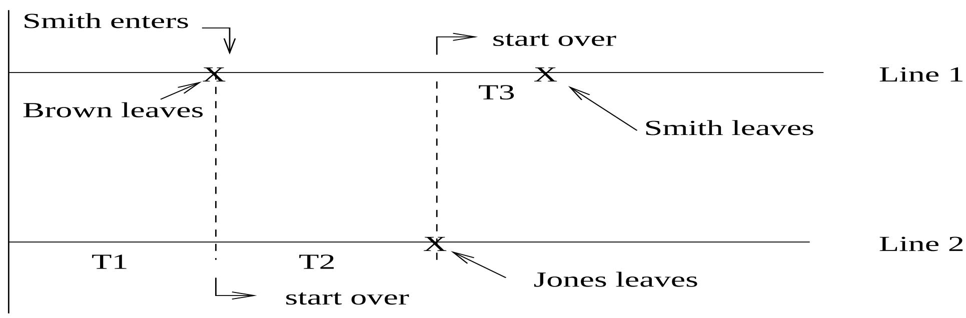
Homework #2 sketch of the solutions

Problem 1. Pb 16, Ch 5

Let T_i the time between the ith and (i-1)th departure, i=1,2,3. That is at time T_1 Smith enters and



takes the line of the first customer to leave. So T_1 is the minimum of two 2 independent exponentials $\mathcal{E}(\lambda_1)$ and $\mathcal{E}(\lambda_2$, and so is $\mathcal{E}(\lambda_1+\lambda_2$. At time T_1, the other customer hasn't left yet, and his remaining time is then again exponential by the memoryless property, but $\mathcal{E}(\lambda_1)$ or $\mathcal{E}(\lambda_2)$ depending of who he is. However, the time T_2 is again $\mathcal{E}(\lambda_1+\lambda_2$ since the min of $\mathcal{E}(\lambda_1)$ and $\mathcal{E}(\lambda_2)$ regardless of the which lines customers are assigned to. However, T_3 is either $\mathcal{E}(\lambda_1)$ or $\mathcal{E}(\lambda_2)$ depending if it involves line 1 or line 2 (not a minimum since there are only one customer left). It will be on the line 1 if line 2 goes first (that's the case in the picture) or in line 2 if line 1 goes first. The probability that it's line 1 is then $\frac{\lambda_1}{\lambda_1+\lambda_2}$ and line 2 with probability $\frac{\lambda_2}{\lambda_1+\lambda_2}$, by the property of the minimum of exponentials. Let T the total time, then

$$\begin{aligned}
E(T) &= E(T_1)+E(T_2)+E(T_3) \\
E(T_1)=E(T_2) &= \frac{1}{\lambda_1+\lambda_2} \\
E(T_3) &= E(T_3|\text{line 1 first})P(\text{line 1 first})+E(T_3|\text{line 2 first})P(\text{line 2 first}) \\
&= \frac{1}{\lambda_2}\times\frac{\lambda_1}{\lambda_1+\lambda_2}+\frac{1}{\lambda_1}\times\frac{\lambda_2}{\lambda_1+\lambda_2} \\
\text{so} & \\
E(T) &= \frac{1}{\lambda_1+\lambda_2}\Big(2+\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda 1}\Big)=\frac{1}{\lambda_1}+\frac{1}{\lambda_2}
\end{aligned}$$

Problem 2. Pb. 25

a) This is the same problem as example 5.8 seen in class but with 3 clerks instead of 2.We follow the same argument and notation than in that example. Denote by $F_i, \imath=1,2,3$ the event that clerk i is available first, and note that in that example the conditioning is really on F_i (for instance there $\{R_1<R_2\}=F_1$) Then

$$P(F_i) = \frac{\lambda_i}{\lambda_1+\lambda_2+\lambda_3}$$

Also,

$$\begin{aligned} E(T|F_i) &= E(R_i + S|F_i) \\ &= E(R_i|F_i) + E(S|F_i) \\ &= \frac{1}{\lambda_1+\lambda_2+\lambda_3} + \frac{1}{\lambda_i} \end{aligned}$$

for the same reason as in the example. Continuing, we have

$$\begin{aligned} E(T) &= \sum_{i=1}^{3} E(T|F_i)P(F_i) \\ &= \sum_{i=1}^{3} \left[\frac{1}{\lambda_1+\lambda_2+\lambda_3} + \frac{1}{\lambda_i}\right] \frac{\lambda_i}{\lambda_1+\lambda_2+\lambda_3} \\ &= \frac{\lambda_1+\lambda_2+\lambda_3}{(\lambda_1+\lambda_2+\lambda_3)^2} + \frac{3}{\lambda_1+\lambda_2+\lambda_3} \\ &= \frac{4}{\lambda_1+\lambda_2+\lambda_3} \end{aligned}$$

b) This time is like problem 1 above (problem 16), but with 3 servers.

<u>Problem 3.</u> Pb. 37

This is just e^{-3t} for $t = 2/60,\ 5/60,\ 10/60,\ 20/60$.