Law of the Minimal Price

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Law of One Price

“All replicating portfolios of a payoff have the same price!”

Debreu (1959), Sharpe (1964), Lintner (1965),
Merton (1973a, 1973b), Ross (1976), Harrison & Kreps (1979),
Cochrane (2001), . . .

will be, in general, violated under the benchmark approach.
Figure 1: Logarithms of savings bond, fair zero coupon bond and savings account.
Two Asset Market Example

- trading times $t_i = i h, \ h > 0$

- risky asset $S^1$ (S&P500 accumulation index)
  
  asset ratio $A^1_{t_i,h} = \frac{S^1_{t_i+h}}{S^1_{t_i}} \in (0, \infty)$
  
  can reach any finite strictly positive value

- savings account $S^0$ (US savings account)

  asset ratio $A^0_{t_i,h} = \frac{S^0_{t_i+h}}{S^0_{t_i}} > 0$
Figure 2: Discounted S&P500.
Figure 3: $\ln(S&P500$ accumulation index) and $\ln(savings$ account).
• portfolio $S^\delta$

  strict positivity of $S^\delta \iff \pi^0_{\delta, t_i} = \delta^0_{t_i} \frac{S^0_{t_i}}{S^\delta_{t_i}} \in [0, 1]$

  \[ A^\delta_{t_i, h} = \frac{S^\delta_{t_i} + h}{S^\delta_{t_i}} = \pi^0_{\delta, t_i} A^0_{t_i, h} + \left(1 - \pi^0_{\delta, t_i}\right) A^1_{t_i, h} \]

• find best performing strictly positive portfolio
• expected growth

\[
g_{\delta, h}^{t_i} = E_{t_i} \left( \ln \left( A_{t_i, h}^{\delta} \right) \right)
\]

\[
\frac{\partial g_{t_i, h}^{\delta}}{\partial \pi_{\delta, t_i}^{0}} = E_{t_i} \left( \frac{A_{t_i, h}^{0} - A_{t_i, h}^{1}}{A_{t_i, h}^{\delta}} \right)
\]

second derivative negative

\implies \text{ one genuine maximum at } \pi_{\delta, t_i}^{0} \in (-\infty, \infty) \text{ for }

\[
\frac{\partial g_{t_i, h}^{\delta}}{\partial \pi_{\delta, t_i}^{0}} = 0
\]
• Three cases:

(i) \( \pi_{0,t_i}^{\delta} \in [0, 1] \) classical case

(ii) \( \pi_{0,t_i}^{\delta} < 0 \) savings account performs poorly, risk premium

\[ \implies \text{index is best performing portfolio} \]

(iii) \( \pi_{0,t_i}^{\delta} > 1 \) index performs poorly

\[ \implies \text{savings account is best performing portfolio} \]
• classical case (i): \( \pi^0_{\delta,t_i} \in [0, 1] \)

\[ E_{t_i} \left( \frac{A_{t_i,h}}{A_{\delta,t_i,h}} \right) = E_{t_i} \left( \frac{A^0_{t_i,h}}{A^\delta_{t_i,h}} \right) = 1 \]

\[ \implies \] genuine maximum \( g^\delta_{t_i,h} = g^\delta_{t_i,h} \)

\( S^0_{t_i} \) and \( S^1_{t_i} \) are constituents of \( S^\delta_{t_i} \)

\( \frac{S^0_{t_i}}{S^\delta_{t_i}} \) and \( \frac{S^1_{t_i}}{S^\delta_{t_i}} \) are martingales \( \equiv \) fair

\[ \implies \] all benchmarked portfolios are fair

\[ \implies \] Law of One Price holds
• other cases

rewrite first order condition

\[
\frac{\partial g_{t_i, h}^\delta}{\partial \pi_{\delta, t_i}^0} = E_{t_i} \left( Q_{t_i, h} - \pi_{\delta, t_i}^0 \frac{(Q_{t_i, h})^2}{1 + \pi_{\delta, t_i}^0 Q_{t_i, h}} \right) = 0
\]

with

\[
Q_{t, h} = \frac{S^0_{t+h} S^1_{t+h}}{S^0_t S^1_t} - 1
\]

\[
\implies \pi_{\delta, t}^0 = \lim_{h \to 0} \frac{1}{h} E_t(Q_{t, h}) \frac{1}{\lim_{h \to 0} \frac{1}{h} E_t \left( \frac{(Q_{t, h})^2}{1 + \pi_{\delta, t}^0 Q_{t, h}} \right)}
\]
Figure 4: US-benchmarked savings account.
• annualized returns of benchmarked savings account

\[ n = 1052 \]

\[
\hat{\mu} = \frac{1}{n + 1} \sum_{i=0}^{n} \frac{1}{h} Q_{t_i, h} \approx -0.0396
\]

\[
\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=0}^{n} \frac{1}{h} (Q_{t_i, h})^2} \approx 0.19
\]

negative fraction \( \pi_{\delta, t}^0 = \frac{\hat{\mu}}{\hat{\sigma}^2} \approx -1.1 \)

\[ \implies \text{savings account unlikely to be fair} \]
Extreme Maturity Bond

- need model with downward trending \( \frac{S^0_t}{S^\delta_*} \)
  to reflect reality

- assume short rate deterministic

- savings bond

  \[
P^*(t, T) = \frac{S^0_t}{S^0_T}
  \]

- select index \( S^1 \) as numeraire portfolio \( S^\delta_* \)
• benchmarked fair zero coupon bond

\[
\hat{P}(t, T) = \frac{P(t, T)}{S^\delta_t} = E_t \left( \frac{1}{S^\delta_T} \right)
\]

martingale

\[\implies \text{real world pricing formula}\]

\[
P(t, T) = S^\delta_t E_t \left( \frac{1}{S^\delta_T} \right)
\]
- discounted numeraire portfolio

\[
S_{t}^{\delta*} = \frac{S_{t}^{\delta*}}{S_{0}^{0}}
\]

for continuous market generally satisfies SDE

\[
dS_{t}^{\delta*} = \alpha_{t} \, dt + \sqrt{S_{t}^{\delta*}} \, \alpha_{t} \, dW_{t}
\]

is time transformed squared Bessel process
• model the drift of discounted index as

\[ \alpha_t = \alpha \frac{\exp\{\eta t\}}{\eta} \]

\[ \implies \text{Minimal Market Model} \]

MMM, see Pl. & Heath (2006)

• net growth rate

\[ \eta \approx 0.0511 \text{ with } R^2 \text{ of } 0.88 \]
Figure 5: Logarithm of discounted index.
• normalized index

\[ Y_t = \frac{\bar{S}_t^{\delta_*}}{\alpha_t} \]

\[ dY_t = (1 - \eta Y_t) \, dt + \sqrt{Y_t} \, dW_t \]

• quadratic variation of \( \sqrt{Y_t} \)

\[ d\sqrt{Y_t} = \ldots + \frac{1}{2} dW_t \]

\[ V_{t,h} = \sum_{\ell=1}^{i_t} \left( \sqrt{Y_{t\ell}} - \sqrt{Y_{t\ell-1}} \right)^2 \approx \left[ \sqrt{Y} \right]_t = \frac{t}{4} \]

• scaling parameter \( \alpha \approx 0.01429 \) with \( R^2 \) of 0.995
Figure 6: Quadratic Variation of $\sqrt{Y_t}$. 
• **fair zero coupon bond** (MMM)

\[
P(t, T) = P^*(t, T) \left(1 - \exp \left\{ - \frac{2 \eta \bar{S}_t^\delta^*}{\alpha (\exp{\eta T} - \exp{\eta t})} \right\} \right)
\]

• **initial prices** in 1920:

\[
P^*(0, T) = 0.025496
\]

\[
P(0, T) = 0.000795
\]

\[
\frac{P(0, T)}{P^*(0, T)} < 0.0312 \implies 3.12\%
\]
Figure 7: Benchmarked savings bond and benchmarked fair zero coupon bond.
Figure 8: Savings bond, fair zero coupon bond and savings account.
• **self-financing hedge portfolio** hedge ratio

\[
\delta^*_t = \frac{\partial P(t, T)}{\partial \bar{S}^\delta_t^*}
\]

\[
= P^*(t, T) \exp \left\{ \frac{-2 \eta \bar{S}^\delta_t^*}{\alpha (\exp\{\eta T\} - \exp\{\eta t\})} \right\} \times \frac{2 \eta}{\alpha (\exp\{\eta T\} - \exp\{\eta t\})}
\]
Figure 9: Logarithm of zero coupon bond and self-financing hedge portfolio.
Figure 10: Benchmarked P&L.
Figure 11: Fraction invested in the index.
Financial Market

- $j$th primary security account
  \[ S_t^j \]
  \[ j \in \{0, 1, \ldots, d\} \]

- savings account
  \[ S_t^0 \]
  \[ t \geq 0 \]
• strategy

\[ \delta = \{ \delta_t = (\delta_{t}^0, \delta_{t}^1, \ldots, \delta_{t}^d)^\top, \ t \geq 0 \} \]

predictable

• portfolio

\[ S_t^\delta = \sum_{j=0}^{d} \delta_t^j S_t^j \]

• self-financing

\[ dS_t^\delta = \sum_{j=0}^{d} \delta_t^j dS_t^j \]
Numeraire Portfolio

Definition 1  \( S^{\delta*} \in \mathcal{V}_x^+ \) numeraire portfolio if

\[
E_t \left( \frac{\frac{S_{t+h}^{\delta}}{S_{t+h}^{\delta*}}}{\frac{S_{t}^{\delta}}{S_{t}^{\delta*}}} - 1 \right) \leq 0
\]

for all nonnegative \( S^{\delta} \) and \( t, h \in [0, \infty) \).

- \( S^{\delta*} \) “best” performing portfolio

Main Assumption of the Benchmark Approach

Assumption 2  \( \text{There exists a numeraire portfolio } S^{\delta*} \in \mathcal{N}^+ \).
Supermartingale Property

- benchmarked value

\[ \hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta^*}} \]

Corollary 3

For nonnegative \( S^\delta \)

\[ \hat{S}_t^\delta \geq E_t \left( \hat{S}_s^\delta \right) \]

\( 0 \leq t \leq s < \infty \)

nonnegative \( \hat{S}^\delta \) supermartingale
Definition 4  Price is fair if, when benchmarked, forms martingale.

\[ \hat{S}_t^\delta = E_t \left( \hat{S}_s^\delta \right) \]

for \( 0 \leq t \leq s < \infty \).
Lemma 5  The minimal nonnegative supermartingale that reaches a given benchmarked contingent claim is a martingale.

see Pl. & Heath (2006)
Law of the Minimal Price

Theorem 6    If a fair portfolio replicates a nonnegative payoff, then this represents the minimal replicating portfolio.

- least expensive
- minimal hedge
- economically correct price in a competitive market
Figure 12: Benchmarked savings bond and benchmarked fair zero coupon bond.
• claim

\[ H_T \]

\[ E_0 \left( \frac{H_T}{S_{δ^*}_T} \right) < \infty \]

**Corollary 7**

*Minimal price for replicable* \( H_T \) *is given by*

**real world pricing formula**

\[ S_{δ_H}^t = S_{δ^*}^t \ E_t \left( \frac{H_T}{S_{δ^*}^T} \right). \]
- normalized benchmarked savings account

\[ \Lambda_T = \frac{\hat{S}_T^0}{\hat{S}_0^0} \]

\[ 1 = \Lambda_0 \geq E_0(\Lambda_T) \]

- real world pricing formula \[ \implies \]

\[ S_0^{\delta_H} = E_0 \left( \Lambda_T \frac{S_0^0}{S_T^0} H_T \right) \]

\[ \implies \]

\[ S_0^{\delta_H} \leq \frac{E_0 \left( \Lambda_T \frac{S_0^0}{S_T^0} H_T \right)}{E_0(\Lambda_T)} \]

similar for any numeraire
Figure 13: Candidate Radon-Nikodym derivative of hypothetical risk neutral measure of real market.
• **special case when savings account is fair:**

\[ \Lambda_T = \frac{dQ}{dP} \text{ forms martingale; } E_0(\Lambda_T) = 1; \]

equivalent risk neutral probability measure \( Q \) exists;

Bayes’ formula \[ \implies \text{risk neutral pricing formula} \]

\[ S_0^{\delta_H} = E_0^Q \left( \frac{S_0}{S_T^0} H_T \right) \]

Harrison & Kreps (1979), Ingersoll (1987),

Constatinides (1992), Duffie (2001), Cochrane (2001), \ldots

• otherwise “risk neutral price” \( \geq \) real world price
• long term growth rate

\[ g^\delta = \limsup_{t \to \infty} \frac{1}{t} \ln \left( \frac{S^\delta_t}{S^\delta_0} \right) \]

Theorem 8  For \( S^\delta \in \mathcal{V}^+_x \)

\[ g^\delta \leq g^{\delta*} \]

• “pathwise best” in the long run

Karatzas & Shreve (1998), Pl. (2004),

Karatzas & Kardaras (2007)
Definition 9  Nonnegative portfolio $S^\delta$ outperforms systematically $S^{\tilde{\delta}}$ if

(i) $S^\delta_0 = S^{\tilde{\delta}}_0$;

(ii) $P \left( S^\delta_t \geq S^{\tilde{\delta}}_t \right) = 1$

(iii) $P \left( S^\delta_t > S^{\tilde{\delta}}_t \right) > 0$.

• “relative arbitrage”   Fernholz & Karatzas (2005)
Figure 14: Logarithms of fair zero coupon bond and savings account.
Theorem 10: *Numeraire portfolio cannot be outperformed systematically.*
• portfolio ratio

\[ A_{t,h}^\delta = \frac{S_{t+h}^\delta}{S_t^\delta} \]

• expected growth

\[ g_{t,h}^\delta = E_t \left( \ln \left( A_{t,h}^\delta \right) \right) \]

\[ t, h \geq 0 \]
• derivative of expected growth

\[ S^\delta \in \mathcal{V}_x^+ \]

\[ S^\delta \] nonnegative

\[ S^{\delta\varepsilon} \] perturbed \( S^\delta \)

\[ A^{\delta\varepsilon}_{t,h} = \varepsilon A^\delta_{t,h} + (1 - \varepsilon) A^\delta_{t,h} \]

for \( t, h \geq 0, \varepsilon > 0 \)

\[ \left. \frac{\partial g^\delta_{t,h}}{\partial \varepsilon} \right|_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( g^{\delta\varepsilon}_{t,h} - g^\delta_{t,h} \right) \]
Definition 11 \( S^\delta \) growth optimal if
\[
\left. \frac{\partial g_{t,h}^{\delta \varepsilon}}{\partial \varepsilon} \right|_{\varepsilon=0} \leq 0
\]
for all \( t, h \geq 0 \) and nonnegative \( S^\delta \).

- alternative definition to expected log-utility
  
  Kelly (1956)
  Hakansson (1971)
  Merton (1973a)
  Roll (1973)
  Markowitz (1976)

Theorem 12 The numeraire portfolio is growth optimal.
Strong Arbitrage

- market participants can only exploit arbitrage
- limited liability

\[ \implies \text{nonnegative total wealth of each market participant} \]

Definition 13 \( A \) nonnegative \( S^\delta \) is a strong arbitrage if \( S_0^\delta = 0 \) and

\[ P ( S_t^\delta > 0 ) > 0. \]

Pl. (2002)-mathematical arguments

Loewenstein & Willard (2000)-economic arguments
Theorem 14 \hspace{0.5cm} \textit{There is no strong arbitrage.}
there is no pricing based on strong arbitrage

- Delbaen & Schachermayer (1998)
  
  free lunches with vanishing risk (FLVR) may exist

- Loewenstein & Willard (2000)
  
  free snacks & cheap thrills may exist
Figure 15: $P(t, T)$ minus savings account.
Figure 16: $P(t, T)$ minus savings account.
• Under BA candidate risk neutral $Q$ may not be equivalent to $P$ since its Radon-Nikodym derivative may be a strict supermartingale.

• Under BA existence of equivalent risk neutral probability measure is more a mathematical convenience than an economic necessity.
References


