Homework #4 solutions

Problem 1 (pb. 3, ch. 6)
Define 5 states as follows:

$$\begin{aligned} 0 &= \text{both down, 1 is serviced} \\ 1 &= \text{both down, 2 is serviced} \\ 2 &= \text{1 down, 2 working} \\ 3 &= \text{2 down, 1 working} \\ 4 &= \text{both working} \end{aligned}$$

To find the rates: a jump from one state to another is possible only if it involve a single event (break, service finished).

$$Q = \begin{pmatrix} -\mu & 0 & 0 & \mu & 0 \\ 0 & -\mu & \mu & 0 & 0 \\ \mu_2 & 0 & -(\mu+\mu_2) & 0 & \mu \\ 0 & \mu_1 & 0 & -(\mu+\mu_1) & \mu \\ 0 & 0 & \mu_1 & \mu_2 & -(\mu_1+\mu_2) \end{pmatrix}$$

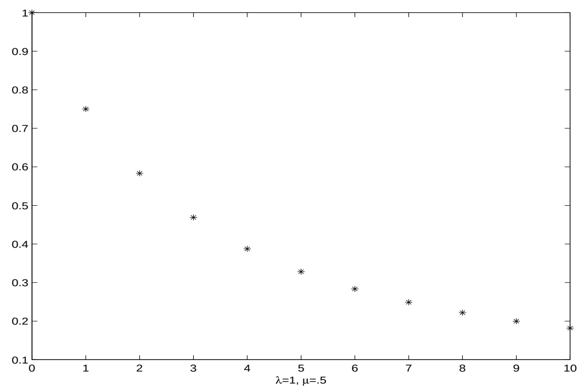
Problem 2 (pb. 6, ch. 6)
Using the formulas for $E(T_i)$ that we have seen in class: $E(T_i) = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} E(T_{i-1})$ we get

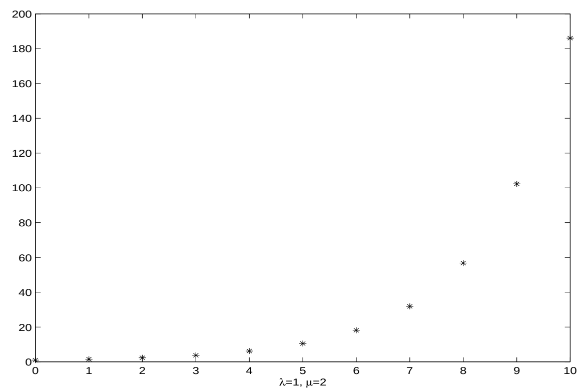
$$\begin{aligned} E(T_0) &= \frac{1}{\lambda} \\ E(T_1) &= \frac{1}{2\lambda} + \frac{\mu}{2\lambda}\frac{1}{\lambda} = \frac{1}{2\lambda}\Big[1 + \frac{\mu}{\lambda}\Big] \\ E(T_2) &= \frac{1}{3\lambda} + \frac{2\mu}{3\lambda}\frac{1}{2\lambda}\big(1 + \frac{\mu}{\lambda}\big) = \frac{1}{3\lambda}\Big[1 + \frac{\mu}{\lambda} + \big(\frac{\mu}{\lambda}\big)^2\Big] \\ E(T_3) &= \frac{1}{4\lambda} + \frac{3\mu}{4\lambda}\frac{1}{3\lambda}\big(1 + \frac{\mu}{\lambda} + \big(\frac{\mu}{\lambda}\big)^2\big) = \frac{1}{4\lambda}\Big[1 + \frac{\mu}{\lambda} + \big(\frac{\mu}{\lambda}\big)^2 + \big(\frac{\mu}{\lambda}\big)^3\Big] \\ E(T_4) &= \frac{1}{5\lambda}\Big[1 + \frac{\mu}{\lambda} + \big(\frac{\mu}{\lambda}\big)^2 + \big(\frac{\mu}{\lambda}\big)^3 + \big(\frac{\mu}{\lambda}\big)^4\Big] \end{aligned}$$

Using the sumation formula for geometric series we can get the general formula

$$\begin{aligned} E(T_i) &= \frac{1}{(i+1)(\lambda-\mu)}\Big[1 - \big(\frac{\mu}{\lambda}\big)^{i+1}\Big] & \mu \neq \lambda \\ E(T_i) &= \frac{1}{\lambda} & \mu = \lambda \end{aligned}$$

Let's see that these answers make sense: If $\lambda = \mu$, by the form of λ_i and μ_i, this leads to a ballanced dynamics, where the process goes up or down is the same probability on each states, and so the expected time to go up doesn't change. On the other hand, if $\lambda > \mu$, should go faster and faster to the state above, so the expected time should decrease with i. Conversely, if $\lambda < \mu$, the process tends to go down and so the time to go up should be larger as i increase. This can be cheked in the next 2 pictures, taht show a plot of the expected time as a function of i in these two cases:





Then The expected time to go from 0 to 4 is $E(T_0) + E(T_1) + E(T_2) + E(T_3)$ and from 1 to 5 is $E(T_1) + E(T_2) + E(T_3) + E(T_4)$

<u>Problem 3</u> (pb. 10, ch. 6)
Define a Markov chain $X(t)$ with state space defined by:
1= machine 1 only ON
2=machine 1 only ON
3= both ON
4= both failled

Then $X(t)$ is indded a Markov chain due to the memoryless property of the exponential. The rate matrix Q is given by

$$Q = \begin{pmatrix} -\mu_2 - \lambda_1 & 0 & \mu_2 & \lambda_1 \\ 0 & -\mu_1 - \lambda_2 & \mu_1 & \lambda_2 \\ \lambda_1 & \lambda_2 & -\lambda_1 - \lambda_2 & 0 \\ \mu_1 & \mu_2 & 0 & -\mu_1 - \mu_2 \end{pmatrix}$$

For example, the rate from 1 to 3 corresponds to 1 ON $\rightarrow$ both ON so 2 is repaired, rate is μ_2. The rate from 1 to 2 is 0 since it's corresponds to 1 only ON to 2 only ON, so 1 needs to fail, 2 be repaired, this involves 2 events so the rate is 0. All the rest can be found the same way.

<u>Problem 4</u> (pb. 13, ch. 6)
Let $X(t) = \#$ customers at time t, so the state space is 0,1,2 and the process is a birth and death with

$$\lambda_0 = \lambda_1 = 3, \quad \mu_1 = \mu_2 = 4$$

Using the equations for the equilibrum probabilities we get

$$P_1 = \frac{3}{4}P_0, \quad P_2 = \left(\frac{3}{4}\right)^2 P_0, \quad P_0 = \frac{1}{1+3/4+(3/4)^2} = \frac{16}{37}$$

so we get:
a) average number of customers in the shop:

$$P_1 + 2P_2 = \frac{30}{37}$$

b) proportion of potential customers is the ratio of customer that find seats over the total number of customers who arrive. A customer gets a seat only if there are not already 2 customers inside, so proportion is:

$$\frac{\lambda(1-P_2)}{\lambda} = 1 - P_2 = \frac{28}{37}$$

c) Now $\mu = 8$. going back to the equations for the limiting probabilities we get:

$$P_0 = \frac{64}{97}, \qquad P_2 = \frac{9}{97}$$

Note that the waiting room has less chance of beeing full now that he working faster. The business he's doing is the number of customer who show up and get a seat so is $\lambda \times (1 - P_2)$ so the additional business due to working twice a s fast is:

$$\lambda \times (1 - P_2^{\text{fast}}) - \lambda \times (1 - P_2^{\text{slow}}) = \lambda \times (1 - \frac{9}{97}) - \lambda \times (1 - \frac{16}{37}) = .45$$

So he can improve by an additional .45 customer/hours.

<u>Problem 5</u> (pb. 15, ch. 6)
Let $X(t)$ be the number of customer in the system at time t. The state space is $0, 1, 2, 3$, and X is a birth and death process with rates:

$$\lambda_0 = \lambda_1 = \lambda_2 = 3, \quad \lambda_3 = 0, \quad \mu_1 = 2, \quad \mu_2 = \mu_3 = 4, \quad \mu_0 = 0$$

The balance equations are

$$P_1 = \frac{3}{2}P_0, \quad P_2 = \frac{9}{8}P_0, \quad P_3 = \frac{27}{32}P_0, \quad P_0 = \frac{1}{1+\frac{3}{2}+\frac{9}{8}+\frac{27}{32}} = \frac{32}{143}$$

As in the previous problem, the fraction that enters the system is

$$\frac{\lambda(1-P_3)}{\lambda} = 1 - P_3 = 1 - \frac{27}{32} \times \frac{32}{143} = \frac{116}{143} \approx .81$$

b) Define $X(t)$ as in part a), it's still a birth and death but now with rates:

$$\lambda_0 = \lambda_1 = \lambda_2 = 3, \quad \lambda_3 = 0, \quad \mu_1 = \mu_2 = \mu_3 = 4, \quad \mu_0 = 0$$

and the balance equations now are

$$P_1 = \frac{3}{4}P_0, \quad P_2 = \left(\frac{3}{4}\right)^2 P_0, \quad P_3 = \left(\frac{3}{4}\right)^3 P_0, \quad P_0 = \frac{1}{1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3} = \frac{64}{175}$$

The fraction that enters the system is

$$1 - P_3 = 1 - \frac{27}{64} \times \frac{64}{175} = \frac{148}{175} \approx .85$$