Simulation of Continuous Random Variables

The starting point of all simulations methods is uniform $U(0,1)$ random variable. Those are obtained from (pseudo) random number generator, which we assume are availlable to us. Note that any uniform random variables $U(a,b)$ can be generated from those by setting $X = a + (b-a) * U$. Also Bernoulli random variables (value 1 with probability p, 0 with probability $1-p$) can be generated by setting $X = 1$ if $U < p$, and $X = 0$ if $U > p$.

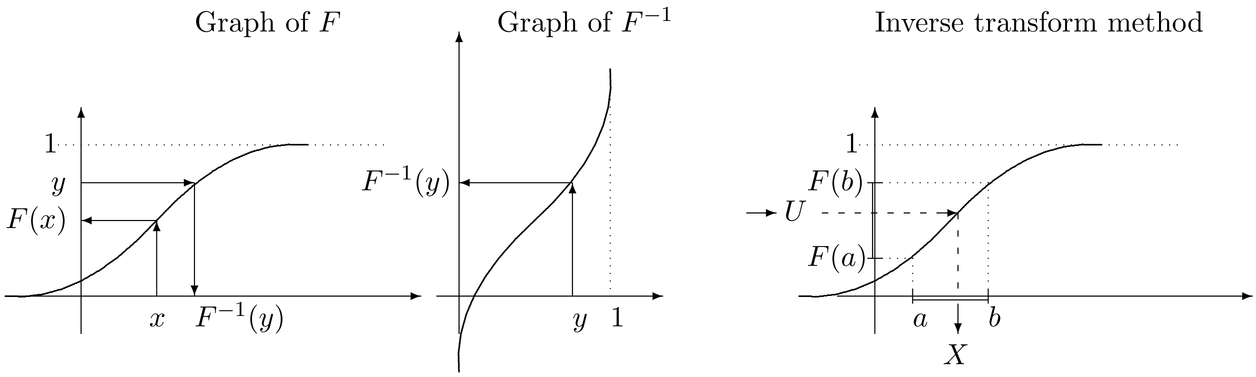
Inverse transform method

Suppose we want to simulate a continuous random variable with C.d.f $F_X(x)$ (here to simplify notation we just write $F(x)$. Since a C.d.f is increasing and here assumed to be continuous, its inverse function F^{-1} is well defined as (see also graphs below). If F is strictly increasing, there is no ambiguity and F^{-1} is given by

$$F^{-1}(y) \quad = \quad x \quad \text{such that } F(x) = y$$

If there are intervals on which F is "flat", then all the x in that interval return the same y. To be well defined, we chose the smallest such x as the value of the inverse.

The inverse transform algorithm just consists of selecting U, a uniform $U(0,1)$ random variable, and set $X = F^{-1}(U)$. Note that if F is constant on some interval, this definition give a correct answer, regardless of the particular choice of the inverse since the probability of a single point for the uniform $U(0,1)$ is 0 anyway.



The inverse transform method can be undertood easily by looking at the graph to the right above. Picking a U uniform on the $y-$axis, which has probability $F(b) - F(a)$ to be between $F(a)$ and $F(b)$ (shaded region that axis), leads to an X, on the $x-$axis, which has the same (and the required) probability to be between a and b (shaded region that axis) To check that the algorithm produce a random variable X with the correct distribution is easy:

$$\begin{aligned} P(a \le X \le b) &= P\big(a \le F^{-1}(U) \le b\big) \\ &= P\big(F(a) \le U \le F(b)\big) \overset{(\text{def})}{=} F(b) - F(a) \end{aligned}$$

The first equality comes from the definition of X. For the second, we just applied the increasing function F to all numbers of the inequality inside the ( ) of the probability, without changing the event, and the fact that $F\big(F^{-1}(y)\big) = y$. The last equality comes from the definition of U as a uniform $U(0,1)$ random variable.

Example The easiest continous random variable to simulate using the inverse transform method (and the most important one for us!), is the exponential distribution. Its C.d.f is given by $F(x) = e^{-\lambda x}$, therefore its inverse is $F^{-1}(y) = \frac{1}{\lambda}\log(y)$.

Rejection Method

Suppose that we want to simulate a continuous random variable X with p.d.f $f_X(x)$ for which the inverse transform method is not applicable. However, there is another random variable Y, with p.d.f f_Y that we can simulate easily (for instance using the inverse transform), and we know that the f_X "is not very different" than f_Y. Roughly, the idea of the rejection method is to simulate values of Y, and accept most of them if it X is more likely than Y, and reject some them for values of X less likely than Y.

Example "Almost uniform". Suppose X has p.d.f

$$f_X(x) = \begin{cases} 2/3 & 0 \le x \le 1 \\ 1/3 & 1 < x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

And we want to simulate it using Y, a $U(0,2)$ random variables. Notice that X is uniform on each interval $[0,1]$ and $(1,2]$, with the first interval twice as likely than the second. Therefore, we could simulate values of Y, accept all the one that fall on $[0,1]$, reject half of those than fall on $(1,2]$

Formally the method described above is given by the following ingredients: The P.d.f of X, $f_X(x)$ we which to simulate, and the p.d.f of Y, $f_Y(x)$ that we know to simulate. We need a condition on f_X and f_Y: There need to exists a constant c such that

$$\begin{aligned} \frac{f_X(x)}{f_Y(x)} &\le c \quad \text{for all } x \\ f_X(x) &\le c f_Y(x) \end{aligned}$$

So it cannot be applied if there is an interval for which f_X is positive, but f_Y is zero. This makes sense: we will never generate the corresponding values of X, since Y can't take those values!

The algorithm is given by:

1) Generate Y (using what ever method availlable)

2) Generate a uniform $U(0,1)$ random number U

3) If $U \le \frac{f_X(x)}{c f_Y(x)}$, accept $X = Y$, otherwise go back to 1) (rejection step)

The graphical interpretation of this method is given below