gamma < 0$ then it is ‘outward drifting’ Ornstein-Uhlenbeck.
By Lemma 13, $Y_t$ is independent of $N$, and thus $P^x(Y_t \in \cdot) = P(Y_t \in \cdot)$ for almost all $x \in \mathbb{R}^d$. It then follows that the decomposition $Z_t = Z_t \oplus Y_t$ as well as (a)-(c) of Theorem 14 are true under $P^x$ too, for almost all $x \in \mathbb{R}^d$.  

We close this section with a theorem and a conjecture on the local behavior of the system.

**Theorem 16.** Let $\{P^x, x \in \mathbb{R}^d\}$ be as in Remark 15. Let $\Rightarrow$ denote convergence in the vague topology. If $\gamma > 0$ (attraction), then, as $n \to \infty$,

\[
2^{-n}Z_n(dy) \Rightarrow \left( \frac{2}{\pi} \right)^{d/2} \exp\left( -\gamma|y - x|^2 \right) dy, \quad P^x - a.s.
\]

for almost all $x \in \mathbb{R}^d$. Consequently,

\[
2^{-n}E Z_n(dy) \Rightarrow f^\gamma(y)dy,
\]

where

\[
f^\gamma(\cdot) = \left(\pi(4 + \gamma^{-1})\right)^{-d/2} \exp \left[ -\frac{|\cdot|^2}{4 + \gamma^{-1}} \right].
\]

**Remark 17.** Concerning the variance term $\Sigma = \left(2 + \frac{1}{\gamma} \right)I_d$ corresponding to $f^\gamma(\cdot)$, it is not surprising that stronger attraction results in smaller variance.

**Conjecture 18.** Let $\gamma < 0$ (repulsion).

1. If $|\gamma| \geq \log_2 \frac{2}{d}$, then $Z$ suffers local extinction:

\[
Z_n(dy) \Rightarrow 0, \quad P - a.s.
\]

2. If $|\gamma| < \log_2 \frac{2}{d}$, then

\[
2^{-n}e^{d|\gamma|n}Z_n(dy) \Rightarrow dy, \quad P - a.s.
\]

Note that in case (2) the rescaled (vague) limit of $Z_n(dy)$ is translation invariant, i.e. Lebesgue (see Example 11 in [2]), and thus the final position of the center of mass plays no role.

Although we will not prove Conjecture 18, we will discuss some of the technicalities in section 8.

6. Proof of Theorem 3

(i) Since $\alpha, \beta$ are constant, the branching is independent of the motion, and therefore $N$ defined by

\[
N_t := e^{-\beta t}\|X_t\|
\]

is a nonnegative martingale (positive on $S$) tending to a limit almost surely. It is straightforward to check that it is uniformly bounded in $L^2$ and is therefore uniformly integrable (UI). Write

\[
X_t = \frac{e^{-\beta t}(x, X_t)}{e^{-\beta t}\|X_t\|} = \frac{e^{-\beta t}(x, X_t)}{N_t}.
\]

We now claim that $N_\infty > 0$ a.s. on $S$. Let $A := \{N_\infty = 0\}$. Clearly $S \mathcal{C} \subset A$, and so if we show that $P(A) = P(S \mathcal{C})$, then we are done. As is well known, $P(S \mathcal{C}) = e^{-\beta/\alpha}$. On the other hand, a standard martingale argument (see the argument after formula (20) in [3]) shows that $0 \leq u(x) := -\log P_{\delta x}(A)$ must solve the equation

\[
\frac{1}{2}\Delta u + \beta u - \alpha u^2 = 0,
\]
but since $P_{\infty}(A) = P(A)$ constant, therefore $-\log P_{\infty}(A)$ solves $\beta u - \alpha u^2 = 0$. Since $N$ is UI, no mass is lost in the limit, giving $P(A) < 1$. So $u > 0$, which in turn implies that $-\log P_{\infty}(A) = \beta/\alpha$.

Once we know that $N_{\infty} > 0$ a.s. on $S$, it is enough to focus on the term $e^{-\beta t}(x, x_t)$.

Recall a particular case of the $H$-transform for the $(L, \beta; \mathbb{R}^d)$-superdiffusion $X$ (see Appendix B in [5]):

**Lemma 19.** Define $X^H$ by

$$X_t^H := H(\cdot, t)X_t \quad \text{that is, } \frac{dX_t^H}{dX_t} = H(\cdot, t), \quad t \geq 0.$$  

If $X$ is an $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusion, and $H(x, t) := e^{-\lambda t}h(x)$, where $h$ is a positive solution of $(L + \beta)h = \lambda h$, then $X^H$ is a $(L + a \frac{\nabla h}{h} \cdot \nabla, 0, e^{-\lambda t}ah; \mathbb{R}^d)$-superdiffusion.

In our case $\beta(\cdot) \equiv \beta$. So choosing $h(\cdot) \equiv 1$ and $\lambda = \beta$, we have $H(t) = e^{-\beta t}$ and $X^H$ is a $(\frac{1}{2}\Delta, 0, e^{-\beta t}ah; \mathbb{R}^d)$-superdiffusion, that is, a critical super-Brownian motion with a clock that is slowing down. One can write

$$e^{-\beta t}\langle x, X_t \rangle = \langle x, X_t^H \rangle.$$  

Let $\{S_{\alpha}\}_{\alpha > 0}$ be the ‘expectation semigroup’ for $X$, that is, the semigroup corresponding to the operator $\frac{1}{2}\Delta + \beta$. The expectation semigroup $\{S_{\alpha}^H\}_{\alpha > 0}$ for $X^H$ satisfies $T_s := S_s^H = e^{-\beta s}S_s$ and thus it corresponds to Brownian motion. In particular then

$$T_s[\text{id}] = \text{id},$$  

where $\text{id}(x) = x$. Therefore $\langle x, X_t^H \rangle$ is a martingale.\footnote{It does not matter that the function is unbounded and changes sign.} If we show that the martingale is UI, we are done. It is enough to show that it is uniformly bounded in $L^2$. To achieve this, define $g_n$ by $g_n(x) = |x| \cdot 1_{\{|x| < n\}}$. Then we have

$$E_{\delta_0}\langle x, X_t^H \rangle^2 = E_{\delta_0}\langle x, X_t^H \rangle^2 \leq E_{\delta_0}\langle |x|, X_t^H \rangle^2,$$

and by the monotone convergence theorem we can continue with

$$= \lim_{n \to \infty} E_{\delta_0}\langle g_n(x), X_t^H \rangle^2.$$  

Since $g_n$ is compactly supported, there is no problem to use the moment formula and continue with

$$= \lim_{n \to \infty} \left( 1 + \int_0^t ds e^{-\beta s}\langle \delta_0, T_s[ag_n^2] \rangle \right) = 1 + \alpha \lim_{n \to \infty} \int_0^t ds e^{-\beta s}T_s[g_n^2](0).$$

Recall that $\{T_s; s \geq 0\}$ is the Brownian semigroup, that is, $T_s[f](x) = E_xf(W_s)$, where $W$ is Brownian motion. Since $g_n(x) \leq |x|$, therefore we can trivially upper estimate the last expression by

$$1 + \alpha \lim_{n \to \infty} \int_0^t ds e^{-\beta s}\mathbb{E}_0(W_s^2) = 1 + \alpha \int_0^t ds se^{-\beta s} = 1 + \frac{\beta}{\alpha}.$$  

Since this upper estimate is independent of $t$, we are done:

$$\sup_{t \geq 0} E_{\delta_0}\langle x, X_t^H \rangle^2 \leq 1 + \frac{\beta}{\alpha}.$$
Cauchy-Schwartz, it is enough to see that an a.s. finite random time where
displayed formula with is mean zero, shifting by $X$
where $B$
i.e. that
and since we have seen in part (i) that
Thus,

Let us start with showing that $E[|X_t|] < \infty$. To see this, first recall that

Thus,

and since we have seen in part (i) that $\langle x, X_t^H \rangle$ is (uniformly) bounded in $L^2$, by
Cauchy-Schwartz, it is enough to see that

$i.e. that $E[\|X_t\|^2; S] < \infty, \forall t > 0$. This is true because $\|X\|$ is a one-dimensional
diffusion on $[0, \infty)$ with generator $x \left( \frac{\partial^2}{\partial x^2} - \beta \frac{\partial}{\partial x} \right)$, which, on $S$, tends to infinity, and
therefore, by Fatou’s Lemma, $\lim_{t \to \infty} E[\|X_t\|^2; S] = 0$. Hence $E[\|X_t\|^2; S] < \infty$
for large $t's$, but then by continuity, $E[\|X_t\|^2; S] < \infty, \forall t > 0$.

Next, since $E[|X_t|] < \infty$, in fact $E|X_t| = 0$ for $t \geq 0$, because $X_t$ is symmetrically
distributed ($X_t \overset{d}{=} -X_t$). Indeed,

where $B^* := \{ \mu^* \mid \mu \in B \}$ and $\mu^*(\cdot) := \mu(-\cdot)$.

We now show that $E(X_t \mid F_s) = X_s$ for $0 \leq s < t$. By the Markov branching
property,

where $X_{\delta_s}$ are independent copies of super-Brownian motions starting at $\delta_x$. Since
$X$ is mean zero, shifting by $x$ and using Fubini’s Theorem, we can continue the last
displayed formula with

Next, we show that $X$ has continuous paths.

Let

where $\epsilon > 0$ is fixed for the rest of the argument. By [6] it follows that there exists
an a.s. finite random time $T = T(\omega)$ such that for all $t > T$,

(This is obviously true on the extinction set.) Hence, for $t > T$,

$$(x, X_t) - (x, X_{t+\Delta t}) = (\langle u(\cdot), t \rangle, X_t) - (\langle u(\cdot), t + \Delta t \rangle, X_{t+\Delta t}) =
(\langle u(\cdot), t \rangle, X_t) - (\langle u(\cdot), X_t\rangle + (\langle u(\cdot), X_{t+\Delta t}\rangle - (\langle u(\cdot), t + \Delta t \rangle, X_{t+\Delta t}\rangle)
= I + II.$$
Now $I \to 0$ as $\Delta t \downarrow 0$ a.s. by the continuity of the paths (the continuity is in the weak topology and $u(\cdot, t)$ is a bounded continuous function for all $t > 0$), and $II \to 0$ as $\Delta t \downarrow 0$ a.s., because

$$II = \langle \gamma(\Delta t), X_{t+\Delta t} \rangle,$$

where $|\gamma(\Delta t)| \leq (\sqrt{2}\beta + \epsilon)\Delta t$ a.s. So

$$II \leq (\sqrt{2}\beta + \epsilon)\Delta t \cdot \|X_{t+\Delta t}\| \text{ a.s.,}$$

and we are done since

$$\lim_{\Delta t \downarrow 0} \|X_{t+\Delta t}\| = \|X_t\| \text{ a.s.,}$$

again because of the path continuity.

This proves right continuity of the paths of $\overline{X}$; the left continuity is similar.

Once we know that the center of mass is a continuous martingale, we can utilize Knight’s multidimensional version of the Dambis-Dubins-Schwarz Representation Theorem, which we cite below. (For the theorem and its proof see e.g. Theorems V. 1.9 and V. 1.10 in [7].)

**Proposition 20** (Theorem KDDS). Let $M = (M_1, ..., M_d)$ be a continuous local martingale in $\mathbb{R}^d$ with $M_0 = 0$, such that $[M^i, M^j] = 0$, $i \neq j$ holds almost surely, where $[M^i, M^j]$ denotes the quadratic covariation. Define

$$T^k_t := \inf\{s \geq 0 \mid [M^k, M^i]_s > t\}, \quad V_k := [M^k, M^i]_\infty.$$

Then there exists in $\mathbb{R}^d$ a Brownian motion $\beta$ independent of $M$, such that the process $B$ defined by

$$B^k_t := M^k_{T^k_t} \text{ for } t \in [0, V_k), \quad B^k_t := M^k_{\infty} + \beta(0 - V_k) \text{ for } t \in [V_k, \infty)$$

is a $d$-dimensional Brownian motion.\(^7\)

In order to finish the proof of Theorem 3(ii), let $x = (x_1, x_2, ..., x_d)$ and $M^i := \langle x_i, \tilde{X} \rangle$, where $\tilde{X} := \frac{X}{\|X\|}$ is the ‘angular part’ of $X$. Then

$$\overline{X}_t = \langle x, \tilde{X}_t \rangle = \langle x_1, \tilde{X}_t \rangle, \langle x_2, \tilde{X}_t \rangle, ..., \langle x_d, \tilde{X}_t \rangle = (M^1, M^2, ..., M^d).$$

To check that $[M^i, M^j] = 0$ for $i \neq j$, use polarization:

$$[M^i, M^j] = \frac{1}{4} \left( [M^i + M^j, M^i + M^j] - [M^i - M^j, M^i - M^j] \right).$$

In our case $[M^i + M^j, M^i + M^j] = [M^i - M^j, M^i - M^j]$, that is,

$$[x_i + x_j, \tilde{X}], (x_i + x_j, \tilde{X}) = [(x_i - x_j, \tilde{X}), (x_i - x_j, \tilde{X})],$$

because changing $x_j$ to $-x_j$ simply means reversing the direction of one of the coordinate axes, and the distribution of $\tilde{X}$ is invariant under this transformation.

Hence, by Proposition 20, the process $B$ defined by

$$B^k_t := M^k_{T^k_t} \text{ for } t \in [0, V_k), \quad B^k_t := M^k_{\infty} + \beta(0 - V_k) \text{ for } t \in [V_k, \infty)$$

is a $d$-dimensional Brownian motion.

Since we already know from part (i) that $\overline{X}_t$ has a finite a.s. limit, it follows that $V_k < \infty$, $1 \leq k \leq d$, $P_0$-a.s., and $M^k_{\infty} = B^k_{V_k}$.

---

\(^7\)If $V_k = \infty$ for all $k$, then of course, there is no $\beta$ in the statement.
We now prove that the time changes are a.s. the same for each coordinate, that is
\[ [M^i, M^j] = [M^j, M^i] \text{ a.s. for } i \neq j. \]
Again, by polarization, this is equivalent to \([M_i + M_j, M_i - M_j] = 0 \text{ a.s.}, \] that is to
\[ ([x_i + x_j, \tilde{X}], (x_i - x_j, \tilde{X})] = 0, \text{ a.s.} \]
Since the change of the coordinates \((x_i, x_j) \mapsto (1/\sqrt{2})(x_i + x_j, x_i - x_j)\) is an orthogonal transformation, under which the distribution of \(\tilde{X}\) is invariant, (6.3) follows from the already proven orthogonality relation \([M^i, M^j] = 0. \]

7. Proof of Theorem 16

Once we have Theorem 14 and Remark 15, we can try to put them together with the Strong Law of Large Numbers for the local mass from [2] for the process \(Y\). If the components of \(Y\) were independent and the branching rate were exponential, Theorem 6 of [2] would be readily applicable. However, since the \(2^n\) components of \(Y\) are not independent (as we have seen, their degree of freedom is \(2^n - 1\) and since, unlike in [2], we now have unit time branching, the method of [2] must be adapted to our setting. This adaptation will require some extra work.

We will need the following result.

Claim 21. \(Y = (Y_i; t \geq 0)\) is independent of \(T\), the tail \(\sigma\)-algebra of \(Z\).

Proof. Let \(A \in \mathcal{T}\). It is enough to show that \((Y_{t_1}, ..., Y_{t_k})\) is independent of \(A\) for \(0 \leq t_1 < ... < t_k\) and \(k \geq 1\) (because it is easy to show that the events that are independent of \(A\) form a \(\sigma\)-algebra). Similarly, since \(A \in \mathcal{T} \subset \sigma(\tilde{Z}_s; s \geq t_k)\), we only need that \((Y_{t_1}, ..., Y_{t_k})\) is independent of \((\tilde{Z}_s; s \geq t_k)\). Note that by Lemma 13, \(Y_{t_k}\) is independent of \(\tilde{Z}_{t_k}\). To complete the proof of the claim, recall Lemma 5 and its proof and notice that the distribution of \((\tilde{Z}_s; s \geq t_k)\) depends on its starting point \(\tilde{Z}_{t_k}\) only, as it is that of a Brownian path appropriately slowed down, whatever \((Y_s; s \leq t_k)\) (or even \((Z_s; s \leq t_k)\)) is. Since, as we have seen, \(Y_{t_k}\) is independent of \(\tilde{Z}_{t_k}\), we are done.

\[ \lim_{n \to \infty} 2^{-n} \langle g, Y_n \rangle = \langle g, \tilde{f} \rangle, \quad P^x - \text{a.s.} \]
This is because \(P^x - \text{a.s.}, \)
\[ \lim_{n \to \infty} 2^{-n} \langle g, Z_n \rangle = \lim_{n \to \infty} 2^{-n} \langle g, Y_n + \tilde{Z}_n \rangle = \lim_{n \to \infty} 2^{-n} \langle g(\cdot + \tilde{Z}_n), Y_n \rangle = I + II, \]
where
\[ I := \lim_{n \to \infty} 2^{-n} \langle g(\cdot + x), Y_n \rangle \]
and
\[ II := \lim_{n \to \infty} 2^{-n} \langle g(\cdot + \tilde{Z}_n) - g(\cdot + x), Y_n \rangle. \]
Now, (7.1) implies that \(I = \langle g(\cdot + x), \tilde{f}(\cdot)\rangle\), while the compact support of \(g\), and Heine’s Theorem yields \(II = 0\). So \(\lim_{n \to \infty} 2^{-n} \langle Z_n, g \rangle = \langle g(\cdot), \tilde{f}(\cdot - x) \rangle\), and since \(2^{-n} \langle Z_n, g \rangle \leq \|g\|_\infty\), it follows that
\[ \lim_{n \to \infty} E2^{-n} \langle Z_n, g \rangle = \int_{\mathbb{R}^d} E^x \left( \lim_{n \to \infty} 2^{-n} \langle Z_n, g \rangle \right) Q(dx) = \int_{\mathbb{R}^d} \langle \tilde{f}(\cdot - x), g \rangle Q(dx), \]
where $Q \sim \mathcal{N}(0, 2I_d)$. Now, if $\hat{f} \sim \mathcal{N}(0, 2I_d)$ then $f^\gamma := \hat{f} * \tilde{f} \sim \mathcal{N}\left(0, \left(2 + \frac{1}{2\gamma}\right) I_d\right)$ and

$$
\int_{\mathbb{R}^d} \langle f(\cdot - x), g \rangle Q(dx) = \langle g, f^\gamma \rangle,
$$

yielding (5.2).

Note that by Claim 21,

$$
P_x \left(\lim_{n \to \infty} 2^{-n} \langle g, Y_n \rangle = \langle g, \tilde{f} \rangle\right) = \mathbb{P} \left(\lim_{n \to \infty} 2^{-n} \langle g, Y_n \rangle = \langle g, \tilde{f} \rangle \mid N = x\right),
$$

and thus, it is sufficient to prove (7.1) with $P^x$ replaced by $P$. To this end, we are going to modify an argument given in [2] (the proof of Lemma 18 in [2]) and so the reader should have [2] handy when reading the rest of this proof.

The setting of Lemma 18 in [2] differs from our setting in three ways. In our setting

(i) We have unit time branching (as opposed to exponential branching considered in [2]).

(ii) The particles of $Y$ are dependent, unlike in [2].

(iii) $\sigma_m < 1$ (but $\lim_{m} \sigma_m = 1$).

We will see that (i) in fact makes the proof easier than in [2] and that (iii) is a very minor issue. The real difficulty is handling (ii). We will see that we do not need a change of measure, unlike in [2], where it was an essential ingredient. Thus, in [2], the only point in the proof that uses the independence of the particles is the crucial inequality for independent and mean zero variables $Z_i$ (inequality (11)) due to Biggins. For $p = 2$ however, it reduces to

$$
E \left(\sum_{i=1}^{n} Z_i\right)^2 \leq 4 \sum_{i=1}^{n} E Z_i^2,
$$

which is trivial as $E \left(\sum_{i=1}^{n} Z_i\right)^2 = \sum_{i=1}^{n} E Z_i^2$.

In our case, instead of independence, we only have weak dependence for the system $Y$. Similarly to [2], let $\{Y_i : i = 1, \ldots, 2^n\}$ describe the configuration of particles at time $n$ and let

$$
U_n := 2^{-n} \langle 1_B, Y_n \rangle.
$$

Note that by the branching property,

$$
U_{n+m} = \sum_{i=1}^{2^n} 2^{-n} U_m^{(i)},
$$

where given $F_n$, the collection $\{U_m^{(i)} : i = 1, \ldots, 2^n\}$ are equal in distribution to $U_m$ under $P_{\delta_{Y_i}}$ respectively.

Let $\{m_n\}$ be sequence of positive integers tending to infinity as $n \to \infty$. We will suppress the dependence on $n$ and simply write $m$. In order for the proof to work, it is necessary that for $Z_i := U^{(i)} - E(U^{(i)} | F_n)$,

$$
E \left(\sum_{i=1}^{n} Z_i\right)^2 \leq K \sum_{i=1}^{n} E Z_i^2, \tag{7.2}
$$

\(^8\)Now $2^n$ is playing the role of the factor $e^{\lambda c t}$. 


holds with some $K > 0$. Although $E(Z_i) = 0$ is still true, it is no longer the case that conditional on $\mathcal{F}_n$, the $Z_i$ are independent. Nevertheless, we can use (4.1)-(4.2) and obtain that (7.2), that is the inequality

$$\sum_{1 \leq i \leq j \leq 2^n} EZ_i Z_j \leq (K - 1) \sum_{i=1}^{2^n} EZ_i^2 =: C \sum_{i=1}^{2^n} EZ_i^2$$

is tantamount to

$$2^n (2^n - 1) \frac{2^{-n}}{(1 - 2^{-n})^2} \leq C 2^n (1 - 2^{-n}), \quad n \geq 1.$$ 

This means that $C \geq (1 - 1/2)^{-2} = 4$, or $K \geq 5$. Hence (7.2) holds with $K = 5$.

We now explain why, as mentioned above, we do not need a spine change of measure. This is because, an estimate analogous to the one in [2] (see the estimates appearing between formulae (11) and (12)) yields

$$E \left( \left| U_{n+m} - E(U_{n+m} \mid \mathcal{F}_n) \right|^2 \mid \mathcal{F}_n \right) \leq \text{const} \cdot 2^{-2n} \sum_{i=1}^{2^n} 1 = \text{const} \cdot 2^{-n},$$

where we used that $U_m \leq 1$. In [2], the analysis was more complicated, because the corresponding upper estimate involved the analogous term $U_n$, which, however, was not upper bounded. Therefore in [2] we proceeded with a spine change of measure and some further calculations. That part of the work is saved now. (Note also, that the analogous term to the martingale by which the change of measure was defined in [2], is now $2^{-n} \langle 1, Y_n \rangle = 1$.)

Then, since $E \left( \left| U_{n+m} - E(U_{n+m} \mid \mathcal{F}_n) \right|^2 \mid \mathcal{F}_n \right)$ is summable, just like in the proof of Lemma 18 in [2], a Borel-Cantelli argument gives:

$$\lim_{n \to \infty} (U_{n+m} - E(U_{n+m} \mid \mathcal{F}_n)) = 0, \quad \text{P-a.s.}$$

The rest of the proof follows some ideas from the proof of Theorem 6 along lattice times in [2], as follows. First note that if $q(x, B, t)$ denotes the transition kernel corresponding to the operator $\frac{1}{2} \Delta - \gamma x \cdot \nabla$, then by ergodicity, $\lim_{n \to \infty} q(x, B, t) = \int_B f(y) dy$. Recall that in our case the underlying diffusion is only asymptotically Ornstein-Uhlenbeck, that is $\sigma_n^2 = 1 - 2^{-n}$ (and not $\sigma_n^2 = 1$ as in [2]), and let the transition probabilities $q_n$ be defined by $q_n(x, B, k) := P(Y^1_k \in B \mid Y^1_n = x)$. By the branching property,

$$E(U_{n+m} \mid \mathcal{F}_n) = \sum_{i=1}^{2^n} 2^{-n} E(U_{m+i}^{(i)}) = 2^{-n} \sum_{i=1}^{2^n} q_n(X_i, B, n + m).$$

Let

$$A_n := \{ \text{supp}(Z_n) \not\subset B_n(0) \}.$$ 

Then $\lim_{n \to \infty} 1_{A_n} = 0$ (see Example 10 in [2]). Using this, the proof will be finished by showing that

$$\lim_{n \to \infty} \sup_{|x| \leq n} \left| q(x, B, n + m) - \int_B \tilde{f}(y) dy \right| = 0,$$

and

$$\lim_{n \to \infty} \sup_{|x| \leq n} |q(x, B, n + m) - q_n(x, B, n + m)| = 0.$$
holds with both \( m = m_n := n \) and \( m = m_n := n + 1 \), taking care of the limit along even and odd sequences (\( U_{2n} \) and \( U_{2n+1} \), respectively).

For (7.3), see Example 10 in [2]. For (7.4), first note that

\[
q_n(x_0, B, n + m) = P(Y_{n+m}^1 \in B \mid Y_n^1 = x_0) = \int_{\mathbb{R}^d} P_{x_0}(Y_m^* \in B - y)Q(dy),
\]

where \( Y_n^* \) under \( P_{x_0} \) is an Ornstein-Uhlenbeck process corresponding to the operator \( \frac{1}{2}\Delta - \gamma x \cdot \nabla \) and starting at \( x_0 \), while \( Q(dy) \) is the distribution of the time \( n + m \) position of the time-inhomogeneous diffusion in the time interval \([n, n + m)\), with infinitesimal generator satisfying

\[
L = \frac{1}{2}(1 - \sigma_k^2)\Delta = \frac{1}{2}2^{-k}\Delta, \quad \text{for } t \in [k, k + 1), \quad k = n, n + 1, \ldots, n + m - 1.
\]

So \( Q(dy) \) is the distribution of

\[
2^{-n}B_0(1) \oplus 2^{-(n+1)}B_1(1) \oplus \ldots \oplus 2^{-(n+m-1)}B_{m-1}(1),
\]

where \( B_0, B_1, \ldots, B_{m-1} \) are standard Brownian motions. Let \( m = n \) (the \( m = n + 1 \) case is similar). Then, by Brownian scaling and elementary computation, \( Q(dy) \) is the distribution of a standard Brownian motion at time \( t_n := \frac{4}{3}\left(1 - 4^{-(n+1)}\right) \cdot 4^{-n} \). Using the explicit Gaussian densities for Brownian motion and the Ornstein-Uhlenbeck process, (7.5) yields (7.4) by a straightforward calculation. \( \square \)

8. **On a possible proof of Conjecture 18**

In this section we provide some discussion for the reader familiar with [2] and interested in a possible way of proving Conjecture 18.

The main difference relative to the attractive case is that, as we have discussed, in that case one does not need the spine change of measure from [2] due to the upper bound \( U_n \leq 1 \). In the repulsive case however, one cannot bypass the spine change of measure. Essentially, an \( h \)-transform transforms the outward Ornstein-Uhlenbeck process into an inward Ornstein-Uhlenbeck process, and in the exponential branching clock setting (and with independent particles), this inward Ornstein-Uhlenbeck process becomes the ‘spine.’ A possible way of proving Conjecture 18 would be to try to adapt the spine change of measure to unit time branching and dependent particles.

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